Interval estimation for the generalized inverted exponential distribution under progressive first failure censoring

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Abstract

In this paper, we consider the inferential procedures for the generalized inverted exponential distribution under progressive first failure censoring. The exact confidence interval for the scale parameter is derived. The generalized confidence intervals (GCIs) for the shape parameter and reliability characteristics of interest are explored. Then the proposed procedure is extended to the prediction interval for the future measurement. The GCIs for the reliability of the stress-strength model are discussed under both equal scale and unequal scale scenarios. Extensive simulations are used to demonstrate the performance of the proposed GCIs and prediction interval. Finally, an example is used to illustrate the proposed methods.

Keywords: progressive first failure censoring; generalized confidence interval; prediction interval; stress-strength model.

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1 Introduction

Censoring schemes are very common in life tests. The type-I and type-II censoring schemes are the most common censoring schemes, but the conventional type-I and type-II censoring schemes do not have the flexibility of allowing the removal of units at points other than the terminal point of the experiment. The progressive censoring scheme, which includes conventional type-I and type-II censoring as special cases, is of importance in the field of reliability and life-testing experiments. It allows the experimenter to remove units from a life test at various stages during the experiment. A book dedicated to progressive censoring was published by Balakrishnan and Aggarwala [?]. Besides the progressive censoring scheme, there is another censoring scheme called first failure

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censoring scheme. Under such censoring scheme, experimenters group the test units into several sets, then perform all the test units simultaneously until the occurrence of the first failure in each set, see Johnson [**?**] and Balasooriya et al. [**?**].

In order to combine the advantages of progressive censoring and first failure censoring, Wu and Kus [**?**] proposed a mixed censoring called progressive first failure censoring. The progressive first failure censoring scheme can be implemented as follows: suppose that n independent groups with k items in each group are put on a life test simultaneously at the initial time $t_0 = 0$. At the first failure time $X_{1:m:n:k}$, R_1 groups are randomly selected and removed along with the group which contains the first failure item. At the second failure time $X_{2:m:n:k}$, R_2 groups are randomly selected and removed along with the group which contains the second failure item and so on, until the mth failure time $X_{m:m:n:k}$ observed in the remaining groups, and then all the remaining R_{m+1} groups are removed. The observed failure times, $X_{1:m:n:k} < X_{2:m:n:k} < \cdots < X_{m:m:n:k}$, are called progressively first failure-censored sample with the progressive censoring scheme $R = (R_1, R_2, \ldots, R_m)$. Here, $n = m + R_1 + R_2 + \cdots + R_m$, and (m, n, k, R) must be pre-specified. It is obvious that if $R_1 = R_2 = \ldots = R_m = 0$, the progressively first-failure censoring reduces to first failure censoring scheme; if $k = 1$, the scheme becomes progressively type II censoring scheme; if $k = 1$ and $R_1 = R_2 = \ldots = R_{m-1} = 0, R_m = n - m$, the scheme reduces to type II censoring scheme.

The generalized inverted exponential distribution (GIED) was proposed by Abouammoh and Alshingiti [**?**]. The probability density function (PDF) and cumulative distribution function of the GIED are given by

$$f(x) = \frac{\alpha\lambda}{x^2} e^{-\lambda/x}(1 - e^{-\lambda/x})^{\alpha-1}, \quad x > 0, \tag{1}$$

and

$$F(x) = 1 - (1 - e^{-\lambda/x})^{\alpha}, \quad x > 0, \tag{2}$$

respectively. Here, $\alpha > 0$ and $\lambda > 0$ are the shape and scale parameters, respectively. The GIED (2) will be denoted by GIED(α, λ). It has been observed that the GIED can be quite effective to analyze skewed data set and is a good alternative to other skewed distribution, such as the Weibull distribution, the gamma distribution, the generalized exponential distribution, etc. Abouammoh and Alshingiti [**?**] found that the GIED has good statistical and reliability properties. They obtained maximum likelihood estimators (MLEs) and least square estimators for the GIED based on a complete sample. For the complete sample, Dey and Dey [**?**] compared the performance of different estimators by Monte Carlo simulation. Kishna and Kumar [?] discussed the point estimation for the GIED based on a progressively type-II censored sample. Dey and Pradhan [?] derived the MLEs and Bayes estimates for the GIED under hybrid censoring scheme. Singh et al. [?] proposed a two-stage group acceptance sampling plan for the GIED under truncated life test. Dey et al. [?] considered Bayesian inference for the GIED under type-II censoring scheme. Ahmed [?] obtained the MLEs, Bayes estimates and prediction interval for the GIED under progressive first failure censoring. Krishna et al. [?] discussed point estimation and interval estimation for the GIED stress-strength model based on the progressively first failure censored samples.

Considering the significant role of stress-strength model in reliability engineering, we discuss the stress-strength reliability $\delta = P(Y < X)$ when the strength X of a system and the stress Y have the GIEDs. The stress-strength model was first introduced by Birnbaum [?]. Henceforward this model has been used in many areas such as mechanical, aerospace system and reliability engineering. Kotz et al. [?] provided some excellent information on past and current developments in the area.

