# LETE

## Fundamentos de Escoamentos Reativos Turbulentos
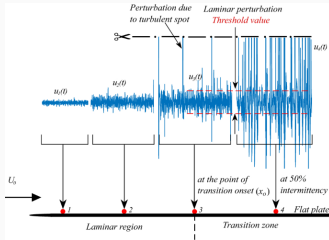
Aula 2-3

Prof. Dr. Guenther Carlos Krieger Filho

28 de setembro de 2020

Escola Politécnica da USP - LETE/CRC - Combustion Research Centre

# Descrição de escoamentos turbulentos



**Ergodicidade**

Média conjunto = Média no tempo

(Média estatística)

**Decomposição de Reynolds**

$$\Phi_{(t)} = \overline{\Phi} + \Phi'_{(t)}$$

**Média no tempo**

$$\overline{\Phi} = \frac{1}{\Delta t} \int_0^{\Delta t} \Phi_{(t)} dt$$

$$\frac{1}{\Delta t} \int_0^{\Delta t} \Phi_{(t)} dt = \frac{1}{\Delta t} \int_0^{\Delta t} \overline{\Phi} dt + \frac{1}{\Delta t} \int_0^{\Delta t} \Phi' dt$$

$$\overline{\Phi} = \overline{\Phi} + \overline{\Phi'} \rightarrow \boxed{\overline{\Phi'} = 0}$$

## Descrição de escoamentos turbulentos

**Variância**

$$\overline{(\Phi')^2} = \frac{1}{\Delta t} \int_0^{\Delta t} (\Phi')^2 dt$$

**Energia Cinética Turbulenta**

$$K = \frac{1}{2} \left[ \overline{(u')^2} + \overline{(v')^2} + \overline{(w')^2} \right]$$

**RMS/Desvio Padrão**

$$\Phi_{RMS} = \sqrt{\overline{(\Phi')^2}}$$

**Intensidade Turbulenta**

$$I = \frac{\left(\frac{2}{3}K\right)^{1/2}}{U_{\text{ref}}}$$

## Momentos (2ªordem) de 2 variáveis

$$\Phi = \overline{\Phi} + \Phi' \; ; \; \Psi = \overline{\Psi} + \Psi' \; ; \; \overline{\Phi'} = \overline{\Psi'} = 0 \; ; \; \boxed{\overline{\Phi'\Psi'} = \frac{1}{\Delta t} \int_0^{\Delta t} \Phi'\Psi' dt}$$

Para velocidade: $\begin{cases} u'v' \\ u'w' \quad \text{Se} = 0 \; \Rightarrow \text{grandezas não correlacionadas} \\ v'w' \end{cases}$

**Autocorrelação temporal**

$$R_{\Phi'\Phi'(\tau)} = \overline{\Phi'_{(t)}\Phi'_{(t+\tau)}} = \frac{1}{\Delta t} \int_0^{\Delta t} \Phi'_{(t)}\Phi'_{(t+\tau)} dt$$

**Autocorrelação espacial**

$$R_{\Phi'\Phi'(\tau)} = \overline{\Phi'_{(t)}\Phi'_{(t+\tau)}} = \frac{1}{\Delta t} \int_0^{\Delta t} \Phi'_{(t)}\Phi'_{(t+\tau)} dt$$

**Lembrando...**

$$\rho = cte \; ; \; \text{decomposição de Reynolds: } \Phi_{(t)} = \overline{\Phi} + \Phi'_{(t)}$$

- $\overline{\dfrac{\partial \Phi}{\partial s}} \equiv \dfrac{1}{\Delta t} \int_0^{\Delta t} \dfrac{\partial \Phi}{\partial s} dt = \dfrac{\partial \overline{\Phi}}{\partial s}$

- $\overline{\int \Phi ds} = \int \overline{\Phi} ds$

- $\overline{\Phi + \Psi} = \overline{\Phi} + \overline{\Psi}$

- $\overline{\Phi \Psi} = \overline{(\overline{\Phi} + \Phi')(\overline{\Psi} + \Psi')} = \overline{\overline{\Phi}\,\overline{\Psi}} + \overline{\Phi'\Psi'} + \underbrace{\overline{\overline{\Phi}\Psi'}}_{=0} + \underbrace{\overline{\Phi'\overline{\Psi}}}_{=0}$

