Fundamentos de Escoamentos Reativos Turbulentos

Aula 2-3

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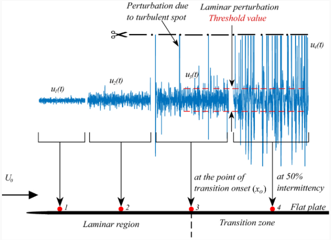
Descrição de escoamentos turbulentos



Ergodicidade

Média conjunto = Média no tempo

(Média estatística)

Decomposição de Reynolds

$$\Phi_{(t)} = \overline{\Phi} + \Phi'_{(t)}$$

Média no tempo

$$\overline{\Phi} = \frac{1}{\Delta t}\int_0^{\Delta t} \Phi_{(t)}\,dt$$

$$\frac{1}{\Delta t}\int_0^{\Delta t} \Phi_{(t)}\,dt = \frac{1}{\Delta t}\int_0^{\Delta t} \overline{\Phi}\,dt + \frac{1}{\Delta t}\int_0^{\Delta t} \Phi'\,dt$$

$$\overline{\Phi} = \overline{\Phi} + \overline{\Phi'} \rightarrow \boxed{\overline{\Phi'} = 0}$$

Descrição de escoamentos turbulentos

Variância

$$\overline{(\Phi')^2} = \frac{1}{\Delta t}\int_0^{\Delta t}(\Phi')^2 dt$$

Energia Cinética Turbulenta

$$K = \frac{1}{2}\left[\overline{(u')^2} + \overline{(v')^2} + \overline{(w')^2}\right]$$

RMS/Desvio Padrão

$$\Phi_{RMS} = \sqrt{\overline{(\Phi')^2}}$$

Intensidade Turbulenta

$$I = \frac{\left(\frac{2}{3}K\right)^{1/2}}{U_{\mathrm{ref}}}$$

Momentos (2ªordem) de 2 variáveis

$$\Phi = \overline{\Phi} + \Phi' \;;\; \Psi = \overline{\Psi} + \Psi' \;;\; \overline{\Phi'} = \overline{\Psi'} = 0 \;;\; \boxed{\overline{\Phi'\Psi'} = \frac{1}{\Delta t}\int_0^{\Delta t} \Phi'\Psi' dt}$$

$$\text{Para velocidade: } \begin{cases} u'v' \\ u'w' \quad \text{Se} = 0 \;\Rightarrow \text{grandezas não correlacionadas} \\ v'w' \end{cases}$$

Autocorrelação temporal

$$R_{\Phi'\Phi'(\tau)} = \overline{\Phi'_{(t)}\Phi'_{(t+\tau)}} = \frac{1}{\Delta t}\int_0^{\Delta t} \Phi'_{(t)}\Phi'_{(t+\tau)} dt$$

Autocorrelação espacial

$$R_{\Phi'\Phi'(\tau)} = \overline{\Phi'_{(t)}\Phi'_{(t+\tau)}} = \frac{1}{\Delta t}\int_0^{\Delta t} \Phi'_{(t)}\Phi'_{(t+\tau)} dt$$

Reynolds Averaged Navier Stokes

Lembrando...

$$\rho = cte \text{ ; decomposição de Reynolds: } \Phi_{(t)} = \overline{\Phi} + \Phi'_{(t)}$$

- $\overline{\dfrac{\partial \Phi}{\partial s}} \equiv \dfrac{1}{\Delta t} \int_0^{\Delta t} \dfrac{\partial \Phi}{\partial s} dt = \dfrac{\partial \overline{\Phi}}{\partial s}$
- $\overline{\int \Phi ds} = \int \overline{\Phi} ds$
- $\overline{\Phi + \Psi} = \overline{\Phi} + \overline{\Psi}$
- $\overline{\Phi \Psi} = \overline{(\overline{\Phi} + \Phi')(\overline{\Psi} + \Psi')} = \overline{\overline{\Phi}}\ \overline{\overline{\Psi}} + \overline{\Phi' \Psi'} + \underbrace{\overline{\overline{\Phi} \Psi'}}_{=0} + \underbrace{\overline{\Phi' \overline{\Psi}}}_{=0}$