The rest of this paper is organized as follows. In Section 2, we derive the exact interval for the scale parameter. we also obtain the generalized confidence intervals (GCIs) for the shape parameter, the pth quantile and reliability function of the GIED. Moreover, the generalized prediction interval (GPI) for the future measurement is derived. The performance of the proposed GCIs and GPI is assessed by Monte Carlo simulation. In Section 3, we consider the GIED stress-strength model and propose the generalized interval estimation procedures for the reliability of the stress-strength model. A simulation study is provided to evaluate the performance of the proposed GCIs. Finally, a real example is used to illustrate the proposed methods.

2 Estimation of the GIED

In this section, the exact confidence interval for the scale parameter λ of the GIED(α, λ) is derived in the presence of progressively first failure censoring scheme. Then the GCIs for the shape parameter α and other quantities of the GIED, such as its quantiles and reliability function are introduced. The prediction interval for the new future measure is also obtained.

In order to derive interval estimation of the GIED, the following results are needed. Lemmas

2 and 3 can be found in Yu et al. [?], respectively.

Lemma 1 *Suppose that* $\boldsymbol{X} = (X^R_{1:m:n:k}, X^R_{2:m:n:k}, \dots, X^R_{m:m:n:k})$ *is a progressively first failure censored sample from the standard exponential distribution* $EXP(1)$ *with censoring scheme* $R = (R_1, \dots, R_m)$*. Let* $W_1 = knX^R_{1:m:n:k}$*,* $W_i = k[n - \sum_{j=1}^{i-1}(R_i+1)](X^R_{i:m:n:k} - X^R_{i-1:m:n:k}), i = 2, ..., m.$ *Then* $W_1, W_2, \dots, W_m$ *are independent standard exponential random variates.*

Proof. From Wu and Kus [?], we have that the joint PDF of the sample $\mathbf{X}$ is

$$\begin{aligned} f(\mathbf{x}) &= Ck^m \prod_{j=1}^{m} f(x^R_{j:m:n:k})[1 - F(x^R_{j:m:n:k})]^{k(R_j+1)-1} \qquad (3) \\ &= Ck^m \exp\left(\sum_{i=1}^{m} k(R_i+1)x^R_{i:m:n:k}\right),\ x^R_{m:m:n:k} > \cdots > x^R_{1:m:n:k} > 0, \end{aligned}$$

where $C = n\prod_{i=2}^{m}\left[n - \sum_{j=1}^{i-1}(R_j+1)\right]$.

Notice that the Jacobian of the transformation $W_1, W_2, \dots, W_m$ is Ck^m, the joint PDF of $W_1, W_2, \dots, W_m$ is then given by

$$f_{W_1,\dots,W_m}(w_1, w_2, \dots, w_m) = \exp(-\sum_{i=1}^{m} w_i),\ w_1 > 0, \dots, w_m > 0.$$

Therefore, $W_1, W_2, \dots, W_m$ are independent standard exponential random variates.

Remark 1: It can be observed from the equation (3) that the progressive first failure censoring with the progressive censoring scheme $R = (R_1, R_2, \dots, R_m)$ is equivalent to the following progressive type II censoring: nk units are placed on test at time $t_0 = 0$. When the first observed failure occurs, $kR_1 + k - 1$ surviving units are removed from the test at random. Then, immediately following the second observed failure, $kR_2 + k - 1$ surviving units are removed from the test at random. This process continues until, at the time of the m^{th} observed failure, the remaining $kR_m + k - 1$ units are all removed from the experiment. In addition, Wu, Kus [?] and Wu, Huang [?] found that $\mathbf{X}$ can be viewed as a progressively type II censored sample from a population with distribution function $1 - (1 - F(x))^k$.

Lemma 2 *Suppose that* $Z_1, ..., Z_n$ *is a random sample from the exponential distribution with mean* θ*. Let* $S_i = \sum_{j=1}^{i} Z_j$*,* $i = 1, ..., n$*, and* $T = 2\sum_{i=1}^{n-1}\log(S_n/S_i)$*. Then (1)* T *and* S_n *are independent; (2)* $T \sim \chi^2(2n-2)$ *and* $2S_n/\theta \sim \chi^2(2n)$*.*

Lemma 3 *Let* $g(\lambda) = \log(1 - e^{-b\lambda})/\log(1 - e^{-a\lambda})$*, where* $b > a > 0$ *are constants. Then* $g(\lambda)$ *is strictly decreasing on* $(0, +\infty)$*.*

2.1 Interval estimation

Let $\mathbf{X} = (X^R_{1:m:n:k}, X^R_{2:m:n:k}, \ldots, X^R_{m:m:n:k})$ be a progressively first failure censored sample from the GIED(α, λ) with censoring scheme $R = (R_1, \ldots, R_m)$, and $\mathbf{x} = (x^R_{1:m:n:k}, x^R_{2:m:n:k}, \ldots, x^R_{m:m:n:k})$ be the observed value of $\mathbf{X}$. Then

$$V^R_{i:m:n:k} = -\log(1 - F(X^R_{i:m:n:k})) = -\alpha \log\left(1 - e^{-\lambda/x^R_{i:m:n:k}}\right),\ i = 1, 2, \ldots, m$$

is a progressively first failure censored sample from the standard exponential distribution.

