

## Wave propagation

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## Maxwell's Equations

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#### 2.2.1 General case

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## Maxwell's Equations

The Maxwell's Equations describe electromagnetic phenomena. They exist in differential and integral form<sup>a</sup>:

$$\operatorname{div} \mathbf{D} = \rho \iff \iint_{\partial V} \mathbf{D} \cdot d\mathbf{A} = \iiint_V \rho dV$$

$$\operatorname{div} \mathbf{B} = 0 \iff \iint_{\partial V} \mathbf{B} \cdot d\mathbf{A} = 0$$

$$\operatorname{rot} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \iff \oint_{\partial A} \mathbf{E} \cdot d\mathbf{s} = -\iint_A \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}$$

$$\operatorname{rot} \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \iff \oint_{\partial A} \mathbf{H} \cdot d\mathbf{s} = \iint_A \left[ \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right] \cdot d\mathbf{A}$$

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\*Using the integral theorem of Gauss and Stokes allows to switch from one domain to the other

## Symbols

- ▷  $\rho$ : Charge density (Ladungsdichte) in  $C/m^3$  ( $As/m^3$ )
- ▷  $\mathbf{D}$ : Electric displacement field (Elektrische Flussdichte) in  $C/m^2$  ( $As/m^2$ )
- ▷  $\mathbf{E}$ : Electric field (Elektrische Feldstärke) in  $V m^{-1}$
- ▷  $\mathbf{H}$ : Magnetic Field (Magnetische Feldstärke) in  $A m^{-1}$
- ▷  $\mathbf{B}$ : Magnetic flux density (Magnetische Flussdichte) in  $T$  ( $N A^{-1} m^{-1}$ )
- ▷  $\mathbf{J}$ : Current density (Stromdichte) in  $A/m^2$
- ▷  $\frac{\partial \mathbf{D}}{\partial t}$ : Displacement current density (Verschiebungsstromdichte) in  $A/m^2$

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- ▷  $\mathbf{H}$ : Magnetic Field (Magnetische Feldstärke) in  $A\cdot m^{-1}$
- ▷  $\Phi$ : Magnetic flux density (Magnetische Flussdichte) in  $T$  ( $N\cdot A^{-1}\cdot m^{-1}$ )
- ▷  $\mathbf{J}$ : Current density (Strömdichte) in  $A/m^2$
- ▷  $\mathbf{D}_0$ : Displacement current density (Verschiebungstromdichte) in  $A/m^2$

**Maxwell's Equations**

The Maxwell's Equations describe electromagnetic phenomena. They exist in differential and integral form\*.

$$\begin{aligned} \text{div } \mathbf{D} = \rho &\iff \iiint_{\text{Vol}} \mathbf{D} \cdot d\mathbf{A} = \iiint_{\text{Vol}} \rho dV \\ \text{div } \mathbf{B} = 0 &\iff \iint_{\text{Surf}} \mathbf{B} \cdot d\mathbf{A} = 0 \\ \text{rot } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} &\iff \oint_{\text{Surf}} \mathbf{E} \cdot d\mathbf{x} = - \iint_{\text{Surf}} \frac{\partial \mathbf{B}}{\partial n} \cdot d\mathbf{A} \\ \text{rot } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} &\iff \oint_{\text{Surf}} \mathbf{H} \cdot d\mathbf{x} = \iint_{\text{Surf}} \left[ \mathbf{J} + \frac{\partial \mathbf{D}}{\partial n} \right] \cdot d\mathbf{A} \end{aligned}$$

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## Maxwell's Equations

The Maxwell's Equations only describe the electromagnetic phenomena. They are not the solutions themselves. These are differential equations whose solutions depend on the boundary conditions. Solution methods are e.g.:

- ▷ numerical methods (e.g. finite integration, finite difference method, finite element method, integral equation method)
- ▷ analytic methods

Analytical solutions can only be determined for special cases. These include, for example, electromagnetic waves.

## Maxwell's Equations

### 2.2 Maxwell's Equations

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2.2.2 Divergence and rotation

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2.2.6 Time harmonic fields

## Definition

Given is a function  $\mathbf{f}(x, y, z)$ . The **divergence** is defined as follow:

$$\begin{aligned}\operatorname{div}(\mathbf{f}) &= \mathbf{e}_x \frac{\partial f_x(x, y, z)}{\partial x} + \mathbf{e}_y \frac{\partial f_y(x, y, z)}{\partial y} + \mathbf{e}_z \frac{\partial f_z(x, y, z)}{\partial z} \\ &= \nabla \cdot \mathbf{f}(x, y, z),\end{aligned}$$

with

$$\nabla = \left( \begin{array}{c} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{array} \right).$$

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$$= \nabla \cdot \mathbf{f}(x, y, z),$$

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## Electrical field

$$\operatorname{div} \mathbf{D} = \rho$$

The electric field is **not** source free.

