

Radar Systems

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**Fachhochschule
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University of Applied Sciences and Arts

November 14, 2024

Outline

Introduction

- 1. Introduction
- 2. Wave propagation
- 3. Block diagram

Signal Processing

- 4. Spectral analysis
- 5. Index

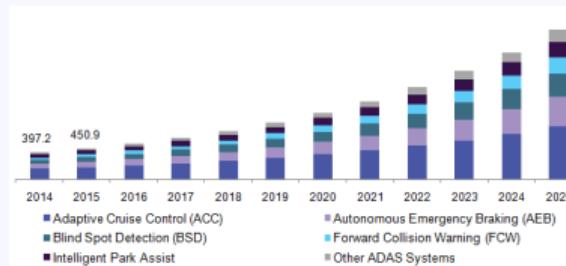
Introduction

1. Introduction
2. Wave propagation
3. Block diagram

Introduction

- 1.1 Motivation
- 1.2 Content
- 1.3 Literature
- 1.4 Outline
- 1.5 Exam

Market forecast



U.S. automotive radar market, by application, in million USD. Source: [Grand View Research](#)

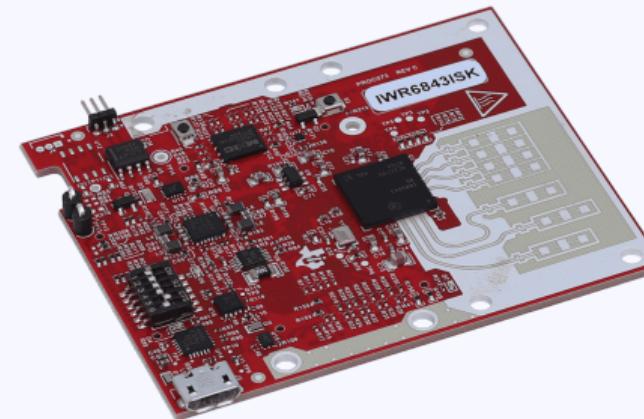


Radar system market forecast. Source: [Yole](#)



Drivers

- ▷ Integration
- ▷ Reference designs
- ▷ Price drop



Source: [Texas Instruments](#)

Introduction

1.1 Motivation

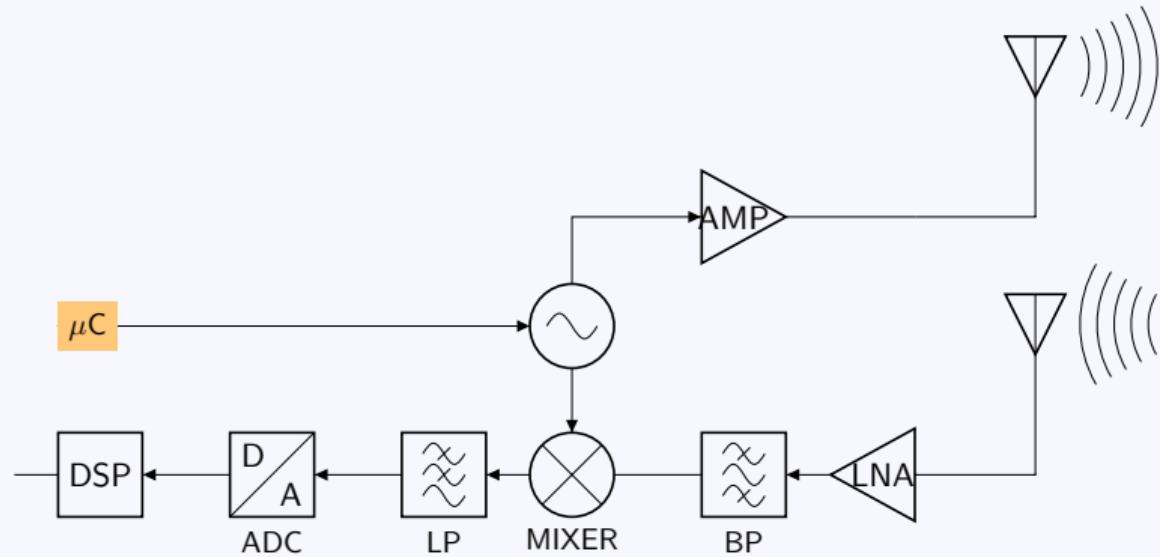
1.2 Content

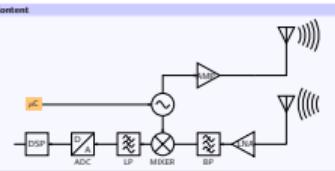
1.3 Literature

1.4 Outline

1.5 Exam

Content





Content

- ▷ Wave propagation and antennas
- ▷ Block diagram
- ▷ Modulation
- ▷ Spectral analysis
- ▷ Point cloud processing
- ▷ Current trends in radar signal processing
- ▷ Applications

Introduction

1.1 Motivation

1.2 Content

1.3 Literature

1.4 Outline

1.5 Exam

Literature

- ▷ Stergiopoulos, Advanced Signal Processing, CRC Press, 2009
- ▷ Kay, S.; Fundamentals of Statistical Signal Processing, Vol. I: Estimation Theory, Prentice Hall, 1993
- ▷ Mahafza, Radar Signal Analysis and Processing using Matlab, CRC Press, 2016
- ▷ Winner, Handbuch Fahrerassistenzsysteme, Springer, 2015

Introduction

1.1 Motivation

1.2 Content

1.3 Literature

1.4 Outline

1.5 Exam

Preliminary outline

Week	Unit	Topic
41	1	Introduction, wave propagation
42	2	Block diagram
43	na	vacation
44	3	Spectral Analysis
45	4	Spectral Analysis
46	5	Angle finding
47	na	Block week
48	6	Angle finding
49 – 51	7 – 9	State Estimation and Tracking
2 – 4	10 – 12	Current trends & joint implementations

Introduction

1.1 Motivation

1.2 Content

1.3 Literature

1.4 Outline

1.5 Exam

Exam

Assessment of the course: Written Exam (60 min, planned for TBA) at the end of the course (50%) and homework (50%) with demonstration/presentation. Homework deals with aspects of signal processing for uses cases in automotive or robotics. Homework is teamwork and can be based upon demonstration boards and/or Matlab/Python and public dataset.

Introduction

1. Introduction
2. Wave propagation
3. Block diagram

Wave propagation

- 2.1 Introduction
- 2.2 Maxwell's Equations
- 2.3 Electromagnetic waves
- 2.4 Fields of current distributions
- 2.5 Reflection, diffraction and damping of plane waves
- 2.6 Micro Strip lines, (coplanar) waveguides

Content

- ▷ Basics of related electromagnetic wave theory
- ▷ Basics of antennas

Study goals

- ▷ Understand basic wave propagation phenomena
- ▷ Understand basic antenna properties

Wave propagation

2.1 Introduction

2.2 Maxwell's Equations

2.2.1 General case

2.2.2 Divergence and rotation

2.2.3 Material equations and continuity equation

2.2.4 Stationary fields

2.2.5 Quasi-stationary fields

2.2.6 Time harmonic fields

2.3 Electromagnetic waves

2.4 Fields of current distributions

2.5 Reflection, diffraction and damping of plane waves

2.6 Micro Strip lines, (coplanar) waveguides

Maxwell's Equations

2.2 Maxwell's Equations

2.2.1 General case

- 2.2.2 Divergence and rotation
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- 2.2.4 Stationary fields
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- 2.2.6 Time harmonic fields

Maxwell's Equations

The Maxwell's Equations describe electromagnetic phenomena. They exist in differential and integral form^a:

$$\operatorname{div} \mathbf{D} = \rho \iff \iint_{\partial V} \mathbf{D} \cdot d\mathbf{A} = \iiint_V \rho dV$$

$$\operatorname{div} \mathbf{B} = 0 \iff \iint_{\partial V} \mathbf{B} \cdot d\mathbf{A} = 0$$

$$\operatorname{rot} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \iff \oint_{\partial A} \mathbf{E} \cdot d\mathbf{s} = -\iint_A \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}$$

$$\operatorname{rot} \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \iff \oint_{\partial A} \mathbf{H} \cdot d\mathbf{s} = \iint_A \left[\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right] \cdot d\mathbf{A}$$

^aUsing the integral theorems of Gauss and Stokes allows to switch from one domain to the other.

