An Introduction to Spectral Methods in Python

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The Method of Lines

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- Suppose we want to solve the linear advection equation:

$$\frac{\partial u}{\partial t} = c\frac{\partial u}{\partial x}$$

- Discretize *space only*

$$\frac{\partial}{\partial t}u_i(t) = c\frac{u_i(t) - u_{i-1}(t)}{\Delta x}$$

- This is a *coupled system* of ODEs:

$$\frac{\partial}{\partial t}\mathbf{u} = D\mathbf{u}, \text{ for some matrix } D$$

- Solve using ODE methods discussed earlier such as RK2

What do we Discretize?

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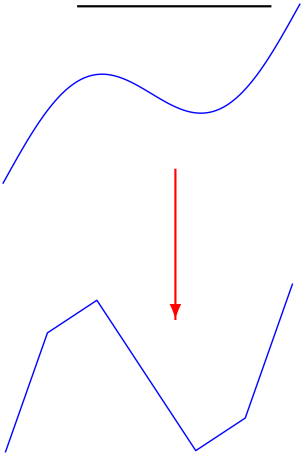
Equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0$$

Function



Discretizing the Solution: The Function-Space Picture

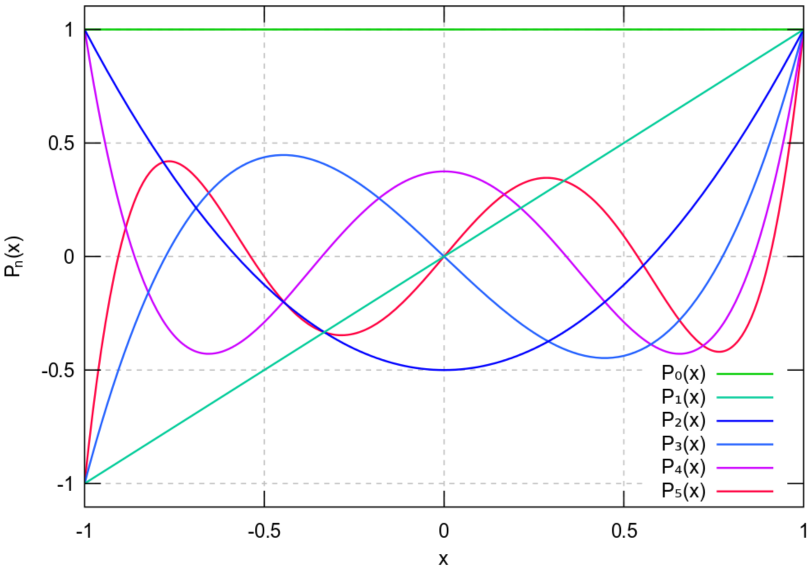
- Represent solution as infinite-dimensional vector in Sobolev space

$$u(x) = \sum_{i=0}^{\infty} u_i \phi_i(x)$$

- To approximate, restrict to finite-dimensional subspace:

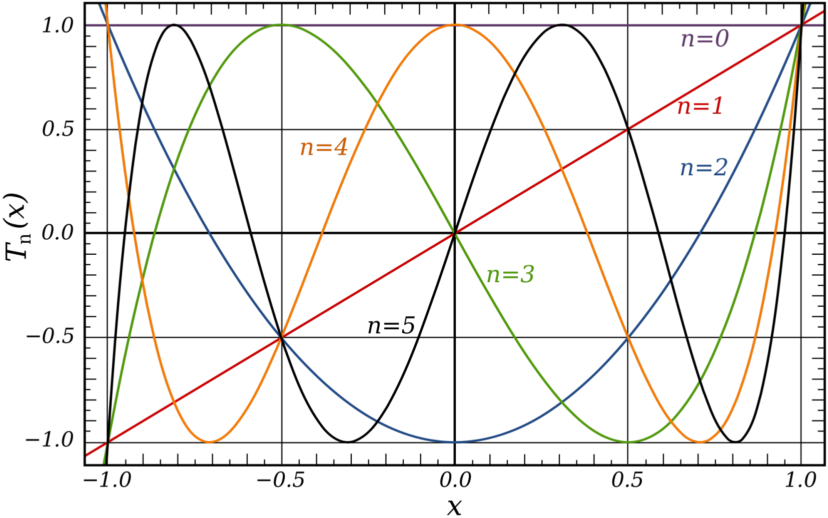
$$u(x) \approx \tilde{u}(x) = \sum_{i=0}^{N} u_i \phi_i(x) \text{ where } N \in \mathbb{N}$$