Let

$$W_1 = knV_{1:m:n:k}, \quad W_i = k[n - \sum_{j=1}^{i-1}(R_j + 1)](V^R_{i:m:n:k} - V^R_{i-1:m:n:k}), i = 2, \ldots, m. \tag{4}$$

Then we have from Lemma 1 that $W_1, \ldots, W_m$ are independent standard exponential random variates. Further, let

$$S_i = \sum_{j=1}^{i} W_j = \sum_{j=1}^{i} k(R_j + 1)\log(1 - e^{-\lambda/x^R_{j:m:n:k}}) + k[n - \sum_{j=1}^{i}(R_j + 1)]\log(1 - e^{-\lambda/x^R_{i:m:n:k}}).$$

Then from Lemma 2, we have

$$T(\lambda) = 2\sum_{i=1}^{m-1} \log \frac{S_m}{S_i} \sim \chi^2(2m - 2). \tag{5}$$

Let $Q(j, i) = \ln\left(1 - e^{-\lambda/x^R_{j:m:n:k}}\right)/\ln\left(1 - e^{-\lambda/x^R_{i:m:m:k}}\right)$. Notice that

$$\frac{S_m}{S_i} = 1 + \frac{\sum_{j=i+1}^{m}(R_j + 1)Q_{(j,i)} - [n - \sum_{j=1}^{i}(R_j + 1)]}{\sum_{j=1}^{i}(R_j + 1)Q_{(j,i)} + n - \sum_{j=1}^{i}(R_j + 1)}, \tag{6}$$

we have from Lemma 3 that $T(\lambda)$ is strictly increasing on $(0, +\infty)$, and

$$\lim_{\lambda \to 0+} T(\lambda) = 0, \lim_{\lambda \to +\infty} T(\lambda) = +\infty.$$

Therefore, the $1 - \gamma$ exact confidence interval (CI) for λ is given by

$$\left[T^{-1}\{\chi^2_{\gamma/2}(2m - 2)\}, \quad T^{-1}\{\chi^2_{1-\gamma/2}(2m - 2)\}\right],$$

where $\chi^2_\gamma(v)$ is the γ percentile of the $\chi^2(v)$, and $T^{-1}(t)$ is the solution of the equation $T(\lambda) = t(> 0)$.

Now we derive the GCI for α. Let $g(T_0, \mathbf{X})$ be the unique solution of the equation $T(\lambda) = T_0$, where $T_0 \sim \chi^2(2m-2)$. Since $V = -2\alpha S_m \sim \chi^2(2m)$, we get the equation $\alpha = -V/(2S_m)$. Based on the substitution method given by Weerahandi [?], we substitute $g(T_0, \mathbf{X})$ for λ in the expression of α and obtain the following generalized pivotal quantity for α:

$$Y_1 = -\frac{V}{2k\sum_{i=1}^{m}(R_i+1)\log(1-e^{-g(T_0,\mathbf{x})/x^R_{i:m:n:k}})} \tag{7}$$

If $Y_{1,\gamma}$ denotes the γ percentile of Y_1, then $Y_{1,\gamma}$ and $Y_{1,1-\gamma}$ are the $1-\gamma$ generalized lower and upper confidence limits for α, respectively.

Notice that the pth quantile $(0 < p < 1)$ and reliability function of the GIED are given by $x_p = -\lambda/\log\left[1-(1-p)^{1/\alpha}\right]$ and $R(x_0) = \left[1-\exp\left(-\lambda/x_0\right)\right]^{\alpha}$, along the same lines as the derivation of Y_1 for λ, the generalized pivotal quantities Y_2 and Y_3 for x_p and $R(x_0)$ are given by

$$\begin{aligned} Y_2 &= -\frac{g(T_0,\mathbf{x})}{\log\left[1-(1-p)^{1/Y_1}\right]}, && (8) \\ Y_3 &= \left(1-e^{-g(T_0,\mathbf{x})/x_0}\right)^{Y_1}, && (9) \end{aligned}$$

respectively.

Let $Y_{2,\gamma}$ and $Y_{3,\gamma}$ be the γ percentiles of Y_2 and Y_3. Then $Y_{2,\gamma}$ and $Y_{3,\gamma}$ are the $1-\gamma$ lower confidence limits for x_p and $R(x_0)$, respectively. The percentiles of Y_1, Y_2 and Y_3 can be obtained by the following Monte Carlo algorithm.

Algorithm 1: The percentiles for Y_1, Y_2 and Y_3.

(1) For a given data set $(m, n, k, R, \mathbf{x})$, generate $T_0 \sim \chi^2(2m-2)$ and $V \sim \chi^2(2m)$, independently. Using these values, compute $g(T_0, \mathbf{x})$ from the equation $T(\lambda) = T_0$.

(2) Compute values of Y_1, Y_2 and Y_3 using (7)–(9).

(3) Repeat the steps 1-2 $B(\geq 10000)$ times. Then there are the B values of $Y_i, i = 1, 2, 3$.

(4) Arrange all Y_i values in ascending order: $Y_{i,(1)} < Y_{i,(2)} < ... < Y_{i,(B)}$. Then $Y_{i,\gamma}$ can be estimated by $Y_{i,(B\gamma)}$.

2.2 Prediction interval

In this subsection, we extend the proposed generalized confidence interval procedure to construct prediction interval for a single future measurement.

