CS-E4850 Computer Vision
Exercise Round 1

Alex Herrero Pons

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Exercise 1. Homogeneous coordinates.
a) The equation of a line in the plane is

$$ax + by + c = 0$$

Show that by using homogeneous coordinates this can be written as

$$\mathbf{x}^\top \mathbf{l} = 0$$

where $\mathbf{l} = (a \ \ b \ \ c)^\top$.

Solution. We know that using homogeneous coordinates $\mathbf{x}^\top = (x \ \ y \ \ 1)$ and that $\mathbf{l} = (a \ \ b \ \ c)^\top$. Hence,

$$\mathbf{x}^\top \mathbf{l} = 0 \rightarrow (x \ \ y \ \ 1)(a \ \ b \ \ c)^\top = 0 \rightarrow ax + by + c = 0$$

$\square$

b) Show that the intersection of two lines $\mathbf{l}$ and $\mathbf{l}'$ is the point $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$.

Solution. To find the intersection of the lines $\mathbf{l}$ and $\mathbf{l}'$ we have to define the vector $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$ where $\times$ represents the cross product. Then from the triple scalar product identity $\mathbf{l}.(\mathbf{l} \times \mathbf{l}') = \mathbf{l}'.(\mathbf{l} \times \mathbf{l}') = 0$ we see that $\mathbf{l}^\top \mathbf{x} = \mathbf{l}'^\top \mathbf{x} = 0$. Hence, we can see that if we think of $\mathbf{x}$ as a point, then $\mathbf{x}$ lies in both lines $\mathbf{l}$ and $\mathbf{l}'$, therefore $\mathbf{x}$ is the intersection point.

$\square$

c) Show that the line through two points $\mathbf{x}$ and $\mathbf{x}'$ is $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$.

Solution. As done in the previous section defining the vector $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$ and using the triple scalar product identity $\mathbf{x}.(\mathbf{x} \times \mathbf{x}') = \mathbf{x}'.(\mathbf{x} \times \mathbf{x}') = 0$ we can observe that $\mathbf{x}^\top \mathbf{l} = \mathbf{x}'^\top \mathbf{l} = 0$. Thus, if $\mathbf{l}$ is thought of as representing a line, then the line $\mathbf{l}$ goes through both points $\mathbf{x}$ and $\mathbf{x}'$.

$\square$

d) Show that for all $\alpha \in \mathbb{R}$ the point $\mathbf{y} = \alpha\mathbf{x} + (1-\alpha)\mathbf{x}'$ lies on the line through points $\mathbf{x}$ and $\mathbf{x}'$.

Solution. Let $\mathbf{l}$ be the line defined as $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$. Knowing that $\mathbf{x} = (x\ ,\ y\ ,\ 1)^\top$ and $\mathbf{x}' = (x'\ ,\ y'\ ,\ 1)^\top$ we we have

$$\mathbf{l} = \mathbf{x} \times \mathbf{x}' = (y - y'\ ,\ x' - x\ ,\ xy' - x'y)^\top$$

. In the other hand we know that using homogeneous coordinates

$$\mathbf{y} = \alpha\mathbf{x} + (1-\alpha)\mathbf{x}' = (\alpha x + (1-\alpha)x'\ ,\ \alpha y + (1-\alpha)y'\ ,\ 1)^\top$$

Now to see that $\mathbf{y}$ belongs to $\mathbf{l}$ que have to prove that $\mathbf{l}^\top\mathbf{y} = 0$.

$$\mathbf{l}^\top\mathbf{y} = (y - y'\ ,\ x' - x\ ,\ xy' - x'y) \cdot (\alpha x + (1-\alpha)x'\ ,\ \alpha y + (1-\alpha)y'\ ,\ 1)^\top =$$

$$= (y - y')(\alpha x + (1-\alpha)x') + (x' - x)(\alpha y + (1-\alpha)y') + xy' - x'y =$$

$$= -x'y' + x'b' + x'y'\alpha - y'x\alpha - x'y\alpha + xy\alpha + x'y' - y'x - x'y'\alpha + y'x\alpha + x'y\alpha - xy\alpha + xy' - x'y = 0$$

□

Exercise 2. Transformations in 2D.
a) Use homogeneous coordinates and give the matrix representations of the following transformation groups: translation, Euclidean transformation (rotation+translation),similarity transformation (scaling+rotation+translation), affine transformation, projective transformation.

