Radar Systems

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Outline

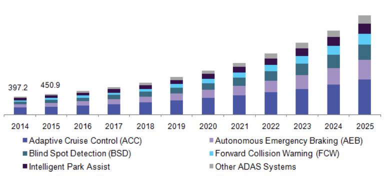
Table of Contents

Introduction

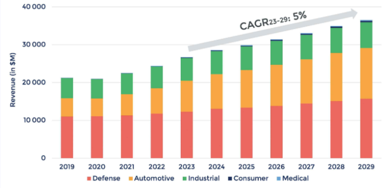
Table of Contents

Introduction

Market forecast

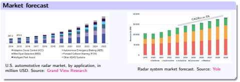


U.S. automototive radar market, by application, in million USD. Source: Grand View Research



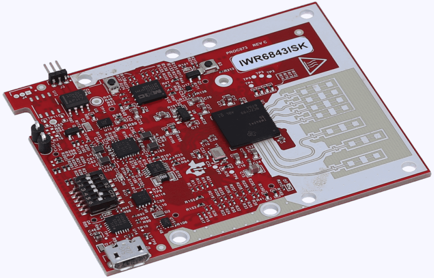
Radar system market forecast. Source: Yole

Motivation



Drivers

- Integration
- Reference designs
- Price drop



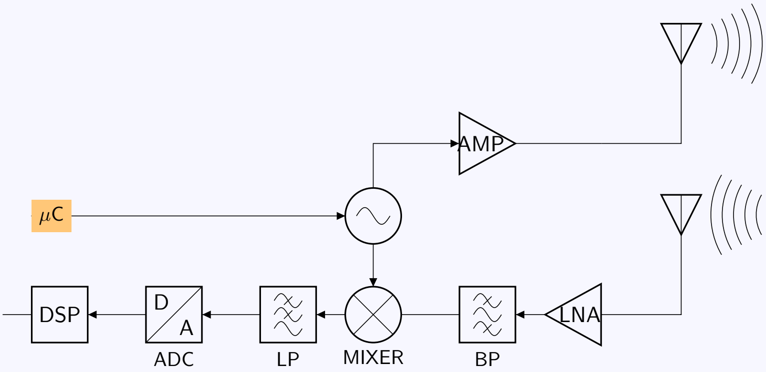
Source: Texas Instruments

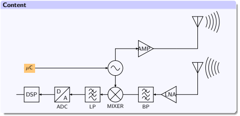
Table of Contents

Introduction

General remarks

Content





Content

- ▷ Wave propagation and antennas
- ▷ Block diagram
- ▷ Modulation
- ▷ Spectral analysis
- ▷ Point cloud processing
- ▷ Current trends in radar signal processing
- ▷ Applications

Table of Contents

Introduction

Literature

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- Stergiopoulos, Advanced Signal Processing, CRC Press, 2009
- Kay, S.; Fundamentals of Statistical Signal Processing, Vol. I: Estimation Theory, Prentice Hall,1993
- Mahafza, Radar Signal Analysis and Processing using Matlab, CRC Press, 2016
- Winner, Handbuch Fahrerassistenzsysteme, Springer, 2015

Table of Contents

Introduction

Preliminary outline

Preliminary outline

Week	Unit	Topic
41	1	Introduction, wave propagation
42	2	Block diagram
43	na	vacation
44	3	Spectral Analysis
45	4	Spectral Analysis
46	5	Angle finding
47	na	Block week
48	6	Angle finding
49 – 51	7 – 9	State Estimation and Tracking
2 – 4	10 – 12	Current trends & joint implementations

Table of Contents

Introduction

Exam

Exam

Assessment of the course: Written Exam (60 min, planned for TBA) at the end of the course (50%) and homework (50%) with demonstration/presentation. Homework deals with aspects of signal processing for uses cases in automotive or robotics. Homework is teamwork and can be based upon demonstration boards and/or Matlab/Python and public dataset.

Table of Contents

Introduction

Table of Contents

Wave propagation

Introduction

Content

- Basics of related electromagnetic wave theory
- Basics of antennas

Study goals

- Understand basic wave propagation phenomena
- Understand basic antenna properties

Table of Contents

Wave propagation

Maxwell's Equations

Maxwell's Equations

Maxwell's Equations

The Maxwell's Equations describe electromagnetic phenomena. They exist in differential and integral form[a]:

$$\operatorname{div} \boldsymbol{D} = \rho \quad \Longleftrightarrow \quad \oiint_{\partial V} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{A} = \iiint_V \rho \mathrm{d} V$$

$$\operatorname{div} \boldsymbol{B} = 0 \quad \Longleftrightarrow \quad \oiint_{\partial V} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{A} = 0$$

$$\operatorname{rot} \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad \Longleftrightarrow \quad \oint_{\partial A} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = -\iint_A \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{A}$$

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Symbols

- ρ: Charge density (Ladungsdichte) in $\mathrm{C/m^3}$ ($\mathrm{As/m^3}$)
- $\boldsymbol{D}$: Electric displacement field (Elektrische Flussdichte) in $\mathrm{C/m^2}$ ($\mathrm{As/m^2}$)
- $\boldsymbol{E}$: Electric field (Elektrische Feldstärke) in $\mathrm{V\,m^{-1}}$
- $\boldsymbol{H}$: Magnetic Field (Magnetische Feldstärke) in $\mathrm{A\,m^{-1}}$
- $\boldsymbol{B}$: Magnetic flux density (Magnetische Flussdichte) in T ($\mathrm{N\,A^{-1}\,m^{-1}}$)
- $\boldsymbol{J}$: Current density (Stromdichte) in $\mathrm{A/m^2}$
- $\frac{\partial \boldsymbol{D}}{\partial t}$: Displacement current density (Verschiebungsstromdichte) in $\mathrm{A/m^2}$

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\end{aligned}$$

[a]Using the integral theorems of Gauss and Stokes allows to switch from one domain to the other.

Maxwell's Equations

The Maxwell's Equations only describe the electromagnetic phenomena. They are not the solutions themselves. These are differential equations whose solutions depend on the boundary conditions. Solution methods are e.g.:

- numerical methods (e.g. finite integration, finite difference method, finite element method, integral equation method)
- analytic methods

Analytical solutions can only be determined for special cases. These include, for example, electromagnetic waves.

Table of Contents

Maxwell's Equations

Divergence

Definition

Given is a function $\boldsymbol{f}(x, y, z)$. The divergence is defined as follow:

$$\begin{aligned} \operatorname{div}(\boldsymbol{f}) &= \boldsymbol{e}_x \frac{\partial f_x(x,y,z)}{\partial x} + \boldsymbol{e}_y \frac{\partial f_y(x,y,z)}{\partial y} + \boldsymbol{e}_z \frac{\partial f_z(x,y,z)}{\partial z} \\ &= \boldsymbol{\nabla} \cdot \boldsymbol{f}(x,y,z), \end{aligned}$$

with

$$\boldsymbol{\nabla} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}.$$

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Electrical field

$$\mathrm{div}\, \boldsymbol{D} = \rho$$

The electric field is **not** source free.

Magnetic field

$$\mathrm{div}\, \boldsymbol{B} = 0$$

The magnetic field is source free.

Curl

Electrical field

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The curl is the defined as follows:

$$\begin{aligned} \operatorname{curl} \boldsymbol{f} = \operatorname{rot} \boldsymbol{f} &= \boldsymbol{\nabla} \times \boldsymbol{f}(x,y,z) \\ &= \boldsymbol{e}_x \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z} \right) + \boldsymbol{e}_y \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x} \right) + \boldsymbol{e}_z \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right). \end{aligned}$$

If the curl[a] of a vector field is zero everywhere, then the vector field is vortex-free (also called irrotational).

---

[a] The notation curl is common in North America. In the rest of the world, rot is commonly used. Using the cross product with the del-operator (also known as nabla-operator) avoids confusion. See Wikipedia as well,

Rotation

Definition

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Electical field

$$\operatorname{rot} \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

The static electrical field ($\frac{\partial \boldsymbol{B}}{\partial t} = 0$) is vortex-free.

Magnetic field

$$\operatorname{rot} \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

The magnetic field is not vortex-free.

Maxwell's Equations

Maxwell's Equations

Material equations

$$\begin{aligned} \boldsymbol{D} &= \varepsilon_0 \varepsilon_r \boldsymbol{E} \\ \boldsymbol{B} &= \mu_0 \mu_r \boldsymbol{H} \end{aligned}$$

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Electric current and charge

$$\text{rot}\, \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

Taking the divergence of both sides results in

$$\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \mathbf{J} + \frac{\partial (\nabla \cdot \mathbf{D})}{\partial t}.$$

Using the fact that the divergence of a curl is zero:

$$\nabla \cdot \mathbf{J} + \frac{\partial (\nabla \cdot \mathbf{D})}{\partial t} = 0.$$

With Gauss's law ($\nabla \cdot \mathbf{D} = \rho$) one gets

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Material equations

$$\begin{aligned} \boldsymbol{D} &= \varepsilon_0 \varepsilon_r \boldsymbol{E} \\ \boldsymbol{B} &= \mu_0 \mu_r \boldsymbol{H} \end{aligned}$$

Continuity equation

Relationship according to the continuity equation:

$$\nabla \cdot \boldsymbol{J}(\boldsymbol{r}\,, t) + \frac{\partial}{\partial t}\rho(\boldsymbol{r}\,, t) = 0.$$

Interpretation:

> *Current is the movement of charge. The continuity equation says that if charge is moving out of a differential volume (i.e., divergence of current density is positive) then the amount of charge within that volume is going to decrease, so the rate of change of charge density is negative. Therefore, the continuity equation amounts to a conservation of charge.*

Wikipedia

Maxwell's Equations

Maxwell's Equations

Stationary case

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Maxwell's Equations

Stationary case

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Stationary case

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Magnetic fields:

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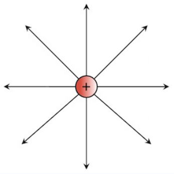
Spherical charge

Given its spherical charge Q with the homogeneous charge density ρ and the radius r_0. For points outside the charge:

$$\boldsymbol{D}(r) = \boldsymbol{e}_r \frac{Q}{4\pi r}.$$



Radial field. Source: physikunterricht-online.de

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Spherical charge - derivation

For reasons of symmetry, it can be concluded that the electric flux density $\boldsymbol{D}$ points radially outwards ($\boldsymbol{D} = \boldsymbol{e}_r D(r)$). Thus, for $r > r_0$:

$$\begin{aligned} \oiint_{\partial V} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{A} &= \iiint_v \rho \, \mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \\ &= 4\pi r D(r) = Q \end{aligned}$$

and

$$D(r) = \frac{Q}{4\pi r},$$

respectively.

Gradient

Spherical charge - derivation

For reasons of symmetry, it can be concluded that the electric flux density $\boldsymbol{D}$ points radially outwards ($\boldsymbol{D} = \boldsymbol{e}_r D(r)$). Thus, for $r > r_0$:

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Given its spherical charge Q with the homogeneous charge density ρ and the radius r_0. For points outside the charge:

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Radial field. Source: physikunterricht-online.de

Stationary case

Electrical field:

$$\begin{aligned} \operatorname{div} \boldsymbol{D} = \rho \quad &\Longleftrightarrow \quad \oiint_{\partial V} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{A} = \iiint_v \rho \mathrm{d}V \\ \operatorname{rot} \boldsymbol{E} = 0 \quad &\Longleftrightarrow \quad \oint_{\partial A} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = 0 \end{aligned}$$

Magnetic fields:

$$\begin{aligned} \operatorname{div} \boldsymbol{B} = 0 \quad &\Longleftrightarrow \quad \oiint_{\partial V} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{A} = 0 \\ \operatorname{rot} \boldsymbol{H} = \boldsymbol{J} \quad &\Longleftrightarrow \quad \oint_{\partial A} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{s} = \iint_A \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{A} \end{aligned}$$

Definition

The gradient of a function $f(x, y, z)$ is defined as

$$\begin{aligned} \operatorname{grad} f(x,y,z) &= \boldsymbol{e}_x \frac{\partial f(x,y,z)}{\partial x} + \boldsymbol{e}_y \frac{\partial f(x,y,z)}{\partial y} + \boldsymbol{e}_z \frac{\partial f(x,y,z)}{\partial z} \\ &= \boldsymbol{\nabla} f(x,y,z) \end{aligned}$$

Properties

$$\operatorname{rot}\left(\operatorname{grad} f(x,y,z)\right) = \boldsymbol{\nabla} \times \boldsymbol{\nabla} f(x,y,z) = 0$$

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Electrical potential

If the electric field is represented as a (negative) gradient of a function $\Phi(x,y,z)$ then

$$\operatorname{rot} \boldsymbol{E} = 0 \iff \oint_{\partial A} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = 0$$

is fulfilled *automatically*. Thus one needs to determine *only* a function $\Phi(x,y,z)$.

- ▷ Field strength: Unit $\mathrm{V\,m^{-1}}$. Specifies the force that would act on a test charge in the electric field ($\boldsymbol{F} = q\boldsymbol{E}$)
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Electrical potential

Obviously, charge densities ρ are the sources of the electrical fields:

$$\operatorname{div} \boldsymbol{D} = -\operatorname{div} \epsilon \ \operatorname{grad} \Phi = \rho$$

Attention

When using the div-operator (and the rot-operator), one needs to take care for vector and derivation-rules. E.g.

$$\operatorname{div} \epsilon \operatorname{grad} \Phi = \epsilon \operatorname{div} \operatorname{grad} \Phi + \operatorname{grad} \Phi \operatorname{grad} \epsilon$$

Poisson-equation

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Electrical potential

In the homogeneous space:

$$\operatorname{div} \boldsymbol{D} = -\epsilon \ \operatorname{div} \ \operatorname{grad} \ \Phi = \rho$$

In cartesian coordinates:

$$\frac{\partial^2 \Phi(x,y,z)}{\partial x^2} + \frac{\partial^2 \Phi(x,y,z)}{\partial y^2} + \frac{\partial^2 \Phi(x,y,z)}{\partial z^2} = -\frac{\rho}{\epsilon}.$$

This equation is called Poisson-equation of the electrical field.