## Magnetic field

$$\operatorname{div} \mathbf{B} = 0$$

The magnetic field is source free.

**Electrical field**

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$$\begin{aligned}\operatorname{curl} \mathbf{f} = \operatorname{rot} \mathbf{f} &= \nabla \times \mathbf{f}(x, y, z) \\ &= \mathbf{e}_x \left( \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z} \right) + \mathbf{e}_y \left( \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x} \right) + \mathbf{e}_z \left( \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right).\end{aligned}$$

If the **curl<sup>a</sup>** of a vector field is zero everywhere, then the vector field is **vortex-free** (also called **irrotational**).

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<sup>a</sup>The notation curl is common in North America. In the rest of the world, rot is commonly used. Using the cross product with the **del-operator** (also known as **nabla-operator**) avoids confusion. See [Wikipedia](#) as well.

**Definition**

Given is a function  $\mathbf{f}(x, y, z) = \mathbf{e}_x f_x(x, y, z) + \mathbf{e}_y f_y(x, y, z) + \mathbf{e}_z f_z(x, y, z)$ .

The **curl** is defined as follows:

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If the **curl\*** of a vector field is zero everywhere, then the vector field is **vortex-free** (also called **irrotational**).

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## Electrical field

$$\operatorname{rot} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

The static electrical field ( $\frac{\partial \mathbf{B}}{\partial t} = 0$ ) is vortex-free.

**Electrical field**

$$\operatorname{div} \mathbf{D} = \rho$$

The electric field is **not** source free.

**Magnetic field**

$$\operatorname{div} \mathbf{B} = 0$$

The magnetic field is source free.

## Magnetic field

$$\operatorname{rot} \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

The magnetic field is not vortex-free.

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## Material equations

$$\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E}$$

$$\mathbf{B} = \mu_0 \mu_r \mathbf{H}$$

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## Electric current and charge

$$\text{rot } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Taking the divergence of both sides results in

$$\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \mathbf{J} + \frac{\partial(\nabla \cdot \mathbf{D})}{\partial t}.$$

Using the fact that the divergence of a curl is zero:

$$\nabla \cdot \mathbf{J} + \frac{\partial(\nabla \cdot \mathbf{D})}{\partial t} = 0.$$

With Gauss's law ( $\nabla \cdot \mathbf{D} = \rho$ ) one gets

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## Material equations

$$\begin{aligned}\mathbf{D} &= \epsilon_0 \epsilon_r \mathbf{E} \\ \mathbf{B} &= \mu_0 \mu_r \mathbf{H}\end{aligned}$$

## Continuity equation

Relationship according to the **continuity equation**:

$$\nabla \cdot \mathbf{J}(\mathbf{r}, t) + \frac{\partial}{\partial t} \rho(\mathbf{r}, t) = 0.$$

Interpretation:

*Current is the movement of charge. The continuity equation says that if charge is moving out of a differential volume (i.e., divergence of current density is positive) then the amount of charge within that volume is going to decrease, so the rate of change of charge density is negative. Therefore, the continuity equation amounts to a conservation of charge.*

[Wikipedia](#)

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## Stationary case

Electrical field:

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Magnetic fields:

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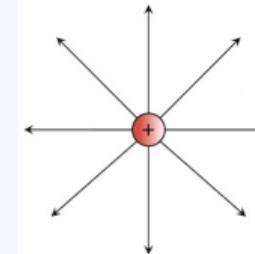
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## Spherical charge

Given its spherical charge  $Q$  with the homogeneous charge density  $\rho$  and the radius  $r_0$ . For points outside the charge:

$$\mathbf{D}(r) = \epsilon_r \frac{Q}{4\pi r}.$$



Radial field. Source: [physikunterricht-online.de](http://physikunterricht-online.de)

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## Spherical charge - derivation

For reasons of symmetry, it can be concluded that the electric flux density  $\mathbf{D}$  points radially outwards ( $\mathbf{D} = \mathbf{e}_r D(r)$ ). Thus, for  $r > r_0$ :

$$\begin{aligned} \iint_{\partial V} \mathbf{D} \cdot d\mathbf{A} &= \iiint_V \rho dx' dy' dz' \\ &= 4\pi r D(r) = Q \end{aligned}$$

and

$$D(r) = \frac{Q}{4\pi r},$$

respectively.