Suppose that X_{new} is a future measurement from the same GIED. Notice that $F(X_{new}) \sim U(0,1)$, then for a given $U \sim U(0,1)$, $X_{new} = -\lambda/\log(1-U^{1/\alpha})$. Therefore, using the substitution method given by Weerahandi [?], we obtain the following generalized prediction quantity:

$$Y_4 = -\frac{g(T_0, \mathbf{x})}{\log(1 - U^{1/Y_1})}. \tag{10}$$

Let $Y_{4,\gamma}$ denote the γ percentile of Y_4. Then $[Y_{4,\gamma/2},\ Y_{4,1-\gamma/2}]$ is a $1-\gamma$ generalized prediction interval (GPI) for X_{new}. Similar to the shape parameter α case, the percentiles of Y_4 can be obtained by the Monte Carlo simulation algorithm.

Algorithm 2: The percentiles for X_{new}.

(1) For a given data set $(m, n, k, R, \mathbf{x})$, generate $T_0 \sim \chi^2(2m-2)$ and $V \sim \chi^2(2m)$, independently. Using these values, compute $g(T_0, \mathbf{x})$ from the equation $T(\lambda) = T_0$.

(2) Compute value of Y_1 using (7).

(3) Generate $U \sim U(0,1)$. Then compute Y_4 based on the equation (10).

(4) Repeat the steps 1-3 $B(\geq 10000)$ times. Then there are the B values of Y_4.

(5) Arrange all Y_4 values in ascending order: $Y_{4,(1)} < Y_{4,(2)} < ... < Y_{4,(B)}$. Then $Y_{4,\gamma}$ can be estimated by $Y_{4,(B\gamma)}$.

2.3 Simulation study

In order to evaluate the performance of the proposed interval estimation and prediction interval, a Monte Carlo simulation is conducted to study the coverage probabilities and the interval lengths of the proposed GCIs and GPIs.

Because λ is the scale parameter, without loss of generality, we take $\lambda = 1$ in our simulation study. The shape parameter considers 2, 4. The confidence level considers 0.9 and 0.95. Notice that $1-(1-F(x))^k = 1-[1-\exp(-\lambda/x)]^{k\alpha}$, then we have from Remark that $\mathbf{X}$ is a progressively type II censored sample from the generalized inverted exponential distribution GIED$(k\alpha, \lambda)$. Hence, without loss of generality, we take $k = 1$. For different progressively first failure censoring schemes, we generated progressively first failure censored samples from the GIED by using the equation (4). Table 1 reports the simulation results of the CI for the parameter λ. Tables 2 and 3 provide the simulation results of the proposed GCIs. All the results are based on 5,000 replications with $B = 10,000$.

Table 1: The coverage percentages and the average lengths (in parentheses) of the CI for λ with nominal levels 0.9, 0.95, based on 5000 replications

		$\alpha=2$		$\alpha=4$		$\alpha=6$	
(m,n)	R	0.9	0.95	0.9	0.95	0.9	0.95
(10,15)	(0,...,0,5)	0.8936	0.9474	0.9070	0.9532	0.9024	0.9526
		(1.3141)	(1.5655)	(1.2105)	(1.4438)	(1.1807)	(1.4092)
	(0,1,...,0,1)	0.9014	0.9526	0.8972	0.9492	0.8950	0.9494
		(1.2189)	(1.4508)	(1.1110)	(1.3235)	(1.0776)	(1.2843)
	(5,0,...,0)	0.8994	0.9488	0.8944	0.9500	0.8986	0.9500
		(1.1606)	(1.3846)	(1.0405)	(1.2417)	(0.9925)	(1.1847)
(10,20)	(0,...,0,10)	0.9076	0.9510	0.9002	0.9512	0.9086	0.9550
		(1.3038)	(1.5545)	(1.2250)	(1.4624)	(1.1924)	(1.4243)
	(1,...,1)	0.8936	0.9468	0.9018	0.9516	0.9018	0.9532
		(1.1567)	(1.3772)	(1.0780)	(1.2846)	(1.0415)	(1.2416)
	(10,0,...,0)	0.8996	0.9486	0.9020	0.9546	0.9034	0.9490
		(1.0771)	(1.2870)	(0.9750)	(1.1650)	(0.9333)	(1.1153)
(15,20)	(0,...,0,5)	0.8998	0.9524	0.8918	0.9498	0.9024	0.9518
		(1.0230)	(1.2184)	(0.9345)	(1.1140)	(0.9091)	(1.0841)
	(1,1,0,...,0,1,1,1)	0.9016	0.9510	0.8992	0.9500	0.9086	0.9560
		(0.9920)	(1.1817)	(0.8977)	(1.0700)	(0.8591)	(1.0243)
	(5,0,...,0)	0.8938	0.9464	0.9012	0.9520	0.8984	0.9466
		(0.9551)	(1.1383)	(0.8481)	(1.0109)	(0.8124)	(0.9685)

Table 2: The coverage percentages and the average lengths (in parentheses) of the GCIs for $1/\alpha, x_{0.1}, R(1.2)$ and X_{new} with nominal levels 0.9, 0.95 when $\alpha=2$, based on 5000 replications

