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Wave propagation

Maxwell's Equations

Maxwell's Equations

Maxwell's Equations

The Maxwell's Equations describe electromagnetic phenomena. They exist in differential and integral form[a]:

$$\operatorname{div} \boldsymbol{D} = \rho \iff \oiint_{\partial V} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{A} = \iiint_{V} \rho \mathrm{d}V$$

$$\operatorname{div} \boldsymbol{B} = 0 \iff \oiint_{\partial V} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{A} = 0$$

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$$\operatorname{rot} \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \iff \oint_{\partial A} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{s} = \iint_{A} \left[\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}\right] \cdot \mathrm{d}\boldsymbol{A}$$

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[a] Using the integral theorems of Gauss and Stokes allows to switch from one domain to the other.

Symbols

- ρ: Charge density (Ladungsdichte) in $\mathrm{C/m^3}$ ($\mathrm{As/m^3}$)
- $\boldsymbol{D}$: Electric displacement field (Elektrische Flussdichte) in $\mathrm{C/m^2}$ ($\mathrm{As/m^2}$)
- $\boldsymbol{E}$: Electric field (Elektrische Feldstärke) in $\mathrm{V\,m^{-1}}$
- $\boldsymbol{H}$: Magnetic Field (Magnetische Feldstärke) in $\mathrm{A\,m^{-1}}$
- $\boldsymbol{B}$: Magnetic flux density (Magnetische Flussdichte) in T ($\mathrm{N\,A^{-1}\,m^{-1}}$)
- $\boldsymbol{J}$: Current density (Stromdichte) in $\mathrm{A/m^2}$
- $\frac{\partial \boldsymbol{D}}{\partial t}$: Displacement current density (Verschiebungsstromdichte) in $\mathrm{A/m^2}$

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Maxwell's Equations

The Maxwell's Equations only describe the electromagnetic phenomena. They are not the solutions themselves. These are differential equations whose solutions depend on the boundary conditions. Solution methods are e.g.:

- numerical methods (e.g. finite integration, finite difference method, finite element method, integral equation method)
- analytic methods

Analytical solutions can only be determined for special cases. These include, for example, electromagnetic waves.

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Maxwell's Equations

Divergence

Definition

Given is a function $\boldsymbol{f}(x, y, z)$. The divergence is defined as follow:

$$\begin{aligned} \operatorname{div}(\boldsymbol{f}) &= \boldsymbol{e}_x \frac{\partial f_x(x,y,z)}{\partial x} + \boldsymbol{e}_y \frac{\partial f_y(x,y,z)}{\partial y} + \boldsymbol{e}_z \frac{\partial f_z(x,y,z)}{\partial z} \\ &= \boldsymbol{\nabla} \cdot \boldsymbol{f}(x,y,z), \end{aligned}$$

with

$$\boldsymbol{\nabla} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}.$$

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Electrical field

$$\mathrm{div}\, \boldsymbol{D} = \rho$$

The electric field is **not** source free.

Magnetic field

$$\mathrm{div}\, \boldsymbol{B} = 0$$

The magnetic field is source free.

Curl

Electrical field

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Given is a function $\boldsymbol{f}(x,y,z) = \boldsymbol{e}_x f_x(x,y,z) + \boldsymbol{e}_y f_y(x,y,z) + \boldsymbol{e}_z f_z(x,y,z)$.
The curl is the defined as follows:

$$\begin{aligned} \operatorname{curl} \boldsymbol{f} = \operatorname{rot} \boldsymbol{f} &= \boldsymbol{\nabla} \times \boldsymbol{f}(x,y,z) \\ &= \boldsymbol{e}_x \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z} \right) + \boldsymbol{e}_y \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x} \right) + \boldsymbol{e}_z \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right). \end{aligned}$$

If the curl[a] of a vector field is zero everywhere, then the vector field is vortex-free (also called irrotational).

[a] The notation curl is common in North America. In the rest of the world, rot is commonly used. Using the cross product with the del-operator (also known as nabla-operator) avoids confusion. See Wikipedia as well,

Rotation

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Electrical field

$$\operatorname{div}\boldsymbol{D} = \rho$$

The electric field is **not** source free.

Magnetic field

$$\operatorname{div}\boldsymbol{B} = 0$$

The magnetic field is source free.

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Electical field

$$\operatorname{rot}\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}$$

The static electrical field ($\frac{\partial\boldsymbol{B}}{\partial t} = 0$) is vortex-free.

Magnetic field

$$\operatorname{rot}\boldsymbol{H} = \boldsymbol{J} + \frac{\partial\boldsymbol{D}}{\partial t}$$

The magnetic field is not vortex-free.

