An Introduction to Spectral Methods in Python

Jonah M. Miller

Perimeter Institute

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- Suppose we want to solve the linear advection equation:

$$\frac{\partial u}{\partial t} = c\frac{\partial u}{\partial x}$$

- Discretize *space only*

$$\frac{\partial}{\partial t}u_i(t) = c\frac{u_i(t) - u_{i-1}(t)}{\Delta x}$$

- This is a *coupled system* of ODEs:

$$\frac{\partial}{\partial t}\mathbf{u} = D\mathbf{u}, \text{ for some matrix } D$$

- Solve using ODE methods discussed earlier such as RK2

What do we Discretize?

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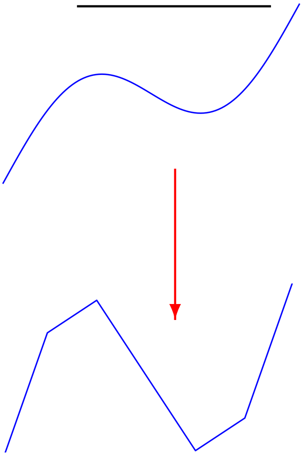
Equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0$$

Function



Discretizing the Solution: The Function-Space Picture

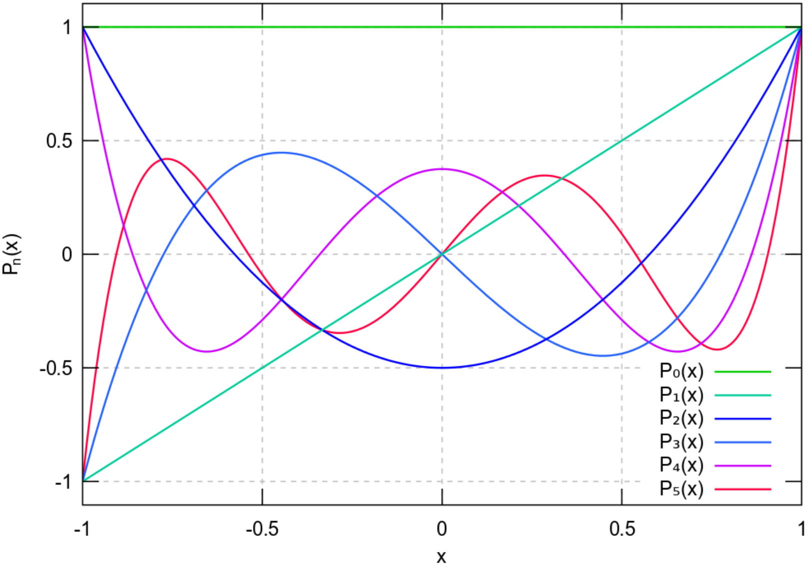
- Represent solution as infinite-dimensional vector in Sobolev space

$$u(x) = \sum_{i=0}^{\infty} u_i \phi_i(x)$$

- To approximate, restrict to finite-dimensional subspace:

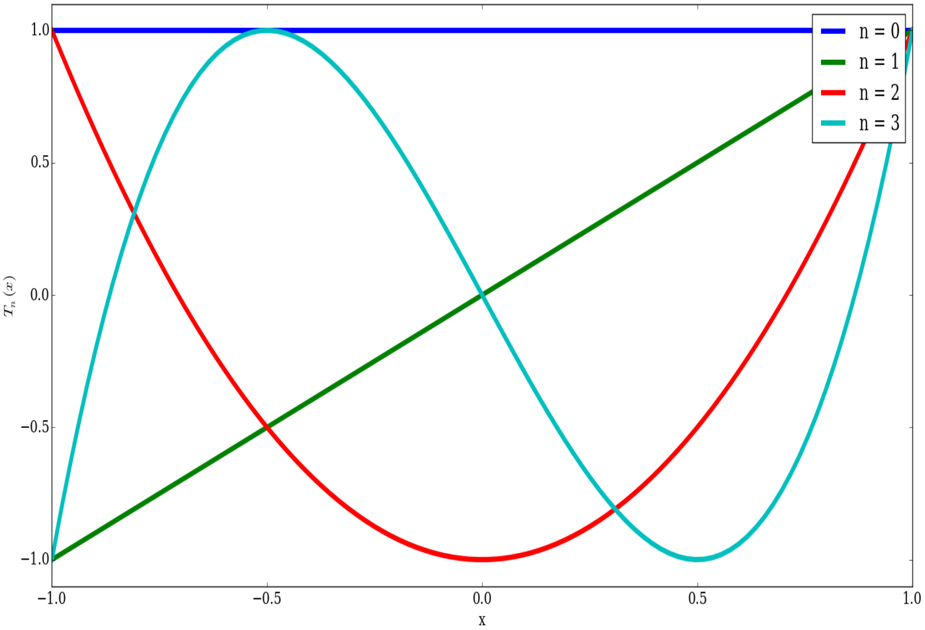
$$u(x) \approx \tilde{u}(x) = \sum_{i=0}^{N} u_i \phi_i(x) \text{ where } N \in \mathbb{N}$$