General electrostatic soluation

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Electrical potential

If the charge density ρ is known everywhere, then the general solution is given as follows:

$$\begin{aligned}
\Phi(\boldsymbol{r}) &= \frac{1}{4\pi\varepsilon_0\,\varepsilon_r} \iiint_V \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\, dx'\, dy'\, dz' \\
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\end{aligned}$$

Stationary magnetic fields

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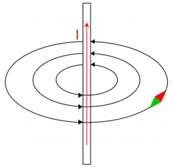
Current-carrying conductor

Given an infinitely long conductor with the diameter d and an orientation in the $z-$direction through which a current I flows. The well known result:

$$\boldsymbol{H}(r) \quad = \quad [-\boldsymbol{e_x}\cos(\varphi)+\boldsymbol{e_y}\cos(\varphi)]\,\frac{I}{2\pi r} = I\,\frac{\boldsymbol{e}_\varphi}{2\pi r}$$



Source: Physikunterricht-online.de

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This equation is called Poisson-equation of the electrical field.

Derivation: Current-carrying conductor

If A is a disk with radius r, then for reasons of symmetry it can be concluded that

$$\boldsymbol{H}(r) = -\boldsymbol{e}_x \sin(\varphi) f(r) + \boldsymbol{e}_y \cos(\varphi) f(r).$$

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$$\boldsymbol{H}(r) = -\boldsymbol{e}_x \sin(\varphi) f(r) + \boldsymbol{e}_y \cos(\varphi) f(r)$$

and thus

$$\oint_{\partial A} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{s} = \iint_A \boldsymbol{J} \mathrm{d}\boldsymbol{A}$$

simplifies as follows:

$$\oint_{\partial A} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{s} = I.$$

Stationary magnetic fields

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$$H(r) = \left[-e_x \cos(\varphi) + e_y \cos(\varphi)\right] \frac{I}{2\pi r} = I \frac{e_\varphi}{2\pi r}$$

Source: Physikunterricht-online.de

Derivation: Current-carrying conductor

Now the circular disc is parameterized using

$$\gamma(t) = r\boldsymbol{e}_x \cos(t) + r\boldsymbol{e}_y \sin(t)$$

and thus

$$\begin{aligned} I &= \oint_{\partial A} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{s} \\ &= \int_0^{2\pi} \left[-\boldsymbol{e}_x \sin(\varphi) f(r) + \boldsymbol{e}_y \cos(\varphi)\right] f(r) \\ &\quad \cdot r \left[-\boldsymbol{e}_x \sin(\varphi) + \boldsymbol{e}_y \cos(\varphi)\right] d\varphi \\ &= f(r) r \int_0^{2\pi} \sin^2(\varphi) + \cos^2(\varphi) d\varphi = f(r) 2\pi r. \end{aligned}$$

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Magnetic vector potential

Analogously to the electric potential, the magnetic vector potential can be introduced as an aid:

$$\boldsymbol{B} = \mathrm{rot}\,\boldsymbol{A}.$$

in case of a **known** current density, $\boldsymbol{A}$ can be calculated directly:

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{r}) &= \frac{\mu_0}{4\pi}\iiint\frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,dx'\,dy'\,dz', \\ \boldsymbol{B}(\boldsymbol{r}) &= \frac{\mu_0}{4\pi}\iiint\boldsymbol{J}(\boldsymbol{r}')\times\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,dx'\,dy'\,dz'. \end{aligned}$$

Hint: The last equation is also known as the Biot-Savart law.

Summary

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Stationary fields

- ▷ Point charges and infinitely long currents help to evaluate scenarios.
- ▷ Introduction of auxiliary quantities (potentials) facilitates the calculation.
- ▷ If the currents and charges are known, the fields can be calculated (often only numerically).
- ▷ Similar to the concept input signal / transfer function / output signal[a].
- ▷ Determining the currents and charges can be very difficult (e.g. charge distribution on semiconductor interfaces when a voltage is applied).

[a] See also Green's functions

Maxwell's Equations

Definition

Definition

Quasi-stationary fields are fields with a fixed spatial distribution and time-dependent intensity. These are cases in which the displacement current density $\frac{\partial \boldsymbol{D}}{\partial t}$ can be neglected compared to the power current density $\boldsymbol{J}$.

Eigenschaften

- In the close range of antennas, fields are quasi-static. A common designation for quasi-static fields is therefore also near fields.
- The near range is the range that is much smaller than the wavelength.
- The alternating magnetic field belongs to the near fields.

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Maxwell's Equations

$$\operatorname{div} \boldsymbol{D} = \rho \iff \oiint_{\partial V} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{A} = \iiint_V \rho \mathrm{d}V$$

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$$\operatorname{rot} \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \iff \oint_{\partial A} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = -\iint_A \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{A}$$

$$\operatorname{rot} \boldsymbol{H} = \boldsymbol{J} + \cancel{\frac{\partial \boldsymbol{D}}{\partial t}} \iff \oint_{\partial A} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{s} = \iint_A \left[\boldsymbol{J} + \cancel{\frac{\partial \boldsymbol{D}}{\partial t}}\right] \cdot \mathrm{d}\boldsymbol{A}$$

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The law of induction states that a time-varying magnetic field is linked to an electric field.

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Properties

- ▷ Approaches from AC engineering can often be applied (e.g. consideration at a fixed frequency, introduction of inductances)

Resonant circuit

Definition

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Maxwell's Equations

$$\operatorname{div} \boldsymbol{D} = \rho \iff \oint\!\!\oint_{\partial V} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{A} = \iiint_V \rho \mathrm{d}V$$
$$\operatorname{div} \boldsymbol{B} = 0 \iff \oint\!\!\oint_{\partial V} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{A} = 0$$
$$\operatorname{rot} \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \iff \oint_{\partial A} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = -\iint_A \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{A}$$
$$\operatorname{rot} \boldsymbol{H} = \boldsymbol{J} + \cancel{\frac{\partial \boldsymbol{D}}{\partial t}} \iff \oint_{\partial A} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{s} = \iint_A \left[\boldsymbol{J} + \cancel{\frac{\partial \boldsymbol{D}}{\partial t}}\right] \cdot \mathrm{d}\boldsymbol{A}$$

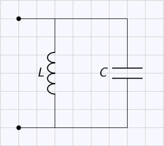
Definition

Quasi-stationary fields are fields with a fixed spatial distribution and time-dependent intensity. These are cases in which the displacement current density $\frac{\partial \boldsymbol{D}}{\partial t}$ can be neglected compared to the power current density $\boldsymbol{J}$.

Eigenschaften

- In the close range of antennas, fields are quasi-static. A common designation for quasi-static fields is therefore also near fields.
- The near range is the range that is much smaller than the wavelength.
- The alternating magnetic field belongs to the near fields.

A simple resonant circuit



Input impedance:

$$Z_{in} = \frac{j\omega L}{1 - \omega^2 LC}$$

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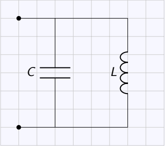
A simple resonant circuit



Current when applying a voltage U_0 (unit jump):

$$\begin{aligned}
I(s) &= U_0 \frac{1}{s} \frac{sL}{1 + s^2 LC} = U_0 \frac{L}{1 + s^2 LC} = U_0 \frac{L\omega_0^2}{\omega_0^2 + s^2} \\
i(t) &= U_0 L \omega_0 \sin(\omega_0 t) u(t),
\end{aligned}$$

with $\omega_0 = \frac{1}{\sqrt{LC}}$.

Resonant circuit

A simple resonant circuit

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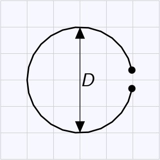
Simple inductance



Inductance of loop with diameter D and diameter of wire d[a]:

$$L \approx \mu_0 \mu_r \frac{D}{2}\left(\ln\left(\frac{8D}{d}\right) - 2\right)$$

Example: $D = 1\,\mathrm{m}$, $d = 0.1\,\mathrm{cm}$: $L = 2.94\,\mu\mathrm{H}$.

[a] see e.g. eeweb und FH Aachen

Resonant circuit

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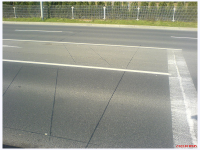

Source: Wikipedia (Wolf Meusel)

Assumption

$$\begin{aligned} L &= 2.94\,\mu\text{H} \\ C &= 4.7\,\mu\text{F} \\ f_0 &= \frac{1}{2\pi\sqrt{LC}} = 42.8\,\text{kHz} \end{aligned}$$

Resonant circuit

Simple inductance

Source: Wikipedia (Wolf Meusel)

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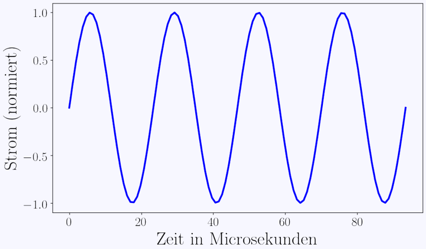
Step response



Calculation of the step response using inverse Laplace transform.

Maxwell's Equations

Wave equation

Differential form of Maxwell's equations

$$
\begin{aligned}
\text{div}\,\boldsymbol{D} &= \rho && \text{Gauss's law} \\
\text{div}\,\boldsymbol{B} &= 0 && \text{Gauss's law for magnetic fields} \\
\text{rot}\,\boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t} && \text{Maxwell–Faradayequation} \\
\text{rot}\,\boldsymbol{H} &= \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} && \text{Ampère's law}
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Time-harmonic field

Assumption: Temporal relation according to[a] $e^{j\omega t}$ and

$$\boldsymbol{E}(\boldsymbol{r}, t) = \mathrm{Re}\left(\underline{\boldsymbol{E}}(\boldsymbol{r})\, e^{j\omega t}\right)$$

$$\begin{aligned} \operatorname{div} \underline{\boldsymbol{D}} &= \underline{\rho} \\ \operatorname{div} \underline{\boldsymbol{B}} &= 0 \\ \operatorname{rot} \underline{\boldsymbol{E}} &= -j\omega \underline{\boldsymbol{B}} \\ \operatorname{rot} \underline{\boldsymbol{H}} &= \underline{\boldsymbol{J}} + j\omega \underline{\boldsymbol{D}} \end{aligned}$$

In the following, we will not explicitly write the underscore.

[a] Note that in physics, one commonly uses $e^{-j\omega t}$

Table of Contents

Wave propagation

Electromagnetic waves

Wave equation

Source free area

$$\begin{aligned} \operatorname{div} \boldsymbol{D} &= 0 \\ \operatorname{div} \boldsymbol{B} &= 0 \\ \operatorname{rot} \boldsymbol{E} &= -j\omega \boldsymbol{B} \\ \operatorname{rot} \boldsymbol{H} &= j\omega \boldsymbol{D} \end{aligned}$$

Goal: Find equation which contains only electric or only magnetic fields.

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Wave equation

$$\begin{aligned} \nabla \times \boldsymbol{E} &= -j\omega \boldsymbol{B} \\ \nabla \times \boldsymbol{H} &= j\omega \boldsymbol{D} \end{aligned}$$

Subst. second eq. into first:

$$\frac{1}{j\omega\epsilon} \nabla \times \nabla \times \boldsymbol{H} = -j\omega\mu \boldsymbol{H}.$$

And thus:

$$\frac{1}{\omega^2 \mu\epsilon} \nabla \times \nabla \times \boldsymbol{H} = \boldsymbol{H}.$$

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Source free area

$$\frac{1}{\omega^2\mu\epsilon}\nabla \times \nabla \times \boldsymbol{H} = \boldsymbol{H}$$

Using speed of light[a]

$$c = \frac{1}{\sqrt{\epsilon\mu}}$$

leads to

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[a] in vacuum: $c = 299\,792\,458\,\mathrm{m\,s^{-1}}$

Wave equations

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Source free area

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Source free area

With wavelength[a] λ and wavenumber

$$k = \frac{2\pi}{\lambda} = \frac{2\pi f}{c}$$

one gets:

$$\frac{c^2}{\omega^2}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{H}$$

and thus one gets the wave equation:

$$\boxed{\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{H} = k^2\boldsymbol{H}.}$$

We are now looking for solutions.

[a] $\lambda = 0.3\,\mathrm{m\,GHz^{-1}}$. Example: $\lambda = 0.3\,\mathrm{m}$ @ $1\,\mathrm{GHz}$

Table of Contents

Electromagnetic waves

Example

Ansatz

For the solution of the equation

$$\nabla \times \nabla \times \boldsymbol{H} \quad = \quad \boldsymbol{H} k^2$$

we are using the Ansatz

$$\boldsymbol{H} = \boldsymbol{e}_x e^{-jky}.$$

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Properties: $\boldsymbol{H} = \boldsymbol{e}_x e^{-jky}$

- $$\boldsymbol{H}(\boldsymbol{r}, t) = \boldsymbol{e}_x \cos\left(\omega t - ky\right):$$
- The magnetic field $\boldsymbol{H}$ is perpendicular to the direction of propagation $\boldsymbol{n} = \boldsymbol{e}_y$
- The corresponding electrical field is given by:

$$\boldsymbol{E} = \frac{1}{Z_f} \boldsymbol{n} \times \boldsymbol{H},$$

with

$$Z_f = \sqrt{\frac{\mu}{\varepsilon}}$$

being the intrinsic wave impedance of the medium.

Example

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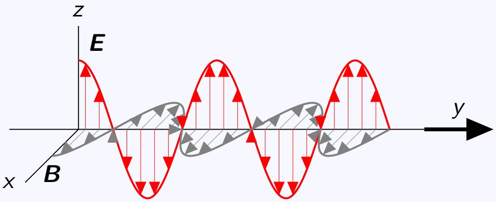
Fields for $t = 0$

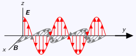


based upon a blog entry of Henri Menke

The direction of the electric field is called polarisation. Here the wave is linear polarized. Other polarizations are elliptic and circular.