Maxwell's Equations

Maxwell's Equations

Material equations

$$
\begin{aligned}
\boldsymbol{D} &= \varepsilon_0 \varepsilon_r \boldsymbol{E} \\
\boldsymbol{B} &= \mu_0 \mu_r \boldsymbol{H}
\end{aligned}
$$

Material equations

$$D = \varepsilon_0 \varepsilon_r E$$
$$B = \mu_0 \mu_r H$$

Electric current and charge

$$\text{rot}\, \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

Taking the divergence of both sides results in

$$\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \mathbf{J} + \frac{\partial (\nabla \cdot \mathbf{D})}{\partial t}.$$

Using the fact that the divergence of a curl is zero:

$$\nabla \cdot \mathbf{J} + \frac{\partial (\nabla \cdot \mathbf{D})}{\partial t} = 0.$$

With Gauss's law ($\nabla \cdot \mathbf{D} = \rho$) one gets

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$$\begin{aligned} \boldsymbol{D} &= \varepsilon_0 \varepsilon_r \boldsymbol{E} \\ \boldsymbol{B} &= \mu_0 \mu_r \boldsymbol{H} \end{aligned}$$

Continuity equation

Relationship according to the continuity equation:

$$\nabla \cdot \boldsymbol{J}(\boldsymbol{r}\,, t) + \frac{\partial}{\partial t}\rho(\boldsymbol{r}\,, t) = 0.$$

Interpretation:

> *Current is the movement of charge. The continuity equation says that if charge is moving out of a differential volume (i.e., divergence of current density is positive) then the amount of charge within that volume is going to decrease, so the rate of change of charge density is negative. Therefore, the continuity equation amounts to a conservation of charge.*

Wikipedia

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Stationary case

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Stationary case

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Static electric fields

Stationary case

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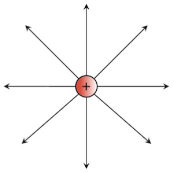
Spherical charge

Given its spherical charge Q with the homogeneous charge density ρ and the radius r_0. For points outside the charge:

$$\boldsymbol{D}(r)=\boldsymbol{e}_r\frac{Q}{4\pi r}.$$



Radial field. Source: physikunterricht-online.de

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Spherical charge - derivation

For reasons of symmetry, it can be concluded that the electric flux density $\boldsymbol{D}$ points radially outwards ($\boldsymbol{D} = \boldsymbol{e}_r D(r)$). Thus, for $r > r_0$:

$$\begin{aligned} \oiint_{\partial V} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{A} &= \iiint_v \rho \, \mathrm{d}x' \mathrm{d}y' \mathrm{d}z' \\ &= 4\pi r D(r) = Q \end{aligned}$$

and

$$D(r) = \frac{Q}{4\pi r},$$

respectively.

Gradient

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Radial field. Source: physikunterricht-online.de

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Definition

The gradient of a function $f(x, y, z)$ is defined as

$$\begin{aligned} \operatorname{grad} f(x,y,z) &= \boldsymbol{e}_x \frac{\partial f(x,y,z)}{\partial x} + \boldsymbol{e}_y \frac{\partial f(x,y,z)}{\partial y} + \boldsymbol{e}_z \frac{\partial f(x,y,z)}{\partial z} \\ &= \boldsymbol{\nabla} f(x,y,z) \end{aligned}$$

Properties

$$\operatorname{rot}\left(\operatorname{grad} f(x,y,z)\right) = \boldsymbol{\nabla} \times \boldsymbol{\nabla} f(x,y,z) = 0$$

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Radial field. Source: physikunterricht-online.de

Electrical potential

If the electric field is represented as a (negative) gradient of a function $\Phi(x, y, z)$ then

$$\operatorname{rot} \boldsymbol{E} = 0 \iff \oint_{\partial A} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = 0$$

is fulfilled *automatically*. Thus one needs to determine *only* a function $\Phi(x, y, z)$.

- Field strength: Unit $\mathrm{V\,m^{-1}}$. Specifies the force that would act on a test charge in the electric field ($\boldsymbol{F} = q\boldsymbol{E}$)
- Potential: Unit V. Indicates the potential energy of a test charge in an electric field.

Gradient

Electrical potential

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- ▷ Field strength: Unit $\mathrm{V\,m^{-1}}$. Specifies the force that would act on a test charge in the electric field ($\boldsymbol{F} = q\boldsymbol{E}$)
- ▷ Potential: Unit V. Indicates the potential energy of a test charge in an electric field.