Solution.
Translation:

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Euclidean transformation:

$$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Similarity transformation:

$$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Affine transformation:

$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Projective transformation:

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

b) What is the number of degrees of freedom in these transformations?

Solution.
Translation: 2 dof.
Euclidean transformation: 3 dof.
Similarity transformation: 4 dof.
Affine transformation: 6 dof.
Projective transformation: 8 dof.

c) Why is the number of degrees of freedom in a projective transformation less than the number of elements in a 3×3 matrix?

Solution. Because the projective matrix is homogeneous, so only the ratio of the matrix elements is significant. Amongst the 9 elements of the matrix there are only 8 independent ratios, hence, there are 8 degrees of freedom and $8 < 9$.

$\square$

Exercise 3. Planar projective transformation.
The equation of a line on a plane, $ax + by + c = 0$, can be written as $\mathbf{l}^\top \mathbf{x} = 0$, where $\mathbf{l} = [a\ b\ c]^\top$ and $\mathbf{x}$ are homogeneous coordinates for lines and points, respectively. Under a planar projective transformation, represented with an invertible 3×3 matrix $\mathbf{H}$, points transform as

$$\mathbf{x}' = \mathbf{H}\mathbf{x}.$$

a) Given the matrix $\mathbf{H}$ for transforming points, as defined above, define the line transformation (i.e. transformation that gives $\mathbf{l}'$ which is a transformed version of $\mathbf{l}$).

Solution. The line transformation should be $\mathbf{l}' = \mathbf{H}^{-\top}\mathbf{l}$.

b) A projective invariant is a quantity which does not change its value in the transformation. Using the transformation rules for points and lines, show that two lines, $\mathbf{l}_1$, $\mathbf{l}_2$, and two points, $\mathbf{x}_1$, $\mathbf{x}_2$, not lying on the lines have the following invariant under projective transformation:

$$I = \frac{(\mathbf{l}_1^\top \mathbf{x}_1)(\mathbf{l}_2^\top \mathbf{x}_2)}{(\mathbf{l}_1^\top \mathbf{x}_2)(\mathbf{l}_2^\top \mathbf{x}_1)}.$$

Why similar construction does not give projective invariants with fewer number of points or lines? (Hint: Projective invariants defined via homogeneous coordinates must be invariant also to arbitrary scaling of the homogeneous coordinate vectors with a non-zero scaling factor.)

Solution. With the given formula you can multiply any term per any constant k and the result won't change. That happens because every term is at the same time in the numerator and in the denominator. For example is we multiply every $\mathbf{x}_1$ per a constant k we got that

$$I = \frac{(\mathbf{l}_1^\top \mathbf{x}_1 k)(\mathbf{l}_2^\top \mathbf{x}_2)}{(\mathbf{l}_1^\top \mathbf{x}_2)(\mathbf{l}_2^\top \mathbf{x}_1 k)} = \frac{(\mathbf{l}_1^\top \mathbf{x}_1)(\mathbf{l}_2^\top \mathbf{x}_2)}{(\mathbf{l}_1^\top \mathbf{x}_2)(\mathbf{l}_2^\top \mathbf{x}_1)},$$

and the same thing happens with any other term.

But if we make a similar construction without one of the terms the explanation above doesn't apply. For example if we delete the $\mathbf{l}_2$ we obtain that

$$I = \frac{\mathbf{l}_1^\top \mathbf{x}_1}{\mathbf{l}_1^\top \mathbf{x}_2}.$$