Example

Fields for $t = 0$



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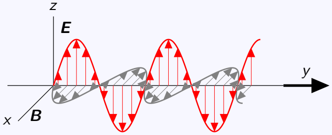
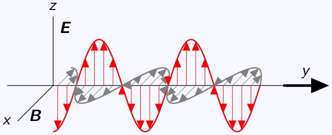
Fields for $t = \frac{T}{4}$



Fields for $t = 2\frac{T}{4}$



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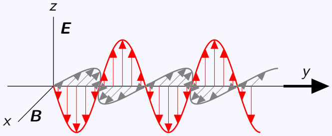
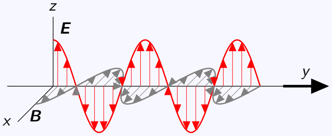


Fields for $t = 4\frac{T}{4}$



Example

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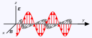
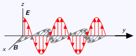


Fields for $t = 3\frac{T}{4}$

Fields for $t = 4\frac{T}{4}$

Fields for $t = 0$



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Proof of correctness

It follows

$$\begin{aligned}
\frac{1}{k^2}\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{H} &= \frac{1}{k^2}\boldsymbol{\nabla} \times \left[\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \boldsymbol{e}_x e^{-jky}\right] \\
&= \frac{1}{k^2}\boldsymbol{\nabla} \times \left[\boldsymbol{e}_z\left(-\frac{\partial}{\partial y} e^{-ky}\right)\right] \\
&= -\frac{1}{k^2}\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \left[\boldsymbol{e}_z k e^{-jky}\right] \\
&= \frac{1}{k}\boldsymbol{e}_x \frac{\partial}{\partial y} e^{-ky} = \boldsymbol{e}_x e^{-jky} = \boldsymbol{H}
\end{aligned}$$

Table of Contents

Wave propagation

Table of Contents

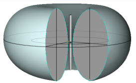
Fields of current distributions

Hertzian dipole

Far field

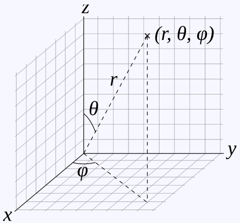
Far field of a very short current element (Hertzian dipole) orientated in $\boldsymbol{e}_z$-direction:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{jk}{4\pi} Ih \frac{e^{-jkr}}{r} \sin\theta \, \boldsymbol{e}_\theta .$$

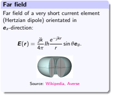


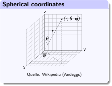
Source: Wikipedia, Averse

Spherical coordinates



Quelle: Wikipedia (Andeggs)



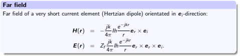


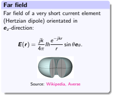
Far field

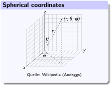
Far field of a very short current element (Hertzian dipole) orientated in $\boldsymbol{e}_i$-direction:

$$\begin{aligned}\boldsymbol{H}(\boldsymbol{r}) &= -\frac{jk}{4\pi} lh \frac{e^{-jkr}}{r} \boldsymbol{e}_r \times \boldsymbol{e}_i \\ \boldsymbol{E}(\boldsymbol{r}) &= Z_f \frac{jk}{4\pi} lh \frac{e^{-jkr}}{r} \boldsymbol{e}_r \times \boldsymbol{e}_r \times \boldsymbol{e}_i.\end{aligned}$$

Herzscher Dipol







Useful equations

$$
\begin{aligned}
\boldsymbol{e}_r \times \boldsymbol{e}_\vartheta &= \boldsymbol{e}_\varphi \\
\boldsymbol{e}_\vartheta \times \boldsymbol{e}_\varphi &= \boldsymbol{e}_r \\
\boldsymbol{e}_r \times \boldsymbol{e}_\varphi &= -\boldsymbol{e}_\vartheta
\end{aligned}
$$

and thus

$$
\begin{aligned}
\boldsymbol{e}_r \times (A_r \boldsymbol{e}_r + A_\vartheta \boldsymbol{e}_\vartheta + A_\varphi \boldsymbol{e}_\varphi) &= (A_\vartheta \boldsymbol{e}_\varphi - A_\varphi \boldsymbol{e}_\vartheta) \\
\boldsymbol{e}_r \times \boldsymbol{e}_r \times (A_r \boldsymbol{e}_r + A_\vartheta \boldsymbol{e}_\vartheta + A_\varphi \boldsymbol{e}_\varphi) &= (-A_\vartheta \boldsymbol{e}_\vartheta - A_\varphi \boldsymbol{e}_\varphi)\,.
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Herzscher Dipol

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\end{aligned}
$$

Far field

Far field of a very short current element (Hertzian dipole) orientated in $\boldsymbol{e}_i$-direction:

$$
\begin{aligned}
\boldsymbol{H}(\boldsymbol{r}) &= -\frac{jk}{4\pi} Ih \frac{e^{-jkr}}{r} \boldsymbol{e}_r \times \boldsymbol{e}_i \\
\boldsymbol{E}(\boldsymbol{r}) &= Z_f \frac{jk}{4\pi} Ih \frac{e^{-jkr}}{r} \boldsymbol{e}_r \times \boldsymbol{e}_r \times \boldsymbol{e}_i.
\end{aligned}
$$

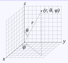
Far field

Far field of a very short current element (Hertzian dipole) orientated in $\boldsymbol{e}_z$-direction:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{jk}{4\pi} Ih \frac{e^{-jkr}}{r} \sin\theta \boldsymbol{e}_\theta.$$

Source: Wikipedia, Averse

Spherical coordinates



Quelle: Wikipedia (Andeggs)

Useful equations

Consequently,

$$
\begin{aligned}
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&= -Z_f \frac{jk}{4\pi} Ih \frac{e^{-jkr}}{r} \left(\boldsymbol{e}_{i,\vartheta} + \boldsymbol{e}_{i,\varphi}\right).
\end{aligned}
$$

Herzscher Dipol

Useful equations

Consequently,

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r}) &= Z_f \frac{jk}{4\pi} Ih \frac{e^{-jkr}}{r} \boldsymbol{e}_r \times \boldsymbol{e}_r \times \boldsymbol{e}_i \\ &= -Z_f \frac{jk}{4\pi} Ih \frac{e^{-jkr}}{r} \left(\boldsymbol{e}_{i,\vartheta} + \boldsymbol{e}_{i,\varphi}\right). \end{aligned}$$

Useful equations

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Useful equations

Vector fields can be converted from cartesian to spherical by making use of (see Wikipedia as well):

$$\begin{aligned} A_r &= A_x \sin\vartheta \cos\varphi + A_y \sin\vartheta \sin\varphi + A_z \cos\vartheta \\ A_\vartheta &= A_x \cos\vartheta \cos\varphi + A_y \cos\vartheta \sin\varphi - A_z \sin\vartheta \\ A_\varphi &= -A_x \sin\varphi + A_y \cos\varphi \end{aligned}$$

Table of Contents

Fields of current distributions

Fields of Electrical Currents

The following considers electrically ideally conductive bodies in otherwise free space. The field radiated by currents[a] on the surface A of these bodies is given by[b]:

$$\boldsymbol{E}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\phi(\boldsymbol{r},t) - \frac{\partial}{\partial t}\boldsymbol{A}(\boldsymbol{r},t).$$

If the currents are known, the fields can be determined. However, determining the currents is generally not trivial.

[a] In certain cases, such as horn antennas, magnetic currents are also considered
[b] See, for example, Blume, Theory of Electromagnetic Fields

Radiation in Free Space

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Potentials

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with

$$r = |\boldsymbol{r} - \boldsymbol{r}'|$$

and

$$t' = t - \frac{r}{c}.$$

Radiation in free space

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Fields of electrical currents

Thus, an integral representation for the electric field can be specified:

$$\boldsymbol{E}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\frac{1}{4\pi\varepsilon}\int_A \frac{\rho(\vec{r}\,',t')}{r}da' - \frac{\partial}{\partial t}\frac{\mu}{4\pi}\int_A \frac{\vec{J}(\vec{r}\,',t')}{r}da'.$$

The relationship between the electric current density $\vec{J}$ and the electric charge density ρ is described by the continuity equation:

$$\boldsymbol{\nabla}\cdot\boldsymbol{J}(\boldsymbol{r}\,',t) + \frac{\partial}{\partial t}\rho(\boldsymbol{r}\,',t) = 0.$$

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Fields of electrical currents

Integral representation of the electric charge density:

$$\rho(\boldsymbol{r}\,',t) = -\int_{-\infty}^{t}\nabla\cdot\boldsymbol{J}(\boldsymbol{r}\,',t)dt.$$

And thus for $\boldsymbol{J}(\boldsymbol{r},t) = 0$ for $t < 0$:

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r},t) &= \nabla\frac{1}{4\pi\varepsilon}\int_A \frac{\int_0^t \nabla\cdot\boldsymbol{J}(\boldsymbol{r}\,',t')dt}{r}da' \\ &\quad - \frac{\partial}{\partial t}\frac{\mu}{4\pi}\int_A \frac{\boldsymbol{J}(\boldsymbol{r}\,',t')}{r}da'.\end{aligned}$$

Note that ∇ is applied to different coordinates (source (primed) and observation point (unprimed)).

Fields of electrical currents

Integral representation of the electric charge density:

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and

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Field of electrical currents

Time harmonic case:

$$\boldsymbol{E}(\boldsymbol{r}) = \nabla \frac{1}{4\pi\varepsilon} \int_A \frac{\nabla \cdot \boldsymbol{J}(\boldsymbol{r}\,')}{j\omega} \frac{e^{-jkr}}{r} da' - \frac{j\omega\mu}{4\pi} \int_A \boldsymbol{J}(\boldsymbol{r}\,') \frac{e^{-jkr}}{r} da',$$

with

$$r = |\boldsymbol{r} - \boldsymbol{r}'|.$$

Table of Contents

Fields of current distributions

Radiation in free space

Field of electrical currents

For large distances between antenna and observation point one can approximate

$$\begin{aligned}
\boldsymbol{E}(\boldsymbol{r}) &= \nabla \frac{1}{4\pi\varepsilon} \int_A \frac{\nabla \cdot \boldsymbol{J}(\boldsymbol{r}\,')}{j\omega} \frac{e^{-jkr}}{r} da' - \frac{j\omega\mu}{4\pi} \int_A \boldsymbol{J}(\boldsymbol{r}\,') \frac{e^{-jkr}}{r} da' \\
&\approx \nabla \frac{1}{4\pi\varepsilon r_0} \int_A \frac{\nabla \cdot \boldsymbol{J}(\boldsymbol{r}\,')}{j\omega} e^{-jkr} da' - \frac{j\omega\mu}{4\pi r_0} \int_A \boldsymbol{J}(\boldsymbol{r}\,') e^{-jkr} da',
\end{aligned}$$

with $r_0 \approx |\boldsymbol{r}|$ being the distance of the observation point to the center of gravity of the antenna.

Field of electrical currents

For large distances between antenna and observation point one can approximate

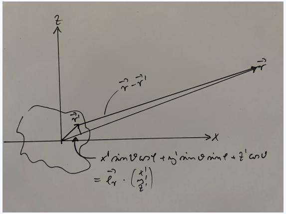
$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r}) &= \nabla \frac{1}{4\pi\varepsilon} \int_A \frac{\nabla \cdot \boldsymbol{J}(\boldsymbol{r}\,')}{j\omega} \frac{e^{-jkr}}{r} da' - \frac{j\omega\mu}{4\pi} \int_A \boldsymbol{J}(\boldsymbol{r}\,') \frac{e^{-jkr}}{r} da' \\ &\approx \nabla \frac{1}{4\pi\varepsilon r_0} \int_A \frac{\nabla \cdot \boldsymbol{J}(\boldsymbol{r}\,')}{j\omega} e^{-jkr} da' - \frac{j\omega\mu}{4\pi r_0} \int_A \boldsymbol{J}(\boldsymbol{r}\,') e^{-jkr} da', \end{aligned}$$

with $r_0 \approx |\boldsymbol{r}|$ being the distance of the observation point to the center of gravity of the antenna.

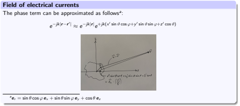
Field of electrical currents

The phase term can be approximated as follows[a]:

$$e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|} \approx e^{-jk|\boldsymbol{r}|} e^{+jk(x' \sin\vartheta \cos\varphi + y' \sin\vartheta \sin\varphi + z' \cos\vartheta)}$$



[a] $\boldsymbol{e}_r = \sin\theta \cos\varphi \, \boldsymbol{e}_x + \sin\theta \sin\varphi \, \boldsymbol{e}_y + \cos\theta \, \boldsymbol{e}_z$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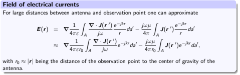
Field of electrical currents
For large distances between antenna and observation point one can approximate
$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r}) &= \nabla\frac{1}{4\pi\varepsilon}\int_A \frac{\nabla\cdot\boldsymbol{J}(\boldsymbol{r}')}{j\omega}\frac{e^{-jkr}}{r}da' - \frac{j\omega\mu}{4\pi}\int_A \boldsymbol{J}(\boldsymbol{r}')\frac{e^{-jkr}}{r}da' \\ &\approx \nabla\frac{1}{4\pi\varepsilon r_0}\int_A \frac{\nabla\cdot\boldsymbol{J}(\boldsymbol{r}')}{j\omega}e^{-jkr}da' - \frac{j\omega\mu}{4\pi r_0}\int_A \boldsymbol{J}(\boldsymbol{r}')e^{-jkr}da',\end{aligned}$$
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Field of electrical currents

Thus, one gets[a]:

$$\begin{aligned}
\boldsymbol{E}(\boldsymbol{r}) \approx\; & \nabla\frac{1}{4\pi\varepsilon\,|\boldsymbol{r}|}e^{-jk|\boldsymbol{r}|}\int_A \frac{\nabla\cdot\boldsymbol{J}(x',y',z')}{j\omega}e^{+jk(x'\sin\vartheta\cos\varphi+y'\sin\vartheta\sin\varphi+z'\cos\vartheta)}dx'dy'dz' \\
& -\frac{j\omega\mu}{4\pi\,|\boldsymbol{r}|}e^{-jk|\boldsymbol{r}|}\int_A \boldsymbol{J}(x',y',z')e^{+jk(x'\sin\vartheta\cos\varphi+y'\sin\vartheta\sin\varphi+z'\cos\vartheta)}dx'dy'dz' \\
=\; & -\frac{jk}{4\pi}\boldsymbol{E}_0(\vartheta,\varphi)\frac{e^{-jk|\boldsymbol{r}|}}{|\boldsymbol{r}|}
\end{aligned}$$

[a] Due to the gradient in spherical coordinates given by

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial f}{\partial\vartheta}\boldsymbol{e}_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial f}{\partial\varphi}\boldsymbol{e}_\varphi$$

the first anti-derivative can be neglected.