A Legendre Basis



Source: Wikipedia

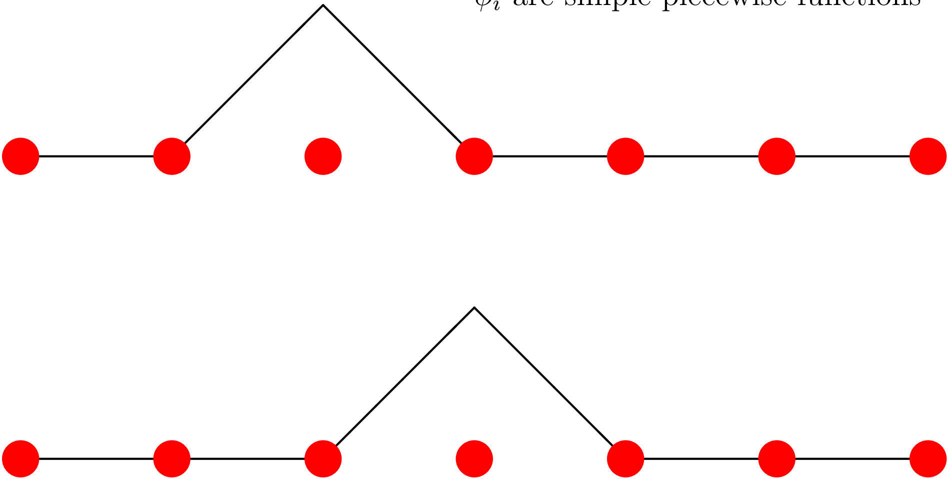
A Chebyshev Basis



Source: Wikipedia

Finite Differences in the Function Space Picture

ϕ_i are simple piecewise functions



Approximating the Equation

- E.g., for

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0,$$

In Finite Differences, demand:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0 \ \forall \ i, n \in \mathbb{Z}$$

In Galerkin Methods, Demand:

$$\int_a^b \left(\frac{\partial \tilde{u}}{\partial t} + c\frac{\partial \tilde{u}}{\partial x}\right) \phi_i w dx = 0 \ \forall \ i \in \mathbb{N}$$

<u>Residual</u>

<u>Error</u>

$$\mathcal{R}_i^n = \frac{u_i^{n+1} - u_i^n}{\Delta t} + c\frac{u_i^n - u_{i-1}^n}{\Delta x}$$

or

$$\epsilon = \|u - \tilde{u}\|$$

$$\mathcal{R}_i^n = \left[\int_a^b \left(\frac{\partial \tilde{u}}{\partial t} + c\frac{\partial \tilde{u}}{\partial x}\right)\phi_i dx\right]_{t=t^n}$$

generically, $\epsilon > 0$

often demand that

$\mathcal{R}_i^n = 0$ identically $\forall\ i, n \in \mathbb{N}$

Exercise

- Solve the equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \text{ with } u(0,x) = \sin\left(e^{-(x-\pi)^2}\right) \text{ for } x \in [0, 2\pi]$$

  with periodic boundary conditions.
- Use a Fourier ansatz:

$$u(t,x) = \sum_{k=0}^{N} [a_k(t)\cos(kx) + b_k(t)\sin(kx)] \text{ for some } N \in \mathbb{N}$$

In no particular order:

- *Numerical Recipes*, by Press, Teukolsky, Vetterling, and Flannery
- *Scientific Computing: An Introductory Survey*, by Heath
- *Introduction to Spectral Methods*, by Grandclement (arXiv:gr-qc/0609020).
- *Spectral Methods: Algorithms, Analysis, and Applications*, by Shen, Tang, and Wang
- *Spectral Methods in MATLAB*, by Trefethen