Reynolds Averaged Navier Stokes

Para um vetor a :

$$a = \overline{a} + a' \text{ ou } a_i = \overline{a_i} + a_i'$$

- $\overline{\text{div } a} = \text{div } \overline{a}$ ou $\overline{\nabla \cdot a} = \nabla \cdot \overline{a}$ ou $\overline{\dfrac{\partial a_i}{\partial x_i}} = \dfrac{\partial \overline{a_i}}{\partial x_i}$

- $\overline{\text{div } \Phi a} = \text{div}\left(\overline{\Phi a}\right) = \text{div}\left(\overline{\Phi}\ \overline{a}\right) + \text{div}\left(\overline{\Phi' a'}\right)$

$$\overline{\frac{\partial(\Phi a_i)}{\partial x_i}} = \frac{\partial}{\partial x_i}\left(\overline{\Phi a_i}\right) = \frac{\partial}{\partial x_i}\left(\overline{\Phi}\ \overline{a_i}\right) + \frac{\partial}{\partial x_i}\left(\overline{\Phi' a_i'}\right)$$

- $\overline{\text{div grad } a} = \text{div grad } \overline{a}$ ou $\overline{\nabla \cdot (\nabla a)} = \nabla \cdot (\nabla \overline{a}) = \nabla^2 \overline{a}$

$$\overline{\frac{\partial}{\partial x_j}\left(\frac{\partial a_i}{\partial x_j}\right)} = \frac{\partial}{\partial x_j}\left(\frac{\partial \overline{a_i}}{\partial x_j}\right) = \frac{\partial^2 \overline{a_i}}{\partial x_j \partial x_j}$$

Reynolds Averaged Navier-Stokes

Continuidade

$$\frac{\partial \overline{u_i}}{\partial x_i} \text{ ou } \nabla \cdot \overline{u} = 0$$

Quantidade de movimento

$$\text{Instantânea: } \frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j) = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

$$\frac{\partial u}{\partial t} + \nabla \cdot (uu) = -\frac{1}{\rho}\nabla p + \nu \nabla^2 u$$

$$\text{Média: } \frac{\partial}{\partial x_j}(\overline{u_i}\,\overline{u_j}) + \frac{\partial}{\partial x_j}\left(\overline{u_i' u_j'}\right) = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{u_i}}{\partial x_j \partial x_j}$$

$$\nabla \cdot (\overline{u}\,\overline{u}) + \nabla \cdot \left(\overline{u' u'}\right) = -\frac{1}{\rho}\nabla \overline{p} + \nu \nabla^2 \overline{u}$$

Reynolds Average Navier-Stokes

$$-\rho\overline{u_i'u_j'} = -\rho\begin{pmatrix} \overline{u'^2} & \overline{u'v'} & \overline{u'w'} \\ \overline{v'u'} & \overline{v'^2} & \overline{v'w'} \\ \overline{w'u'} & \overline{w'v'} & \overline{w'^2} \end{pmatrix} \Rightarrow \text{Tensor de Reynolds}$$

$$\underbrace{\mathrm{Tr}\begin{pmatrix}\text{Tensor de} \\ \text{Reynolds}\end{pmatrix}}_{\text{traço}} = 2K = -2\rho\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

$$K \rightarrow \text{Energia cinética turbulenta}$$

Reynolds Averaged Navier Stokes

Transporte de Escalar

$$\text{Instantânea: } \frac{\partial \Phi}{\partial t} + \frac{\partial}{\partial x_j}(\Phi u_j) = \frac{D}{\rho}\frac{\partial^2 \Phi}{\partial x_j \partial x_j} + S_\Phi$$

$$\frac{\partial \boldsymbol{\Phi}}{\partial t} + \nabla \cdot (\boldsymbol{\Phi u}) = \frac{D}{\rho}\nabla^2 \boldsymbol{\Phi} + S_\Phi$$

$$\text{Média: } \frac{\partial}{\partial x_j}\left(\overline{\Phi}\,\overline{u_j}\right) + \frac{\partial}{\partial x_j}\left(\overline{\Phi' u_j'}\right) = \frac{D}{\rho}\frac{\partial^2 \overline{\Phi}}{\partial x_j \partial x_j} + \overline{S_\Phi}$$

$$\nabla \cdot \left(\overline{\boldsymbol{\Phi}}\,\overline{\boldsymbol{u}}\right) + \nabla \cdot \left(\overline{\boldsymbol{\Phi}' \boldsymbol{u}'}\right) = \frac{D}{\rho}\nabla^2 \overline{\boldsymbol{\Phi}} + \overline{S_\Phi}$$