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*Using the integral theorem of Gauss and Stokes allows to switch from one domain to the other

Symbols

- ▷ ρ : Charge density (Ladungsdichte) in C/m^3 (As/m^3)
- ▷ \mathbf{D} : Electric displacement field (Elektrische Flussdichte) in C/m^2 (As/m^2)
- ▷ \mathbf{E} : Electric field (Elektrische Feldstärke) in $V m^{-1}$
- ▷ \mathbf{H} : Magnetic Field (Magnetische Feldstärke) in $A m^{-1}$
- ▷ \mathbf{B} : Magnetic flux density (Magnetische Flussdichte) in T ($N A^{-1} m^{-1}$)
- ▷ \mathbf{J} : Current density (Stromdichte) in A/m^2
- ▷ $\frac{\partial \mathbf{D}}{\partial t}$: Displacement current density (Verschiebungsstromdichte) in A/m^2

Symbols

- ▷ ρ : Charge density (Ladungsdichte) in C/m^3 (A_s/m^3)
- ▷ \mathbf{D} : Electric displacement field (Elektrische Flussdichte) in C/m^2 (A_s/m^2)
- ▷ \mathbf{E} : Electric field (Elektrische Feldstärke) in V/m
- ▷ \mathbf{H} : Magnetic Field (Magnetische Feldstärke) in $A\cdot m^{-1}$
- ▷ Φ : Magnetic flux density (Magnetische Flussdichte) in T ($N\cdot A^{-1}\cdot m^{-1}$)
- ▷ \mathbf{J} : Current density (Strömdichte) in A/m^2
- ▷ \mathbf{D}_0 : Displacement current density (Verschiebungstromdichte) in A/m^2

Maxwell's Equations

The Maxwell's Equations describe electromagnetic phenomena. They exist in differential and integral form*.

$$\begin{aligned} \text{div } \mathbf{D} = \rho &\iff \iiint_{\text{Vol}} \mathbf{D} \cdot d\mathbf{A} = \iiint_{\text{Vol}} \rho dV \\ \text{div } \mathbf{B} = 0 &\iff \iint_{\text{Surf}} \mathbf{B} \cdot d\mathbf{A} = 0 \\ \text{rot } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} &\iff \oint_{\text{Surf}} \mathbf{E} \cdot d\mathbf{x} = - \iint_{\text{Surf}} \frac{\partial \mathbf{B}}{\partial n} \cdot d\mathbf{A} \\ \text{rot } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} &\iff \oint_{\text{Surf}} \mathbf{H} \cdot d\mathbf{x} = \iint_{\text{Surf}} \left[\mathbf{J} + \frac{\partial \mathbf{D}}{\partial n} \right] \cdot d\mathbf{A} \end{aligned}$$

*Using the integral theorem of Gauss and Stokes allows to switch from one domain to the other.

Maxwell's Equations

The Maxwell's Equations only describe the electromagnetic phenomena. They are not the solutions themselves. These are differential equations whose solutions depend on the boundary conditions. Solution methods are e.g.:

- ▷ numerical methods (e.g. finite integration, finite difference method, finite element method, integral equation method)
- ▷ analytic methods

Analytical solutions can only be determined for special cases. These include, for example, electromagnetic waves.

Maxwell's Equations

2.2 Maxwell's Equations

2.2.1 General case

2.2.2 Divergence and rotation

2.2.3 Material equations and continuity equation

2.2.4 Stationary fields

2.2.5 Quasi-stationary fields

2.2.6 Time harmonic fields

Definition

Given is a function $\mathbf{f}(x, y, z)$. The **divergence** is defined as follow:

$$\begin{aligned}\operatorname{div}(\mathbf{f}) &= \mathbf{e}_x \frac{\partial f_x(x, y, z)}{\partial x} + \mathbf{e}_y \frac{\partial f_y(x, y, z)}{\partial y} + \mathbf{e}_z \frac{\partial f_z(x, y, z)}{\partial z} \\ &= \nabla \cdot \mathbf{f}(x, y, z),\end{aligned}$$

with

$$\nabla = \left(\begin{array}{c} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{array} \right).$$

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$$= \nabla \cdot \mathbf{f}(x, y, z),$$

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Electrical field

$$\operatorname{div} \mathbf{D} = \rho$$

The electric field is **not** source free.

Magnetic field

$$\operatorname{div} \mathbf{B} = 0$$

The magnetic field is source free.

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$$\nabla = \left(\begin{array}{c} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{array} \right).$$

DefinitionGiven is a function $\mathbf{f}(x, y, z) = \mathbf{e}_x f_x(x, y, z) + \mathbf{e}_y f_y(x, y, z) + \mathbf{e}_z f_z(x, y, z)$.The **curl** is the defined as follows:

$$\begin{aligned}\operatorname{curl} \mathbf{f} = \operatorname{rot} \mathbf{f} &= \nabla \times \mathbf{f}(x, y, z) \\ &= \mathbf{e}_x \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z} \right) + \mathbf{e}_y \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x} \right) + \mathbf{e}_z \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right).\end{aligned}$$

If the **curl^a** of a vector field is zero everywhere, then the vector field is **vortex-free** (also called **irrotational**).

^aThe notation curl is common in North America. In the rest of the world, rot is commonly used. Using the cross product with the **del-operator** (also known as **nabla-operator**) avoids confusion. See [Wikipedia](#) as well.

Definition

Given is a function $\mathbf{f}(x, y, z) = \mathbf{e}_x f_x(x, y, z) + \mathbf{e}_y f_y(x, y, z) + \mathbf{e}_z f_z(x, y, z)$.

The **curl** is defined as follows:

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If the **curl*** of a vector field is zero everywhere, then the vector field is **vortex-free** (also called **irrotational**).

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Electrical field

$$\operatorname{rot} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

The static electrical field ($\frac{\partial \mathbf{B}}{\partial t} = 0$) is vortex-free.

Electrical field

$$\operatorname{div} \mathbf{D} = \rho$$

The electric field is **not** source free.

Magnetic field

$$\operatorname{div} \mathbf{B} = 0$$

The magnetic field is source free.

Magnetic field

$$\operatorname{rot} \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

The magnetic field is not vortex-free.

Definition

Given is a function $\mathbf{f}(x, y, z)$. The **divergence** is defined as follow:

$$\begin{aligned}\operatorname{div}(\mathbf{f}) &= \mathbf{e}_x \frac{\partial f_x}{\partial x} + \mathbf{e}_y \frac{\partial f_y}{\partial y} + \mathbf{e}_z \frac{\partial f_z}{\partial z} \\ &= \nabla \cdot \mathbf{f}(x, y, z),\end{aligned}$$

with

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right).$$

Maxwell's Equations

2.2 Maxwell's Equations

2.2.1 General case

2.2.2 Divergence and rotation

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Material equations

$$\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E}$$

$$\mathbf{B} = \mu_0 \mu_r \mathbf{H}$$

Material equations

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$$\mathbf{B} = \mu_0 \mathbf{H}$$

Electric current and charge

$$\text{rot } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Taking the divergence of both sides results in

$$\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \mathbf{J} + \frac{\partial(\nabla \cdot \mathbf{D})}{\partial t}.$$

Using the fact that the divergence of a curl is zero:

$$\nabla \cdot \mathbf{J} + \frac{\partial(\nabla \cdot \mathbf{D})}{\partial t} = 0.$$

With Gauss's law ($\nabla \cdot \mathbf{D} = \rho$) one gets

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

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Material equations

$$\begin{aligned}\mathbf{D} &= \epsilon_0 \epsilon_r \mathbf{E} \\ \mathbf{B} &= \mu_0 \mu_r \mathbf{H}\end{aligned}$$

Continuity equation

Relationship according to the **continuity equation**:

$$\nabla \cdot \mathbf{J}(\mathbf{r}, t) + \frac{\partial}{\partial t} \rho(\mathbf{r}, t) = 0.$$

Interpretation:

Current is the movement of charge. The continuity equation says that if charge is moving out of a differential volume (i.e., divergence of current density is positive) then the amount of charge within that volume is going to decrease, so the rate of change of charge density is negative. Therefore, the continuity equation amounts to a conservation of charge.

[Wikipedia](#)

Maxwell's Equations

2.2 Maxwell's Equations

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Stationary case

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Stationary case

Electrical field:

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Magnetic fields:

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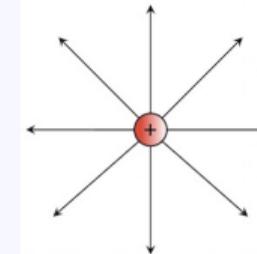
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$$\operatorname{rot} \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \iff \oint_{\partial V} \mathbf{H} \cdot d\mathbf{x} = \iint_V \left[\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right] \cdot d\mathbf{A}$$

Spherical charge

Given its spherical charge Q with the homogeneous charge density ρ and the radius r_0 . For points outside the charge:

$$\mathbf{D}(r) = \epsilon_r \frac{Q}{4\pi r}.$$



Radial field. Source: physikunterricht-online.de

Spherical charge

Given its spherical charge Q with the homogeneous charge density ρ and the radius r_0 . For points outside the charge:

$$\mathbf{D}(r) = \frac{Q}{4\pi r^2} \hat{\mathbf{r}}$$



Radial field. Source: physikunterrichtsraum.de

Spherical charge - derivation

For reasons of symmetry, it can be concluded that the electric flux density \mathbf{D} points radially outwards ($\mathbf{D} = \mathbf{e}_r D(r)$). Thus, for $r > r_0$:

$$\begin{aligned} \iint_{\partial V} \mathbf{D} \cdot d\mathbf{A} &= \iiint_V \rho dx' dy' dz' \\ &= 4\pi r D(r) = Q \end{aligned}$$

and

$$D(r) = \frac{Q}{4\pi r},$$

respectively.

Stationary case

Electrical field:

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Magnetic fields:

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- └ Wave propagation
- └ Maxwell's Equations

Spherical charge - derivation

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and

$$D(r) = \frac{Q}{4\pi r^2}$$

respectively.