Radiation in free space

Field of electrical currents

Thus, one gets[a]:

$$\boldsymbol{E}(\boldsymbol{r}) \approx \nabla \frac{1}{4\pi\varepsilon\,|\boldsymbol{r}|} e^{-jk|\boldsymbol{r}|} \int_A \frac{\nabla\cdot\boldsymbol{J}(x',y',z')}{j\omega} e^{+jk(x'\sin\vartheta\cos\varphi+y'\sin\vartheta\sin\varphi+z'\cos\vartheta)} dx'dy'dz' - \frac{j\omega\mu}{4\pi\,|\boldsymbol{r}|} e^{-jk|\boldsymbol{r}|} \int_A \boldsymbol{J}(x',y',z') e^{+jk(x'\sin\vartheta\cos\varphi+y'\sin\vartheta\sin\varphi+z'\cos\vartheta)} dx'dy'dz' = -\frac{jk}{4\pi}\boldsymbol{E}_0(\vartheta,\varphi)\frac{e^{-jk|\boldsymbol{r}|}}{|\boldsymbol{r}|}$$

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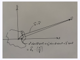

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Field of electrical currents

For large distances between antenna and observation point one can approximate

$$\boldsymbol{E}(\boldsymbol{r}) = \nabla\frac{1}{4\pi\varepsilon}\int_A \frac{\nabla\cdot\boldsymbol{J}(\boldsymbol{r}')}{j\omega}\frac{e^{-jkr}}{r}da' - \frac{j\omega\mu}{4\pi}\int_A \boldsymbol{J}(\boldsymbol{r}')\frac{e^{-jkr}}{r}da' \approx \nabla\frac{1}{4\pi\varepsilon r_0}\int_A \frac{\nabla\cdot\boldsymbol{J}(\boldsymbol{r}')}{j\omega}e^{-jkr}da' - \frac{j\omega\mu}{4\pi r_0}\int_A \boldsymbol{J}(\boldsymbol{r}')e^{-jkr}da',$$

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Field of electrical currents

For large distances between antenna and observation point one can approximate

$$\boldsymbol{E}(\boldsymbol{r}) = -\frac{jk}{4\pi}\boldsymbol{E}_0(\vartheta,\varphi)\frac{e^{-jk|\boldsymbol{r}|}}{|\boldsymbol{r}|},$$

where $\boldsymbol{E}_0(\vartheta,\varphi)$ contains anti-derivatives of the form

$$\int_A \boldsymbol{J}(x',y',z')e^{jk(x'\cos\vartheta\cos\varphi+y'\sin\vartheta\sin\varphi+z'\cos\vartheta)}dx'dy'dz'$$

Compare[a] to Fourier transform: $F(\omega) = \int\limits_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = \mathcal{F}\{f(t)\}$.

[a] See Klark, Antennen und Strahlungsfelder as well (available as ebook in library)

Radiation in free space

Field of electrical currents

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Thus, one gets[a]:

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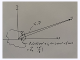


[a] $\boldsymbol{e}_r=\sin\theta\cos\varphi\,\boldsymbol{e}_x+\sin\theta\sin\varphi\,\boldsymbol{e}_y+\cos\theta\,\boldsymbol{e}_z$

Summary

A more detailed examination is beyond the scope here. For further consideration, it is important to note:

- ▷ The use of potentials simplifies calculations (assumption: currents are known).
- ▷ An analysis of the time-harmonic case leads to the use of retarded potentials.
- ▷ The calculation of potentials leads to integrals of the type of Fourier transformation.
- ▷ Similarly to signals and systems, we will later examine exemplary current distributions.

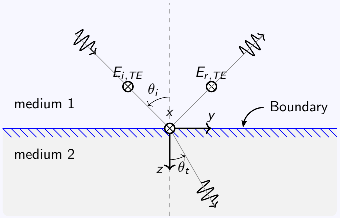
Table of Contents

Wave propagation

Reflection, diffraction and damping of plane waves

Reflection of plane waves

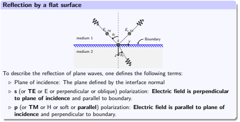
Reflection by a flat surface



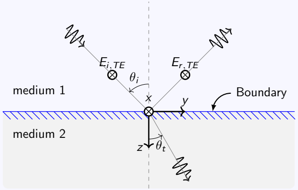
To describe the reflection of plane waves, one defines the following terms:

- Plane of incidence: The plane defined by the interface normal
- **s** (or **TE** or E or perpendicular or oblique) polarization: **Electric field is perpendicular to plane of incidence** and parallel to boundary.
- **p** (or **TM** or H or soft or **parallel**) polarization: **Electric field is parallel to plane of incidence** and perpendicular to boundary.

Reflection of plane waves



Snell's law



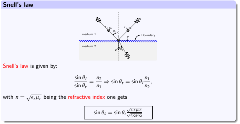
Snell's law is given by:

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{n_2}{n_1} \Rightarrow \sin\theta_t = \sin\theta_i \frac{n_1}{n_2},$$

with $n = \sqrt{\epsilon_r \mu_r}$ being the refractive index one gets

$$\boxed{\sin\theta_t = \sin\theta_i \frac{\sqrt{\epsilon_{r1}\mu_{r1}}}{\sqrt{\epsilon_{r2}\mu_{r2}}}}$$

Reflection of plane waves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$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{n_2}{n_1} \Rightarrow \sin\theta_t = \sin\theta_i \frac{n_1}{n_2},$$

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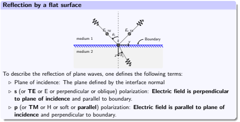
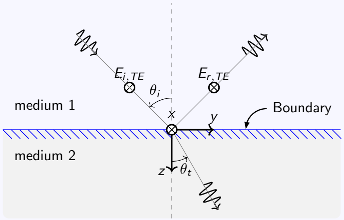

Reflection by a flat surface

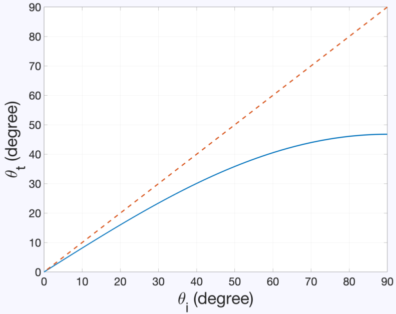
To describe the reflection of plane waves, one defines the following terms:

- Plane of incidence: The plane defined by the interface normal
- **s** (or **TE** or E or perpendicular or oblique) polarization: **Electric field is perpendicular to plane of incidence** and parallel to boundary.
- **p** (or **TM** or H or soft or **parallel**) polarization: **Electric field is parallel to plane of incidence** and perpendicular to boundary.

Snell's law

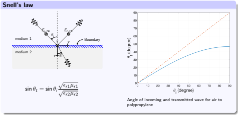


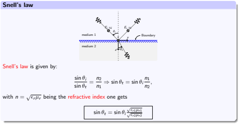
$$\sin\theta_t = \sin\theta_i \frac{\sqrt{\epsilon_{r1}\mu_{r1}}}{\sqrt{\epsilon_{r2}\mu_{r2}}}$$

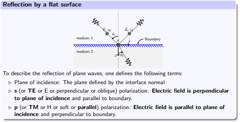


Angle of incoming and transmitted wave for air to polypropylene

Reflection of plane waves







Reflection by a flat surface – TM

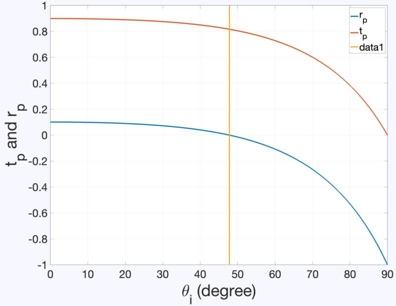
For p polarization (E-Field parallel to plane of incidence – **TM**):

$$r_\mathrm{p} = r_\mathrm{TM} = \frac{n_2 \cos\theta_\mathrm{i} - n_1 \cos\theta_\mathrm{t}}{n_2 \cos\theta_\mathrm{i} + n_1 \cos\theta_\mathrm{t}},$$

$$t_\mathrm{p} = t_\mathrm{TM} = \frac{2 n_1 \cos\theta_\mathrm{i}}{n_2 \cos\theta_\mathrm{i} + n_1 \cos\theta_\mathrm{t}},$$

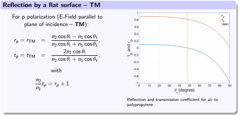
with

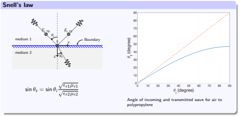
$$\frac{n_2}{n_1} t_p = r_p + 1$$

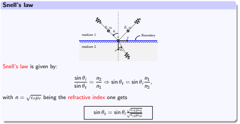


Reflection and transmission coefficient for air to polypropylene

Brewster Angle







Brewster Angle

An electromagnetic wave with p polarization is completely transmitted in case of

$$\theta_B = \operatorname{atan}\left(\frac{n_2}{n_1}\right).$$

Reflection of plane waves

Brewster Angle

An electromagnetic wave with p polarization is completely transmitted in case of

$$\theta_B = \operatorname{atan}\left(\frac{n_2}{n_1}\right).$$

Reflection by a flat surface – TM

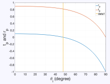
For p polarization (E-Field parallel to plane of incidence – **TM**):

$$r_p = r_{TM} = \frac{n_2\cos\theta_i - n_1\cos\theta_t}{n_2\cos\theta_i + n_1\cos\theta_t},$$

$$t_p = t_{TM} = \frac{2n_1\cos\theta_i}{n_2\cos\theta_i + n_1\cos\theta_t},$$

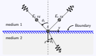
with

$$\frac{n_2}{n_1}t_p = r_p + 1$$

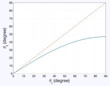


Reflection and transmission coefficient for air to polypropylene

Snell's law



$$\sin\theta_t = \sin\theta_i \frac{\sqrt{\epsilon_{r1}\mu_{r1}}}{\sqrt{\epsilon_{r2}\mu_{r2}}}$$



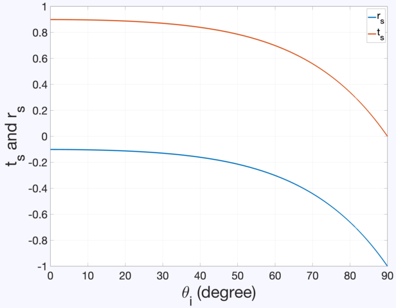
Angle of incoming and transmitted wave for air to polypropylene

Reflection by a flat surface – TE

For s polarization (E-Field perpendicular to plane of incidence – **TE**):

$$r_s = r_{TE} = \frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t},$$

$$t_s = t_{TE} = \frac{2n_1\cos\theta_i}{n_1\cos\theta_i + n_2\cos\theta_t},$$



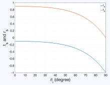
Reflection and transmission coefficient for air to polypropylene

Reflection of plane waves

Reflection by a flat surface – TE

For s polarization (E-Field perpendicular to plane of incidence – **TE**):

$$r_s = r_{TE} = \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t},$$
$$t_s = t_{TE} = \frac{2n_1 \cos\theta_i}{n_1 \cos\theta_i + n_2 \cos\theta_t}.$$



Reflection and transmission coefficient for air to polypropylene

Brewster Angle

An electromagnetic wave with p polarization is completely transmitted in case of

$$\theta_B = \operatorname{atan}\left(\frac{n_2}{n_1}\right).$$

Reflection by a flat surface – TM

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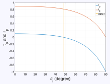


Reflection and transmission coefficient for air to polypropylene

Reflection by a flat perfectly conducting surface

Using a complex permittivity

$$\varepsilon_{r,2} = \varepsilon'_{r,2} + j\varepsilon''_{r,2},$$

with $\varepsilon''_{r,2} \to \infty$ one gets

$$\begin{aligned} r_p = r_{TM} &= 1, \\ t_p = t_{TM} &= 0, \\ r_s = r_{TE} &= -1, \\ t_s = t_{TE} &= 0 \end{aligned}$$

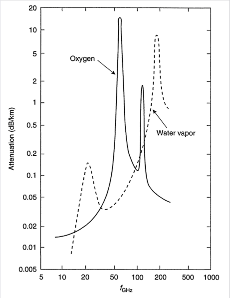
Hint: Negative sign of r_{TE} can be explained by boundary condition (no tangential electrical field).

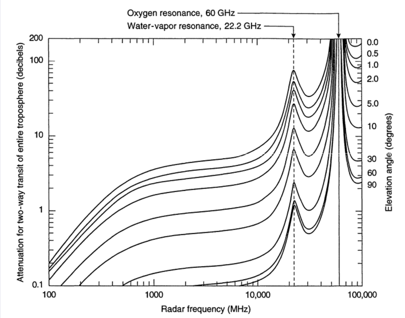
Table of Contents

Reflection, diffraction and damping of plane waves

Damping of plane waves

Road Surface



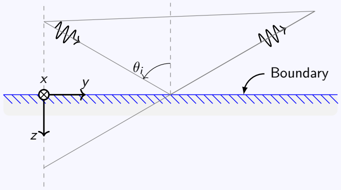


Theoretical attenuation for oxygen and water vapor and wave attenuation in standard atmosphere for target outside the troposphere (source: Peebles, radar principles)

Table of Contents

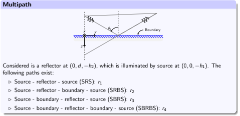
Reflection, diffraction and damping of plane waves

Multipath

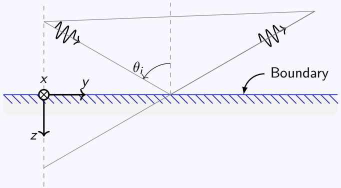


Considered is a reflector at $(0, d, -h_2)$, which is illuminated by source at $(0, 0, -h_1)$. The following paths exist:

- Source - reflector - source (SRS): r_1
- Source - reflector - boundary - source (SRBS): r_2
- Source - boundary - reflector - source (SBRS): r_3
- Source - boundary - reflector - boundary - source (SBRBS): r_4



Multipath



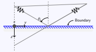
Thus, the overall signal (assuming that the reflector is isotropic):

$$R_x \sim \frac{e^{-jkr_1}}{r_1} + r_{p,s}\frac{e^{-jkr_2}}{r_2} + r_{p,s}\frac{e^{-jkr_3}}{r_3} + r_{p,s}^2\frac{e^{-jkr_4}}{r_4},$$

with $r_1 = 2\sqrt{(h_2 - h_1)^2 + d^2}$, $r_2 = r_3 = \sqrt{(h_2 + h_1)^2 + d^2}$ and $r_4 = 2\sqrt{(h_2 + h_1)^2 + d^2}$, for $h_2 > h_1$.