Exemplo: Placa plana c/ $\rho =$ cte (Turns cap. 11)

Quantidade de movimento em x $\left(\frac{\partial p}{\partial x} = 0\right)$

$$\underbrace{\frac{\partial v_x}{\partial t}}_{1} + \underbrace{\frac{\partial}{\partial x}(v_x v_x)}_{2} + \underbrace{\frac{\partial}{\partial y}(v_x v_y)}_{3} = \underbrace{\nu \frac{\partial^2 v_x}{\partial y \partial y}}_{4}$$

$$1 : \overline{\frac{\partial}{\partial t}(\overline{v_x} + v_x')} = \frac{\partial \overline{v_x}}{\partial t} + \frac{\partial \overline{v_x'}}{\partial t} = 0$$

$$2 : \overline{\frac{\partial}{\partial x}(\overline{v_x} + v_x')(\overline{v_x} + v_x')} = \frac{\partial}{\partial x}(\overline{v_x}\,\overline{v_x}) + \frac{\partial}{\partial x}\left(\overline{v_x'^2}\right)$$

$$3 : \overline{\frac{\partial}{\partial y}(\overline{v_x} + v_x')\left(\overline{v_y} + v_y'\right)} = \frac{\partial}{\partial y}(\overline{v_x}\,\overline{v_y}) + \frac{\partial}{\partial y}\left(\overline{v_x' v_y'}\right)$$

$$4 : \nu \overline{\frac{\partial^2 v_x}{\partial y \partial y}} = \nu \frac{\partial^2 \overline{v_x}}{\partial y \partial y}$$

Exemplo: Placa plana c/ $\rho =$ cte (Turns cap. 11)

Quantidade de movimento em x

$$\frac{\partial}{\partial x}\left(\overline{v_x}\,\overline{v_x}\right)+\frac{\partial}{\partial y}\left(\overline{v_x}\,\overline{v_y}\right)+\frac{\partial}{\partial y}\left(\overline{v_x' v_y'}\right)+\frac{\partial}{\partial x}\left(\overline{v_x' v_x'}\right)=\nu\frac{\partial^2\overline{v_x}}{\partial y\partial y}$$

Continuidade

$$\frac{\partial\overline{v_x}}{\partial x}+\frac{\partial\overline{v_y}}{\partial y}=0$$

- p/ jato (análogo à placa plana)

$$\rho\left(\overline{v_x}\frac{\partial\overline{v_x}}{\partial x}+\overline{v_r}\frac{\partial\overline{v_x}}{\partial r}\right)=\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\overline{v_x}}{\partial r}\right)-\frac{1}{r}\frac{\partial}{\partial r}\left(\rho r\overline{v_x' v_r'}\right)$$

Eddy viscosity ou hipótese de Boussinesq (1877)

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\tau_{\text{lam}} + \tau_{\text{turb}}\right)\right]$$

$$\tau_{\text{lam}} = \mu \frac{\partial \overline{v_x}}{\partial r}$$

$$\tau_{\text{turb}} = -\rho \nu_t \frac{\partial \overline{v_x}}{\partial r} \; ; \; \nu_t = \frac{\mu_t}{\rho}$$

$$\mu_{\text{eff}} = \mu_{\text{molecular}} + \mu_{\text{turb}}$$

1. Como determinar μ_{turb}?
2. $\mu_{\text{molecular}}$ é propriedade termodinâmica de transporte
   μ_{turb} é dependente do "padrão"do escoamento
3. Nem sempre é função apenas do grad da velocidade.