**Para um vetor $a$ :**

$$a = \overline{a} + a' \text{ ou } a_i = \overline{a_i} + a'_i$$

- $\overline{\text{div } a} = \text{div } \overline{a}$ ou $\overline{\nabla \cdot a} = \nabla \cdot \overline{a}$ ou $\overline{\dfrac{\partial a_i}{\partial x_i}} = \dfrac{\partial \overline{a_i}}{\partial x_i}$

- $\overline{\text{div } \Phi a} = \text{div } \left(\overline{\Phi a}\right) = \text{div } \left(\overline{\Phi} \ \overline{a}\right) + \text{div } \left(\overline{\Phi' a'}\right)$

  $\overline{\dfrac{\partial (\Phi a_i)}{\partial x_i}} = \dfrac{\partial}{\partial x_i} \left(\overline{\Phi a_i}\right) = \dfrac{\partial}{\partial x_i} \left(\overline{\Phi} \ \overline{a_i}\right) + \dfrac{\partial}{\partial x_i} \left(\overline{\Phi' a'_i}\right)$

- $\overline{\text{div grad } a} = \text{div grad } \overline{a}$ ou $\overline{\nabla \cdot (\nabla a)} = \nabla \cdot (\nabla \overline{a}) = \nabla^2 \overline{a}$

  $\overline{\dfrac{\partial}{\partial x_j} \left(\dfrac{\partial a_i}{\partial x_j}\right)} = \dfrac{\partial}{\partial x_j} \left(\dfrac{\partial \overline{a_i}}{\partial x_j}\right) = \dfrac{\partial^2 \overline{a_i}}{\partial x_j \partial x_j}$

## Reynolds Averaged Navier-Stokes

**Continuidade**

$$\frac{\partial \overline{u_i}}{\partial x_i} \text{ ou } \nabla \cdot \overline{u} = 0$$

**Quantidade de movimento**

$$\text{Instantânea: } \frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}\left(u_i u_j\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

$$\frac{\partial u}{\partial t} + \nabla \cdot (uu) = -\frac{1}{\rho}\nabla p + \nu \nabla^2 u$$

$$\text{Média: } \frac{\partial}{\partial x_j}\left(\overline{u_i}\,\overline{u_j}\right) + \frac{\partial}{\partial x_j}\left(\overline{u_i' u_j'}\right) = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{u_i}}{\partial x_j \partial x_j}$$

$$\nabla \cdot (\overline{u}\,\overline{u}) + \nabla \cdot \left(\overline{u'u'}\right) = -\frac{1}{\rho}\nabla \overline{p} + \nu \nabla^2 \overline{u}$$

## Reynolds Average Navier-Stokes

$$-\rho\overline{u_i'u_j'} = -\rho \begin{pmatrix} \overline{u'^2} & \overline{u'v'} & \overline{u'w'} \\ \overline{v'u'} & \overline{v'^2} & \overline{v'w'} \\ \overline{w'u'} & \overline{w'v'} & \overline{w'^2} \end{pmatrix} \Rightarrow \text{Tensor de Reynolds}$$

$$\underbrace{\text{Tr}\begin{pmatrix} \text{Tensor de} \\ \text{Reynolds} \end{pmatrix}}_{\text{traço}} = 2K = -2\rho\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

$K \rightarrow$ Energia cinética turbulenta

**Transporte de Escalar**

$$\text{Instantânea: } \frac{\partial \Phi}{\partial t} + \frac{\partial}{\partial x_j}\left(\Phi u_j\right) = \frac{D}{\rho}\frac{\partial^2 \Phi}{\partial x_j \partial x_j} + S_\Phi$$

$$\frac{\partial \Phi}{\partial t} + \nabla \cdot (\Phi u) = \frac{D}{\rho}\nabla^2 \Phi + S_\Phi$$

$$\text{Média: } \frac{\partial}{\partial x_j}\left(\overline{\Phi}\,\overline{u_j}\right) + \frac{\partial}{\partial x_j}\left(\overline{\Phi' u_j'}\right) = \frac{D}{\rho}\frac{\partial^2 \overline{\Phi}}{\partial x_j \partial x_j} + \overline{S_\Phi}$$

$$\nabla \cdot \left(\overline{\Phi}\,\overline{u}\right) + \nabla \cdot \left(\overline{\Phi' u'}\right) = \frac{D}{\rho}\nabla^2 \overline{\Phi} + \overline{S_\Phi}$$