Spherical charge

Given its spherical charge Q with the homogeneous charge density ρ and the radius r_0 . For points outside the charge:

$$D(r) = \rho \frac{Q}{4\pi r^2}$$



Radial field. Source: physikunterrichtsraum.de

Definition

The **gradient** of a function $f(x, y, z)$ is defined as

$$\begin{aligned} \text{grad } f(x, y, z) &= \mathbf{e}_x \frac{\partial f(x, y, z)}{\partial x} + \mathbf{e}_y \frac{\partial f(x, y, z)}{\partial y} + \mathbf{e}_z \frac{\partial f(x, y, z)}{\partial z} \\ &= \nabla f(x, y, z) \end{aligned}$$

Properties

$$\text{rot}(\text{grad } f(x, y, z)) = \nabla \times \nabla f(x, y, z) = 0$$

Stationary case

Electrical field:

$$\begin{aligned} \text{div } \mathbf{D} = \rho &\iff \iint_{\partial V} \mathbf{D} \cdot d\mathbf{A} = \iiint_V \rho dV \\ \text{rot } \mathbf{E} = 0 &\iff \oint_{\partial A} \mathbf{E} \cdot d\mathbf{s} = 0 \end{aligned}$$

Magnetic fields:

$$\begin{aligned} \text{div } \mathbf{B} = 0 &\iff \iint_{\partial V} \mathbf{B} \cdot d\mathbf{A} = 0 \\ \text{rot } \mathbf{H} = \mathbf{J} &\iff \oint_{\partial A} \mathbf{H} \cdot d\mathbf{s} = \iint_A \mathbf{J} \cdot d\mathbf{A} \end{aligned}$$

- └ Wave propagation
- └ Maxwell's Equations

Definition

The **gradient** of a function $f(x, y, z)$ is defined as

$$\begin{aligned}\text{grad } f(x, y, z) &= e_x \frac{\partial f(x, y, z)}{\partial x} + e_y \frac{\partial f(x, y, z)}{\partial y} + e_z \frac{\partial f(x, y, z)}{\partial z} \\ &= \nabla f(x, y, z)\end{aligned}$$

Properties

$$\text{rot}(\text{grad } f(x, y, z)) = \nabla \times \nabla f(x, y, z) = 0$$

Electrical potential

If the electric field is represented as a (negative) gradient of a function $\Phi(x, y, z)$ then

$$\text{rot } \mathbf{E} = 0 \iff \oint_{\partial A} \mathbf{E} \cdot d\mathbf{s} = 0$$

is fulfilled *automatically*. Thus one needs to determine *only* a function $\Phi(x, y, z)$.

- ▷ Field strength: Unit Vm^{-1} . Specifies the force that would act on a test charge in the electric field ($\mathbf{F} = q\mathbf{E}$)
- ▷ Potential: Unit V. Indicates the potential energy of a test charge in an electric field.

Spherical charge - derivation

For reasons of symmetry it can be concluded that the electric flux density \mathbf{D} points radially outwards ($\mathbf{D} = e_r D(r)$). Thus, for $r > r_0$:

$$\begin{aligned}\iint_D \mathbf{D} \cdot d\mathbf{A} &= \iiint_V \rho \, dx \, dy \, dz \\ &= 4\pi r D(r) Q\end{aligned}$$

and

$$D(r) = \frac{Q}{4\pi r^2}$$

respectively.

Spherical charge

Given its spherical charge Q with the homogeneous charge density ρ and the radius r_0 . For points outside the charge

$$D(r) = e_r \frac{Q}{4\pi r^2}$$



Radial field. Source: physikunterrichts online.de

Electrical potential

If the electric field is represented as a (negative) gradient of a function $\Phi(x, y, z)$ then

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Electrical potential

Obviously, charge densities ρ are the sources of the electrical fields:

$$\operatorname{div} \mathbf{D} = -\operatorname{div} \epsilon \operatorname{grad} \Phi = \rho$$

Definition

The **gradient** of a function $f(x, y, z)$ is defined as

$$\operatorname{grad} f(x, y, z) = \mathbf{e}_x \frac{\partial f(x, y, z)}{\partial x} + \mathbf{e}_y \frac{\partial f(x, y, z)}{\partial y} + \mathbf{e}_z \frac{\partial f(x, y, z)}{\partial z}$$

$$= \nabla f(x, y, z)$$

Properties

$$\operatorname{rot}(\operatorname{grad} f(x, y, z)) = \nabla \times \nabla f(x, y, z) = 0$$

Attention

When using the div-operator (and the rot-operator), one needs to take care for vector and derivation-rules. E.g.

$$\operatorname{div} \epsilon \operatorname{grad} \Phi = \epsilon \operatorname{div} \operatorname{grad} \Phi + \operatorname{grad} \Phi \operatorname{div} \epsilon$$

Spherical charge - derivation

For reasons of symmetry, it can be concluded that the electric flux density \mathbf{D} points radially outwards ($\mathbf{D} = \epsilon_0 \mathbf{D}(r) \hat{r}$). Thus, for $r > r_0$:

$$\iint_{\partial V} \mathbf{D} \cdot d\mathbf{A} = \iiint_V \rho \frac{\partial r}{\partial r} dr dz dx$$

$$= 4\pi r_0^2 \mathbf{D}(r) \hat{r} \cdot \hat{r} = Q$$

and

$$\mathbf{D}(r) = \frac{Q}{4\pi r^2} \hat{r}$$

respectively

Electrical potential

Obviously, charge densities ρ are the sources of the electrical fields:
 $\operatorname{div} \mathbf{D} = -\operatorname{div} \epsilon \operatorname{grad} \Phi = \rho$

Attention

When using the div-operator (and the rot-operator), one needs to take care for vector and derivation-rules. E.g.

$$\operatorname{div} \epsilon \operatorname{grad} \Phi = \epsilon \operatorname{div} \operatorname{grad} \Phi + \operatorname{grad} \Phi \cdot \operatorname{grad} \epsilon$$

Electrical potential

If the electric field is represented as a (negative) gradient of a function $\Phi(x, y, z)$ then

$$\operatorname{rot} \mathbf{E} = 0 \iff \oint_{\partial V} \mathbf{E} \cdot d\mathbf{x} = 0$$

is fulfilled automatically. Thus one needs to determine only a function $\Phi(x, y, z)$.

▷ Field strength: Unit V/m^2 . Specifies the force that would act on a test charge in the electric field ($\mathbf{F} = \epsilon \mathbf{E}$)

▷ Potential: Unit V . Indicates the potential energy of a test charge in an electric field.

Electrical potential

In the homogeneous space:

$$\operatorname{div} \mathbf{D} = -\epsilon \operatorname{div} \operatorname{grad} \Phi = \rho$$

In cartesian coordinates:

$$\frac{\partial^2 \Phi(x, y, z)}{\partial x^2} + \frac{\partial^2 \Phi(x, y, z)}{\partial y^2} + \frac{\partial^2 \Phi(x, y, z)}{\partial z^2} = -\frac{\rho}{\epsilon}.$$

This equation is called **Poisson-equation** of the electrical field.

Definition

The **gradient** of a function $f(x, y, z)$ is defined as

$$\operatorname{grad} f(x, y, z) = \epsilon_x \frac{\partial f(x, y, z)}{\partial x} + \epsilon_y \frac{\partial f(x, y, z)}{\partial y} + \epsilon_z \frac{\partial f(x, y, z)}{\partial z}$$

$$= \nabla f(x, y, z)$$

Properties

$$\operatorname{rot} [\operatorname{grad} f(x, y, z)] = \nabla \times \nabla f(x, y, z) = 0$$

Electrical potential

In the homogeneous space:

$$\operatorname{div} \mathbf{D} = -\operatorname{div} \operatorname{grad} \phi = \rho$$

In cartesian coordinates:

$$\frac{\partial^2 \phi(x, y, z)}{\partial x^2} + \frac{\partial^2 \phi(x, y, z)}{\partial y^2} + \frac{\partial^2 \phi(x, y, z)}{\partial z^2} = \frac{\rho}{\epsilon_0}$$

This equation is called **Poisson-equation** of the electrical field.