Eddy viscosity ou hipótese de Boussinesq (1877)

Comprimento de mistura de Prandtl l_m

$$\mu_t = \rho \nu_t = \rho l_m v_{turb} = \rho l_m^2 \left| \frac{\partial v_x}{\partial x_r} \right|$$

- Jatos livres: $v_{turb} \propto \overline{v}_{x,max} - \overline{v}_{x,min}$

Comprimento de mistura de Prandtl l_m

$$\mu_t = \underbrace{0,1365}_{experimental} \rho l_m^2 \left(\overline{v}_{x,max} - \overline{v}_{x,min} \right)$$

Eddy viscosity ou hipótese de Boussinesq (1877)

- Jatos livres: $v_{turb} \propto \overline{v}_{x,max} - \overline{v}_{x,min}$

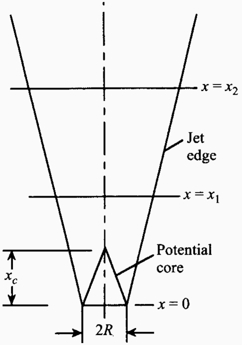
Quantidade de movimento

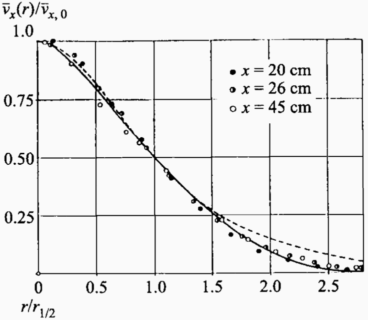
$$\rho\left(\overline{v_x}\frac{\partial\overline{v_x}}{\partial x} + \overline{v_r}\frac{\partial\overline{v_x}}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left[r(\mu+\mu_t)\frac{\partial\overline{v_x}}{\partial r}\right]$$

Continuidade

$$\frac{\partial}{\partial x}\left(\overline{v}_x r\right) + \frac{\partial}{\partial r}\left(\overline{v}_r r\right) = 0$$

Eddy viscosity ou hipótese de Boussinesq (1877)





$$J_e = \rho_e v_e^2 \pi R^2 \quad ; \quad l_m = 0,075\ \delta_{99\%}$$

$$\xi = \left[\frac{3J_e}{16\rho_e\pi}\right]\frac{1}{\nu_t}\frac{r}{x}$$

$\delta_{99\%} \rightarrow$ raio no qual $\dfrac{\overline{v}_x(r)}{\overline{v}_{x,0}} = 1\%$

a) $\nu_t = 0,0102\ \delta_{99\%}\ \overline{v}_{x,0}(x) \approx cte$

b) $r_{1/2} \propto x$

$\overline{v}_{x,0} \propto x^{-1}$

Eddy viscosity ou hipótese de Boussinesq (1877)

Aplicando a solução analítica p/ jato laminar:

$$\mu_{lam} \Rightarrow \rho\nu_t$$

$$\overline{v}_x = \frac{3}{8\pi}\frac{J_e}{\rho\nu_t x}\left[1+\frac{\xi^2}{4}\right]^{-2} \qquad \overline{v}_r = \left[\frac{3J_e}{16\pi\rho_e}\right]^{1/2}\frac{1}{x}\frac{\xi-\xi^3/4}{[1+\xi^2/4]^2}$$

Fazendo $\dfrac{\overline{v}_x}{v_e} = 0,375(v_e R/\nu_t)(x/R)^{-1}[1+\xi^2/4]^{-2}$, no centro
$r = 0 \Rightarrow \xi = 0$

$$\frac{\overline{v}_x}{v_e} = 0,375(v_e R/\nu_t)(x/R)^{-1}[1+\xi^2/4]^{-2} \tag{1}$$

$$\frac{\overline{v}_x}{\overline{v}_{x,0}} = \frac{1}{2} = \left[1+\xi^2/4\right]^{-2} \rightarrow \xi = 1,287 = \frac{3}{16}\left[\frac{\rho_e v_e^2 \pi R^2}{\rho_e \pi}\right]^{1/2}\frac{1}{\nu_t}\frac{r_{1/2}}{x}$$

$$\rightarrow 1,287 = 0,43 v_e R\frac{1}{\nu_t}\frac{r_{1/2}}{x} \rightarrow r_{1/2} = 2,97\left(\frac{v_e R}{\nu_t x}\right)^{-1} \tag{2}$$

Eddy viscosity ou hipótese de Boussinesq (1877)

Resolvendo 1, 2 e a)

$$\frac{\overline{v}_{x,o}}{v_e} = 13,15(x/R)^{-1} \qquad \nu_t = 0,0285 v_e R$$
$$\frac{r_{1/2}}{x} = 0,08468$$

Lembrando no caso laminar $\frac{\overline{v}_{x,0}}{v_e} \propto Re_{jet}$ e $\frac{r_{1/2}}{x} \propto Re_{jet}^{-1}$. Portanto no caso turbulento existe independência do número de Reynolds.