**Stationary case**

## Electrical field:

$$\begin{aligned} \operatorname{div} \mathbf{D} = \rho &\iff \iint_{\partial V} \mathbf{D} \cdot d\mathbf{A} = \iiint_V \rho dV \\ \operatorname{rot} \mathbf{E} = 0 &\iff \oint_{\partial A} \mathbf{E} \cdot d\mathbf{s} = 0 \end{aligned}$$

## Magnetic fields:

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- └ Wave propagation
- └ Maxwell's Equations

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Radial field. Source: physikunterrichtsraum.de

**Definition**

The **gradient** of a function  $f(x, y, z)$  is defined as

$$\begin{aligned} \text{grad } f(x, y, z) &= \mathbf{e}_x \frac{\partial f(x, y, z)}{\partial x} + \mathbf{e}_y \frac{\partial f(x, y, z)}{\partial y} + \mathbf{e}_z \frac{\partial f(x, y, z)}{\partial z} \\ &= \nabla f(x, y, z) \end{aligned}$$

**Properties**

$$\text{rot}(\text{grad } f(x, y, z)) = \nabla \times \nabla f(x, y, z) = 0$$

**Stationary case**

## Electrical field:

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**Properties**

$$\text{rot}(\text{grad } f(x, y, z)) = \nabla \times \nabla f(x, y, z) = 0$$

## Electrical potential

If the electric field is represented as a (negative) gradient of a function  $\Phi(x, y, z)$  then

$$\text{rot } \mathbf{E} = 0 \iff \oint_{\partial A} \mathbf{E} \cdot d\mathbf{s} = 0$$

is fulfilled *automatically*. Thus one needs to determine *only* a function  $\Phi(x, y, z)$ .

- ▷ Field strength: Unit  $\text{Vm}^{-1}$ . Specifies the force that would act on a test charge in the electric field ( $\mathbf{F} = q\mathbf{E}$ )
- ▷ Potential: Unit V. Indicates the potential energy of a test charge in an electric field.

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and

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respectively.

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Given its spherical charge  $Q$  with the homogeneous charge density  $\rho$  and the radius  $r_0$ . For points outside the charge

$$D(r) = \epsilon_0 \frac{Q}{4\pi r^2}$$



Radial field. Source: physikunterrichts online.de

**Electrical potential**

If the electric field is represented as a (negative) gradient of a function  $\Phi(x, y, z)$  then

$$\operatorname{rot} \mathbf{E} = 0 \iff \oint_{\partial V} \mathbf{E} \cdot d\mathbf{s} = 0$$

is fulfilled automatically. Thus one needs to determine only a function  $\Phi(x, y, z)$ .

▷ Field strength: Unit  $\text{V m}^{-1}$ . Specifies the force that would act on a test charge in the electric field ( $\mathbf{F} = \epsilon_0 \mathbf{E}$ )

▷ Potential: Unit  $\text{V}$ . Indicates the potential energy of a test charge in an electric field.

## Electrical potential

Obviously, charge densities  $\rho$  are the sources of the electrical fields:

$$\operatorname{div} \mathbf{D} = -\operatorname{div} \epsilon \operatorname{grad} \Phi = \rho$$

**Definition**

The **gradient** of a function  $f(x, y, z)$  is defined as

$$\operatorname{grad} f(x, y, z) = \mathbf{e}_x \frac{\partial f(x, y, z)}{\partial x} + \mathbf{e}_y \frac{\partial f(x, y, z)}{\partial y} + \mathbf{e}_z \frac{\partial f(x, y, z)}{\partial z}$$

$$= \nabla f(x, y, z)$$

**Properties**

$$\operatorname{rot}(\operatorname{grad} f(x, y, z)) = \nabla \times \nabla f(x, y, z) = 0$$

## Attention

When using the div-operator (and the rot-operator), one needs to take care for vector and derivation-rules. E.g.

$$\operatorname{div} \epsilon \operatorname{grad} \Phi = \epsilon \operatorname{div} \operatorname{grad} \Phi + \operatorname{grad} \Phi \operatorname{div} \epsilon$$

**Spherical charge - derivation**

For reasons of symmetry, it can be concluded that the electric flux density  $\mathbf{D}$  points radially outwards ( $\mathbf{D} = \epsilon_0 \mathbf{D}(r) \hat{r}$ ). Thus, for  $r > r_0$ :

$$\iint_{\partial V} \mathbf{D} \cdot d\mathbf{A} = \iiint_V \rho \, dr' dy' dz'$$

$$= 4\pi r_0^2 \mathbf{D}(r) \hat{r} \cdot \hat{r} = Q$$

and

$$\mathbf{D}(r) = \frac{Q}{4\pi r^2} \hat{r}$$

respectively

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**Electrical potential**

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is fulfilled automatically. Thus one needs to determine only a function  $\Phi(x, y, z)$ .

▷ Field strength: Unit  $V/m^2$ . Specifies the force that would act on a test charge in the electric field ( $\mathbf{F} = \epsilon \mathbf{E}$ )

▷ Potential: Unit  $V$ . Indicates the potential energy of a test charge in an electric field.