Electrical potential

If the charge density ρ is known everywhere, then the general solution is given as follows:

$$\begin{aligned}\Phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0\epsilon_r} \iiint_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dx' dy' dz' \\ \mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0\epsilon_r} \iiint_V \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} dx' dy' dz'\end{aligned}$$

Electrical potentialObviously, charge densities ρ are the sources of the electrical fields:

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Attention

When using the div-operator (and the rot-operator), one needs to take care for vector and derivation-rules. E.g.

$$\operatorname{div} (\mathbf{v} \operatorname{grad} \phi) = \mathbf{v} \operatorname{div} \operatorname{grad} \phi + \operatorname{grad} \phi \operatorname{div} \mathbf{v}$$

Electrical potentialIf the electric field is represented as a (negative) gradient of a function $\Phi(x, y, z)$ then

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Electrical potential

If the charge density ρ is known everywhere, then the general solution is given as follows:

$$\begin{aligned}\phi(r) &= \frac{1}{4\pi\epsilon_0 r} \iiint_V \frac{\rho(r')}{|r - r'|} dr' dy' dz' \\ \mathbf{E}(r) &= \frac{1}{4\pi\epsilon_0 r^2} \iiint_V \rho(r') \frac{r - r'}{|r - r'|^3} dr' dy' dz'\end{aligned}$$

Electrical potential

In the homogeneous space:

$$\operatorname{div} \mathbf{D} = -\epsilon \operatorname{curl} \operatorname{grad} \phi = \rho$$

In cartesian coordinates:

$$\frac{\partial^2 \phi(x, y, z)}{\partial x^2} + \frac{\partial^2 \phi(x, y, z)}{\partial y^2} + \frac{\partial^2 \phi(x, y, z)}{\partial z^2} = -\frac{\rho}{\epsilon}$$

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Obviously, charge densities ρ are the sources of the electrical fields:

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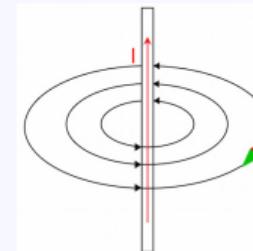
When using the div-operator (and the rot-operator), one needs to take care for vector and derivation-rules. E.g.

$$\operatorname{div} (\epsilon \operatorname{grad} \phi) = \epsilon \operatorname{div} \operatorname{grad} \phi + \operatorname{grad} \phi \cdot \operatorname{grad} \epsilon$$

Current-carrying conductor

Given an infinitely long conductor with the diameter d and an orientation in the z -direction through which a current I flows. The well known result:

$$\mathbf{H}(r) = [-\mathbf{e}_x \cos(\varphi) + \mathbf{e}_y \cos(\varphi)] \frac{I}{2\pi r} = I \frac{\mathbf{e}_\varphi}{2\pi r}$$



Source: Physikunterricht-online.de

Current-carrying conductor

Given an infinitely long conductor with the diameter d and an orientation in the z -direction through which a current I flows. The well known result:

$$\mathbf{H}(r) = [-\mathbf{e}_x \cos(\varphi) + \mathbf{e}_y \sin(\varphi)] \frac{I}{2\pi d} = I \frac{\mathbf{e}_\theta}{2\pi d}$$



Source: Physikunterrichts online.de

Electrical potential

If the charge density ρ is known everywhere, then the general solution is given as follows:

$$\begin{aligned}\mathbf{E}(r) &= \frac{1}{4\pi\epsilon_0 r^2} \iiint_V \frac{\rho(r')}{|r - r'|} dr' dy' dz' \\ \mathbf{E}(r) &= \frac{1}{4\pi\epsilon_0 r^2} \iiint_V \rho(r') \frac{r - r'}{|r - r'|^3} dr' dy' dz'\end{aligned}$$

Derivation: Current-carrying conductor

If A is a disk with radius r , then for reasons of symmetry it can be concluded that

$$\mathbf{H}(r) = -\mathbf{e}_x \sin(\varphi) f(r) + \mathbf{e}_y \cos(\varphi) f(r).$$

Electrical potential

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$$\frac{\partial^2 \phi(x, y, z)}{\partial x^2} + \frac{\partial^2 \phi(x, y, z)}{\partial y^2} + \frac{\partial^2 \phi(x, y, z)}{\partial z^2} = -\frac{\rho}{\epsilon_0}$$

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Derivation: Current-carrying conductor

$$\mathbf{H}(r) = -\mathbf{e}_x \sin(\varphi) f(r) + \mathbf{e}_y \cos(\varphi) f(r)$$

and thus

$$\oint_{\partial A} \mathbf{H} \cdot d\mathbf{s} = \iint_A \mathbf{J} d\mathbf{A}$$

simplifies as follows:

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Given an infinitely long conductor with the diameter d and an orientation in the z -direction through which a current I flows. The well known result:

$$\mathbf{H}(r) = [-\mathbf{e}_x \cos(\varphi) + \mathbf{e}_y \sin(\varphi)] \frac{I}{2\pi r} = I \frac{\mathbf{e}_z}{2\pi r}$$



Source: Physikum.tu-berlin.de

Electrical potential

If the charge density ρ is known everywhere, then the general solution is given as follows:

$$\Phi(r) = \frac{1}{4\pi \epsilon_0 r} \iint_V \frac{\rho(r')}{|r - r'|} dr' dy' dz'$$

$$\mathbf{E}(r) = \frac{1}{4\pi \epsilon_0 r} \iint_V \rho(r') \frac{r - r'}{|r - r'|^3} dr' dy' dz'$$

Derivation: Current-carrying conductor

$$\mathbf{H}(r) = -\mathbf{e}_x \sin(\varphi) f(r) + \mathbf{e}_y \cos(\varphi) f(r)$$

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Derivation: Current-carrying conductor

Now the circular disc is parameterized using

$$\gamma(t) = r \mathbf{e}_x \cos(t) + r \mathbf{e}_y \sin(t)$$

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$$\mathbf{H}(r) = [-\mathbf{e}_x \cos(\varphi) + \mathbf{e}_y \sin(\varphi)] \frac{d}{2\pi r} = I \frac{\mathbf{e}_z}{2\pi r}$$



Source: Physikunterrichts online.de

$$\begin{aligned} I &= \oint_{\partial A} \mathbf{H} \cdot d\mathbf{s} \\ &= \int_0^{2\pi} [-\mathbf{e}_x \sin(\varphi) f(r) + \mathbf{e}_y \cos(\varphi)] f(r) \\ &\quad \cdot r [-\mathbf{e}_x \sin(\varphi) + \mathbf{e}_y \cos(\varphi)] d\varphi \\ &= f(r) r \int_0^{2\pi} \sin^2(\varphi) + \cos^2(\varphi) d\varphi = f(r) 2\pi r. \end{aligned}$$

Derivation: Current-carrying conductor

Now the circular disc is parameterized using

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and thus

$$\begin{aligned} I &= \oint_{\partial A} \mathbf{H} \cdot d\mathbf{x} \\ &= \int_0^{2\pi} [-\mathbf{e}_x \sin(\varphi) t(r) + \mathbf{e}_y \cos(\varphi) t(r) \\ &\quad + r[-\mathbf{e}_x \sin(\varphi) + \mathbf{e}_y \cos(\varphi)]] d\varphi \\ &= t(r) r \int_0^{2\pi} [\sin^2(\varphi) + \cos^2(\varphi)] d\varphi = t(r) 2\pi r. \end{aligned}$$

Derivation: Current-carrying conductor

and thus

$$\mathbf{H}(r) = -\mathbf{e}_x \sin(\varphi) t(r) + \mathbf{e}_y \cos(\varphi) t(r)$$

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$$\oint_{\partial A} \mathbf{H} \cdot d\mathbf{x} = \iint_A \mathbf{J} \cdot \mathbf{A}$$

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And thus the well known result:

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Derivation: Current-carrying conductor

And thus the well known result:

$$\mathbf{H}(r) = \{-\mathbf{a}_x \cos(\varphi) + \mathbf{a}_y \sin(\varphi)\} \frac{1}{2\pi r} - \frac{\mathbf{a}_z}{2\pi r}$$

Magnetic vector potential

Analogously to the electric potential, the magnetic vector potential can be introduced as an aid:

$$\mathbf{B} = \text{rot } \mathbf{A}.$$

in case of a **known** current density, \mathbf{A} can be calculated directly:

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \iiint \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dx' dy' dz', \\ \mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \iiint \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} dx' dy' dz'.\end{aligned}$$

Hint: The last equation is also known as the **Biot-Savart law**.

Derivation: Current-carrying conductor

Now the circular disc is parameterized using

$$\gamma(t) = r\mathbf{a}_x \cos(t) + r\mathbf{a}_y \sin(t)$$

and thus

$$\begin{aligned}I &= \oint_{\text{disc}} \mathbf{H} \cdot d\mathbf{x} \\ &= \int_0^{2\pi} [-\mathbf{a}_x \sin(\varphi) I(r) + \mathbf{a}_y \cos(\varphi) I(r) \\ &\quad \times [-\mathbf{a}_x \cos(\varphi) + \mathbf{a}_y \sin(\varphi)] d\varphi \\ &= I(r) \int_0^{2\pi} [\sin^2(\varphi) + \cos^2(\varphi)] d\varphi = I(r) 2\pi r.\end{aligned}$$

Derivation: Current-carrying conductor

$$\mathbf{H}(r) = -\mathbf{a}_x \sin(\varphi) I(r) + \mathbf{a}_y \cos(\varphi) I(r)$$

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and thus

$$\begin{aligned}I &= \oint_{\text{disc}} \mathbf{H} \cdot d\mathbf{s} \\ &= \int_0^{2\pi} [-\mathbf{e}_x \sin(\varphi) t'(r) + \mathbf{e}_y \cos(\varphi) t'(r) \\ &\quad \times [-\mathbf{e}_x \sin(\varphi) + \mathbf{e}_y \cos(\varphi)]] d\varphi \\ &= t'(r) \int_0^{2\pi} [\sin^2(\varphi) + \cos^2(\varphi)] d\varphi = t'(r) 2\pi r.\end{aligned}$$

Stationary fields

- ▷ Point charges and infinitely long currents help to evaluate scenarios.
- ▷ Introduction of auxiliary quantities (potentials) facilitates the calculation.
- ▷ If the currents and charges are known, the fields can be calculated (often only numerically).
- ▷ Similar to the concept input signal / transfer function / output signal^a.
- ▷ Determining the currents and charges can be very difficult (e.g. charge distribution on semiconductor interfaces when a voltage is applied).