## Exemplo: Placa plana c/ $\rho = $ cte (Turns cap. 11)

**Quantidade de movimento em x $\left(\dfrac{\partial p}{\partial x} = 0\right)$**

$$\underbrace{\frac{\partial v_x}{\partial t}}_{\textcircled{1}} + \underbrace{\frac{\partial}{\partial x}\left(v_x v_x\right)}_{\textcircled{2}} + \underbrace{\frac{\partial}{\partial y}\left(v_x v_y\right)}_{\textcircled{3}} = \underbrace{\nu \frac{\partial^2 v_x}{\partial y \partial y}}_{\textcircled{4}}$$

$\textcircled{1}: \overline{\dfrac{\partial}{\partial t}\left(\overline{v_x} + v_x'\right)} = \dfrac{\partial \overline{v_x}}{\partial t} + \dfrac{\partial \overline{v_x'}}{\partial t} = 0$

$\textcircled{2}: \overline{\dfrac{\partial}{\partial x}\left(\overline{v_x} + v_x'\right)\left(\overline{v_x} + v_x'\right)} = \dfrac{\partial}{\partial x}\left(\overline{v_x}\,\overline{v_x}\right) + \dfrac{\partial}{\partial x}\left(\overline{v_x'^2}\right)$

$\textcircled{3}: \overline{\dfrac{\partial}{\partial y}\left(\overline{v_x} + v_x'\right)\left(\overline{v_y} + v_y'\right)} = \dfrac{\partial}{\partial y}\left(\overline{v_x}\,\overline{v_y}\right) + \dfrac{\partial}{\partial y}\left(\overline{v_x' v_y'}\right)$

$\textcircled{4}: \nu \overline{\dfrac{\partial^2 v_x}{\partial y \partial y}} = \nu \dfrac{\partial^2 \overline{v_x}}{\partial y \partial y}$

## Exemplo: Placa plana c/ $\rho = $ cte (Turns cap. 11)

**Quantidade de movimento em x**

$$\frac{\partial}{\partial x}\left(\overline{v_x}\;\overline{v_x}\right) + \frac{\partial}{\partial y}\left(\overline{v_x}\;\overline{v_y}\right) + \frac{\partial}{\partial y}\left(\overline{v_x'v_y'}\right) + \frac{\partial}{\partial x}\left(\overline{v_x'v_x'}\right) = \nu\frac{\partial^2\overline{v_x}}{\partial y \partial y}$$

**Continuidade**

$$\frac{\partial\overline{v_x}}{\partial x} + \frac{\partial\overline{v_y}}{\partial y} = 0$$

- p/ jato (análogo à placa plana)

$$\rho\left(\overline{v_x}\frac{\partial\overline{v_x}}{\partial x} + \overline{v_r}\frac{\partial\overline{v_x}}{\partial r}\right) = \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\overline{v_x}}{\partial r}\right) - \frac{1}{r}\frac{\partial}{\partial r}\left(\rho r\overline{v_x'v_r'}\right)$$

## Eddy viscosity ou hipótese de Boussinessq (1877)

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\tau_{\text{lam}} + \tau_{\text{turb}}\right)\right]$$

$$\tau_{\text{lam}} = \mu\frac{\partial \overline{v_x}}{\partial r}$$

$$\tau_{\text{turb}} = -\rho\nu_t\frac{\partial \overline{v_x}}{\partial r} \; ; \; \nu_t = \frac{\mu_t}{\rho}$$

$$\mu_{\text{eff}} = \mu_{\text{molecular}} + \mu_{\text{turb}}$$

1. Como determinar $\mu_{\text{turb}}$?

2. $\mu_{\text{molecular}}$ é propriedade termodinâmica de transporte
   $\mu_{\text{turb}}$ é dependente do "padrão" do escoamento

3. Nem sempre é função apenas do grad da velocidade.

# Eddy viscosity ou hipótese de Boussinessq (1877)

**Comprimento de mistura de Prandtl $l_m$**

$$\mu_t = \rho \nu_t = \rho l_m v_{turb} = \rho l_m^2 \left| \frac{\partial v_x}{\partial x_r} \right|$$

- Jatos livres: $v_{turb} \propto \overline{v}_{x,max} - \overline{v}_{x,min}$

**Comprimento de mistura de Prandtl $l_m$**

$$\mu_t = \underbrace{0,1365}_{experimental} \rho l_m^2 \left( \overline{v}_{x,max} - \overline{v}_{x,min} \right)$$

