CS-E4850 Computer Vision
Exercise Round 6

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Exercise 1. Least-squares fitting for affine transformations. (pen & paper problem) A brief overview of affine transformation estimation is presented on slides 17-19 of Lecture 5. Present a derivation and compute an example by performing the following stages:

a) Compute the gradient of the least squares error $E = \sum_{i=1}^{n} ||x_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t}||^2$ with respect to the parameters of the transformation (i.e. elements of matrix $\mathbf{M}$ and vector $\mathbf{t}$).

Solution.

$$VE = -2\sum_{i=1}^{n}\left(\frac{\partial M}{\partial m_1}x_i\right)^\top (x_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t}) \; - \; 2\sum_{i=1}^{n}\left(\frac{\partial M}{\partial m_2}x_i\right)^\top (x_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t}) -$$

$$- 2\sum_{i=1}^{n}\left(\frac{\partial M}{\partial m_3}x_i\right)^\top (x_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t}) \; - \; 2\sum_{i=1}^{n}\left(\frac{\partial M}{\partial m_4}x_i\right)^\top (x_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t}) -$$

$$- 2\sum_{i=1}^{n}(x_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t})$$

b) Show that by setting the aforementioned gradient to zero you will get an equation of the form $\mathbf{S}\mathbf{h} = \mathbf{u}$, where vector $\mathbf{h}$ contains the unknown parameters of the transformation, and 6×6 matrix $\mathbf{S}$ and 6×1 vector $\mathbf{u}$ depend on the coordinates of the point correspondences $\{x_i', x_i\}, i = 1, \ldots, n$.

Solution. The resulting equations obtained setting the gradient above to zero are:

$$\sum_{i=1}^{n}\left(\frac{\partial M}{\partial m_1}x_i\right)^\top (x_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t}) = 0,$$

$$\sum_{i=1}^{n}\left(\frac{\partial M}{\partial m_2}x_i\right)^\top (x_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t}) = 0,$$

$$\sum_{i=1}^{n}\left(\frac{\partial M}{\partial m_3}x_i\right)^\top (x_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t}) = 0,$$

$$\sum_{i=1}^{n}\left(\frac{\partial M}{\partial m_4}x_i\right)^{\top}(x_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t}) = 0,$$

$$\sum_{i=1}^{n}(x_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t}) = 0.$$

Knowing that,

$$\sum_{i=1}^{n}(x_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t}) = \sum_{i=1}^{n}\begin{pmatrix} x_i' \\ y_i' \end{pmatrix} - \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}\begin{pmatrix} x_i \\ y_i \end{pmatrix} - \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \sum_{i=1}^{n}\begin{pmatrix} x_i' - m_1x_i - m_2y_i - t_1 \\ y_i' - m_2x_i - m_4y_i - t_2 \end{pmatrix}.$$

We obtain,

$$\sum_{i=1}^{n}\left(\frac{\partial M}{\partial m_1}x_i\right)^{\top}(x_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t}) = \sum_{i=1}^{n}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_i \\ y_i \end{pmatrix}\right)^{\top}\begin{pmatrix} x_i' - m_1x_i - m_2y_i - t_1 \\ y_i' - m_2x_i - m_4y_i - t_2 \end{pmatrix} =$$

$$= \sum_{i=1}^{n} x_i'x_i - m_1x_i^2 - m_2y_ix_i - t_1x_i = 0.$$

Analogously we do the same for the next three equations and we obtain,

$$\sum_{i=1}^{n}\left(\frac{\partial M}{\partial m_2}x_i\right)^{\top}(x_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t}) = \sum_{i=1}^{n} x_i'y_i - m_1x_iy_i - m_2y_i^2 - t_1y_i = 0,$$

$$\sum_{i=1}^{n}\left(\frac{\partial M}{\partial m_3}x_i\right)^{\top}(x_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t}) = \sum_{i=1}^{n} y_i'x_i - m_3x_i^2 - m_4y_ix_i - t_2x_i = 0,$$

$$\sum_{i=1}^{n}\left(\frac{\partial M}{\partial m_4}x_i\right)^{\top}(x_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t}) = \sum_{i=1}^{n} y_i'y_i - m_3x_iy_i - m_4y_i^2 - t_2y_i = 0.$$