^aSee also [Green's functions](#)

Maxwell's Equations

2.2 Maxwell's Equations

- 2.2.1 General case
- 2.2.2 Divergence and rotation
- 2.2.3 Material equations and continuity equation
- 2.2.4 Stationary fields
- 2.2.5 Quasi-stationary fields**
- 2.2.6 Time harmonic fields

Definition

Quasi-stationary fields are fields with a fixed spatial distribution and time-dependent intensity. These are cases in which the displacement current density $\frac{\partial D}{\partial t}$ can be neglected compared to the power current density J .

Eigenschaften

- ▷ In the close range of antennas, fields are quasi-static. A common designation for quasi-static fields is therefore also near fields.
- ▷ The near range is the range that is much smaller than the wavelength.
- ▷ The alternating magnetic field belongs to the near fields.

Definition

Quasi-stationary fields are fields with a fixed spatial distribution and time-dependent intensity.
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Maxwell's Equations

$$\operatorname{div} \mathbf{D} = \rho \iff \iint_{\partial V} \mathbf{D} \cdot d\mathbf{A} = \iiint_V \rho dV$$

$$\operatorname{div} \mathbf{B} = 0 \iff \iint_{\partial V} \mathbf{B} \cdot d\mathbf{A} = 0$$

$$\operatorname{rot} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \iff \oint_{\partial A} \mathbf{E} \cdot d\mathbf{s} = -\iint_A \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}$$

$$\operatorname{rot} \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \iff \oint_{\partial A} \mathbf{H} \cdot d\mathbf{s} = \iint_A \left[\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right] \cdot d\mathbf{A}$$

Maxwell's Equations

$$\begin{aligned} \operatorname{div} \mathbf{D} = \rho &\iff \iint_{\partial V} \mathbf{D} \cdot d\mathbf{A} = \iiint_V \rho dV \\ \operatorname{div} \mathbf{B} = 0 &\iff \iint_{\partial V} \mathbf{B} \cdot d\mathbf{A} = 0 \\ \operatorname{rot} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} &\iff \oint_{\partial A} \mathbf{E} \cdot d\mathbf{s} = - \iint_A \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A} \\ \operatorname{rot} \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} &\iff \oint_{\partial A} \mathbf{H} \cdot d\mathbf{s} = \iint_A \left[\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right] \cdot d\mathbf{A} \end{aligned}$$

Definition

The **law of induction** states that a time-varying magnetic field is linked to an electric field.

$$\operatorname{rot} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \iff \oint_{\partial A} \mathbf{E} \cdot d\mathbf{s} = - \iint_A \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}$$

Definition

Quasi-stationary fields are fields with a fixed spatial distribution and time-dependent intensity. These are cases in which the displacement current density $\frac{\partial \mathbf{D}}{\partial t}$ can be neglected compared to the power current density \mathbf{J} .

Eigenvalues

- ▷ In the **near range** of antennas, fields are quasi-static. A common designation for quasi-static fields is therefore also **near fields**.
- ▷ The **near range** is the range that is much smaller than the wavelength.
- ▷ The alternating magnetic field belongs to the near fields.

Properties

- ▷ Approaches from AC engineering can often be applied (e.g. consideration at a fixed frequency, introduction of inductances)

Definition

The **law of induction** states that a time-varying magnetic field is linked to an electric field.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \iff \oint_{\text{SA}} \mathbf{E} \cdot d\mathbf{s} = -\iint_{\text{SA}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}$$

Properties

- Approaches from AC engineering can often be applied (e.g. consideration at a fixed frequency, introduction of inductances)

Maxwell's Equations

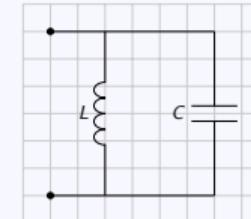
$$\nabla \times \mathbf{D} = \rho \iff \iint_{\text{SA}} \mathbf{D} \cdot d\mathbf{A} = \iint_{\text{V}} \rho dV$$

$$\nabla \cdot \mathbf{B} = 0 \iff \iint_{\text{SA}} \mathbf{B} \cdot d\mathbf{A} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \iff \oint_{\text{SA}} \mathbf{E} \cdot d\mathbf{s} = -\iint_{\text{A}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}$$

$$\nabla \cdot \mathbf{H} = J + \frac{\partial \mathbf{D}}{\partial t} \iff \oint_{\text{SA}} \mathbf{H} \cdot d\mathbf{s} = \iint_{\text{A}} [J + \frac{\partial \mathbf{D}}{\partial t}] \cdot d\mathbf{A}$$

A simple resonant circuit



Input impedance:

$$Z_{in} = \frac{j\omega L}{1 - \omega^2 LC}$$

Definition

Quasi-stationary fields are fields with a fixed spatial distribution and time-dependent intensity. There are cases in which the displacement current density $\frac{\partial \mathbf{D}}{\partial t}$ can be neglected compared to the power current density \mathbf{J} .

Eigenschaften

- In the **close range** of antennas, fields are quasi-static. A common designation for quasi-static fields is therefore also **near fields**.
- The **near range** is the range that is much smaller than the wavelength.
- The alternating magnetic field belongs to the near fields.

A simple resonant circuit

Input impedance:

$$Z_{in} = \frac{j\omega L}{1 - \omega^2 LC}$$

Definition

The law of induction states that a time-varying magnetic field is linked to an electric field.

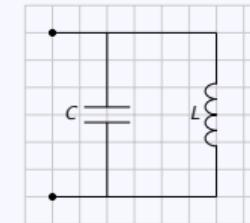
$$\text{rot } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \iff \oint_{\partial A} \mathbf{E} \cdot d\mathbf{s} = -\iint_A \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}$$

Properties

- Approaches from AC engineering can often be applied (e.g. consideration at a fixed frequency, introduction of inductances)

Maxwell's Equations

$$\begin{aligned} \text{div } \mathbf{D} = \rho &\iff \iint_{\partial V} \mathbf{D} \cdot d\mathbf{A} = \iiint_V \rho dV \\ \text{div } \mathbf{B} = 0 &\iff \iint_{\partial V} \mathbf{B} \cdot d\mathbf{A} = 0 \\ \text{rot } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} &\iff \oint_{\partial A} \mathbf{E} \cdot d\mathbf{s} = -\iint_A \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A} \\ \text{rot } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} &\iff \iint_{\partial A} \mathbf{H} \cdot d\mathbf{s} = \iint_A \left[\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right] \cdot d\mathbf{A} \end{aligned}$$

A simple resonant circuitCurrent when applying a voltage U_0 (unit jump):

$$\begin{aligned} I(s) &= U_0 \frac{1}{s} \frac{sL}{1 + s^2 LC} = U_0 \frac{L}{1 + s^2 LC} = U_0 \frac{L\omega_0^2}{\omega_0^2 + s^2} \\ i(t) &= U_0 L \omega_0 \sin(\omega_0 t) u(t), \end{aligned}$$

$$\text{with } \omega_0 = \frac{1}{\sqrt{LC}}.$$

A simple resonant circuit

Current when applying a voltage U_0 (unit jump):

$$\begin{aligned} i(t) &= U_0 \frac{1}{\sqrt{1 + \omega^2 LC}} \cdot \sin(\omega_0 t) = U_0 \frac{1}{\sqrt{1 + \omega^2 LC}} = U_0 \frac{\omega_0}{\sqrt{\omega_0^2 + \omega^2}} \\ i(t) &= U_0 L \omega_0 \sin(\omega_0 t) \end{aligned}$$

with $\omega_0 = \frac{1}{\sqrt{LC}}$

A simple resonant circuit



Input impedance:

$$Z_{in} = \frac{j\omega L}{1 - \omega^2 LC}$$

Definition

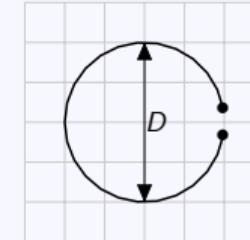
The law of induction states that a time-varying magnetic field is linked to an electric field.

$$rot \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} \Leftrightarrow \oint_{\text{loop}} \mathbf{E} \cdot d\mathbf{s} = - \iint_A \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}$$

Properties

d) Approaches from AC engineering can often be applied (e.g. consideration at a fixed frequency, introduction of inductances)

Simple inductance

Inductance of loop with diameter D and diameter of wire d ^a:

$$L \approx \mu_0 \mu_r \frac{D}{2} \left(\ln \left(\frac{8D}{d} \right) - 2 \right)$$

Example: $D = 1 \text{ m}$, $d = 0.1 \text{ cm}$: $L = 2.94 \mu\text{H}$.^asee e.g. [eeweb](#) und [FH Aachen](#)

Simple inductanceInductance of loop with diameter D and radius of wire d^2 :

$$L \approx \mu_0 \frac{D}{2} \ln\left(\frac{8D}{d}\right) - 2$$

Example: $D = 1\text{ m}$, $d = 0.1\text{ mm}$: $L = 2.94\text{ }\mu\text{H}$.See e.g. [Wikipedia](#) and [FIR Antennas](#)**A simple resonant circuit**Current when applying a voltage U_0 (unit jump):

$$\begin{aligned} i(s) &= U_0 \frac{1}{s + \omega_0^2 LC} = U_0 \frac{1}{1 + s^2 LC} = U_0 \frac{\omega_0^2}{\omega_0^2 + s^2} \\ i(t) &= U_0 \omega_0 \sin(\omega_0 t) u(t), \end{aligned}$$

with $\omega_0 = \frac{1}{\sqrt{LC}}$ **Simple inductance**

Source: Wikipedia (Wolf Meusel)