Quantidade de movimento instantânea

$$\frac{\partial}{\partial x_j}\left(\rho u_i u_j\right) = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \tag{3}$$

Aplicando decomposição de Reynolds e a média temporal

$$\frac{1}{\Delta t}\int_0^t \left[\frac{\partial}{\partial x_j}\rho(\overline{u}_i\overline{u}_j + \overline{u}_i u_j' + u_i'\overline{u}_j + u_i'u_j') = -\frac{\partial}{\partial x_i}\left(\overline{p} + p'\right) + \frac{\partial}{\partial x_j}\left(\overline{\tau}_{ij} + \tau_{ij}'\right)\right] dt$$

$$\frac{\partial}{\partial x_j}\rho(\overline{u}_i\overline{u}_j + \overline{u_i'u_j'}) = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial \overline{\tau}_{ij}}{\partial x_j}$$

Quantidade de movimento média

$$\frac{\partial}{\partial x_j}\left(\rho\overline{u}_i\overline{u}_j\right) = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\overline{\tau}_{ij} - \rho\overline{u_i'u_j'}\right) \tag{4}$$

Equação de transporte p/ $-\rho\overline{u_i'u_j'}$

Reescrevendo o indice repetido como k na equação 3 e multiplicando por $\overline{u}_i$ e $\overline{u}_j$ e somando as duas, lembrando a continuidade $\dfrac{\partial}{\partial x_i}(\rho u_i) = 0$, temos para o termo advectivo

$$u_j \frac{\partial}{\partial x_k}(\rho u_i u_k) + u_i \frac{\partial}{\partial x_k}(\rho u_j u_k) \rightarrow \quad u_j \left[u_i \underbrace{\frac{\partial}{\partial x_k}(\rho u_k)}_{=0} + \rho u_k \frac{\partial u_i}{\partial x_k} \right] + u_i \left[u_j \underbrace{\frac{\partial}{\partial x_k}(\rho u_k)}_{=0} + \rho u_k \frac{\partial u_j}{\partial x_k} \right]$$

$$\rightarrow \rho u_j u_k \frac{\partial u_i}{\partial x_k} + \rho u_i u_k \frac{\partial u_j}{\partial x_k} \rightarrow \rho u_k \left[u_j \frac{\partial u_i}{\partial x_k} + u_i \frac{\partial u_j}{\partial x_k} \right]$$

$$\rightarrow \rho u_k \frac{\partial}{\partial x_k}(u_i u_j) \rightarrow \boxed{\frac{\partial}{\partial x_k}(\rho u_i u_j u_k)} = \rho u_k \frac{\partial}{\partial x_k}(u_i u_j) + \rho u_i u_j \underbrace{\frac{\partial u_k}{\partial x_k}}_{=0}$$

Equação de transporte p/ $-\rho\overline{u_i' u_j'}$

Assim a nova equação resulta

$$\frac{\partial}{\partial x_k}\left(\rho u_i u_j u_k\right) = -u_j \frac{\partial p}{\partial x_i} - u_i \frac{\partial p}{\partial x_j} + u_j \frac{\partial \tau_{ik}}{\partial x_k} + u_i \frac{\partial \tau_{jk}}{\partial x_k}$$

Decomposição de Reynolds

$$\begin{aligned}
\frac{\partial}{\partial x_k}\left[\rho(\overline{u}_i + u_i')(\overline{u}_j + u_j')(\overline{u}_k + u_k')\right] = &- (\overline{u}_j + u_j')\frac{\partial(\overline{p} + p')}{\partial x_i} \\
&- (\overline{u}_i + u_i')\frac{\partial(\overline{p} + p')}{\partial x_j} \\
&+ (\overline{u}_j + u_j')\frac{\partial}{\partial x_k}\left(\overline{\tau}_{ik} + \tau_{ik}'\right) \\
&+ (\overline{u}_i + u_i')\frac{\partial}{\partial x_k}\left(\overline{\tau}_{jk} + \tau_{jk}'\right)
\end{aligned}$$