		$1/\alpha$		$x_{0.1}$		$R(1.2)$		X_{new}	
(m,n)	R	0.9	0.95	0.9	0.95	0.9	0.95	0.9	0.95
(10,15)	(0,...,0,5)	0.9070	0.9536	0.9016	0.9522	0.9004	0.9530	0.8980	0.9508
		(1.2420)	(1.5880)	(0.2558)	(0.3057)	(0.3679)	(0.4316)	(10.2636)	(36.2443)
	(0,1,...,0,1)	0.9070	0.9534	0.9008	0.9516	0.8990	0.9524	0.8992	0.9498
		(1.1379)	(1.4537)	(0.2551)	(0.3055)	(0.3612)	(0.4244)	(8.8577)	(26.7809)
	(5,0,...,0)	0.9066	0.9530	0.9020	0.9522	0.8998	0.9482	0.9006	0.9500
		(0.9827)	(1.2487)	(0.2701)	(0.3234)	(0.3573)	(0.4201)	(7.2783)	(17.9154)
(10,20)	(0,...,0,10)	0.9062	0.9520	0.9020	0.9502	0.9024	0.9532	0.8994	0.9512
		(1.4231)	(1.8288)	(0.2244)	(0.2690)	(0.3948)	(0.4608)	(15.8147)	(84.5349)
	(1,...,1)	0.9058	0.9542	0.9022	0.9506	0.9014	0.9532	0.8988	0.9508
		(1.2214)	(1.5667)	(0.2263)	(0.2718)	(0.3734)	(0.4378)	(10.6488)	(38.3008)
	(10,0,...,0)	0.9070	0.9534	0.9030	0.9514	0.8978	0.9488	0.8992	0.9496
		(0.9661)	(1.2274)	(0.2507)	(0.3005)	(0.3569)	(0.4197)	(7.3278)	(17.9753)
(15,20)	(0,...,0,5)	0.8980	0.9474	0.9064	0.9544	0.8956	0.9468	0.8982	0.9468
		(0.8702)	(1.0843)	(0.2165)	(0.2583)	(0.3012)	(0.3553)	(6.1658)	(13.2799)
	(1,1,0,...,0,1,1,1)	0.8978	0.9484	0.9064	0.9548	0.8976	0.9464	0.8988	0.9464
		(0.8170)	(1.0168)	(0.2186)	(0.2611)	(0.2984)	(0.3522)	(5.8693)	(12.1136)
	(5,0,...,0)	0.8974	0.9482	0.9062	0.9552	0.8962	0.9464	0.8998	0.9468
		(0.7379)	(0.9163)	(0.2265)	(0.2705)	(0.2984)	(0.3522)	(5.5045)	(10.7415)

It is observed from Tables 1, 2 and 3 that the coverage percentages of these intervals are quite close to the nominal coverage probabilities, even for small sample sizes. The simulation results also show that for a fixed n, the average interval lengths decrease as m increases, as one would expect. These findings show that proposed CI, GCIs and GPI work well.

Table 3: The coverage percentages and the average lengths (in parentheses) of the GCIs for $1/\alpha, x_{0.1}, R(1.2)$ and X_{new} with nominal levels 0.9, 0.95 when $\alpha = 4$, based on 5000 replications

		$1/\alpha$		$x_{0.1}$		$R(1.2)$		X_{new}	
(m,n)	R	0.9	0.95	0.9	0.95	0.9	0.95	0.9	0.95
(10,15)	(0,...,0,5)	0.9066	0.9512	0.9008	0.9516	0.9066	0.9526	0.8992	0.9508
		(0.7847)	(1.0154)	(0.1689)	(0.2019)	(0.2879)	(0.3449)	(2.3643)	(4.7300)
	(0,1,...,0,1)	0.9076	0.9530	0.9004	0.9518	0.9046	0.9520	0.8998	0.9504
		(0.7043)	(0.9102)	(0.1678)	(0.2009)	(0.2760)	(0.3317)	(2.1563)	(3.9730)
	(5,0,...,0)	0.9064	0.9540	0.9020	0.9510	0.9036	0.9544	0.8994	0.9498
		(0.5887)	(0.7550)	(0.1761)	(0.2106)	(0.2573)	(0.3102)	(1.9169)	(3.1828)
(10,20)	(0,...,0,10)	0.9062	0.9502	0.9012	0.9498	0.9094	0.9532	0.8990	0.9516
		(0.9136)	(1.1905)	(0.1495)	(0.1793)	(0.3182)	(0.3789)	(3.0073)	(7.4310)
	(1,...,1)	0.9062	0.9518	0.9016	0.9500	0.9058	0.9532	0.8992	0.9522
		(0.7612)	(0.9889)	(0.1500)	(0.1800)	(0.2936)	(0.3520)	(2.3888)	(4.7631)
	(10,0,...,0)	0.9064	0.9534	0.9030	0.9510	0.9036	0.9548	0.8988	0.9498
		(0.5739)	(0.7355)	(0.1644)	(0.1966)	(0.2593	(0.3126)	(1.9202)	(3.1794)
(15,20)	(0,...,0,5)	0.9004	0.9478	0.9058	0.9544	0.8932	0.9490	0.8974	0.9466
		(0.5363)	(0.6737)	(0.1429)	(0.1706)	(0.2269)	(0.2717)	(1.7902)	(2.8542)
	(1,1,0,...,0,1,1,1)	0.8994	0.9474	0.9060	0.9548	0.8932	0.9488	0.8984	0.9464
		(0.4969)	(0.6232)	(0.1439)	(0.1718)	(0.2190)	(0.2628)	(1.7330)	(2.6932)
	(5,0,...,0)	0.8978	0.9462	0.9052	0.9550	0.8952	0.9480	0.8994	0.9464
		(0.4383)	(0.5477)	(0.1482)	(0.1768)	(0.2082)	(0.2501)	(1.6616)	(2.4966)

Remark 2: We found from the simulation results that the expectation of the length of the GCI for α maybe not exist. Thus we provide the average interval lengths of the proposed GCI for $1/\alpha$.