Hence,

$$\sum_{i=1}^{n} x_i'x_i = \sum_{i=1}^{n} m_1x_i^2 + m_2y_ix_i + t_1x_i,$$

$$\sum_{i=1}^{n} x_i'y_i = \sum_{i=1}^{n} m_1x_iy_i + m_2y_i^2 + t_1y_i,$$

$$\sum_{i=1}^{n} y_i'x_i = \sum_{i=1}^{n} m_3x_i^2 + m_4y_ix_i + t_2x_i,$$

$$\sum_{i=1}^{n} y_i'y_i = \sum_{i=1}^{n} m_3x_iy_i + m_4y_i^2 + t_2y_i.$$

From the last equation ($\sum_{i=1}^{n}(x_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t}) = 0$) we can extract,

$$\sum_{i=1}^{n} x_i' - m_1x_i - m_2y_i - t_1 = 0,$$

$$\sum_{i=1}^{n} x_i' = \sum_{i=1}^{n} m_1x_i + m_2y_i + t_1.$$

And also,

$$\sum_{i=1}^{n} y_i' - m_3 x_i - m_3 y_i - t_2 = 0,$$

$$\sum_{i=1}^{n} y_i' = \sum_{i=1}^{n} m_3 x_i + m_4 y_i + t_2.$$

Finally we can express the system $\mathbf{Sh} = \mathbf{u}$ as,

$$\mathbf{Sh} = \sum_{i=1}^{n} \begin{pmatrix} x_i^2 & x_i y_i & 0 & 0 & x_i & 0 \\ x_i y_i & y_i^2 & 0 & 0 & y_i & 0 \\ 0 & 0 & x_i^2 & x_i y_i & 0 & x_i \\ 0 & 0 & x_i y_i & y_i^2 & 0 & y_i \\ x_i & y_i & 0 & 0 & 1 & 0 \\ 0 & 0 & x_i & y_i & 0 & 1 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ t_1 \\ t_2 \end{pmatrix} = \sum_{i=1}^{n} \begin{pmatrix} x_i' x_i \\ x_i' y_i \\ y_i' x_i \\ y_i' y_i \\ x_i' \\ y_i' \end{pmatrix} = \mathbf{u}$$

c) Thus, one may solve the transformation by computing $\mathbf{h} = \mathbf{S}^{-1}\mathbf{u}$. Compute the affine transformation from the following point correspondences $\{(0,0) \to (1,2)\}$, $\{(1,0) \to (3,2)\}$, and $\{(0,1) \to (1,4)\}$.

Solution.

$$n = 3,$$
$$(x_1, y_1) = (0,0) \to (x_1', y_1') = (1,2),$$
$$(x_2, y_2) = (1,0) \to (x_2', y_2') = (3,2),$$
$$(x_3, y_3) = (0,1) \to (x_3', y_3') = (1,4).$$

$$\sum_{i=0}^{n} x_i = x_1 + x_2 + x_3 = (0) + (1) + (0) = 1$$

$$\sum_{i=0}^{n} y_i = y_1 + y_2 + y_3 = (0) + (0) + (1) = 1$$

$$\sum_{i=0}^{n} x_i^2 = x_1^2 + x_2^2 + x_3^2 = (0)^2 + (1)^2 + (0)^2 = 1$$

$$\sum_{i=0}^{n} y_i^2 = y_1^2 + y_2^2 + y_3^2 = (0)^2 + (0)^2 + (1)^2 = 1$$

$$\sum_{i=0}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + x_3 y_3 = (0)(0) + (1)(0) + (0)(1) = 0$$

$$\mathbf{S} = \sum_{i=1}^{n} \begin{pmatrix} x_i^2 & x_i y_i & 0 & 0 & x_i & 0 \\ x_i y_i & y_i^2 & 0 & 0 & y_i & 0 \\ 0 & 0 & x_i^2 & x_i y_i & 0 & x_i \\ 0 & 0 & x_i y_i & y_i^2 & 0 & y_i \\ x_i & y_i & 0 & 0 & 1 & 0 \\ 0 & 0 & x_i & y_i & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 & 3 \end{pmatrix}$$

$$\mathbf{S}^{-1} = \begin{pmatrix} 2 & 1 & 0 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 & 0 & -1 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{pmatrix}$$

$$\mathbf{u} = \sum_{i=1}^{n} \begin{pmatrix} x_i'x_i \\ x_i'y_i \\ y_i'x_i \\ y_i'y_i \\ x_i' \\ y_i' \end{pmatrix} = \begin{pmatrix} (1)(0)+(3)(1)+(1)(0) \\ (1)(0)+(3)(0)+(1)(1) \\ (2)(0)+(2)(1)+(4)(0) \\ (2)(0)+(2)(0)+(4)(1) \\ (1)+(3)+(1) \\ (2)+(2)+(4) \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 4 \\ 5 \\ 8 \end{pmatrix}$$

$$\mathbf{h} = \mathbf{S}^{-1}\mathbf{u} =$$

$$= \begin{pmatrix} 2 & 1 & 0 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 & 0 & -1 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 2 \\ 4 \\ 5 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \\ 1 \\ 2 \end{pmatrix}$$