Multipath

Multipath



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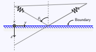
Multipath



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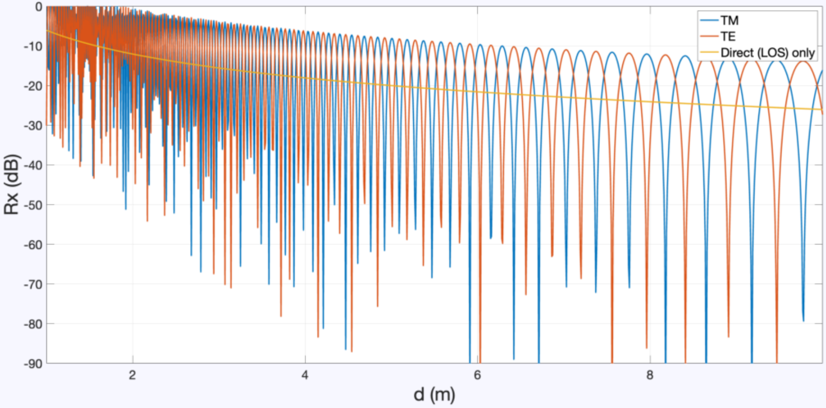
- Source - boundary - reflector - boundary - source (SBRBS): r_4

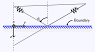
Perfectly conducting surface



$h_2 = 0.6\,\mathrm{m},\ h_1 = 0.5\,\mathrm{m},\ f = 76\,\mathrm{GHz}$

Multipath

Multipath

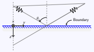


Thus, the overall signal (assuming that the reflector is isotropic):

$$R_x \sim \frac{e^{-jkr_1}}{r_1} + r_{p,s}\frac{e^{-jkr_2}}{r_2} + r_{p,s}\frac{e^{-jkr_3}}{r_3} + r_{p,s}^2\frac{e^{-jkr_4}}{r_4},$$

with $r_1 = 2\sqrt{(h_2 - h_1)^2 + d^2}$, $r_2 = r_3 = \sqrt{(h_2 + h_1)^2 + d^2}$ and $r_4 = 2\sqrt{(h_2 + h_1)^2 + d^2}$, for $h_2 > h_1$.

Multipath



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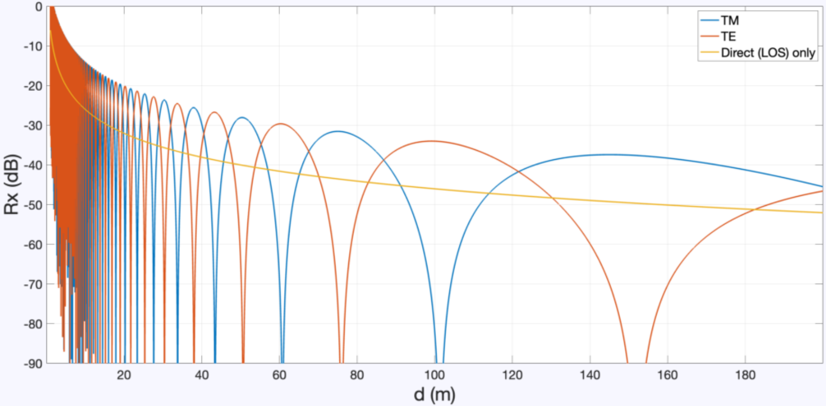
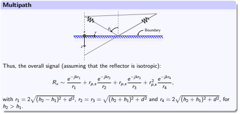
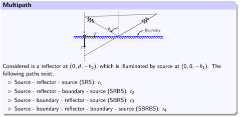
Perfectly conducting surface



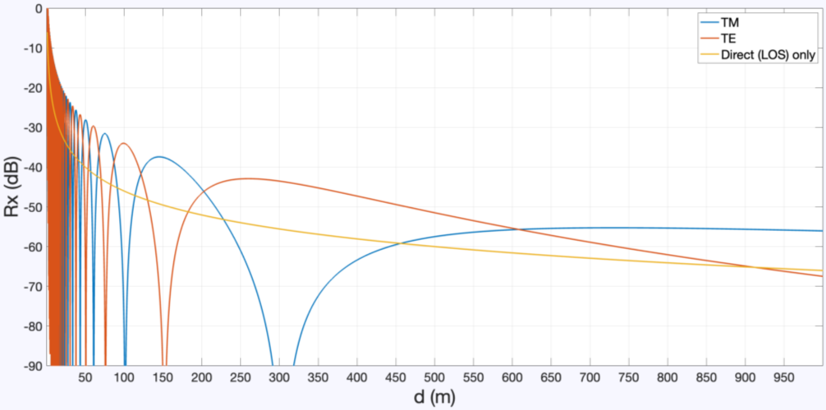
$h_2 = 0.6\,\mathrm{m},\ h_1 = 0.5\,\mathrm{m},\ f = 76\,\mathrm{GHz}$

Multipath



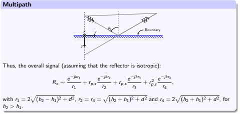


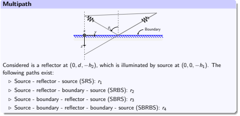
Perfectly conducting surface



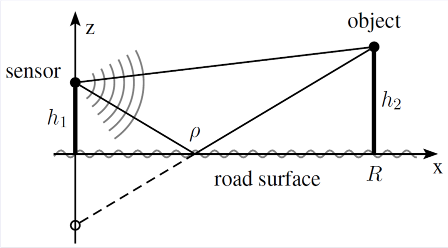
$h_2 = 0.6\,\text{m},\ h_1 = 0.5\,\text{m},\ f = 76\,\text{GHz}$

Multipath





Road Surface



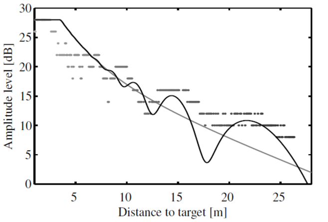


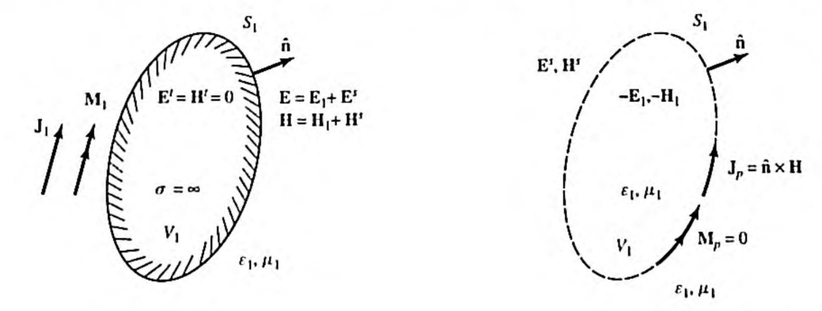
Fig. 4. Measured (dots) and simulated (line) radar amplitude of measurement with Opel Vectra as target

Source: Büren, Yang, Characterization of Automotive Radar Targets from 22 to 29 GHz

Table of Contents

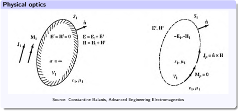
Reflection, diffraction and damping of plane waves

Physical optics

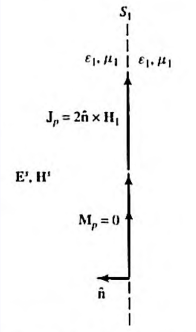


Source: Constantine Balanis, Advanced Engineering Electromagnetics

Physical optics



Physical optics



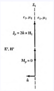
Source: Constantine Balanis, Advanced Engineering Electromagnetics

Approximation

$$\boldsymbol{J} = 2\boldsymbol{n} \times \boldsymbol{H}.$$

Edge diffraction

Physical optics

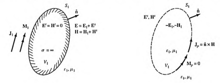


Source: Constantine Balanis, Advanced Engineering Electromagnetics

Approximation

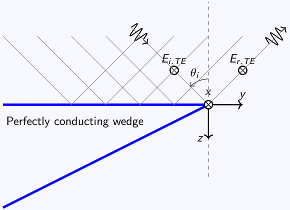
$J = 2n \times H.$

Physical optics

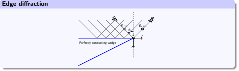


Source: Constantine Balanis, Advanced Engineering Electromagnetics

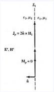
Edge diffraction



Edge diffraction



Physical optics



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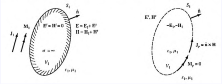
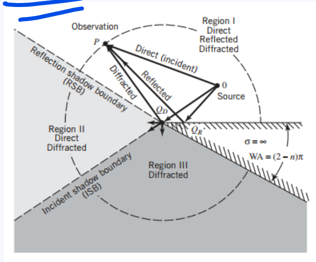
$J = 2n \times H$.

Physical optics



Source: Constantine Balanis, Advanced Engineering Electromagnetics

Edge diffraction



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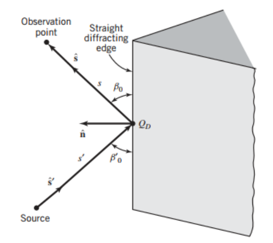
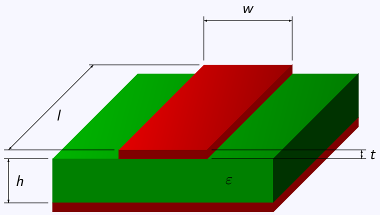


Source: Constantine Balanis, Advanced Engineering Electromagnetics

Table of Contents

Wave propagation

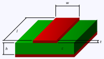
Micro Strip lines



Figure

Coplanar waveguide

Micro Strip lines



Figure

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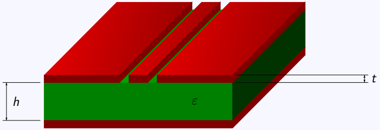


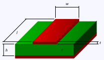
Figure

Notes

Coplanar waveguide

Figure

Micro Strip lines



Figure

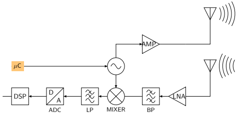
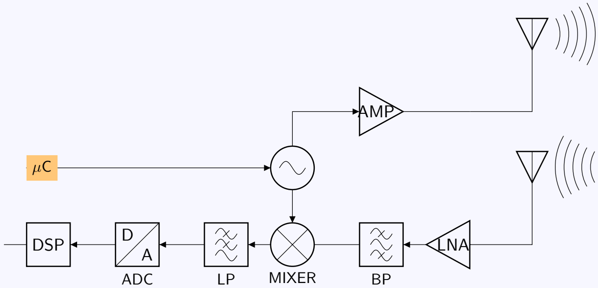


Table of Contents

Introduction

Block diagram

Blockdiagramm



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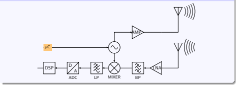

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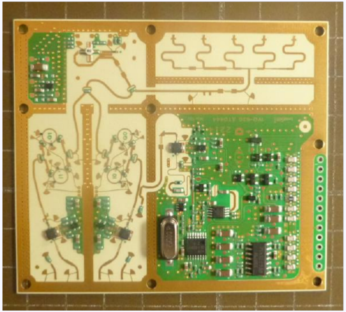

Source: FCC

Blockdiagramm

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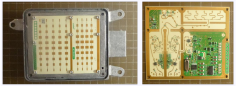

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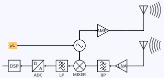
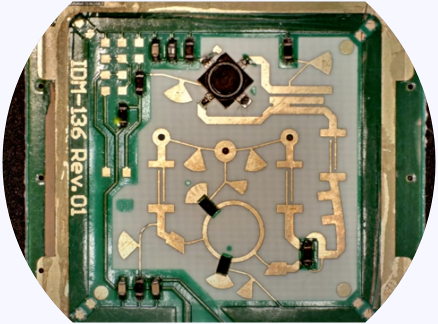


CDM 324



Source: The Signal Path (Youtube)

Table of Contents

Block diagram

Key performance indicators

- ▷ Phase stability
- ▷ Large electrical tuning range
- ▷ Linearity of frequency versus control voltage
- ▷ Large gain factor
- ▷ Capability for accepting wideband modulation
- ▷ Low cost
- ▷ Load pulling

Notes

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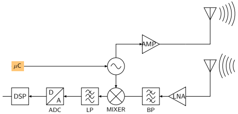


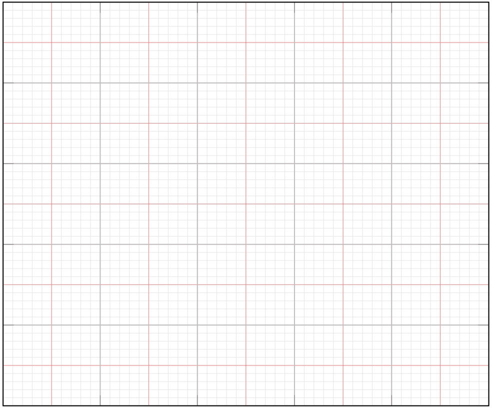
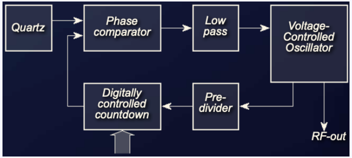


Table of Contents

Block diagram

PLL



Source: Radartutorial

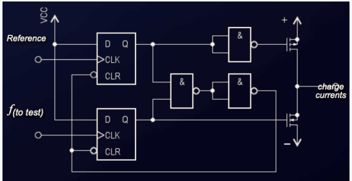


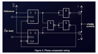
Figure 2: Phase comparator wiring

Source: Radartutorial

Notes

PLL

Source: Radartutorial



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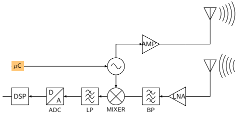


Table of Contents

Block diagram

Notes

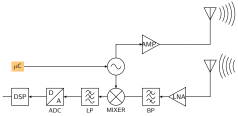
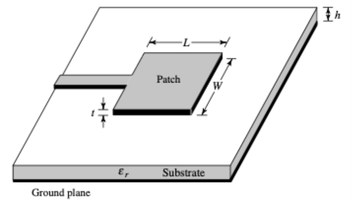