A Legendre Basis



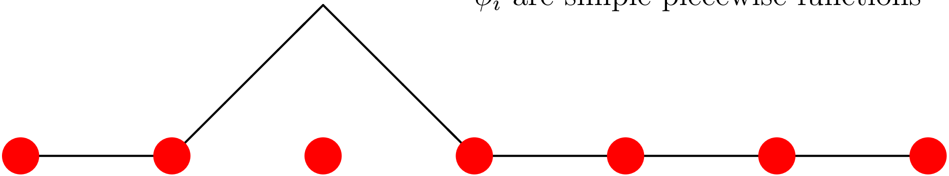
Source: Wikipedia

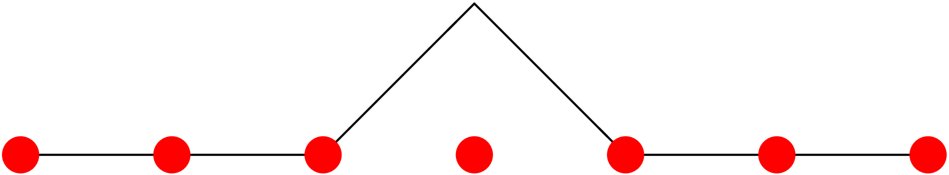
A Chebyshev Basis



Finite Differences in the Function Space Picture

ϕ_i are simple piecewise functions





Approximating the Equation

- E.g., for

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0,$$

In Finite Differences, demand:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0 \; \forall \; i, n \in \mathbb{Z}$$

In Galerkin Methods, Demand:

$$\int_a^b \left(\frac{\partial \tilde{u}}{\partial t} + c\frac{\partial \tilde{u}}{\partial x}\right) \phi_i \; w \; dx = 0 \; \forall \; i \in \mathbb{N}$$

Measuring Goodness

<u>Error</u>

$$\epsilon(t,x) = u(t,x) - \tilde{u}(t,x),$$

where $L[u] = 0$

generically, $\epsilon(x) > 0$

<u>Residual</u>

$$R(t,x) = \tilde{L}[\tilde{u}]$$

often demand that

$\mathcal{R}(t,x) = 0$ identically

Error

$$\epsilon_i^n = u(t,x)\Big|_{x=x^i}^{t=t^n} - u_i^{n+1},$$

or

$$\epsilon_i^n = \left[\int_a^b \left(u(t,x) - \tilde{u}(t,x)\right)\phi_i dx\right]_{t=t^n}$$

generically, $\epsilon_i^n > 0$

Residual

$$\mathcal{R}_i^n = \frac{u_i^{n+1} - u_i^n}{\Delta t} + c\frac{u_i^n - u_{i-1}^n}{\Delta x}$$

or

$$\mathcal{R}_i^n = \left[\int_a^b \left(\frac{\partial \tilde{u}}{\partial t} + c\frac{\partial \tilde{u}}{\partial x}\right)\phi_i dx\right]_{t=t^n}$$

often demand that

$$\mathcal{R}_i^n = 0 \text{ identically } \forall\ i, n \in \mathbb{N}$$

Example: The Galerkin Scheme

Suppose we want to solve

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} F[u] = 0$$

where F is an arbitrary (*nonlinear*) functional of u using the ansatz:

$$u(t,x) \approx \tilde{u} = \sum_{i=1}^{N} u_i(t)\phi_i(x) \;\; N \in \mathbb{N}$$

and

$$F[u] \approx \tilde{F} = \sum_{i=1}^{N} F_i[u]\phi_i(x)$$

Exercise

- Solve the equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \text{ with } u(0,x) = \sin\left(e^{-(x-\pi)^2}\right) \text{ for } x \in [-\pi, \pi]$$

  with periodic boundary conditions.
- Use a Fourier ansatz:

$$u(t,x) = a_0 + \sum_{k=1}^{N} \left[a_k(t)\cos(kx) + b_k(t)\sin(kx)\right] \text{ for some } N \in \mathbb{N}$$

- Assume the following inner product:

$$\langle a, b\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} a(x)b(x)dx$$

In no particular order:

- *Numerical Recipes*, by Press, Teukolsky, Vetterling, and Flannery
- *Scientific Computing: An Introductory Survey*, by Heath
- *Introduction to Spectral Methods*, by Grandclement (arXiv:gr-qc/0609020).
- *Spectral Methods in MATLAB*, by Trefethen