## Electrical potential

In the homogeneous space:

$$\operatorname{div} \mathbf{D} = -\epsilon \operatorname{div} \epsilon \operatorname{grad} \Phi = \rho$$

In cartesian coordinates:

$$\frac{\partial^2 \Phi(x, y, z)}{\partial x^2} + \frac{\partial^2 \Phi(x, y, z)}{\partial y^2} + \frac{\partial^2 \Phi(x, y, z)}{\partial z^2} = -\frac{\rho}{\epsilon}.$$

This equation is called **Poisson-equation** of the electrical field.

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**Properties**

$$\operatorname{rot} [\operatorname{grad} f(x, y, z)] = \nabla \times \nabla f(x, y, z) = 0$$

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This equation is called **Poisson-equation** of the electrical field.

## Electrical potential

If the charge density  $\rho$  is known everywhere, then the general solution is given as follows:

$$\begin{aligned}\Phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0\epsilon_r} \iiint_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dx' dy' dz' \\ \mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0\epsilon_r} \iiint_V \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} dx' dy' dz'\end{aligned}$$

**Electrical potential**Obviously, charge densities  $\rho$  are the sources of the electrical fields:

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When using the div-operator (and the rot-operator), one needs to take care for vector and derivation-rules. E.g.

$$\operatorname{div} (\mathbf{v} \operatorname{grad} \phi) = \mathbf{v} \operatorname{div} \operatorname{grad} \phi + \operatorname{grad} \phi \operatorname{div} \mathbf{v}$$

**Electrical potential**If the electric field is represented as a (negative) gradient of a function  $\Phi(x, y, z)$  then

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**Electrical potential**

If the charge density  $\rho$  is known everywhere, then the general solution is given as follows:

$$\begin{aligned}\phi(r) &= \frac{1}{4\pi\epsilon_0 r} \iiint_V \frac{\rho(r')}{|r - r'|} dr' dy' dz' \\ \mathbf{E}(r) &= \frac{1}{4\pi\epsilon_0 r^2} \iiint_V \rho(r') \frac{r - r'}{|r - r'|^3} dr' dy' dz'\end{aligned}$$

**Electrical potential**

In the homogeneous space:

$$\operatorname{div} \mathbf{D} = -\epsilon \operatorname{curl} \operatorname{grad} \phi = \rho$$

In cartesian coordinates:

$$\frac{\partial^2 \phi(x, y, z)}{\partial x^2} + \frac{\partial^2 \phi(x, y, z)}{\partial y^2} + \frac{\partial^2 \phi(x, y, z)}{\partial z^2} = -\frac{\rho}{\epsilon}$$

This equation is called **Poisson-equation** of the electrical field.

**Electrical potential**

Obviously, charge densities  $\rho$  are the sources of the electrical fields:

$$\operatorname{div} \mathbf{D} = -\epsilon \operatorname{curl} \operatorname{grad} \phi = \rho$$

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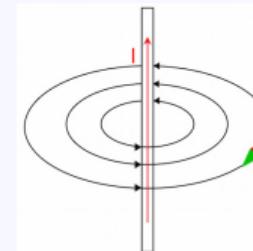
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$$\operatorname{div} (\epsilon \operatorname{grad} \phi) = \epsilon \operatorname{div} \operatorname{grad} \phi + \operatorname{grad} \phi \cdot \operatorname{grad} \epsilon$$

## Current-carrying conductor

Given an infinitely long conductor with the diameter  $d$  and an orientation in the  $z$ -direction through which a current  $I$  flows. The well known result:

$$\mathbf{H}(r) = [-\mathbf{e}_x \cos(\varphi) + \mathbf{e}_y \cos(\varphi)] \frac{I}{2\pi r} = I \frac{\mathbf{e}_\varphi}{2\pi r}$$



Source: [Physikunterricht-online.de](http://Physikunterricht-online.de)

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Source: Physikunterrichts online.de

**Electrical potential**

If the charge density  $\rho$  is known everywhere, then the general solution is given as follows:

$$\begin{aligned}\mathbf{E}(r) &= \frac{1}{4\pi\epsilon_0 r^2} \iiint_V \frac{\rho(r')}{|r - r'|} dr' dy' dz' \\ \mathbf{E}(r) &= \frac{1}{4\pi\epsilon_0 r^2} \iiint_V \rho(r') \frac{r - r'}{|r - r'|^3} dr' dy' dz'\end{aligned}$$

**Derivation: Current-carrying conductor**

If  $A$  is a disk with radius  $r$ , then for reasons of symmetry it can be concluded that

$$\mathbf{H}(r) = -\mathbf{e}_x \sin(\varphi) f(r) + \mathbf{e}_y \cos(\varphi) f(r).$$