## Eddy viscosity ou hipótese de Boussinessq (1877)

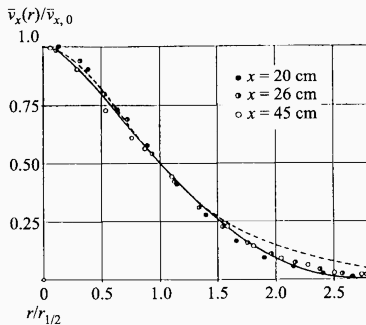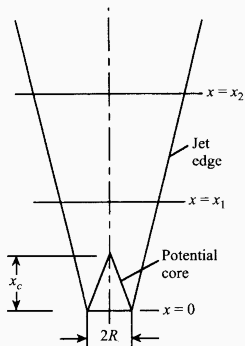- Jatos livres: $v_{turb} \propto \overline{v}_{x,max} - \overline{v}_{x,min}$

**Quantidade de movimento**

$$\rho \left( \overline{v_x} \frac{\partial \overline{v_x}}{\partial x} + \overline{v_r} \frac{\partial \overline{v_x}}{\partial r} \right) = \frac{1}{r} \frac{\partial}{\partial r} \left[ r(\mu + \mu_t) \frac{\partial \overline{v_x}}{\partial r} \right]$$

**Continuidade**

$$\frac{\partial}{\partial x} \left( \overline{v}_x r \right) + \frac{\partial}{\partial r} \left( \overline{v}_r r \right) = 0$$

## Eddy viscosity ou hipótese de Boussinessq (1877)



$$J_e = \rho_e v_e^2 \pi R^2 \quad ; \quad l_m = 0,075 \, \delta_{99\%}$$

$$\xi = \left[ \frac{3J_e}{16\rho_e \pi} \right] \frac{1}{\nu_t} \frac{r}{x}$$

$\delta_{99\%} \rightarrow$ raio no qual $\dfrac{\overline{v}_x(r)}{\overline{v}_{x,0}} = 1\%$



a) $\nu_t = 0,0102 \, \delta_{99\%} \, \overline{v}_{x,0}(x) \approx cte$

b) $r_{1/2} \propto x$
   $\overline{v}_{x,0} \propto x^{-1}$

14

## Eddy viscosity ou hipótese de Boussinessq (1877)

Aplicando a solução analítica p/ jato laminar:

$$\mu_{lam} \Rightarrow \rho \nu_t$$

$$\overline{v}_x = \frac{3}{8\pi} \frac{J_e}{\rho \nu_t x} \left[ 1 + \frac{\xi^2}{4} \right]^{-2} \qquad \overline{v}_r = \left[ \frac{3 J_e}{16 \pi \rho_e} \right]^{1/2} \frac{1}{x} \frac{\xi - \xi^3/4}{[1 + \xi^2/4]^2}$$

Fazendo $\dfrac{\overline{v}_x}{v_e} = 0,375(v_e R/\nu_t)(x/R)^{-1}[1 + \xi^2/4]^{-2}$, no centro
$r = 0 \Rightarrow \xi = 0$

$$\frac{\overline{v}_x}{v_e} = 0,375(v_e R/\nu_t)(x/R)^{-1}[1 + \xi^2/4]^{-2} \tag{1}$$

$$\frac{\overline{v}_x}{\overline{v}_{x,0}} = \frac{1}{2} = \left[ 1 + \xi^2/4 \right]^{-2} \rightarrow \xi = 1,287 = \frac{3}{16} \left[ \frac{\rho_e v_e^2 \pi R^2}{\rho_e \pi} \right]^{1/2} \frac{1}{\nu_t} \frac{r_{1/2}}{x}$$

$$\rightarrow 1,287 = 0,43 v_e R \frac{1}{\nu_t} \frac{r_{1/2}}{x} \rightarrow r_{1/2} = 2,97 \left( \frac{v_e R}{\nu_t x} \right)^{-1} \tag{2}$$

Resolvendo 1, 2 e a)

$$\frac{\overline{v}_{x,o}}{v_e} = 13,15(x/R)^{-1} \qquad \nu_t = 0,0285 v_e R$$

$$\frac{r_{1/2}}{x} = 0,08468$$

Lembrando no caso laminar $\frac{\overline{v}_{x,0}}{v_e} \propto Re_{jet}$ e $\frac{r_{1/2}}{x} \propto Re_{jet}^{-1}$. Portanto no caso turbulento existe independência do número de Reynolds.