3 Inference for the stress-strength model

In this section, we discuss the inferential procedure for the stress-strength model $P(X_1 < X_2)$ when the stress X_1 and the strength X_2 are independent and $X_i \sim \text{GIED}(\alpha_i, \lambda_i), i = 1, 2$. The reliability of the stress-strength model is given by

$$\begin{aligned}\delta &= P(X_1 < X_2) = E[P(X_1 < X_2|X_2)] \\ &= 1 - \int_0^\infty \frac{\alpha_2\lambda_2}{x^2} e^{-\lambda_2/x}(1-e^{-\lambda_2/x})^{\alpha_2-1}(1-e^{-\lambda_1/x})^{\alpha_1} dx \\ &= 1 - \alpha_2 \int_0^1 (1-t)^{\alpha_2-1}(1-t^{\lambda_1/\lambda_2})^{\alpha_1} dt. \end{aligned} \tag{11}$$

In particular, when $\lambda_1 = \lambda_2$, the above equation can be simplified as

$$\delta = \frac{\alpha_1}{\alpha_1+\alpha_2}.$$

Since it is easy to obtain the maximum likelihood estimation (MLE) of δ, the main aim of this section is to obtain the GCI for δ.

3.1 The GCIs for δ

Suppose that $\mathbf{X}_i = (X_{1:m_i:n_i:k_i}, X_{2:m_i:n_i:k_i}, \ldots, X_{m_i:m_i:n_i:k_i})$ is the progressively first failure censored sample from the GIED(α_i, λ_i) with the censoring scheme $R_i = (R_{i,1}, R_{i,2}, \ldots, R_{i,m_i})$, and that $\mathbf{x}_i = (x_{1:m_i:n_i:k_i}, x_{2:m_i:n_i:k_i}, \ldots, x_{m_i:m_i:n_i:k_i})$ be the observed value of $\mathbf{X}_i$. Furthermore, let

$$S_{i,j} = \sum_{l=1}^{j} k_i(R_{i,l}+1)\log(1-e^{-\lambda/x^{R_i}_{l:m_i:n_i:k_i}}) + k_i[n_i - \sum_{l=1}^{j}(R_{i,l}+1)]\log(1-e^{-\lambda_i}/x^{R_i}_{j:m_i:n_i:k_i}).$$

$$T_i(\lambda_i) = 2\sum_{j=1}^{m_i-1} \log\frac{S_{i,m_i}}{S_{i,j}},\ i = 1, 2.$$

Then $T_i(\lambda_i)$ and S_{i,m_i} are independent, and $T_i(\lambda_i) \sim \chi^2(2m_i - 2)$, $V_i = 2\alpha_i S_{i,m_i} \sim \chi^2(2m_i)$.

We first consider GCI for δ when $\lambda_1 = \lambda_2 \hat{=} \lambda$. In this case, it is obvious that $T_3(\lambda) = T_1(\lambda) + T_2(\lambda) \sim \chi^2(2m_1 + 2m_2 - 4)$. Let $g(T_0, \mathbf{x}_1, \mathbf{x}_2)$ be the unique solution of the equation $T_3(\lambda) = T_0$ for $T_0 \sim \chi^2(2m_1 + 2m_2 - 4)$. Similar to the derivation of Y_1 in Section 2.1, the generalized pivotal quantity for δ is given by

$$Y_5 = \frac{Y_{1,1}}{Y_{1,1}+Y_{2,1}}, \tag{12}$$

where $Y_{i,1} = -V_i/[k_i\sum_{j=1}^{m_i}(R_{i,j}+1)\log(1-e^{-g(T_0,\mathbf{x}_1,\mathbf{x}_2)/x^{R_i}_{j:m_i:n_i:k_i}})]$, $i = 1, 2$.

Let $Y_{5,\gamma}$ denote the γ percentile of Y_5. Then $[Y_{5,\gamma/2},\ Y_{5,1-\gamma/2}]$ is the $1-\gamma$ GCI of δ when $\lambda_1 = \lambda_2$. Just as in the case of Y_1, the percentiles of Y_5 can be obtained by Monte Carlo simulations.

Algorithm 3: GCI for the reliability δ with the common scale parameters.

(1) Generate $T_0 \sim \chi^2(2m_1 + 2m_2 - 4)$. Then for given progressively first failure censored samples $\mathbf{x_1}, \mathbf{x_2}$, obtain $g(T_0, \mathbf{x_1}, \mathbf{x_2})$ from the equation $T_3(\lambda) = T_0$.

(2) Generate $V_i \sim \chi^2(2m_i)$, $i = 1, 2$. Then compute Y_5 based on the equation (12).

(3) Repeat the steps (1) and (2) B times. Then there are the B values of Y_5.

(4) Arrange all Y_5 values in ascending order: $Y_{5,(1)} < Y_{5,(2)} < ... < Y_{5,(B)}$. Then a $1 - \beta$ GCI of the reliability δ is given by $[Y_{5,(B\gamma/2)}, Y_{5,(B-B\gamma/2)}]$.