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**Derivation: Current-carrying conductor**

$$\mathbf{H}(r) = -\mathbf{e}_x \sin(\varphi) f(r) + \mathbf{e}_y \cos(\varphi) f(r)$$

and thus

$$\oint_{\partial A} \mathbf{H} \cdot d\mathbf{s} = \iint_A \mathbf{J} d\mathbf{A}$$

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Source: Physikunterrichtsraum.de

**Electrical potential**If the charge density  $\rho$  is known everywhere, then the general solution is given as follows:

$$\Phi(r) = \frac{1}{4\pi \epsilon_0 r} \iint_V \frac{\rho(r')}{|r - r'|} dr' dy' dz'$$

$$\mathbf{E}(r) = \frac{1}{4\pi \epsilon_0 r} \iint_V \rho(r') \frac{r - r'}{|r - r'|^3} dr' dy' dz'$$

**Derivation: Current-carrying conductor**

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**Derivation: Current-carrying conductor**

Now the circular disc is parameterized using

$$\gamma(t) = r \mathbf{e}_x \cos(t) + r \mathbf{e}_y \sin(t)$$

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**Derivation: Current-carrying conductor**If  $A$  is a disk with radius  $r$ , then for reasons of symmetry it can be concluded that

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Source: Physikunterrichts online.de

$$\begin{aligned} I &= \oint_{\partial A} \mathbf{H} \cdot d\mathbf{s} \\ &= \int_0^{2\pi} [-\mathbf{e}_x \sin(\varphi) f(r) + \mathbf{e}_y \cos(\varphi)] f(r) \\ &\quad \cdot r [-\mathbf{e}_x \sin(\varphi) + \mathbf{e}_y \cos(\varphi)] d\varphi \\ &= f(r) r \int_0^{2\pi} \sin^2(\varphi) + \cos^2(\varphi) d\varphi = f(r) 2\pi r. \end{aligned}$$

**Derivation: Current-carrying conductor**

Now the circular disc is parameterized using

$$\gamma(t) = r\mathbf{e}_x \cos(t) + r\mathbf{e}_y \sin(t)$$

and thus

$$\begin{aligned} I &= \oint_{\text{disc}} \mathbf{H} \cdot d\mathbf{x} \\ &= \int_0^{2\pi} [-\mathbf{e}_x \sin(\varphi) t(r) + \mathbf{e}_y \cos(\varphi) t(r) \\ &\quad + r[-\mathbf{e}_x \sin(\varphi) + \mathbf{e}_y \cos(\varphi)]] d\varphi \\ &= t(r) r \int_0^{2\pi} [\sin^2(\varphi) + \cos^2(\varphi)] d\varphi = t(r) 2\pi r. \end{aligned}$$

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$$\oint_{\text{disc}} \mathbf{H} \cdot d\mathbf{x} = \iint_A \mathbf{J} dA$$

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**Derivation: Current-carrying conductor**

And thus the well known result:

$$\mathbf{H}(r) = [-\mathbf{e}_x \cos(\varphi) + \mathbf{e}_y \cos(\varphi)] \frac{1}{2\pi r} = \frac{\mathbf{e}_\varphi}{2\pi r}.$$

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## Derivation: Current-carrying conductor

And thus the well known result:

$$\mathbf{H}(\mathbf{r}) = \{-\mathbf{a}_x \cos(\varphi) + \mathbf{a}_y \sin(\varphi)\} \frac{1}{2\pi r} - \frac{\mathbf{a}_z}{2\pi r}$$

## Derivation: Current-carrying conductor

Now the circular disc is parameterized using

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## Magnetic vector potential

Analogously to the electric potential, the magnetic vector potential can be introduced as an aid:

$$\mathbf{B} = \operatorname{rot} \mathbf{A}.$$

in case of a **known** current density,  $\mathbf{A}$  can be calculated directly:

$$\begin{aligned} \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \iiint \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dx' dy' dz', \\ \mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \iiint \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} dx' dy' dz'. \end{aligned}$$

Hint: The last equation is also known as the **Biot-Savart law**.