Organizando e aplicando a média

$$\frac{\partial}{\partial x_k}\left[\overline{\rho\overline{u}_i\overline{u}_j\overline{u}_k}+\overline{\overline{u}_i\rho u_j'u_k'}+\overline{\overline{u}_j\rho u_i'u_k'}+\overline{\overline{u}_k\rho u_i'u_j'}+\overline{\rho u_i'u_j'u_k'}\right]$$
$$=-\overline{\overline{u}_j\frac{\partial\overline{p}}{\partial x_i}}-\overline{\overline{u}_j\frac{\partial p'}{\partial x_i}}-\overline{u_j'\frac{\partial\overline{p}}{\partial x_i}}-\overline{u_j'\frac{\partial p'}{\partial x_i}}-\overline{\overline{u}_i\frac{\partial\overline{p}}{\partial x_j}}-\overline{\overline{u}_i\frac{\partial p'}{\partial x_j}}-\overline{u_i'\frac{\partial\overline{p}}{\partial x_j}}-\overline{u_i'\frac{\partial p'}{\partial x_j}}$$
$$-\overline{\overline{u}_j\frac{\partial\overline{\tau}_{ik}}{\partial x_k}}-\overline{\overline{u}_j\frac{\partial\tau_{ik}'}{\partial x_k}}-\overline{u_j'\frac{\partial\overline{\tau}_{ik}}{\partial x_k}}-\overline{u_j'\frac{\partial\tau_{ik}'}{\partial x_k}}-\overline{\overline{u}_i\frac{\partial\overline{\tau}_{jk}}{\partial x_k}}-\overline{\overline{u}_i\frac{\partial\tau_{jk}'}{\partial x_k}}-\overline{u_i'\frac{\partial\overline{\tau}_{jk}}{\partial x_k}}-\overline{u_i'\frac{\partial\tau_{jk}'}{\partial x_k}}$$

$$\frac{\partial}{\partial x_k}[\rho\overline{u}_i\overline{u}_j\overline{u}_k+\overbrace{\rho\overline{u}_i\overline{u_j'u_k'}}^{A}+\rho\overline{u}_j\overline{u_i'u_k'}+\rho\overline{u}_k\overline{u_i'u_j'}+\rho\overline{u_i'u_j'u_k'}]$$
$$=-\overline{u}_j\frac{\partial\overline{p}}{\partial x_i}-\overline{u_j'\frac{\partial p'}{\partial x_i}}-\overline{u}_i\frac{\partial\overline{p}}{\partial x_j}-\overline{u_i'\frac{\partial p'}{\partial x_j}}-\overline{u}_j\frac{\partial\overline{\tau}_{ik}}{\partial x_k}-\overline{u_j'\frac{\partial\tau_{ik}'}{\partial x_k}}-\overline{u}_i\frac{\partial\overline{\tau}_{jk}}{\partial x_k}-\overline{u_i'\frac{\partial\tau_{jk}'}{\partial x_k}} \quad (5)$$

Equação de transporte p/ $-\rho\overline{u_i'u_j'}$

Reescrevendo o indice repetido como k na equação 4 e multiplicando por $\overline{u}_i$ e por $\overline{u}_j$ e então somando as duas equações resultantes temos

$$\frac{\partial}{\partial x_k}\left(\rho\overline{u}_i\overline{u}_j\overline{u}_k\right) = -\overline{u}_j\frac{\partial\overline{p}}{\partial x_i} - \overline{u}_i\frac{\partial\overline{p}}{\partial x_j} + \overline{u}_j\frac{\partial}{\partial x_k}\left(\overline{\tau}_{ik} - \rho\overline{u_i'u_k'}\right) + \overline{u}_i\frac{\partial}{\partial x_k}(\overline{\tau}_{jk} - \overbrace{\rho\overline{u_j'u_k'}}^{\text{A'}}) \tag{6}$$

Subtraindo 6 de 5, sendo $\text{A} = \text{A'} - \rho\overline{u_j'u_k'}\dfrac{\partial\overline{u}_i}{\partial x_k}$