Exercise 2. Similarity transformation from two point correspondences. (pen & paper) A similarity transformation consists of rotation, scaling and translation and is defined in two dimensions as follows:

$$\mathbf{x}' = s\mathbf{R}\mathbf{x} + \mathbf{t} \quad \Leftrightarrow \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = s \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix} \tag{1}$$

Describe a method for solving the parameters s, θ, t_x, t_y of a similarity transformation from two point correspondences $\{\mathbf{x}_1 \to \mathbf{x}'_1\}$,$\{\mathbf{x}_2 \to \mathbf{x}'_2\}$ using the following stages:

a) Compute the vectors $\mathbf{v}' = \mathbf{x}'_2 - \mathbf{x}'_1$ and $\mathbf{v} = \mathbf{x}_2 - \mathbf{x}_1$ and present a formula to recover the rotation angle θ from the corresponding unit vectors.

Solution.

$$\mathbf{v} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$$

$$\mathbf{v}' = \begin{pmatrix} x'_2 - x'_1 \\ y'_2 - y'_1 \end{pmatrix} = s \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$$

Knowing that,

$$\cos(\theta) = \frac{v' \cdot v}{||v'|| \cdot ||v||},$$

we can obtain,

$$\theta = \cos^{-1}\left(\frac{(x'_2 - x'_1)(x_2 - x_1) + (y'_2 - y'_1)(y_2 - y_1)}{\sqrt{(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \right).$$

b) Compute the scale factor $\mathbf{s}$ as the ratio of the norms of vectors $\mathbf{v}'$ and $\mathbf{v}$.

Solution.

$$\mathbf{s} = \frac{||\mathbf{v}'||}{||\mathbf{v}||} = \frac{\sqrt{(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^1}}.$$

c) After solving s and θ compute $\mathbf{t}$ using equation (1) and either one of the two point correspondences.

Solution.

$$x' = s\cos(\theta)x - s\sin(\theta)y + t_x$$
$$y' = s\sin(\theta)x + s\cos(\theta)y + t_y$$

$$t_x = x' - s\cos(\theta)x + s\sin(\theta)y$$
$$t_y = y' - s\sin(\theta)x - s\cos(\theta)y$$

d) Use the procedure to compute the transformation from the following point correspondences: $\{(\frac{1}{2},0) \to (0,0)\}$,$\{(0,\frac{1}{2}) \to (-1,-1)\}$.

Solution.

$$(x_1,y_1) = (\frac{1}{2},0), \quad (x_1',y_1') = (0,0),$$

$$(x_2,y_2) = (0,\frac{1}{2}), \quad (x_2',y_2') = (-1,-1).$$

$$\mathbf{v} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix},$$

$$\mathbf{v}' = \begin{pmatrix} x_2' - x_1' \\ y_2' - y_1' \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

$$\theta = \cos^{-1}\left(\frac{(x_2'-x_1')(x_2-x_1)+(y_2'-y_1')(y_2-y_1)}{\sqrt{(x_2'-x_1')^2+(y_2'-y_1')^2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}\right) =$$

$$= \cos^{-1}\left(\frac{(-1)(-\frac{1}{2})+(-1)(\frac{1}{2})}{\dots}\right) = \cos^{-1}(0) = \frac{\pi}{2}.$$

$$\mathbf{s} = \frac{||\mathbf{v}'||}{||\mathbf{v}||} = \frac{\sqrt{(x_2'-x_1')^2+(y_2'-y_1')^2}}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^1}} =$$

$$= \sqrt{\frac{(-1)^2+(-1)^2}{(-\frac{1}{2})^2+\frac{1}{2})^2}} = \sqrt{\frac{2}{\frac{1}{2}}} = \sqrt{4} = 2.$$

$$t_x = x' - s\cos(\theta)x + s\sin(\theta)y = 0 - 2(0)(\frac{1}{2}) + 2(1)(0) = 0,$$

$$t_y = y' - s\sin(\theta)x - s\cos(\theta)y = 0 - 2(1)(\frac{1}{2}) - 2(0)(0) = -1.$$

So the resulting equation for the transformation is,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = 2\begin{pmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$