Table of Contents

Block diagram

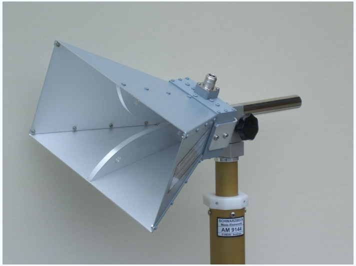
Antennas

Example



Source: Balanis, antenna theory

Example



Source: Wikipedia (Schwarzbeck Mess-Elektronik)

Antennas

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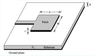

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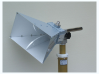

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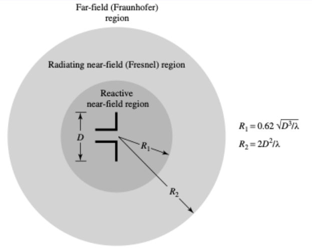
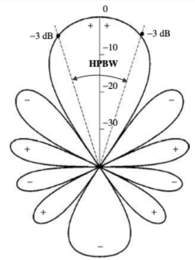
Far field and near field



Source: Balanis, antenna theory

Pattern



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Antennas

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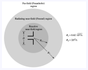

Source: Balanis, antenna theory

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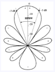

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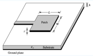

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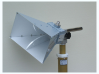

Source: Wikipedia (Schwarzbeck Mess-Elektronik)

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- Directivity: Ratio of radiation density in a given direction over that of an isotropic source
- Gain: Taking efficiency into account
- Polarization
- Bandwidth / Input resistance

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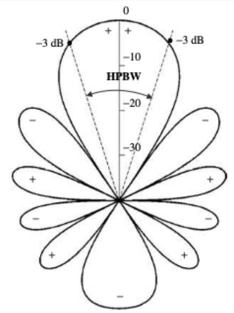

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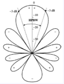

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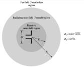

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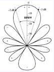

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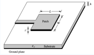

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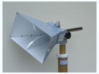

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- Micro strip (e.g. Bosch MRR)
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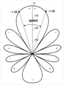

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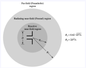

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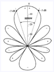

Source: Balanis, antenna theory

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Antenna	Directivity (logarithmic scale)	Directivity (linear scale)
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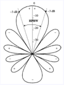

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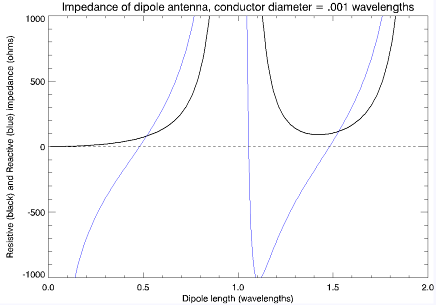

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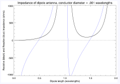

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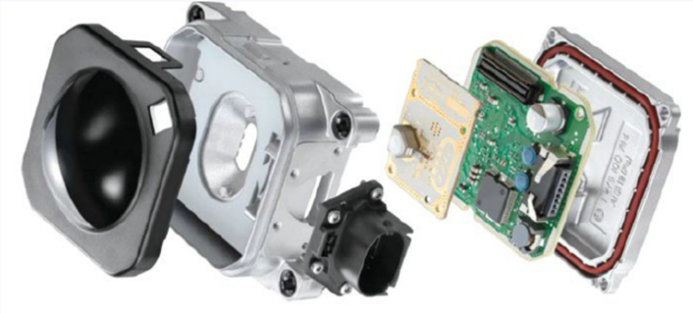
Bosch MRR



Source: Handbuch Fahrerassistenzsysteme

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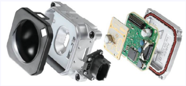

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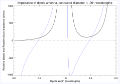

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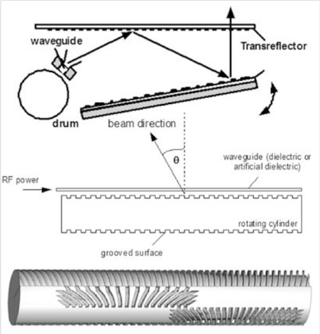
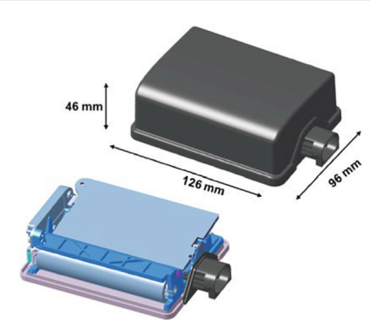
Continental





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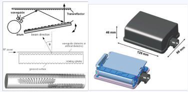
Continental



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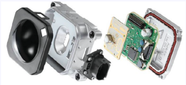
Bosch MRR



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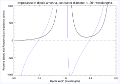
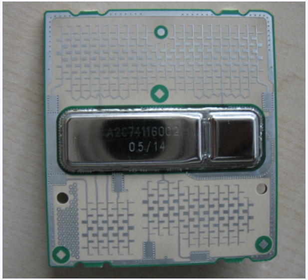
Dipole antenna input impedance



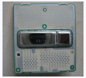
Source: Wikipedia

Continental ARS 4B



E.g. used in Tesla 3. Source: FCC

Continental ARS 4B



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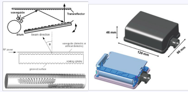
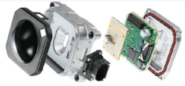
Continental



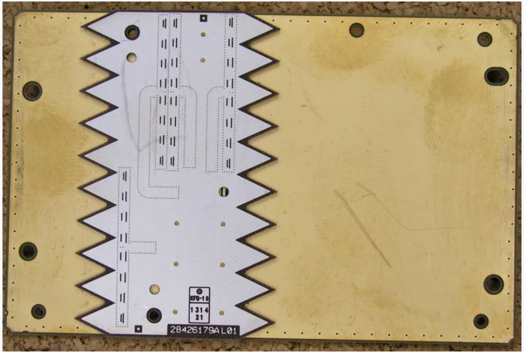
Source: Handbuch Fahrerassistenzsysteme

Bosch MRR



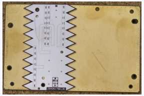
Source: Handbuch Fahrerassistenzsysteme

Delphi (APTIV) SRR2



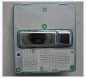
SRR 5 (Source: FCC)

Delphi (APTIV) SRR2



SRR 5 (Source: FCC)

Continental ARS 4B



E.g. used in Tesla 3. Source: FCC

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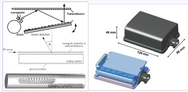
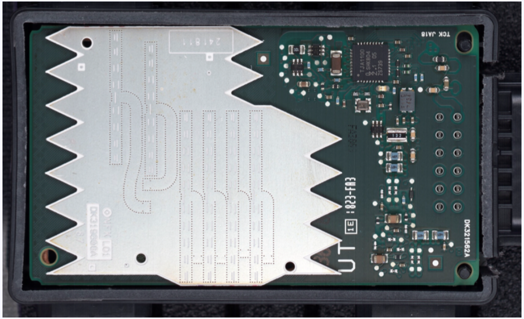


Source: Handbuch Fahrerassistenzsysteme

Aptiv SRR



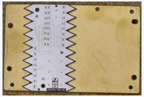
SRR 5 (Source: FCC)

Antennas

Aptiv SRR

SRR 5 (Source: FCC)

Delphi (APTIV) SRR2



SRR 5 (Source: FCC)

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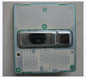
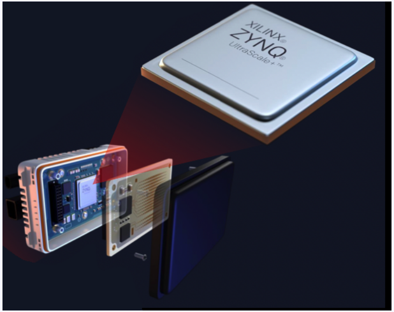


E.g. used in Tesla 3. Source: FCC

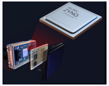
Continental ARS 540



ARS 540 with 12x TX + 16x RX antennas. Source: Xilinx (modifiziert)

Antennas

Continental ARS 540



ARS 540 with 12x TX + 16x RX antennas. Source: Xilinx (modifiziert)

Aptiv SRR

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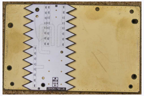

SRR 5 (Source: FCC)

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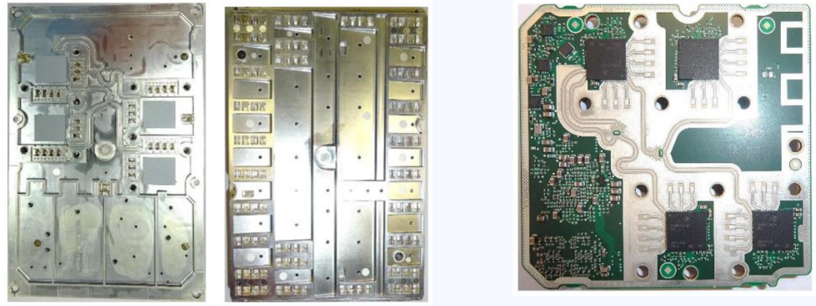

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Antennas

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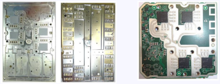

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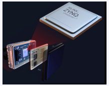

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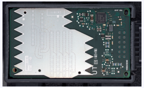
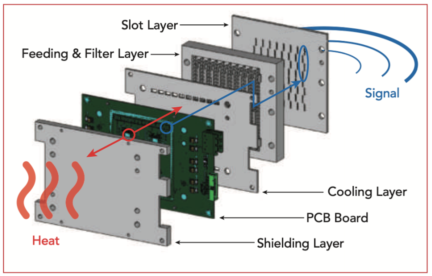
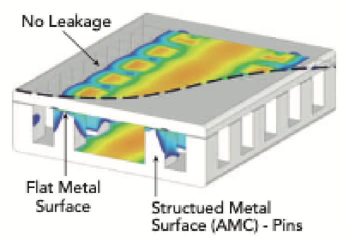


SRR 5 (Source: FCC)

Gapwaves





Source: Microwave Journal Feb 2019. Corner radars exclusively for Hella (SOP 2024, source: Hella)

Antennas

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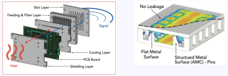

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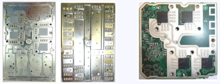

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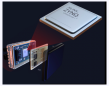
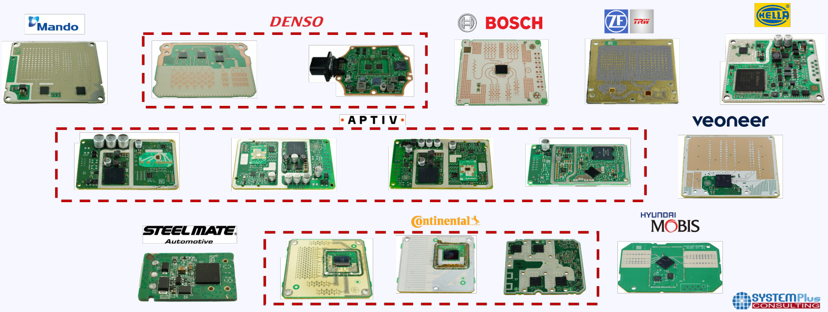


ARS 540 with 12x TX + 16x RX antennas. Source: Xilinx (modifiziert)

Tier 1 - Overview



Source: Systemplus

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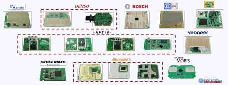
EIRP

Tier 1 - Overview



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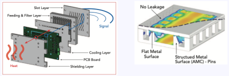
Gapwaves



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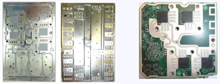
Continental



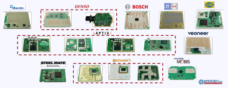
Source: FCC. See Huber+Suhner as well.

EIRP

Further Readings

EIRP

Tier 1 - Overview



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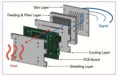

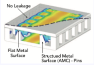

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Reports & Links

Comprehensive Survey of 77, 79 GHz Automotive Radar Companies – Sensors and ICs

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Notes

Reports & Links

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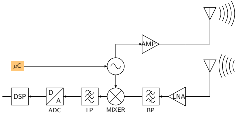

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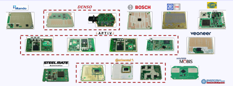


Source: Systemplus

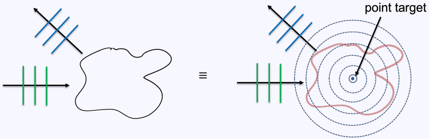
Table of Contents

Antennas

RCS

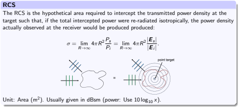
The RCS is the hypothetical area required to intercept the transmitted power density at the target such that, if the total intercepted power were re-radiated isotropically, the power density actually observed at the receiver would be produced produced:

$$\sigma = \lim_{R\to\infty} 4\pi R^2 \frac{P_s}{P_i} = \lim_{R\to\infty} 4\pi R^2 \frac{|\boldsymbol{E}_s|}{|\boldsymbol{E}_i|}.$$



Unit: Area (m^2). Usually given in dBsm (power: Use $10\log_{10} x$).

Radar Cross Section



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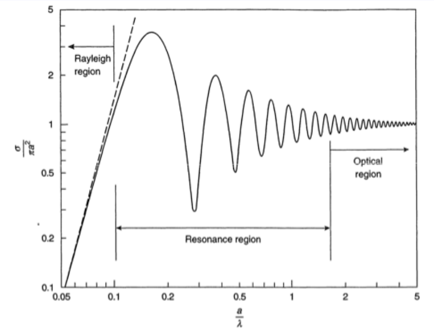
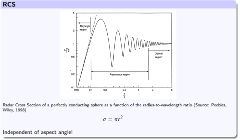


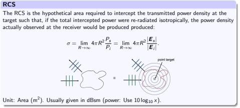
Radar Cross Section of a perfectly conducting sphere as a function of the radius-to-wavelength ratio (Source: Peebles, Wiley, 1998)

$$\sigma = \pi r^2$$

Independent of aspect angle!