**Magnetic vector potential**

Analogously to the electric potential, the magnetic vector potential can be introduced as an all:

$$\mathbf{B} = \nabla \times \mathbf{A}$$

In case of a known current density,  $\mathbf{A}$  can be calculated directly:

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**Derivation: Current-carrying conductor**

And thus the well known result:

$$\mathbf{H}(r) = [-\mathbf{e}_x \cos(\varphi) + \mathbf{e}_y \cos(\varphi)] \frac{1}{2\pi r} - \frac{\mathbf{e}_z}{2\pi r}$$

**Derivation: Current-carrying conductor**

Now the circular disc is parameterized using

$$\gamma(t) = r\mathbf{e}_x \cos(t) + r\mathbf{e}_y \sin(t)$$

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$$\begin{aligned} I &= \oint_{\text{disc}} \mathbf{H} \cdot d\mathbf{s} \\ &= \int_0^{2\pi} [ -\mathbf{e}_x \sin(\varphi) t'(r) + \mathbf{e}_y \cos(\varphi) t'(r) \\ &\quad - \mathbf{e}_z \sin(\varphi) + \mathbf{e}_z \cos(\varphi) ] d\varphi \\ &= t'(r) \int_0^{2\pi} [\sin^2(\varphi) + \cos^2(\varphi)] d\varphi = t'(r) 2\pi r. \end{aligned}$$

## Stationary fields

- ▷ Point charges and infinitely long currents help to evaluate scenarios.
- ▷ Introduction of auxiliary quantities (potentials) facilitates the calculation.
- ▷ If the currents and charges are known, the fields can be calculated (often only numerically).
- ▷ Similar to the concept input signal / transfer function / output signal<sup>a</sup>.
- ▷ Determining the currents and charges can be very difficult (e.g. charge distribution on semiconductor interfaces when a voltage is applied).

<sup>a</sup>See also [Green's functions](#)

## Maxwell's Equations

### 2.2 Maxwell's Equations

- 2.2.1 General case
- 2.2.2 Divergence and rotation
- 2.2.3 Material equations and continuity equation
- 2.2.4 Stationary fields
- 2.2.5 Quasi-stationary fields**
- 2.2.6 Time harmonic fields

## Definition

Quasi-stationary fields are fields with a fixed spatial distribution and time-dependent intensity. These are cases in which the displacement current density  $\frac{\partial D}{\partial t}$  can be neglected compared to the power current density  $J$ .

## Eigenschaften

- ▷ In the close range of antennas, fields are quasi-static. A common designation for quasi-static fields is therefore also near fields.
- ▷ The near range is the range that is much smaller than the wavelength.
- ▷ The alternating magnetic field belongs to the near fields.

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## Maxwell's Equations

$$\operatorname{div} \mathbf{D} = \rho \iff \iint_{\partial V} \mathbf{D} \cdot d\mathbf{A} = \iiint_V \rho dV$$

$$\operatorname{div} \mathbf{B} = 0 \iff \iint_{\partial V} \mathbf{B} \cdot d\mathbf{A} = 0$$

$$\operatorname{rot} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \iff \oint_{\partial A} \mathbf{E} \cdot d\mathbf{s} = -\iint_A \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}$$

$$\operatorname{rot} \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \iff \oint_{\partial A} \mathbf{H} \cdot d\mathbf{s} = \iint_A \left[ \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right] \cdot d\mathbf{A}$$

**Maxwell's Equations**

$$\begin{aligned} \operatorname{div} \mathbf{D} = \rho &\iff \iint_{\partial V} \mathbf{D} \cdot d\mathbf{A} = \iiint_V \rho dV \\ \operatorname{div} \mathbf{B} = 0 &\iff \iint_{\partial V} \mathbf{B} \cdot d\mathbf{A} = 0 \\ \operatorname{rot} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} &\iff \oint_{\partial A} \mathbf{E} \cdot d\mathbf{s} = - \iint_A \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A} \\ \operatorname{rot} \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} &\iff \oint_{\partial A} \mathbf{H} \cdot d\mathbf{s} = \iint_A \left[ \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right] \cdot d\mathbf{A} \end{aligned}$$

**Definition**

The **law of induction** states that a time-varying magnetic field is linked to an electric field.

$$\operatorname{rot} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \iff \oint_{\partial A} \mathbf{E} \cdot d\mathbf{s} = - \iint_A \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}$$

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**Eigenschaften**

- ▷ In the **far range** of antennas, fields are quasi-static. A common designation for quasi-static fields is therefore also **near fields**.
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- ▷ The alternating magnetic field belongs to the near fields.