## Equação de transporte p/ $-\rho\overline{u_i' u_j'}$

**Quantidade de movimento instantânea**

$$\frac{\partial}{\partial x_j}\left(\rho u_i u_j\right) = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \tag{3}$$

Aplicando decomposição de Reynolds e a média temporal

$$\frac{1}{\Delta t}\int_0^t \left[\frac{\partial}{\partial x_j}\rho(\overline{u}_i\overline{u}_j + \overline{u}_i u_j' + u_i'\overline{u}_j + u_i'u_j') = -\frac{\partial}{\partial x_i}\left(\overline{p} + p'\right) + \frac{\partial}{\partial x_j}\left(\overline{\tau}_{ij} + \tau_{ij}'\right)\right] dt$$

$$\frac{\partial}{\partial x_j}\rho(\overline{u}_i\overline{u}_j + \overline{u_i'u_j'}) = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial \overline{\tau}_{ij}}{\partial x_j}$$

**Quantidade de movimento média**

$$\frac{\partial}{\partial x_j}\left(\rho\overline{u}_i\overline{u}_j\right) = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\overline{\tau}_{ij} - \rho\overline{u_i'u_j'}\right) \tag{4}$$

## Equação de transporte p/ $-\rho\overline{u_i'u_j'}$

Reescrevendo o indice repetido como $k$ na equação 3 e multiplicando por $\overline{u}_i$ e $\overline{u}_j$ e somando as duas, lembrando a continuidade $\dfrac{\partial}{\partial x_i}(\rho u_i) = 0$ , temos para o termo advectivo

$$u_j\frac{\partial}{\partial x_k}(\rho u_i u_k) + u_i\frac{\partial}{\partial x_k}(\rho u_j u_k) \rightarrow \quad u_j\left[u_i\underbrace{\frac{\partial}{\partial x_k}(\rho u_k)}_{=0} + \rho u_k\frac{\partial u_i}{\partial x_k}\right]$$

$$+ u_i\left[u_j\underbrace{\frac{\partial}{\partial x_k}(\rho u_k)}_{=0} + \rho u_k\frac{\partial u_j}{\partial x_k}\right]$$

$$\rightarrow \rho u_j u_k\frac{\partial u_i}{\partial x_k} + \rho u_i u_k\frac{\partial u_j}{\partial x_k} \rightarrow \rho u_k\left[u_j\frac{\partial u_i}{\partial x_k} + u_i\frac{\partial u_j}{\partial x_k}\right]$$

$$\rightarrow \rho u_k\frac{\partial}{\partial x_k}(u_i u_j) \rightarrow \boxed{\frac{\partial}{\partial x_k}(\rho u_i u_j u_k)} = \rho u_k\frac{\partial}{\partial x_k}(u_i u_j) + \rho u_i u_j\underbrace{\frac{\partial u_k}{\partial x_k}}_{=0}$$

## Equação de transporte p/ $-\rho\overline{u_i' u_j'}$

Assim a nova equação resulta

$$\frac{\partial}{\partial x_k}\left(\rho u_i u_j u_k\right) = -u_j\frac{\partial p}{\partial x_i} - u_i\frac{\partial p}{\partial x_j} + u_j\frac{\partial \tau_{ik}}{\partial x_k} + u_i\frac{\partial \tau_{jk}}{\partial x_k}$$

Decomposição de Reynolds

$$\frac{\partial}{\partial x_k}\left[\rho(\overline{u}_i + u_i')(\overline{u}_j + u_j')(\overline{u}_k + u_k')\right] = -\left(\overline{u}_j + u_j'\right)\frac{\partial(\overline{p} + p')}{\partial x_i}$$

$$-\left(\overline{u}_i + u_i'\right)\frac{\partial(\overline{p} + p')}{\partial x_j}$$

$$+\left(\overline{u}_j + u_j'\right)\frac{\partial}{\partial x_k}\left(\overline{\tau}_{ik} + \tau_{ik}'\right)$$

$$+\left(\overline{u}_i + u_i'\right)\frac{\partial}{\partial x_k}\left(\overline{\tau}_{jk} + \tau_{jk}'\right)$$