Radar Cross Section





RCS

The maximum of the RCS of a perfectly conducting flat plate:

$$\sigma = 4\pi w^2 h^2/\lambda^2$$

This is only the maximum (perpendicular). Rule of thumb: License plate up to 30 dBsm at 76 GHz (far field, near field: reflect energy away) Example:

$$\sigma = 4\pi 0.3^2 0.1^2/0.0039^2 \approx 743$$

and thus

$$\mathrm{dBsm} = 10\log_{10} 743 = 28.7\mathrm{dB}$$

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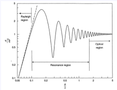

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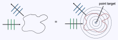

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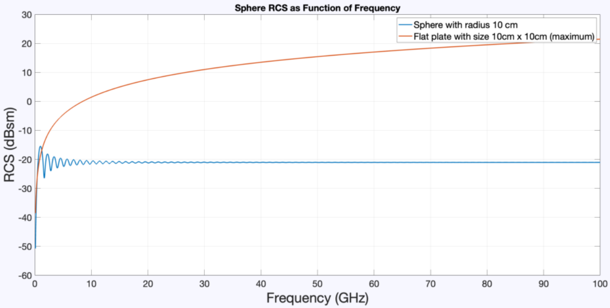
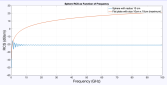


Radar Cross Section of a perfectly conducting sphere compared to plate

Radar Cross Section

RCS



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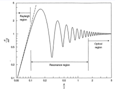

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$$\sigma = \pi r^2$$

Independent of aspect angle!

RCS

The received power usually fluctuates. The figure is for a single frequency (CW) and performed at 76 GHz for a Mercedes C Class. Note: Pedestrian approx. -8 dBsm with strong fluctuation

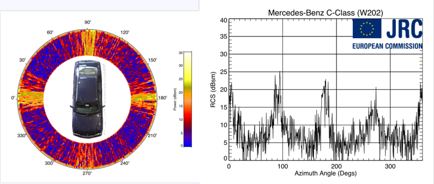

Table of Contents

Block diagram

Radar equation

Radar equation

The received power P_r is given by the equation

$$P_r = \frac{P_t G_t G_r \lambda^2 \sigma F^4}{(4\pi)^3 R^4},$$

where P_t is the transmitter power, G_t is the antenna gain of the transmitting antenna, λ is the transmitted wavelength, G_r is the gain of receiving antenna, σ is the radar cross section of the target, F is the pattern propagation factor and R is distance from the radar to the target (transmitter and receiver at the same location).

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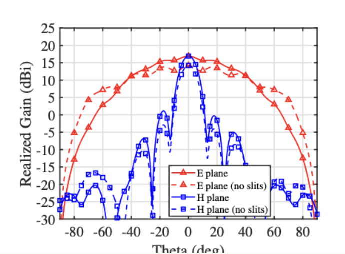
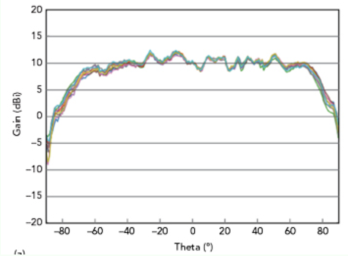
Assume that a radar has a maximum range of 100 m at boresight for a 20 dBsm target and the patterns shown below. Plot the coverage for a 20 dBsm and a 0 dBsm target.



Pattern. Source: Ren et al. Gapwaveguide Automotive Imaging Radar Antenna With Launcher in Package Technology



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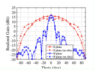

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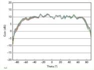

Pattern. Source: Microwave Journal

Table of Contents

Block diagram

Noise figure

Noise figure

The noise figure F is given by

$$F = \frac{\mathrm{SNR}_{\mathrm{in}}}{\mathrm{SNR}_{\mathrm{out}}} = \frac{1}{G} \cdot \frac{N_2}{N_1},$$

with gain factor G of the amplifier..

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Cascading

If several two-ports are connected in series, the following applies according to the so-called Friis formula:

$$F_{\mathrm{total}} = \frac{\mathrm{SNR}_{\mathrm{in}}}{\mathrm{SNR}_{\mathrm{out}}} = 1 + (F_1 - 1) + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1 \cdot G_2} + \cdots + \frac{F_n - 1}{G_1 G_2 G_3 \cdots G_{n-1}}$$

Placing

- Placing a filter, like any a device, before the first LNA negatively impacts system noise figure. Ideal position for a filter is after the LNA to preserve noise figure integrity
- If filter is placed after the LNA, risk exist that strong out-of-band signals may cause LNA compression
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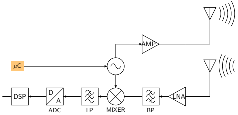

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Table of Contents

Block diagram

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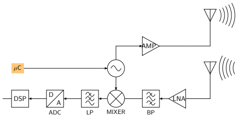

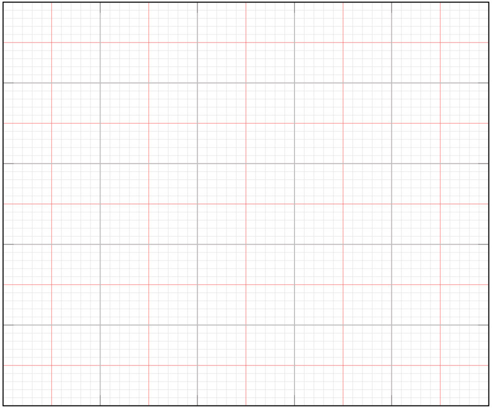

Table of Contents

Block diagram

Mixer

Frequency conversion

$$\cos(\omega_1 t)\cos(\omega_2 t) = \frac{1}{2}\left(\cos(\omega_1 t + \omega_2 t) + \cos(\omega_1 t - \omega_2 t)\right)$$

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Diode

Shockley-Equation[a] (see e.g. Wikipedia):

$$\begin{aligned} I_\mathrm{D} &= I_\mathrm{S}\left(\mathrm{e}^{\frac{U_\mathrm{D}}{n\,U_\mathrm{T}}} - 1\right) \\ &\approx I_\mathrm{S}\left(1 + \frac{U_\mathrm{D}}{n\,U_\mathrm{T}} + \frac{1}{2}\left(\frac{U_\mathrm{D}}{n\,U_\mathrm{T}}\right)^2 + \cdots\right) \end{aligned}$$

[a] with $I_\mathrm{S} \approx 10^{-12} \ldots 10^{-6}\,\mathrm{A}$ dem Sättigungsstrom, n dem Emissionskoeffizient ($n \approx 1 \ldots 2$), $U_\mathrm{T} = \frac{k \cdot T}{q} \approx 25\,\mathrm{mV}$ (bei Raumtemperatur) der Temperaturspannung, der absoluten Temperatur T und der Boltzmannkonstante $k = 1{,}381 \cdot 10^{-23}\,\mathrm{Ws/K}$ sowie der Elementarladung $q = 1{,}602 \cdot 10^{-19}\,\mathrm{As}$

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Frequency conversion

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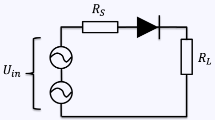
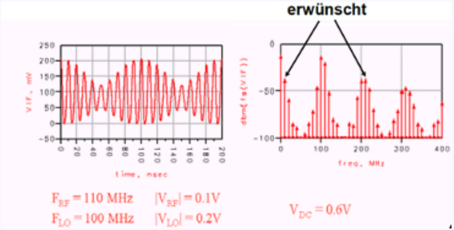
Schematic



Schematic

Measurement

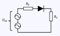


Source: Roland Pfeiffer

$$U_L = c_0 + c_1 U_{in} + c_2 U_{in}^2 + \cdots ,$$

with $(a+b)^2 = a^2 + 2ab + b^2$ one gets "spurs".

Schematic



Schematic

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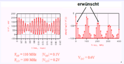

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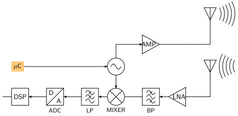
Key performance indicators

- ▷ Linearity
- ▷ Conversion gain
- ▷ Noise figure
- ▷ Power consumption
- ▷ Bias

Notes

Key performance indicators

- Linearity
- Conversion gain
- Noise figure
- Power consumption
- Bias



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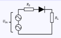

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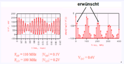

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Block diagram

ADC

TI AWR 1843

- ▷ 25 MSPS
- ▷ 4 ADC on board (for receive antennas, further ADC on board)
- ▷ 12 Bit resolution
- ▷ Source: TI

ADC

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- 25 MSPS
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TI AWR 2243

- up to 45 MSPS (22.5 complex)
- 4 ADC on board (for receive antennas, further ADC on board)
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Receiver dynamics

TI AWR 2243
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Receiver dynamics:

$$D = \frac{P_{\mathrm{max}}}{P_{\mathrm{min}}}$$

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Receiver dynamics

RCS in dBsm	RCS	R	$\frac{1}{R^4}$	$\frac{\sigma}{R^4}$
0	1	1	1	1
-8	0.1585	1	1	0.1585
30	1000	10	0.0001	0.1
30	1000	80	2.44e-08	2.44e-05

Compare to $1/2^{12} = 2.44\text{e-}04$

Notes

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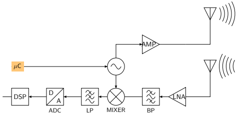

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Table of Contents

Signal Processing

Table of Contents

Spectral analysis

Content

- Modulation schemes (CW, FMCW, Chirp sequence, Tx Phase coding)
- IQ mixer
- CFAR
- Range and range rate estimation

Study goals

- Implement simple simulators
- Implement simple CFAR
- Implement range and range rate estimators
- Implement simple target simulators

Table of Contents

Spectral analysis

Modulation Schemes

Exercise (#4.1)

Implement a simulator in Matlab that simulates the receive signal for a CW radar in case of a single target.

Exercise

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Implement a simulator in Matlab that simulates the receive signal for a CW radar in case of a single target.

Table of Contents

Modulation Schemes

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Phase and frequency

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Phase is the integral of the frequency. I.e. if

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then

$$\varphi = \omega_0 t + 0.5\Delta\omega t^2.$$

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Given is a target with radial distance $R(t)$ and a transmit signal of the form

$$\omega_{\text{rx}}(t) = \omega_0 + \Delta\omega \cdot t$$

and thus

$$s_{\text{tx}} = A_{\text{tx}} \sin(\omega_0\, t + 0.5\, \Delta\omega\, t^2)$$

and for the down-mixed and filtered signal:

$$s_{\text{rx}}(t) = A_{\text{rx}} \sin\left(\omega_0\, t + 0.5\, \Delta\omega\, t^2 - \left(\omega_0 \left(t - \frac{2\, R(t)}{c}\right) + 0.5\, \Delta\omega \left(t - \frac{2\, R(t)}{c}\right)\right)^2\right),$$

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Receive signal

Thus it follows that

$$\begin{aligned} s_{\text{rx}}(t) &= A_{\text{rx}} \sin\left(0.5\, \Delta\omega\, t^2 + \omega_0 \frac{2\,R(t)}{c} - 0.5\, \Delta\omega\, t^2 + \Delta\omega\, \frac{2\,R(t)}{c}\, t - \Delta\omega\, \frac{2\,R^2(t)}{c^2}\right). \\ &= A_{\text{rx}} \sin\left(\frac{2\,\omega_0}{c} R(t) + \frac{2\Delta\omega}{c} R(t)\, t - \frac{2\,\Delta\omega}{c^2} R^2(t)\right) \end{aligned}$$

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Receive signal for a moving target

Expanding

$$R(t) = R_0 + \dot{R}t$$

into

$$s_{\text{rx}}(t) = A_{\text{rx}} \sin\left(\frac{2\,\omega_0}{c} R(t) + \frac{2\Delta\omega}{c} R(t)\,t - \frac{2\,\Delta\omega}{c^2} R^2(t)\right)$$

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respectively.

Receive signal for a moving target

Sorting, and neglecting terms with c^2:

$$s_{rx}(t) = A_{rx}\sin\left(\frac{2\Delta\omega}{c}R_0\,t + \frac{2\,\omega_0}{c}\dot{R}t + \frac{2\Delta\omega}{c}\dot{R}t^2\right)$$

And thus

$$s_{rx}(t) = A_{rx}\sin\left(2\pi\,C_1 R_0\,t + 2\pi\,C_2\dot{R}t + \frac{2\Delta\omega}{c}\dot{R}t^2\right),$$

with

$$2\pi\,C_1 = \frac{2\Delta\omega}{c} \Rightarrow C_1 = \frac{\Delta\omega}{\pi c} = \frac{2\Delta f}{c} = \frac{2}{\Delta\lambda}$$

and

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respectively (Note: Doppler shift $f_d = \frac{2\dot{R}}{\lambda_0}$)

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Receive signal for a moving target

Expanding

$$R(t) = R_0 + \dot{R}t$$

into

$$s_{rx}(t) = A_{rx}\sin\left(\frac{2\,\omega_0}{c}R(t) + \frac{2\Delta\omega}{c}R(t)\,t - \frac{2\,\Delta\omega}{c^2}R^2(t)\right)$$

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Neglecting constant terms (A_{rx} might be complex) and expanding:

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Receive signal

Thus it follows that

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Resolution

Considering

$$s_{rx}(t) \approx A_{rx}\sin\left(2\pi\,C_1 R_0\,t + 2\pi\,C_2\dot{R}\right),$$

with $C_2 = \frac{2}{\lambda_0}$, an observation interval of T and

$$\Delta f = \frac{\Delta f_{\mathrm{BW}}}{T} \Rightarrow C_1 = \frac{2\Delta f}{c} = \frac{2\Delta f_{\mathrm{BW}}}{Tc} = \frac{2}{T\lambda_{\mathrm{BW}}}$$

leads to a frequency resolution of $\Delta f = \frac{1}{T}$ and thus

$$\begin{aligned} \frac{1}{T} &= \Delta R\frac{2}{T\lambda_{\mathrm{BW}}} \Rightarrow \Delta R = \frac{\lambda_{\mathrm{BW}}}{2} \\ \frac{1}{T} &= \Delta\dot{R}\frac{2}{\lambda_0} \Rightarrow \Delta\dot{R} = \frac{\lambda_0}{2\,T} \end{aligned}$$

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Receive signal for a moving target

Sorting, and neglecting terms with c^2:

$$s_{\mathrm{rx}}(t) = A_{\mathrm{rx}} \sin\left(\frac{2\Delta\omega}{c} R_0\, t + \frac{2\,\omega_0}{c}\dot{R}t + \frac{2\Delta\omega}{c}\dot{R}t^2\right)$$