Definition

The gradient of a function $f(x, y, z)$ is defined as

$$\begin{aligned} \operatorname{grad} f(x,y,z) &= \boldsymbol{e}_x \frac{\partial f(x,y,z)}{\partial x} + \boldsymbol{e}_y \frac{\partial f(x,y,z)}{\partial y} + \boldsymbol{e}_z \frac{\partial f(x,y,z)}{\partial z} \\ &= \boldsymbol{\nabla} f(x,y,z) \end{aligned}$$

Properties

$$\operatorname{rot}(\operatorname{grad} f(x,y,z)) = \boldsymbol{\nabla} \times \boldsymbol{\nabla} f(x,y,z) = 0$$

Spherical charge - derivation

For reasons of symmetry, it can be concluded that the electric flux density $\boldsymbol{D}$ points radially outwards ($\boldsymbol{D} = \boldsymbol{e}_r D(r)$). Thus, for $r > r_0$:

$$\begin{aligned} \oiint_{\partial V} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{A} &= \iiint_v \rho \,\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \\ &= 4\pi r D(r) = Q \end{aligned}$$

and

$$D(r) = \frac{Q}{4\pi r},$$

respectively.

Electrical potential

Obviously, charge densities ρ are the sources of the electrical fields:

$$\operatorname{div} \boldsymbol{D} = -\operatorname{div} \epsilon \;\operatorname{grad} \Phi = \rho$$

Attention

When using the div-operator (and the rot-operator), one needs to take care for vector and derivation-rules. E.g.

$$\operatorname{div} \epsilon \operatorname{grad} \Phi = \epsilon \operatorname{div} \operatorname{grad} \Phi + \operatorname{grad} \Phi \operatorname{grad} \epsilon$$

Poisson-equation

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Electrical potential

If the electric field is represented as a (negative) gradient of a function $\Phi(x, y, z)$ then

$$\operatorname{rot} \boldsymbol{E} = 0 \iff \oint_{\partial A} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = 0$$

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Properties

$$\operatorname{rot}(\operatorname{grad} f(x,y,z)) = \boldsymbol{\nabla} \times \boldsymbol{\nabla} f(x,y,z) = 0$$

Electrical potential

In the homogeneous space:

$$\operatorname{div} \boldsymbol{D} = -\epsilon \ \operatorname{div} \ \operatorname{grad} \ \Phi = \rho$$

In cartesian coordinates:

$$\frac{\partial^2 \Phi(x,y,z)}{\partial x^2} + \frac{\partial^2 \Phi(x,y,z)}{\partial y^2} + \frac{\partial^2 \Phi(x,y,z)}{\partial z^2} = -\frac{\rho}{\epsilon}.$$

This equation is called Poisson-equation of the electrical field.

General electrostatic soluation

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If the charge density ρ is known everywhere, then the general solution is given as follows:

$$\Phi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0\,\varepsilon_r} \iiint_V \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\, dx'\, dy'\, dz'$$

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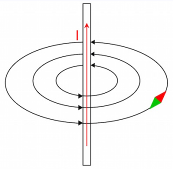
Current-carrying conductor

Given an infinitely long conductor with the diameter d and an orientation in the $z-$direction through which a current I flows. The well known result:

$$\mathbf{H}(r) = [-\mathbf{e_x}\cos(\varphi) + \mathbf{e_y}\cos(\varphi)]\,\frac{I}{2\pi r} = I\,\frac{\mathbf{e}_\varphi}{2\pi r}$$



Source: Physikunterricht-online.de

Stationary magnetic fields

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Now the circular disc is parameterized using

$$\gamma(t) = r\boldsymbol{e}_x \cos(t) + r\boldsymbol{e}_y \sin(t)$$

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$$\begin{aligned} I &= \oint_{\partial A} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{s} \\ &= \int_0^{2\pi} [-\boldsymbol{e}_x \sin(\varphi) f(r) + \boldsymbol{e}_y \cos(\varphi)] f(r) \\ &\quad \cdot r [-\boldsymbol{e}_x \sin(\varphi) + \boldsymbol{e}_y \cos(\varphi)] d\varphi \\ &= f(r) r \int_0^{2\pi} \sin^2(\varphi) + \cos^2(\varphi) d\varphi = f(r) 2\pi r. \end{aligned}$$

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Stationary magnetic fields

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Magnetic vector potential

Analogously to the electric potential, the magnetic vector potential can be introduced as an aid:

$$\boldsymbol{B} = \mathrm{rot}\,\boldsymbol{A}.$$

in case of a **known** current density, $\boldsymbol{A}$ can be calculated directly:

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{r}) &= \frac{\mu_0}{4\pi}\iiint \frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\, dx'\, dy'\, dz', \\ \boldsymbol{B}(\boldsymbol{r}) &= \frac{\mu_0}{4\pi}\iiint \boldsymbol{J}(\boldsymbol{r}') \times \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\, dx'\, dy'\, dz'. \end{aligned}$$

Hint: The last equation is also known as the Biot-Savart law.

Summary

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Stationary fields

- Point charges and infinitely long currents help to evaluate scenarios.
- Introduction of auxiliary quantities (potentials) facilitates the calculation.
- If the currents and charges are known, the fields can be calculated (often only numerically).
- Similar to the concept input signal / transfer function / output signal[a].
- Determining the currents and charges can be very difficult (e.g. charge distribution on semiconductor interfaces when a voltage is applied).