Organizando e aplicando a média

$$\frac{\partial}{\partial x_k}\left[\overline{\rho\overline{u_i}\,\overline{u_j}}\,\overline{u}_k + \overline{\overline{u}_i\rho u_j' u_k'} + \overline{\overline{u}_j\rho u_i' u_k'} + \overline{\overline{u}_k\rho u_i' u_j'} + \overline{\rho u_i' u_j' u_k'}\right]$$

$$= -\overline{u}_j\frac{\overline{\partial\overline{p}}}{\partial x_i} - \overline{u}_j\frac{\overline{\partial p'}}{\partial x_i} - \overline{u_j'\frac{\partial\overline{p}}{\partial x_i}} - \overline{u_j'\frac{\partial p'}{\partial x_i}} - \overline{u}_i\frac{\overline{\partial\overline{p}}}{\partial x_j} - \overline{u}_i\frac{\overline{\partial p'}}{\partial x_j} - \overline{u_i'\frac{\partial\overline{p}}{\partial x_j}} - \overline{u_i'\frac{\partial p'}{\partial x_j}}$$

$$-\overline{u}_j\frac{\overline{\partial\overline{\tau}_{ik}}}{\partial x_k} - \overline{u}_j\frac{\overline{\partial\tau_{ik}'}}{\partial x_k} - \overline{u_j'\frac{\partial\overline{\tau}_{ik}}{\partial x_k}} - \overline{u_j'\frac{\partial\tau_{ik}'}{\partial x_k}} - \overline{u}_i\frac{\overline{\partial\overline{\tau}_{jk}}}{\partial x_k} - \overline{u}_i\frac{\overline{\partial\tau_{jk}'}}{\partial x_k} - \overline{u_i'\frac{\partial\overline{\tau}_{jk}}{\partial x_k}} - \overline{u_i'\frac{\partial\tau_{jk}'}{\partial x_k}}$$

$$\frac{\partial}{\partial x_k}[\rho\overline{u}_i\overline{u}_j\overline{u}_k + \overbrace{\rho\overline{u}_i\overline{u_j' u_k'}}^{\text{A}} + \rho\overbrace{\overline{u}_j\overline{u_i' u_k'}}^{\text{B}} + \rho\overline{u}_k\overline{u_i' u_j'} + \rho\overline{u_i' u_j' u_k'}]$$

$$= -\overline{u}_j\frac{\partial\overline{p}}{\partial x_i} - \overline{u_j'\frac{\partial p'}{\partial x_i}} - \overline{u}_i\frac{\partial\overline{p}}{\partial x_j} - \overline{u_i'\frac{\partial p'}{\partial x_j}} - \overline{u}_j\frac{\partial\overline{\tau}_{ik}}{\partial x_k} - \overline{u_j'\frac{\partial\tau_{ik}'}{\partial x_k}} - \overline{u}_i\frac{\partial\overline{\tau}_{jk}}{\partial x_k} - \overline{u_i'\frac{\partial\tau_{jk}'}{\partial x_k}}$$

$$(5)$$

## Equação de transporte p/ $-\rho\overline{u'_i u'_j}$

Reescrevendo o indice repetido como $k$ na equação 4 e multiplicando por $\overline{u}_i$ e por $\overline{u}_j$ e então somando as duas equações resultantes temos

$$\frac{\partial}{\partial x_k}\left(\rho\overline{u}_i\overline{u}_j\overline{u}_k\right) = -\overline{u}_j\frac{\partial\overline{p}}{\partial x_i} - \overline{u}_i\frac{\partial\overline{p}}{\partial x_j} + \overline{u}_j\frac{\partial}{\partial x_k}(\overline{\tau}_{ik} - \overbrace{\rho\overline{u'_i u'_k}}^{B'}) + \overline{u}_i\frac{\partial}{\partial x_k}(\overline{\tau}_{jk} - \overbrace{\rho\overline{u'_j u'_k}}^{A'})$$

(6)

Subtraindo 6 de 5, sendo $\overline{A} = \overline{A'} - \rho\overline{u'_j u'_k}\dfrac{\partial\overline{u}_i}{\partial x_k}$, e similar $\overline{B}$ e $\overline{B'}$