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Expanding

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$$s_{\mathrm{rx}}(t) = A_{\mathrm{rx}} \sin\left(\frac{2\,\omega_0}{c}\left(R_0 + \dot{R}t\right) + \frac{2\Delta\omega}{c}\left(R_0 + \dot{R}t\right) t - \frac{2\,\Delta\omega}{c^2}\left(R_0 + \dot{R}t\right)^2\right)$$

Neglecting constant terms (A_{rx} might be complex) and expanding:

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Exercise (#4.2)

Implement a simulator in Matlab that simulates the receive signal for a FMCW radar in case of a single target (w/wo IQ-mixer)

Exercise

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Considering

$$s_{\mathrm{rx}}(t) \approx A_{\mathrm{rx}} \sin\left(2\pi\, C_1 R_0\, t + 2\pi\, C_2 \dot{R}\right),$$

with $C_2 = \frac{2}{\lambda_0}$, an observation interval of T and

$$\Delta f = \frac{\Delta f_{\mathrm{BW}}}{T} \Rightarrow C_1 = \frac{2\Delta f}{c} = \frac{2\Delta f_{\mathrm{BW}}}{Tc} = \frac{2}{T\lambda_{\mathrm{BW}}}$$

leads to a frequency resolution of $\Delta f = \frac{1}{T}$ and thus

$$\frac{1}{T} = \Delta R \frac{2}{T\lambda_{\mathrm{BW}}} \Rightarrow \Delta R = \frac{\lambda_{\mathrm{BW}}}{2}$$

$$\frac{1}{T} = \Delta \dot{R} \frac{2}{\lambda_0} \Rightarrow \Delta \dot{R} = \frac{\lambda_0}{2T}$$

Receive signal for a moving target

Sorting, and neglecting terms with c^2:

$$s_{\mathrm{rx}}(t) = A_{\mathrm{rx}} \sin\left(\frac{2\Delta\omega}{c} R_0\, t + \frac{2\,\omega_0}{c}\dot{R}t + \frac{2\Delta\omega}{c}\dot{R}t^2\right)$$

And thus

$$s_{\mathrm{rx}}(t) = A_{\mathrm{rx}} \sin\left(2\pi\, C_1 R_0\, t + 2\pi\, C_2 \dot{R}t + \frac{2\Delta\omega}{c}\dot{R}t^2\right),$$

with

$$2\pi\, C_1 = \frac{2\Delta\omega}{c} \Rightarrow C_1 = \frac{\Delta\omega}{\pi c} = \frac{2\Delta f}{c} = \frac{2}{\Delta\lambda}$$

and

$$2\pi\, C_2 = \frac{2\,\omega_0}{c} \Rightarrow C_2 = \frac{\omega_0}{\pi c} = \frac{2}{\lambda_0},$$

respectively (Note: Doppler shift $f_d = \frac{2\dot{R}}{\lambda_0}$)

Exercise (#4.3)

Implement a simulator in Matlab that simulates the receive signal for a FMCW radar in case of a single target (w/wo IQ-mixer) plus a distributed target (guardrail and/or walking person).

THIS IS A DRAFT VERSION / DISTRIBUTION PROHIBITED

Exercise

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Table of Contents

Modulation Schemes

Exercise (#4.4)

Implement a simulator in Matlab that simulates the receive signal for a chirp sequence radar in case of a single target.

Exercise

Exercise (#4.4)

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Modulation Schemes

Table of Contents

Modulation Schemes

Table of Contents

Spectral analysis

Table of Contents

Spectral analysis

Angle of Arrival

Steering vector

Uniform linear array

Assuming N-antennas with identical directivity on the $z - axis$ with positions $z_n = n \cdot d$ and an incoming plane wave, the steering vector can be described as follows:

$$\boldsymbol{a}(\vartheta) = \begin{bmatrix} 1 & e^{jkd\cos\vartheta} & e^{j2kd\cos\vartheta} & \dots & e^{j(N-1)d\cos\vartheta} \end{bmatrix}^T.$$

Adding all receive signals and using

$$\sum_{k=0}^{N-1} q^k = \frac{1-q^N}{1-q}$$

leads to

$$|\boldsymbol{a}(\vartheta)| = \left| \frac{1 - e^{jkNd\cos\vartheta}}{1 - e^{jkd\cos\vartheta}} \right| = \frac{\left|\sin\left(\frac{1}{2}Nkd\cos\vartheta\right)\right|}{\left|\sin\left(\frac{1}{2}kd\cos\vartheta\right)\right|}$$

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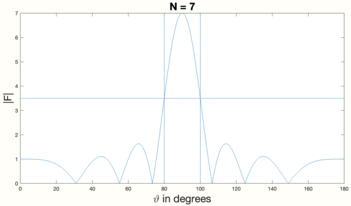
Properties

- Maximum of N.
- For $d = \lambda/2$: First zero at

$$\begin{aligned} \frac{1}{2}\pi N \cos_0 \vartheta &= \pi \\ \Leftrightarrow \cos(\vartheta_0) &= 2/N. \end{aligned}$$

- Thus[a]



[a] Linearization around $\pi/2$: $\Delta\vartheta \approx 2/N$

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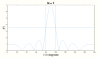

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Uniform linear array: $\lambda/2$-spacing

Uniform linear array with array spacing $\lambda/2$ located on z-axis. Phase of element n:

$$e^{jkn\frac{\lambda}{2}\cos(\vartheta)} = e^{j\pi\cos(\vartheta)n}.$$

Compare to sampled signal (nT) of frequency f_0:

$$f_n = e^{j2\pi f n T}.$$

This leads to a spacial frequency

$$f_{\text{spat}} = 0.5\cos(\vartheta).$$

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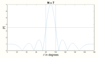

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[a] $T = 1$

Exercise

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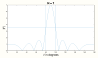

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Exercise (#4.5)

Implement a ULA in MATLAB and numerically determine the 3 dB−beamwidth.

Exercise

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Table of Contents

Angle of Arrival

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Signal model

Signal model

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$$\boldsymbol{x}(t) = \boldsymbol{A}(\vartheta)\boldsymbol{s}(t) + \boldsymbol{n}(t),$$

where $\boldsymbol{A}$ is a $N_a \times N_s$ matrix, with

$$N_a: \ \text{number of antennas}$$

and

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Weighting vector

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Weighting vector

Weighting vector

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Signal model

Weighting vector

Use a weighting vector

$$\boldsymbol{w}(\vartheta)$$

and calculate the power:

$$\begin{aligned} P(\vartheta) &= \left[\boldsymbol{w}^H(\vartheta)\boldsymbol{s}\right]\left[\boldsymbol{w}^H(\vartheta)\boldsymbol{s}\right]^T \\ &= \boldsymbol{w}^H(\vartheta)\boldsymbol{s}\boldsymbol{s}^T\boldsymbol{w}. \end{aligned}$$

Task: Maximize the receive power:

$$\arg\max_{\vartheta} \{P(\vartheta)\}$$

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The spatial covariance matrix is then given by:

$$\begin{aligned} \boldsymbol{R} &= E\left\{\boldsymbol{x}(t)\boldsymbol{x}^H(t)\right\} \\ &= \boldsymbol{A}\boldsymbol{P}\boldsymbol{A}^H + \sigma^2\boldsymbol{I}, \end{aligned}$$

with

$$\boldsymbol{P} = E\left\{\boldsymbol{s}(t)\boldsymbol{s}(t)^H\right\}$$

being the source covariance matrix. Thus, the receive power becomes:

$$P(\vartheta) = \boldsymbol{w}^H(\vartheta)\boldsymbol{s}\boldsymbol{s}^T\boldsymbol{w} = \boldsymbol{w}^H(\vartheta)\boldsymbol{R}\boldsymbol{w}.$$

Classical beamforming (Bartellt)

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Spatial covariance matrix

Classical beamformer

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$$\boldsymbol{w}(\vartheta) = \boldsymbol{a}$$

and normalizing $\boldsymbol{w}(\vartheta)$ using

$$\boldsymbol{w}(\vartheta) = \frac{\boldsymbol{a}(\vartheta)}{\sqrt{\boldsymbol{a}^H(\vartheta)\boldsymbol{a}(\vartheta)}}$$

leads to

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Target separation: Approx.

$$\Delta\varphi = \frac{2}{N},$$

assuming ULA with $\lambda/2$ spacing.
See e.g. *Two decades of array signal processing research: The parametric approach, page 73.*

Exercise

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Implement a ULA in MATLAB with $d = \frac{\lambda}{2}$ and $N = 6$. Simulate three targets at 30°, 90° and 110°, respectively. Implement a delay-and-sum beamformer and an FFT based approach. Change N to be 5,

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Capon beamforming

Spatial spectrum:

$$P(\vartheta) = \frac{1}{\boldsymbol{a}^H(\vartheta)\boldsymbol{R}^{-1}\boldsymbol{a}(\vartheta)}$$

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Eigendecomposition of spatial covariance matrix

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where $\boldsymbol{\lambda}_s$ contains the N_s largest Eigenvalues, $\boldsymbol{U}_s$ contains the corresponding Eigenvectors, $\boldsymbol{U}_n$ contains the EIgenvectors related to noise and $\boldsymbol{\lambda}_s$ the Eigenvalues related to noise.

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Music beamforming

Music beamforming

The receiver power becomes

$$P(\vartheta) = \mathbf{w}^H(\vartheta)\mathbf{s}\mathbf{s}^T\mathbf{w}.$$

This becomes minimal in case of a steering vector pointing into the direction of the noise subspace (orthogonal to signal subspace). Taking the reciprocal and normalizing:

$$P(\vartheta) = \frac{1}{\mathbf{a}^H(\vartheta)\mathbf{U}_n\mathbf{U}_n^H\mathbf{a}(\vartheta)} \Rightarrow P_{\text{norm}}(\vartheta) = \frac{\mathbf{a}^H(\vartheta)\mathbf{a}}{\mathbf{a}^H(\vartheta)\mathbf{U}_n\mathbf{U}_n^H\mathbf{a}(\vartheta)}.$$

Notes:

- Does not work in case of correlated signals. Signals can be de-correlated e.g. by making use of spatial smoothing.

Capon beamforming

Spatial spectrum:

$$P(\vartheta) = \frac{1}{\mathbf{a}^H(\vartheta)\mathbf{R}^{-1}\mathbf{a}(\vartheta)}$$

Table of Contents

Angle of Arrival

Table of Contents

Angle of Arrival

OMP

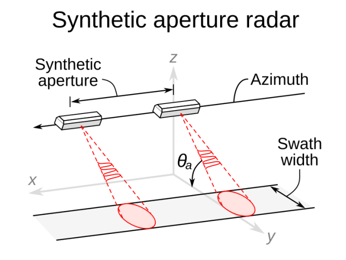
Signal decomposition

- ▷ use a large, highly redundant dictionary that spans the signal space
- ▷ choose the smallest subset

Table of Contents

Spectral analysis

SAR



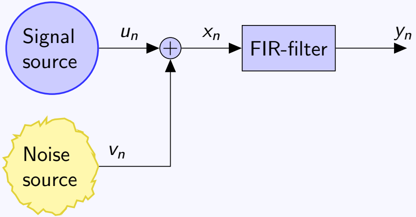
Source: Wikipedia

Related concepts

Eigenfilter

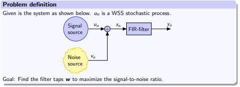
Problem definition

Given is the system as shown below. u_n is a WSS stochastic process.



Goal: Find the filter taps $\boldsymbol{w}$ to maximize the signal-to-noise ratio.

Eigenfilter



Corollary

The average power of the signal output:

$$P_y = \boldsymbol{w}^H \boldsymbol{R} \boldsymbol{w},$$

with $\boldsymbol{R}$ beeing the autocorrelation matrix of the input signal u_n. The average power of the noise component:

$$N_y = \sigma^2 \boldsymbol{w}^H \boldsymbol{w}.$$

Signal to noise ratio:

$$(SNR)_y = \frac{\boldsymbol{w}^H \boldsymbol{R} \boldsymbol{w}}{\sigma^2 \boldsymbol{w}^H \boldsymbol{w}}$$

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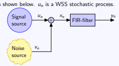

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Corollary

The filter values w_n of the FIR-filter for maximum signal to noise ratio subject to the contraint $\boldsymbol{w}^H \boldsymbol{w} = 1$ can be calculated as follows:

1. Perform an eigendecomposition of $\boldsymbol{R}$
2. Find the largest eigenvalue λ_m and the corresponding eigenvector $\boldsymbol{q}_m$.
3. $\boldsymbol{q}_m$ defines the impulse response of the optimum filter.

And thus:

$$(SNR)_{y,max} = \frac{\lambda_m}{\sigma^2},$$

with σ^2 beeing the noise covariance.

Eigenfilter

Corollary

The filter values w_n of the FIR-filter for maximum signal to noise ratio subject to the contraint $\boldsymbol{w}^H\boldsymbol{w} = 1$ can be calculated as follows:

1. Perform an eigendecomposition of $\boldsymbol{R}$
2. Find the largest eigenvalue λ_m and the corresponding eigenvector $\boldsymbol{q}_m$.
3. $\boldsymbol{q}_m$ defines the impulse response of the optimum filter.

And thus:

$$(SNR)_{y,max} = \frac{\lambda_m}{\sigma^2},$$

with σ^2 beeing the noise covariance.

Corollary

The average power of the signal output:

$$P_y = \boldsymbol{w}^H\boldsymbol{R}\boldsymbol{w},$$

with $\boldsymbol{R}$ beeing the autocorrelation matrix of the input signal u_n. The average power of the noise component:

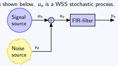
$$N_y = \sigma^2\boldsymbol{w}^H\boldsymbol{w}.$$

Signal to noise ratio:

$$(SNR)_y = \frac{\boldsymbol{w}^H\boldsymbol{R}\boldsymbol{w}}{\sigma^2\boldsymbol{w}^H\boldsymbol{w}}$$

Problem definition

Given is the system as shown below. u_n is a WSS stochastic process.



Goal: Find the filter taps $\boldsymbol{w}$ to maximize the signal-to-noise ratio.

Corollary

This type of filter can be viewed as the *stochastic* counterpart of the so-called matched filter. The latter maximizes the output signal-to-noise ratio for a known signal subject to additive white noise. Furthermore, it can be understood as an example for principal component analysis.

Table of Contents

index