We now consider GCI for δ when $\lambda_1 \neq \lambda_2$. In this case, let $g_i(T_{i,0}, \mathbf{x}_i)$ be the solution of the equation $T_i(\lambda_i) = T_{i,0}$, where $T_{i,0} \sim \chi^2(2m_i)$. Using the substitution method, the generalized pivotal quantity for δ is given by

$$Y_6 = 1 - Y_{2,2} \int_0^1 (1-t)^{Y_{2,2}-1} (1 - t^{g_1(T_{1,0}, \mathbf{x}_1)/g_2(T_{2,0}, \mathbf{x}_2)})^{Y_{1,2}} dt, \tag{13}$$

where $Y_{i,2} = -V_i / [k_i \sum_{j=1}^{m_i} (R_{i,j} + 1) \log(1 - e^{-g(T_{i,0}, \mathbf{x}_i)/x_{j:m_i:n_i:k_i}^{R_i}})]$, $i = 1, 2$.

If $Y_{6,\gamma}$ denotes the γ percentile of Y_6, then $[Y_{6,\gamma/2}, Y_{6,1-\gamma/2}]$ is a $1 - \gamma$ GCI for the reliability δ. Similarly, the value $Y_{6,\gamma}$ can be obtained by Monte Carlo simulation. Because our simulation results show that the coverage probabilities of the proposed generalized confidence lower limit (GCLL) based on (13) are larger than the nominal coverage probabilities (These simulation results are not provided for saving space.), we propose a modified generalized pivotal quantity. The modified generalized pivotal quantity is given by

$$Y_7 = \left| \log \frac{1 + Y_6}{1 - Y_6} - RZ \right|,$$

where $\widehat{\delta}$ is the MLE of δ, $RZ = \log[(1 + \widehat{\delta})/(1 - \widehat{\delta})]$.

If $Y_{7,\gamma}$ denotes the γ percentile of T_7, then a $1 - \gamma$ modified GCI (MGCI) for the reliability δ is given by

$$\left[\frac{e^{RZ - Y_{7,1-\gamma}} - 1}{e^{RZ - Y_{7,1-\gamma}} + 1}, \frac{e^{RZ + Y_{7,1-\gamma}} - 1}{e^{RZ + Y_{7,1-\gamma}} + 1} \right],$$

and $(e^{RZ - Y_{7,1-2\gamma}} - 1)/(e^{RZ - Y_{7,1-2\gamma}} + 1)$ is a $1 - \gamma$ modified GCLL for the reliability δ. Similarly, the value $Y_{7,\gamma}$ can be obtained by the following Monte Carlo method.

Algorithm 4: MGCI for the reliability δ with the unequal scale parameters.

(1) Generate $T_{i,0} \sim \chi^2(2m_i - 2)$. Then for given progressively first failure censored sample $\mathbf{x_i}$, obtain $g(T_{i,0}, \mathbf{x_i})$ from the equation $T_i(\lambda_i) = T_{i,0}$, $i = 1, 2$.

(2) Generate V_i from $\chi^2(2m_i)$, $i = 1, 2$. Then compute Y_6 on the basis of (13).

(3) Repeat the steps (1) and (2) B times. Then there are the B values of Y_6.

(5) For given sample $\mathbf{x_i}$, obtain the MLEs $\widehat{\lambda}_i$ and $\widehat{\alpha}_i$, $i = 1, 2$. Then compute the value of $\widehat{\delta}$.

(6) Compute $Y_{7,i} = |\log[(1 + Y_{6,i})/(1 - Y_{6,i})] - RZ|$, $i = 1, 2, ..., B$, where $RZ = \log[(1 + \widehat{\delta})/(1 - \widehat{\delta})]$.

(7) Arrange all Y_7 values in ascending order: $Y_{7,(1)} < Y_{7,(2)} < ... < Y_{7,(B)}$. Then a $1-\gamma$ MGCI for the reliability δ is given by

$$\left[\frac{e^{RZ-Y_{7,(B-B\gamma)}} - 1}{e^{RZ-Y_{7,(B-B\gamma)}} + 1}, \frac{e^{RZ+Y_{7,(B-B\gamma)}} - 1}{e^{RZ+Y_{7,(B-B\gamma)}} + 1}\right].$$

3.2 Simulation study

In this subsection, a simulation study is conducted to assess the proposed GCI and MGCI for δ with both equal scale and unequal scale parameters in terms of the coverage percentage and the average interval length. In the simulation study, we consider two scenarios of the GIEDs. The first one is a pair of the GIEDs with the parameters $(\alpha_1, \lambda_1, \alpha_2, \lambda_2) = (4, 1, 2, 1)$ and the parameter $\delta = 2/3$, and the second one is a pair of the GIEDs with the parameters $(\alpha_1, \lambda_1, \alpha_2, \lambda_2) = (4, 1, 2, 2)$ and the parameter $\delta = 0.8810$. In addition, we take $n_1 = n_2 = n = 10, 20$, $k_1 = k_2 = k = 1$, $m_1 = m_2 = m = 10, 15$ and the same progressive censoring schemes, $R_1 = R_2 = R$ for a pair of the GIEDs in the simulation study. For a given combination of the parameters and censoring scheme, the simulation is carried over 5000 simulation runs with $B = 10000$. The simulation results are reported in Table 4.