**Properties**

- ▷ Approaches from AC engineering can often be applied (e.g. consideration at a fixed frequency, introduction of inductances)

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The **law of induction** states that a time-varying magnetic field is linked to an electric field.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \iff \oint_{\text{SA}} \mathbf{E} \cdot d\mathbf{s} = -\iint_{\text{SA}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}$$

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**Maxwell's Equations**

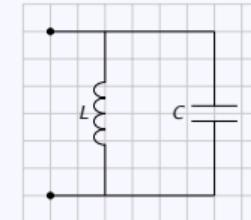
$$\nabla \times \mathbf{D} = \rho \iff \iint_{\text{SA}} \mathbf{D} \cdot d\mathbf{A} = \iint_{\text{V}} \rho dV$$

$$\nabla \cdot \mathbf{B} = 0 \iff \iint_{\text{SA}} \mathbf{B} \cdot d\mathbf{A} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \iff \oint_{\text{SA}} \mathbf{E} \cdot d\mathbf{s} = -\iint_{\text{A}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}$$

$$\nabla \cdot \mathbf{H} = J + \frac{\partial \mathbf{D}}{\partial t} \iff \oint_{\text{SA}} \mathbf{H} \cdot d\mathbf{s} = \iint_{\text{A}} [J + \frac{\partial \mathbf{D}}{\partial t}] \cdot d\mathbf{A}$$

## A simple resonant circuit



Input impedance:

$$Z_{in} = \frac{j\omega L}{1 - \omega^2 LC}$$

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**Quasi-stationary fields** are fields with a fixed spatial distribution and time-dependent intensity. There are cases in which the displacement current density  $\frac{\partial \mathbf{D}}{\partial t}$  can be neglected compared to the power current density  $\mathbf{J}$ .

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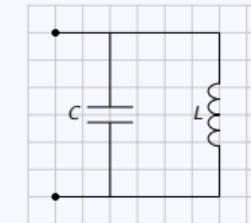
$$\text{rot } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \Leftrightarrow \oint_{\text{lin}} \mathbf{E} \cdot d\mathbf{s} = - \iint_A \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}$$

**Properties**

- Approaches from AC engineering can often be applied (e.g. consideration at a fixed frequency, introduction of inductances)

**Maxwell's Equations**

$$\begin{aligned} \text{div } \mathbf{D} = \rho &\Leftrightarrow \iint_{\text{surf}} \mathbf{D} \cdot d\mathbf{A} = \iiint_V \rho dV \\ \text{div } \mathbf{B} = 0 &\Leftrightarrow \iint_{\text{surf}} \mathbf{B} \cdot d\mathbf{A} = 0 \\ \text{rot } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} &\Leftrightarrow \oint_{\text{lin}} \mathbf{E} \cdot d\mathbf{s} = - \iint_A \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A} \\ \text{rot } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} &\Leftrightarrow \iint_{\text{surf}} \mathbf{H} \cdot d\mathbf{s} = \iint_A \left[ \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right] \cdot d\mathbf{A} \end{aligned}$$

**A simple resonant circuit**Current when applying a voltage  $U_0$  (unit jump):

$$\begin{aligned} I(s) &= U_0 \frac{1}{s} \frac{sL}{1 + s^2 LC} = U_0 \frac{L}{1 + s^2 LC} = U_0 \frac{L\omega_0^2}{\omega_0^2 + s^2} \\ i(t) &= U_0 L \omega_0 \sin(\omega_0 t) u(t), \end{aligned}$$

with  $\omega_0 = \frac{1}{\sqrt{LC}}$ .

## A simple resonant circuit

Current when applying a voltage  $U_0$  (unit jump):

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with  $\omega_0 = \frac{1}{\sqrt{LC}}$ 

## A simple resonant circuit



Input impedance:

$$Z_{in} = \frac{j\omega L}{1 - \omega^2 LC}$$

## Definition

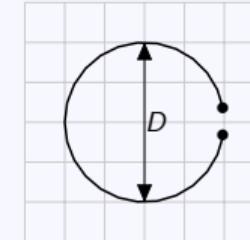
The law of induction states that a time-varying magnetic field is linked to an electric field.

$$rot \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} \Leftrightarrow \oint_{\text{loop}} \mathbf{E} \cdot d\mathbf{s} = - \iint_A \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}$$

## Properties

d) Approaches from AC engineering can often be applied (e.g. consideration at a fixed frequency, introduction of inductances)

## Simple inductance

Inductance of loop with diameter  $D$  and diameter of wire  $d$ <sup>a</sup>:

$$L \approx \mu_0 \mu_r \frac{D}{2} \left( \ln \left( \frac{8D}{d} \right) - 2 \right)$$

Example:  $D = 1 \text{ m}$ ,  $d = 0.1 \text{ cm}$ :  $L = 2.94 \mu\text{H}$ .<sup>a</sup>see e.g. eeweb und FH Aachen