Equação de transporte de $\overline{u_i'u_j'}$

$$\begin{aligned}\frac{\partial}{\partial x_k}\left(\rho\overline{u}_k\overline{u_i'u_j'}\right) = &-\frac{\partial}{\partial x_k}\left(\rho\overline{u_i'u_j'u_k'}\right) - \overline{u_j'\frac{\partial p'}{\partial x_i}} - \overline{u_i'\frac{\partial p'}{\partial x_j}} \\ &+ \overline{u_j'\frac{\partial\tau_{ik}'}{\partial x_k}} + \overline{u_i'\frac{\partial\tau_{jk}'}{\partial x_k}} - \rho\overline{u_i'u_k'}\frac{\partial\overline{u}_i}{\partial x_k} - \rho\overline{u_j'u_k'}\frac{\partial\overline{u}_j}{\partial x_k}\end{aligned} \tag{7}$$

Equação de transporte p/ $-\rho\overline{u_i'u_j'}$

- Termos de tensões:

$$\overline{u_j'\frac{\partial\tau_{ik}'}{\partial x_k}}+\overline{u_i'\frac{\partial\tau_{jk}'}{\partial x_k}}\;\;;\;\;\tau_{ik}'=\mu\frac{\partial u_i'}{\partial x_k}$$

$$\rightarrow \overline{u_j'\frac{\partial}{\partial x_k}\left(\mu\frac{\partial u_i'}{\partial x_k}\right)}+\overline{u_i'\frac{\partial}{\partial x_k}\left(\mu\frac{\partial u_j'}{\partial x_k}\right)}$$

$$\rightarrow \mu\left[\frac{\partial}{\partial x_k}\left(\overline{u_j'\frac{\partial u_i'}{\partial x_k}}\right)-\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}+\frac{\partial}{\partial x_k}\left(\overline{u_i'\frac{\partial u_j'}{\partial x_k}}\right)-\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}\right]$$

$$\rightarrow \mu\left[\frac{\partial}{\partial x_k}\left(\frac{\partial(\overline{u_i'u_j'})}{\partial x_k}\right)-2\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}\right]$$

- Termo de dissipação:

$$\mu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}=\varepsilon_{ik}=\frac{2}{3}\delta_{ik}\varepsilon$$

Equação de transporte p/ $-\rho\overline{u_i'u_j'}$

- Termos de pressão:

$$-\overline{u_j'\frac{\partial p'}{\partial x_i}}-\overline{u_i'\frac{\partial p'}{\partial x_j}}\rightarrow-\frac{\partial}{\partial x_k}\left(\overline{p'\delta_{ki}u_j'}\right)+\overline{p'\frac{\partial u_j'}{\partial x_i}}-\frac{\partial}{\partial x_k}\left(\overline{p'\delta_{kj}u_i'}\right)+\overline{p'\frac{\partial u_i'}{\partial x_j}}$$

$$\rightarrow-\frac{\partial}{\partial x_k}\left(\overline{p'\delta_{ki}u_j'}+\overline{p'\delta_{kj}u_i'}\right)+\overline{p'\left(\frac{\partial u_j'}{\partial x_i}+\frac{\partial u_i'}{\partial x_j}\right)}$$

Finalmente temos

Equação de transporte de $\overline{u_i'u_j'}$

$$\begin{aligned}\frac{\partial}{\partial x_k}\left(\rho\overline{u}_k\overline{u_i'u_j'}\right) = &-\frac{\partial}{\partial x_k}\left(\rho\overline{u_i'u_j'u_k'}\right) - \frac{\partial}{\partial x_k}\left(\overline{p'\delta_{ki}u_j'} + \overline{p'\delta_{kj}u_i'}\right)\\ &+\overline{p'\left(\frac{\partial u_j'}{\partial x_i} + \frac{\partial u_i'}{\partial x_j}\right)} + \mu\left[\frac{\partial}{\partial x_k}\left(\frac{\partial(\overline{u_i'u_j'})}{\partial x_k}\right)\right]\\ &-\frac{2}{3}\delta_{ik}\varepsilon - \rho\overline{u_i'u_k'}\frac{\partial\overline{u}_i}{\partial x_k} - \rho\overline{u_j'u_k'}\frac{\partial\overline{u}_j}{\partial x_k}\end{aligned} \tag{8}$$

A partir da equação 8, chegar na equação da energia cinética turbulenta $K = \left(\frac{1}{2}\overline{u_i'u_i'}\right)$.

Considerações finais

Perguntas?