**Equação de transporte de $\overline{u'_i u'_j}$**

$$\frac{\partial}{\partial x_k}\left(\rho\overline{u}_k\overline{u'_i u'_j}\right) = -\frac{\partial}{\partial x_k}\left(\rho\overline{u'_i u'_j u'_k}\right) - \overline{u'_j\frac{\partial p'}{\partial x_i}} - \overline{u'_i\frac{\partial p'}{\partial x_j}}$$
$$+ \overline{u'_j\frac{\partial\tau'_{ik}}{\partial x_k}} + \overline{u'_i\frac{\partial\tau'_{jk}}{\partial x_k}} - \rho\overline{u'_i u'_k}\frac{\partial\overline{u}_i}{\partial x_k} - \rho\overline{u'_j u'_k}\frac{\partial\overline{u}_j}{\partial x_k}$$

(7)

## Equação de transporte p/ $-\rho\overline{u_i' u_j'}$

- Termos de tensões:

$$\overline{u_j'\frac{\partial \tau_{ik}'}{\partial x_k}} + \overline{u_i'\frac{\partial \tau_{jk}'}{\partial x_k}} \;\; ; \;\; \tau_{ik}' = \mu\frac{\partial u_i'}{\partial x_k}$$

$$\rightarrow \overline{u_j'\frac{\partial}{\partial x_k}\left(\mu\frac{\partial u_i'}{\partial x_k}\right)} + \overline{u_i'\frac{\partial}{\partial x_k}\left(\mu\frac{\partial u_j'}{\partial x_k}\right)}$$

$$\rightarrow \mu\left[\frac{\partial}{\partial x_k}\left(\overline{u_j'\frac{\partial u_i'}{\partial x_k}}\right) - \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} + \frac{\partial}{\partial x_k}\left(\overline{u_i'\frac{\partial u_j'}{\partial x_k}}\right) - \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}\right]$$

$$\rightarrow \mu\left[\frac{\partial}{\partial x_k}\left(\frac{\partial(\overline{u_i' u_j'})}{\partial x_k}\right) - 2\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}\right]$$

- Termo de dissipação:

$$2\mu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} = \varepsilon_{ij} = \frac{2}{3}\delta_{ij}\varepsilon$$

## Equação de transporte p/ $-\rho\overline{u_i'u_j'}$

- Termos de pressão:

$$-\overline{u_j'\frac{\partial p'}{\partial x_i}} - \overline{u_i'\frac{\partial p'}{\partial x_j}} \rightarrow -\frac{\partial}{\partial x_k}\left(\overline{p'\delta_{ki}u_j'}\right) + \overline{p'\frac{\partial u_j'}{\partial x_i}} - \frac{\partial}{\partial x_k}\left(\overline{p'\delta_{kj}u_i'}\right) + \overline{p'\frac{\partial u_i'}{\partial x_j}}$$

$$\rightarrow -\frac{\partial}{\partial x_k}\left(\overline{p'\delta_{ki}u_j'} + \overline{p'\delta_{kj}u_i'}\right) + \overline{p'\left(\frac{\partial u_j'}{\partial x_i} + \frac{\partial u_i'}{\partial x_j}\right)}$$

Finalmente temos

**Equação de transporte de $\overline{u_i' u_j'}$**

$$\frac{\partial}{\partial x_k}\left(\rho \overline{u}_k \overline{u_i' u_j'}\right) = -\frac{\partial}{\partial x_k}\left(\rho \overline{u_i' u_j' u_k'}\right) - \frac{\partial}{\partial x_k}\left(\overline{p' \delta_{ki} u_j'} + \overline{p' \delta_{kj} u_i'}\right)$$
$$+ \overline{p'\left(\frac{\partial u_j'}{\partial x_i} + \frac{\partial u_i'}{\partial x_j}\right)} + \mu\left[\frac{\partial}{\partial x_k}\left(\frac{\partial(\overline{u_i' u_j'})}{\partial x_k}\right)\right]$$
$$- \frac{2}{3}\delta_{ij}\varepsilon - \rho\overline{u_i' u_k'}\frac{\partial \overline{u}_i}{\partial x_k} - \rho\overline{u_j' u_k'}\frac{\partial \overline{u}_j}{\partial x_k} \tag{8}$$

A partir da equação 8, chegar na equação da energia cinética turbulenta $K = \left(\frac{1}{2}\overline{u_i' u_i'}\right)$.

# Considerações finais

**Perguntas?**

This work is licensed under a Creative Commons "Attribution-NonCommercial-ShareAlike 4.0 International" license.