**Simple inductance**Inductance of loop with diameter  $D$  and radius of wire  $d^2$ :

$$L \approx \mu_0 \frac{D}{2} \ln\left(\frac{8D}{d}\right) - 2$$

Example:  $D = 1\text{ m}$ ,  $d = 0.1\text{ mm}$ :  $L = 2.94\text{ }\mu\text{H}$ .See e.g. [Wikipedia](#) and [FIR Antennas](#)**A simple resonant circuit**Current when applying a voltage  $U_0$  (unit jump):

$$\begin{aligned} i(s) &= U_0 \frac{1}{s + \omega_0^2 LC} = U_0 \frac{1}{1 + s^2 LC} = U_0 \frac{\omega_0^2}{\omega_0^2 + s^2} \\ i(t) &= U_0 \omega_0 \sin(\omega_0 t) u(t), \end{aligned}$$

with  $\omega_0 = \frac{1}{\sqrt{LC}}$ **Simple inductance**

Source: Wikipedia (Wolf Meusel)

**Assumption**

$$L = 2.94\text{ }\mu\text{H}$$

$$C = 4.7\text{ }\mu\text{F}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = 42.8\text{ kHz}$$

**A simple resonant circuit**

Input impedance:

$$Z_{in} = \frac{j\omega L}{1 - j\omega LC}$$

**Simple inductance**



**Assumption**

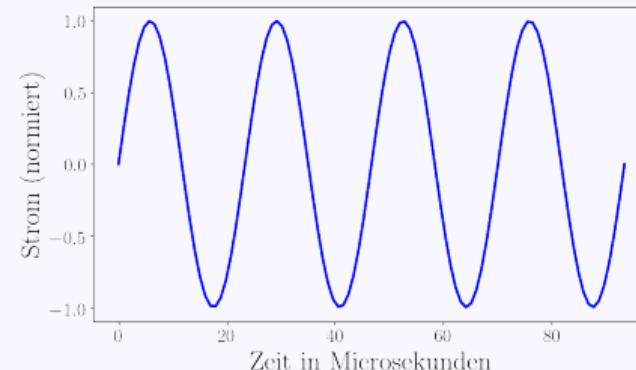
$$L = 2.94 \mu\text{H}$$

$$C = 4.7 \mu\text{F}$$

$$\omega_0 = \frac{1}{2\pi\sqrt{LC}} = 42.8 \text{ kHz}$$

Source: Wikipedia (Wolf Model)

## Step response



**Simple inductance**



Inductance of loop with diameter  $D$  and width  $d$ :

$$L \approx \mu_0 \pi \frac{D}{2} \left( \ln \left( \frac{4D}{d} \right) - 2 \right)$$

Example:  $D = 1 \text{ m}$ ,  $d = 0.3 \text{ cm}$ ;  $L = 2.94 \mu\text{H}$ .

\*See e.g. [Wolfs und Fitt-Ausdruck](#)

**A simple resonant circuit**



Current when applying a voltage  $U_0$  (unit jump):

$$I(t) = U_0 \frac{1}{\omega_0^2 + t^2} \cdot \frac{\omega_0^2}{\omega_0^2 + t^2} = U_0 \frac{1}{\omega_0^2 + t^2} \cdot \frac{L}{LC} = U_0 \frac{L}{m_0^2 + t^2}$$

with  $\omega_0 = \frac{1}{\sqrt{LC}}$

Calculation of the step response using inverse Laplace transform.

## Maxwell's Equations

### 2.2 Maxwell's Equations

- 2.2.1 General case
- 2.2.2 Divergence and rotation
- 2.2.3 Material equations and continuity equation
- 2.2.4 Stationary fields
- 2.2.5 Quasi-stationary fields
- 2.2.6 Time harmonic fields

## Differential form of Maxwell's equations

$$\operatorname{div} \mathbf{D} = \rho \quad \text{Gauss's law}$$

$$\operatorname{div} \mathbf{B} = 0 \quad \text{Gauss's law for magnetic fields}$$

$$\operatorname{rot} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{Maxwell-Faraday equation}$$

$$\operatorname{rot} \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \text{Ampère's law}$$

## Differential form of Maxwell's equations

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$\operatorname{rot} \underline{\mathbf{H}} = \underline{\mathbf{J}} + \frac{\partial \underline{\mathbf{D}}}{\partial t}$	Ampere's law

## Time-harmonic field

Assumption: Temporal relation according to<sup>a</sup>  $e^{j\omega t}$  and

$$\underline{\mathbf{E}}(\mathbf{r}, t) = \operatorname{Re} (\underline{\mathbf{E}}(\mathbf{r}) e^{j\omega t})$$

$$\operatorname{div} \underline{\mathbf{D}} = \rho$$

$$\operatorname{div} \underline{\mathbf{B}} = 0$$

$$\operatorname{rot} \underline{\mathbf{E}} = -j\omega \underline{\mathbf{B}}$$

$$\operatorname{rot} \underline{\mathbf{H}} = \underline{\mathbf{J}} + j\omega \underline{\mathbf{D}}$$

In the following, we will not explicitly write the underscore.

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<sup>a</sup>Note that in physics, one commonly uses  $e^{-j\omega t}$