[a] See also Green's functions

Maxwell's Equations

Definition

Quasi-stationary fields are fields with a fixed spatial distribution and time-dependent intensity. These are cases in which the displacement current density $\frac{\partial \boldsymbol{D}}{\partial t}$ can be neglected compared to the power current density $\boldsymbol{J}$.

Eigenschaften

- In the close range of antennas, fields are quasi-static. A common designation for quasi-static fields is therefore also near fields.
- The near range is the range that is much smaller than the wavelength.
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The law of induction states that a time-varying magnetic field is linked to an electric field.

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Properties

- Approaches from AC engineering can often be applied (e.g. consideration at a fixed frequency, introduction of inductances)

Resonant circuit

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$$\begin{aligned}
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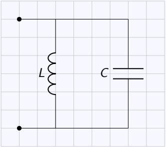
A simple resonant circuit



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$$Z_{in} = \frac{j\omega L}{1 - \omega^2 LC}$$

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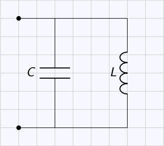

Current when applying a voltage U_0 (unit jump):

$$\begin{aligned}
I(s) &= U_0 \frac{1}{s} \frac{sL}{1+s^2LC} = U_0 \frac{L}{1+s^2LC} = U_0 \frac{L\omega_0^2}{\omega_0^2 + s^2} \\
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\end{aligned}$$

with $\omega_0 = \frac{1}{\sqrt{LC}}$.

A simple resonant circuit

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Input impedance:

$$Z_{in} = \frac{j\omega L}{1-\omega^2 LC}$$

Definition

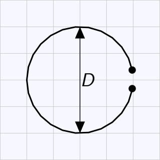
The law of induction states that a time-varying magnetic field is linked to an electric field.

$$\mathrm{rot}\, \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \iff \oint_{\partial A} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = -\iint_A \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{A}$$

Properties

- Approaches from AC engineering can often be applied (e.g. consideration at a fixed frequency, introduction of inductances)

Simple inductance



Inductance of loop with diameter D and diameter of wire d[a]:

$$L \approx \mu_0 \mu_r \frac{D}{2}\left(\ln\left(\frac{8D}{d}\right) - 2\right)$$

Example: $D = 1\,\mathrm{m}$, $d = 0.1\,\mathrm{cm}$: $L = 2.94\,\mu\mathrm{H}$.

[a] see e.g. eeweb und FH Aachen

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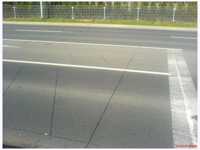

Source: Wikipedia (Wolf Meusel)

Assumption

$$\begin{aligned} L &= 2.94\,\mu\mathrm{H} \\ C &= 4.7\,\mu\mathrm{F} \\ f_0 &= \frac{1}{2\pi\sqrt{LC}} = 42.8\,\mathrm{kHz} \end{aligned}$$

Resonant circuit

Simple inductance

Source: Wikipedia (Wolf Meusel)

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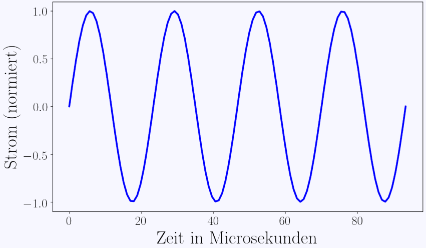
Step response



Calculation of the step response using inverse Laplace transform.

Maxwell's Equations

Wave equation

Differential form of Maxwell's equations

$$
\begin{aligned}
\operatorname{div} \boldsymbol{D} &= \rho && \text{Gauss's law} \\
\operatorname{div} \boldsymbol{B} &= 0 && \text{Gauss's law for magnetic fields} \\
\operatorname{rot} \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t} && \text{Maxwell–Faradayequation} \\
\operatorname{rot} \boldsymbol{H} &= \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} && \text{Ampère's law}
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Time-harmonic field

Assumption: Temporal relation according to[a] $e^{j\omega t}$ and

$$\boldsymbol{E}(\boldsymbol{r}, t) = \operatorname{Re}\left(\underline{\boldsymbol{E}}(\boldsymbol{r})\, e^{j\omega t}\right)$$

$$
\begin{aligned}
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\operatorname{div} \underline{\boldsymbol{B}} &= 0 \\
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\operatorname{rot} \underline{\boldsymbol{H}} &= \underline{\boldsymbol{J}} + j\omega \underline{\boldsymbol{D}}
\end{aligned}
$$

In the following, we will not explicitly write the underscore.

[a] Note that in physics, one commonly uses $e^{-j\omega t}$