Assumption

$$L = 2.94\text{ }\mu\text{H}$$

$$C = 4.7\text{ }\mu\text{F}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = 42.8\text{ kHz}$$

A simple resonant circuit

Input impedance:

$$Z_{in} = \frac{j\omega L}{1 - j\omega LC}$$

Simple inductance



Assumption

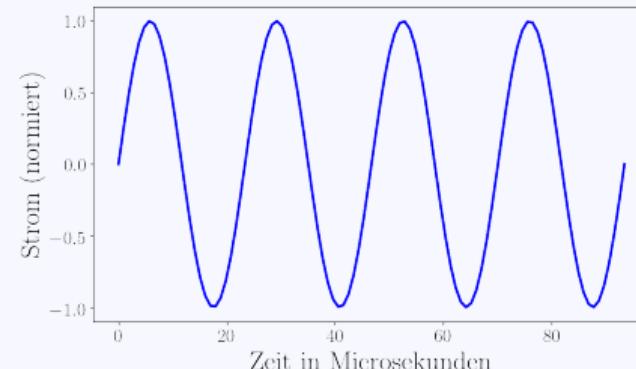
$$L = 2.94 \mu\text{H}$$

$$C = 4.7 \mu\text{F}$$

$$\omega_0 = \frac{1}{2\pi\sqrt{LC}} = 42.8 \text{ kHz}$$

Source: Wikipedia (Wolf Model)

Step response



Simple inductance



Inductance of loop with diameter D and width d :

$$L \approx \mu_0 \pi \frac{D}{2} \left(\ln \left(\frac{4D}{d} \right) - 2 \right)$$

Example: $D = 1 \text{ m}$, $d = 0.3 \text{ cm}$: $L \approx 2.94 \mu\text{H}$.

*See e.g. [Wolfs und Fitt Ausdruck](#)

A simple resonant circuit



Current when applying a voltage U_0 (unit jump):

$$I(t) = U_0 \frac{1}{\omega_0^2 + t^2} \cdot \frac{\omega_0^2}{\omega_0^2 + t^2} = U_0 \frac{1}{\omega_0^2 + t^2} \cdot \frac{L}{LC} = U_0 \frac{L}{\omega_0^2 + \omega_0^2 t^2}$$

$$I(t) = U_0 L \omega_0 \sin(\omega_0 t) e^{-\omega_0 t}$$

with $\omega_0 = \frac{1}{\sqrt{LC}}$

Calculation of the step response using inverse Laplace transform.

Maxwell's Equations

2.2 Maxwell's Equations

- 2.2.1 General case
- 2.2.2 Divergence and rotation
- 2.2.3 Material equations and continuity equation
- 2.2.4 Stationary fields
- 2.2.5 Quasi-stationary fields
- 2.2.6 Time harmonic fields

Differential form of Maxwell's equations

$$\operatorname{div} \mathbf{D} = \rho \quad \text{Gauss's law}$$

$$\operatorname{div} \mathbf{B} = 0 \quad \text{Gauss's law for magnetic fields}$$

$$\operatorname{rot} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{Maxwell-Faraday equation}$$

$$\operatorname{rot} \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \text{Ampère's law}$$

Differential form of Maxwell's equations

$\operatorname{div} \underline{\mathbf{D}} = \rho$	Gauss's law
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$\operatorname{rot} \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t}$	Maxwell-Faraday equation
$\operatorname{rot} \underline{\mathbf{H}} = \underline{\mathbf{J}} + \frac{\partial \underline{\mathbf{D}}}{\partial t}$	Ampere's law

Time-harmonic field

Assumption: Temporal relation according to^a $e^{j\omega t}$ and

$$\underline{\mathbf{E}}(\mathbf{r}, t) = \operatorname{Re} (\underline{\mathbf{E}}(\mathbf{r}) e^{j\omega t})$$

$$\operatorname{div} \underline{\mathbf{D}} = \rho$$

$$\operatorname{div} \underline{\mathbf{B}} = 0$$

$$\operatorname{rot} \underline{\mathbf{E}} = -j\omega \underline{\mathbf{B}}$$

$$\operatorname{rot} \underline{\mathbf{H}} = \underline{\mathbf{J}} + j\omega \underline{\mathbf{D}}$$

In the following, we will not explicitly write the underscore.

^aNote that in physics, one commonly uses $e^{-j\omega t}$

Wave propagation

2.1 Introduction

2.2 Maxwell's Equations

2.3 Electromagnetic waves

2.3.1 Wave equation

2.3.2 Plane waves

2.4 Fields of current distributions

2.5 Reflection, diffraction and damping of plane waves

2.6 Micro Strip lines, (coplanar) waveguides

Electromagnetic waves

2.3 Electromagnetic waves

2.3.1 Wave equation

2.3.2 Plane waves

Source free area

$$\operatorname{div} \mathbf{D} = 0$$

$$\operatorname{div} \mathbf{B} = 0$$

$$\operatorname{rot} \mathbf{E} = -j\omega \mathbf{B}$$

$$\operatorname{rot} \mathbf{H} = j\omega \mathbf{D}$$

Goal: Find equation which contains only electric or only magnetic fields.

Source free area

$$\begin{aligned}\operatorname{div} \mathbf{D} &= 0 \\ \operatorname{div} \mathbf{B} &= 0 \\ \operatorname{rot} \mathbf{E} &= -j\omega \mathbf{B} \\ \operatorname{rot} \mathbf{H} &= j\omega \mathbf{D}\end{aligned}$$

Goal: Find equation which contains only electric or only magnetic fields.

Wave equation

$$\begin{aligned}\nabla \times \mathbf{E} &= -j\omega \mathbf{B} \\ \nabla \times \mathbf{H} &= j\omega \mathbf{D}\end{aligned}$$

Subst. second eq. into first:

$$\frac{1}{j\omega\epsilon} \nabla \times \nabla \times \mathbf{H} = -j\omega\mu \mathbf{H}.$$

And thus:

$$\frac{1}{\omega^2\mu\epsilon} \nabla \times \nabla \times \mathbf{H} = \mathbf{H}.$$

Wave equation

$$\nabla \times \mathbf{E} = -j\omega \mathbf{B}$$

$$\nabla \times \mathbf{H} = j\omega \mathbf{D}$$

Subst. second eq. into first:
 $\frac{1}{j\omega} \nabla \times \nabla \times \mathbf{H} = -j\omega \mathbf{H}$.

And thus:

$$\frac{1}{\omega^2 \mu \epsilon} \nabla \times \nabla \times \mathbf{H} = \mathbf{H}$$

Source free area

$$\frac{1}{\omega^2 \mu \epsilon} \nabla \times \nabla \times \mathbf{H} = \mathbf{H}$$

Using speed of light^a

$$c = \frac{1}{\sqrt{\epsilon \mu}}$$

leads to

$$\frac{c^2}{\omega^2} \nabla \times \nabla \times \mathbf{H} = \mathbf{H}.$$

^ain vacuum: $c = 299\,792\,458 \text{ m s}^{-1}$

Source free area

$$\frac{1}{\omega^2 \mu_0} \nabla \times \nabla \times \mathbf{H} = -\mathbf{H}$$

Using speed of light*

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

leads to

$$\frac{c^2}{2} \nabla \times \nabla \times \mathbf{H} = -\mathbf{H}$$

*in vacuum: $c = 299792458 \text{ m/s}^2$

Wave equation

$$\nabla \times \mathbf{E} = -j\omega \mathbf{B}$$

$$\nabla \times \mathbf{H} = j\omega \mathbf{D}$$

Subst. second eq. into first:

$$\frac{1}{j\omega} \nabla \times \nabla \times \mathbf{H} = -j\omega \mathbf{H}$$

And thus:

$$\frac{1}{\omega^2 \mu_0} \nabla \times \nabla \times \mathbf{H} = -\mathbf{H}$$

Source free area

$$\operatorname{div} \mathbf{D} = 0$$

$$\operatorname{div} \mathbf{B} = 0$$

$$\operatorname{rot} \mathbf{E} = -j\omega \mathbf{B}$$

$$\operatorname{rot} \mathbf{H} = j\omega \mathbf{D}$$

Goal: Find equation which contains only electric or only magnetic fields.

Source free area

With wavelength^a λ and wavenumber

$$k = \frac{2\pi}{\lambda} = \frac{2\pi f}{c}$$

one gets:

$$\frac{c^2}{\omega^2} \nabla \times \nabla \times \mathbf{H} = \mathbf{H}$$

and thus one gets the wave equation:

$$\nabla \times \nabla \times \mathbf{H} = k^2 \mathbf{H}.$$

We are now looking for solutions.

$$^a \lambda = 0.3 \text{ m GHz}^{-1}. \text{ Example: } \lambda = 0.3 \text{ m @ 1 GHz}$$

Electromagnetic waves

2.3 Electromagnetic waves

2.3.1 Wave equation

2.3.2 Plane waves

Ansatz

For the solution of the equation

$$\nabla \times \nabla \times \mathbf{H} = \mathbf{H}k^2$$

we are using the Ansatz

$$\mathbf{H} = \mathbf{e}_x e^{-jky}.$$

This fulfills the wave equation.