Table 4: The coverage percentages and the average lengths (in parentheses) of the GCIs/MGCIs for δ with nominal levels 0.9, 0.95, based on 5000 replications

$(\alpha_1, \lambda_1, \alpha_2, \lambda_2)$	(k, m, n)	R	0.9	0.95
(4, 1, 2, 1)	(1,10,15)	(0,...,0,5)	0.8994 (0.3261)	0.9466 (0.3852)
		(5,0,...,0)	0.8988 (0.3224)	0.9468 (0.3809)
	(1,10,20)	(0,...,0,10)	0.9000 (0.3274)	0.9458 (0.3866)
		(10,0,...,0)	0.8980 (0.3220)	0.9478 (0.3804)
	(1,15,20)	(0,...,0,5)	0.9024 (0.2697)	0.9534 (0.3194)
		(5,0,...,0)	0.9038 (0.2673)	0.9538 (0.3166)
(4, 1, 2, 2)	(1,10,15)	(0,...,0,5)	0.9080 (0.2449)	0.9574 (0.2992)
		(5,0,...,0)	0.9022 (0.2404)	0.9502 (0.2944)
	(1,10,20)	(0,...,0,10)	0.9056 (0.2546)	0.9524 (0.3120)
		(10,0,...,0)	0.8996 (0.2351)	0.9508 (0.2885)
	(1,15,20)	(0,...,0,5)	0.9090 (0.1967)	0.9548 (0.2386)
		(5,0,...,0)	0.9032 (0.1965)	0.9486 (0.2389)

It is seen from Table 4 that for all combinations, the coverage percentages of the GCIs and MGCIs for δ are quite close to the nominal coverage probabilities, even for small sample sizes. The simulation results also show that for a fixed n, as m increases the average interval lengths decrease, as expected. These findings show that the performance of the proposed MGCI is very satisfactory for all cases.

4 An illustrative example

In this section, we use the following progressively first failure censored data to illustrate the methods of inference developed in this paper. These data are the strength of single carbon fibers of 10 mm and 20 mm in gauge length. The original data are provided by Bader and Priest [**?**]. Krishna et al. [**?**] generated two progressively first failure censored samples from those original data. They showed that those data can be fitted by the GIEDs.

Table 5: Two progressively first failure censored samples

X_1	$n_1 = 23, m_1 = 18, k_1 = 3$	$R_1 = (0*6, 1*5, 0*7)$	0.562, 0.564, 0.729, 0.802, 0.950, 1.111, 1.208, 1.247, 1.271, 1.348, 1.429, 1.522, 1.524, 1.551, 1.609, 1.740, 1.976, 2.068
X_2	$n_2 = 21, m_2 = 16, k_3 = 3$	$R_2 = (0*5, 1*5, 0*6)$	1.151, 1.382, 1.453, 1.478, 1.507, 1.600, 1.647, 1.704, 1.768, 1.775, 1.825, 1.866, 1.925, 2.227, 2.375, 2.627

Based on the first censored sample, the MLEs of λ_1 and α_1 are $\widehat{\lambda}_1 = 4.1155$ and $\widehat{\alpha}_1 = 5.1947$, respectively. The 95% exact confidence interval of λ_1 is $[2.1668, 5.4888]$. The GCIs of $\alpha_1, x_{1,0.1}$ and $R_{X_1}(1)$ are given by $[1.0875, 13.6657]$, $[0.8280, 1.2160]$ and $[0.8354, 0.9565]$, respectively. The GPI of $X_{1,new}$ is $[0.7194, 11.8121]$.

Based on the second censored sample, the MLEs of λ_2 and α_2 are $\widehat{\lambda}_2 = 10.1753$ and $\widehat{\alpha}_2 = 60.6688$, respectively. The 95% exact confidence interval of λ_2 is $[6.3857, 13.9296]$. The 95% confidence intervals of λ_1 and λ_2 show $\lambda_1 \neq \lambda_2$. The MLE of $\delta = P(X_1 < X_2)$ is 0.5841 based on the equation (11). The 95% MGCI of δ is $[0.3585, 0.7453]$ based on the generalized pivotal quantity Y_7. The above GCIs and MGCI are based on $B = 100000$. Krishna et al. [**?**] reported that the MLE, the 95% asymptotic, bootstrap-p and bootstrap-t confidence intervals of δ are 0.6310, [0.1770,1], [0.5137,0.9330], [0.2691,0.7484], respectively. Because we observed from the simulation results that our MGCI for δ works well, it is clear that the MGCI of δ outperforms other confidence intervals in terms of the interval lengths of these intervals.

5 Conclusion

The generalized inverted exponential distribution is an important skewed distribution. This study developed interval estimation procedures for the GIED and the GIED stress-strength model. The idea is to transform progressively first failure censored sample from the GIED into random variables following the standard exponential distribution. The exact confidence interval for the scale parameter was proposed. The GCIs for the shape parameter, the percentile and reliability of the GIED were also provided. In addition, the GPI for the future measurement was derived. In the simulation study, we showed that the proposed GCIs and GPI work well.

The GIED stress-strength model was also considered. The proposed procedure was also successfully extended to the GIED stress-strength model with both equal scale and unequal scale parameters. The performance of the proposed MGCIs for the reliability of the GIED stress-strength model was guaranteed using extensive simulations.