- └ Wave propagation
- └ Electromagnetic waves

Properties: $\mathbf{H} = \mathbf{e}_x e^{-jky}$



$$\mathbf{H}(\mathbf{r}, t) = \mathbf{e}_x \cos(\omega t - ky) :$$

- ▷ The magnetic field \mathbf{H} is perpendicular to the direction of propagation $\mathbf{n} = \mathbf{e}_y$
- ▷ The corresponding electrical field is given by:

$$\mathbf{E} = \frac{1}{Z_f} \mathbf{n} \times \mathbf{H},$$

with

$$Z_f = \sqrt{\frac{\mu}{\epsilon}}$$

being the intrinsic **wave impedance** of the medium.

Properties: $H = e_z e^{-jky}$

$$\mathbf{H}(r, t) = \mathbf{e}_z \cos(\omega t - kx)$$

- The magnetic field \mathbf{H} is perpendicular to the direction of propagation $\mathbf{n} = \mathbf{e}_y$.
- The corresponding electrical field is given by:

$$\mathbf{E} = \frac{1}{Z_f} \mathbf{n} \times \mathbf{H},$$

with

$$Z_f = \sqrt{\frac{\mu}{\epsilon}}$$

being the intrinsic wave impedance of the medium.

Properties

In free space, the corresponding electrical field is given by:

$$\mathbf{E} = \frac{1}{Z_f} \mathbf{n} \times \mathbf{H} = \frac{1}{\pi 120 \Omega} \mathbf{e}_z e^{-jky}.$$

Ansatz

For the solution of the equation

$$\nabla \times \nabla \times \mathbf{H} = \mu \mathbf{H}^2$$

we are using the Ansatz

$$\mathbf{H} = \mathbf{e}_z e^{-jky}.$$

This fulfills the wave equation.

- └ Wave propagation
- └ Electromagnetic waves

Properties

In free space, the corresponding electrical field is given by:

$$\mathbf{E} = \frac{1}{Z_0} \mathbf{n} \times \mathbf{H} = \frac{1}{\pi \epsilon_0 \mu_0} \mathbf{e}_x e^{-jkr}$$

Properties: $\mathbf{H} = \mathbf{e}_z e^{-jkz}$

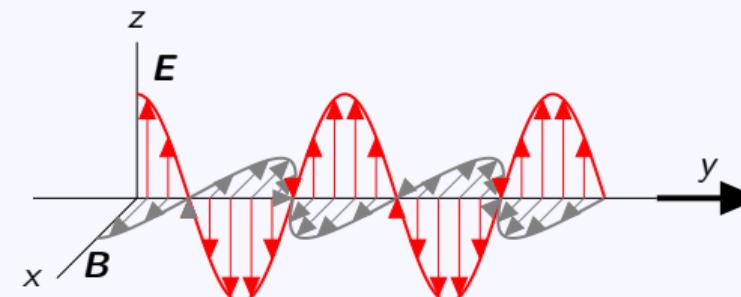
- ▷ $\mathbf{H}(r, t) = \mathbf{e}_z \cos(\omega t - kz)$:
- ▷ The magnetic field \mathbf{H} is perpendicular to the direction of propagation $\mathbf{n} \equiv \mathbf{e}_z$.
- ▷ The corresponding electrical field is given by:

$$\mathbf{E} = \frac{1}{Z_0} \mathbf{n} \times \mathbf{H},$$

with

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

being the intrinsic wave impedance of the medium.

Fields for $t = 0$ 

based upon a blog entry of [Henri Menke](#)

The direction of the electric field is called **polarisation**. Here the wave is linear polarized. Other polarizations are elliptic and circular.

Ansatz

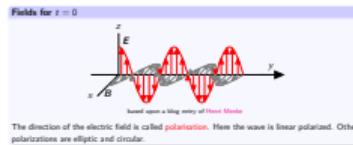
For the solution of the equation

$$\nabla \times (\nabla \times \mathbf{H}) = -\mu_0 \mathbf{E}$$

we are using the Ansatz

$$\mathbf{H} = \mathbf{e}_z e^{-jkz}.$$

This fulfills the wave equation.



Properties
In free space, the corresponding electrical field is given by:

$$\mathbf{E} = \frac{1}{Z_0} \mathbf{a}_z \times \mathbf{H} = \frac{1}{\pi D \mu_0} \mathbf{a}_x e^{-jky}$$

Properties: $\mathbf{H} = \mathbf{a}_x e^{-jky}$
 $\mathbf{H}(r, t) = \mathbf{a}_x \cos(\omega t - ky)$:
 ▷ The magnetic field \mathbf{H} is perpendicular to the direction of propagation $\mathbf{n} = \mathbf{a}_y$,
 ▷ The corresponding electrical field is given by:

$$\mathbf{E} = \frac{1}{Z_0} \mathbf{a}_z \times \mathbf{H},$$

 with

$$Z_0 = \sqrt{\frac{\mu}{\epsilon}}$$

 being the **intrinsic wave impedance** of the medium.

Fields for $t = \frac{T}{4}$



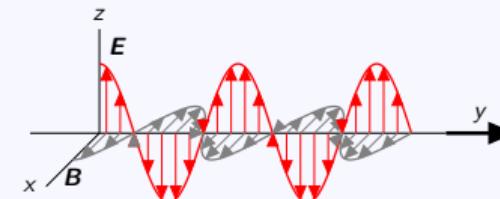
Fields for $t = 2\frac{T}{4}$

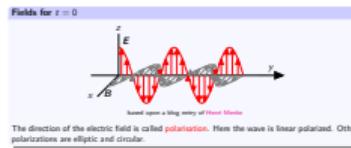
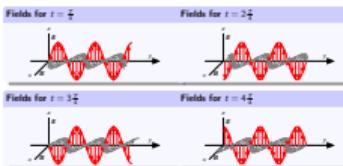


Fields for $t = 3\frac{T}{4}$



Fields for $t = 4\frac{T}{4}$



**Properties**

In free space, the corresponding electrical field is given by:

$$\mathbf{E} = \frac{1}{2\mu_0} \sigma \times \mathbf{H} = \frac{1}{\epsilon_0 \mu_0 c^2} \sigma_x e^{-jky}$$

Proof of correctness

It follows

$$\begin{aligned}
 \frac{1}{k^2} \nabla \times \nabla \times \mathbf{H} &= \frac{1}{k^2} \nabla \times \left[\left(\begin{array}{c} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{array} \right) \times \mathbf{e}_x e^{-jky} \right] \\
 &= \frac{1}{k^2} \nabla \times \left[\mathbf{e}_z \left(-\frac{\partial}{\partial y} e^{-jky} \right) \right] \\
 &= -\frac{1}{k^2} \left(\begin{array}{c} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{array} \right) \times \left[\mathbf{e}_z k e^{-jky} \right] \\
 &= \frac{1}{k} \mathbf{e}_x \frac{\partial}{\partial y} e^{-jky} = \mathbf{e}_x e^{-jky} = \mathbf{H}
 \end{aligned}$$

Wave propagation

2.1 Introduction

2.2 Maxwell's Equations

2.3 Electromagnetic waves

2.4 Fields of current distributions

2.4.1 Hertzian dipole

2.4.2 General solution

2.4.3 Far field of antennas

2.5 Reflection, diffraction and damping of plane waves

2.6 Micro Strip lines, (coplanar) waveguides

Fields of current distributions

2.4 Fields of current distributions

2.4.1 Hertzian dipole

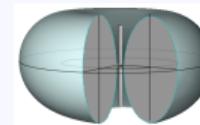
2.4.2 General solution

2.4.3 Far field of antennas

Far field

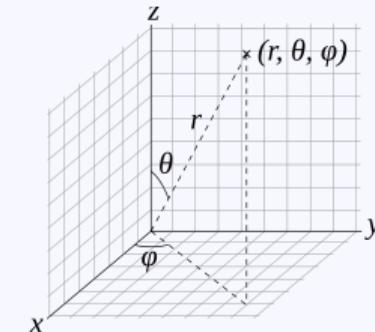
Far field of a very short current element (Hertzian dipole) orientated in \mathbf{e}_z -direction:

$$\mathbf{E}(\mathbf{r}) = \frac{jk}{4\pi} I h \frac{e^{-jkr}}{r} \sin \theta \mathbf{e}_\theta.$$

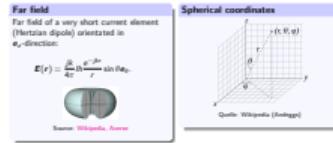


Source: [Wikipedia](#), Averse

Spherical coordinates



Quelle: Wikipedia (Andeggs)



Far field

Far field of a very short current element (Hertzian dipole) orientated in \mathbf{e}_i -direction:

$$\begin{aligned}\mathbf{H}(\mathbf{r}) &= -\frac{jk}{4\pi} lh \frac{e^{-jkr}}{r} \mathbf{e}_r \times \mathbf{e}_i \\ \mathbf{E}(\mathbf{r}) &= Z_f \frac{jk}{4\pi} lh \frac{e^{-jkr}}{r} \mathbf{e}_r \times \mathbf{e}_r \times \mathbf{e}_i.\end{aligned}$$

Far fieldFar field of a very short current element (Hertzian dipole) oriented in \mathbf{e}_r -direction:

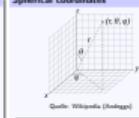
$$\begin{aligned} \mathbf{H}(r) &= -\frac{\mu_0}{4\pi r} \frac{e^{jkr}}{r} \mathbf{e}_r \times \mathbf{e}_t \\ \mathbf{E}(r) &= Z_0 \frac{\mu_0}{4\pi r} \frac{e^{jkr}}{r} \mathbf{e}_r \times \mathbf{e}_t \times \mathbf{e}_z. \end{aligned}$$

Far fieldFar field of a very short current element (Hertzian dipole) oriented in \mathbf{e}_ϑ -direction:

$$\mathbf{E}(r) = \frac{\mu_0}{4\pi} \frac{e^{jkr}}{r} \ln(\lambda r),$$



Source: Wikipedia, Author

Spherical coordinates

Quelle: Wikipedia (Autoren)

Useful equations

$$\mathbf{e}_r \times \mathbf{e}_\vartheta = \mathbf{e}_\varphi$$

$$\mathbf{e}_\vartheta \times \mathbf{e}_\varphi = \mathbf{e}_r$$

$$\mathbf{e}_r \times \mathbf{e}_\varphi = -\mathbf{e}_\vartheta$$

and thus

$$\begin{aligned} \mathbf{e}_r \times (A_r \mathbf{e}_r + A_\vartheta \mathbf{e}_\vartheta + A_\varphi \mathbf{e}_\varphi) &= (A_\vartheta \mathbf{e}_\varphi - A_\varphi \mathbf{e}_\vartheta) \\ \mathbf{e}_r \times \mathbf{e}_r \times (A_r \mathbf{e}_r + A_\vartheta \mathbf{e}_\vartheta + A_\varphi \mathbf{e}_\varphi) &= (-A_\vartheta \mathbf{e}_\vartheta - A_\varphi \mathbf{e}_\varphi). \end{aligned}$$

└ Wave propagation

└ Fields of current distributions

Useful equations

$$\begin{aligned}\mathbf{e}_r \times \mathbf{e}_\theta &= \mathbf{e}_\varphi \\ \mathbf{e}_\theta \times \mathbf{e}_r &= \mathbf{e}_\varphi \\ \mathbf{e}_r \times \mathbf{e}_\varphi &= -\mathbf{e}_\theta\end{aligned}$$

and thus

$$\begin{aligned}\mathbf{e}_r \times (\mathbf{A}_r \mathbf{e}_r + \mathbf{A}_\theta \mathbf{e}_\theta + \mathbf{A}_\varphi \mathbf{e}_\varphi) &= (\mathbf{A}_\theta \mathbf{e}_\varphi - \mathbf{A}_\varphi \mathbf{e}_\theta) \\ \mathbf{e}_\theta \times (\mathbf{A}_r \mathbf{e}_r + \mathbf{A}_\theta \mathbf{e}_\theta + \mathbf{A}_\varphi \mathbf{e}_\varphi) &= (-\mathbf{A}_r \mathbf{e}_\varphi - \mathbf{A}_\varphi \mathbf{e}_r)\end{aligned}$$

Useful equations

Consequently,

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= Z_f \frac{jk}{4\pi} I h \frac{e^{-jkr}}{r} \mathbf{e}_r \times \mathbf{e}_r \times \mathbf{e}_i \\ &= -Z_f \frac{jk}{4\pi} I h \frac{e^{-jkr}}{r} (\mathbf{e}_{i,\vartheta} + \mathbf{e}_{i,\varphi}).\end{aligned}$$

Far fieldFar field of a very short current element (Hertzian dipole) oriented in \mathbf{e}_r -direction:

$$\begin{aligned}\mathbf{H}(\mathbf{r}) &= -\frac{\mu_0}{4\pi} \frac{e^{-jkr}}{r} \mathbf{e}_r \times \mathbf{e}_i \\ \mathbf{E}(\mathbf{r}) &= Z_f \frac{\mu_0}{4\pi} \frac{e^{-jkr}}{r^2} \mathbf{e}_r \times \mathbf{e}_i \times \mathbf{e}_i.\end{aligned}$$

Far fieldFar field of a very short current element (Hertzian dipole) oriented in \mathbf{e}_r -direction:

$$\mathbf{E}(\mathbf{r}) = \frac{\mu_0}{4\pi} I h \frac{e^{-jkr}}{r^2} \sin(\theta) \mathbf{e}_i$$

Source: Wikipedia, Author

Spherical coordinates

Quelle: Wikipedia (Autoren)

Useful equations

Consequently,

$$\begin{aligned} \mathbf{E}(r) &= Z_0 \frac{\mu_0}{4\pi} \frac{e^{jkr}}{r^2} \mathbf{a}_r \times \mathbf{a}_\theta \times \mathbf{a}_\varphi \\ &= -Z_0 \frac{\mu_0}{4\pi} \frac{e^{jkr}}{r^2} (\mathbf{a}_{r,\theta} + \mathbf{a}_{r,\varphi}). \end{aligned}$$

Useful equations

Vector fields can be converted from cartesian to spherical by making use of (see [Wikipedia](#) as well):

$$\begin{aligned} \mathbf{a}_r \times \mathbf{a}_\theta &= \mathbf{a}_\varphi \\ \mathbf{a}_r \times \mathbf{a}_\varphi &= \mathbf{a}_\theta \\ \mathbf{a}_\theta \times \mathbf{a}_\varphi &= -\mathbf{a}_r \end{aligned}$$

and thus

$$\begin{aligned} \mathbf{a}_r \times (\mathbf{A}_x \mathbf{a}_r + \mathbf{A}_\theta \mathbf{a}_\theta + \mathbf{A}_\varphi \mathbf{a}_\varphi) &= (\mathbf{A}_\theta \mathbf{a}_\theta - \mathbf{A}_\varphi \mathbf{a}_\varphi) \\ \mathbf{a}_r \times \mathbf{a}_r \times (\mathbf{A}_x \mathbf{a}_r + \mathbf{A}_\theta \mathbf{a}_\theta + \mathbf{A}_\varphi \mathbf{a}_\varphi) &= (-\mathbf{A}_x \mathbf{a}_r - \mathbf{A}_\varphi \mathbf{a}_\varphi). \end{aligned}$$

$$A_r = A_x \sin \vartheta \cos \varphi + A_y \sin \vartheta \sin \varphi + A_z \cos \vartheta$$

$$A_\theta = A_x \cos \vartheta \cos \varphi + A_y \cos \vartheta \sin \varphi - A_z \sin \vartheta$$

$$A_\varphi = -A_x \sin \varphi + A_y \cos \varphi$$

Far fieldFar field of a very short current element (Hertzian dipole) oriented in \mathbf{a}_r -direction:

$$\begin{aligned} \mathbf{H}(r) &= -\frac{\mu_0}{4\pi} \frac{g^2 e}{r} \mathbf{a}_r \times \mathbf{a}_\theta \\ \mathbf{E}(r) &= Z_0 \frac{\mu_0}{4\pi} \frac{g^2 e}{r} \mathbf{a}_r \times \mathbf{a}_\theta \times \mathbf{a}_\varphi. \end{aligned}$$

Fields of current distributions

2.4 Fields of current distributions

2.4.1 Hertzian dipole

2.4.2 General solution

2.4.3 Far field of antennas

Fields of Electrical Currents

The following considers electrically ideally conductive bodies in otherwise free space. The field radiated by currents^a on the surface A of these bodies is given by^b:

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\phi(\mathbf{r}, t) - \frac{\partial}{\partial t}\mathbf{A}(\mathbf{r}, t).$$

If the currents are known, the fields can be determined. However, determining the currents is generally not trivial.

^aIn certain cases, such as horn antennas, magnetic currents are also considered

^bSee, for example, Blume, Theory of Electromagnetic Fields

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Potentials

$$\begin{aligned}\mathbf{A}(\mathbf{r}, t) &= \frac{\mu}{4\pi} \int_A \frac{\mathbf{J}(\mathbf{r}', t')}{r} da', \\ \phi(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon} \int_A \frac{\rho(\mathbf{r}', t')}{r} da',\end{aligned}$$

with

$$r = |\mathbf{r} - \mathbf{r}'|$$

and

$$t' = t - \frac{r}{c}.$$

Potentials

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu}{4\pi r} \int_A \frac{\mathbf{J}(\mathbf{r}', t')}{r'} d\mathbf{a}'$$

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi r} \int_A \frac{\rho(\mathbf{r}', t')}{r'} d\mathbf{a}'$$

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$$r = |\mathbf{r} - \mathbf{r}'|$$

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Fields of electrical currents

Thus, an integral representation for the electric field can be specified:

$$\mathbf{E}(\mathbf{r}, t) = -\nabla \frac{1}{4\pi\epsilon} \int_A \frac{\rho(\mathbf{r}', t')}{r} d\mathbf{a}' - \frac{\partial}{\partial t} \frac{\mu}{4\pi} \int_A \frac{\mathbf{J}(\mathbf{r}', t')}{r} d\mathbf{a}'.$$

The relationship between the electric current density \vec{J} and the electric charge density ρ is described by the continuity equation:

$$\nabla \cdot \mathbf{J}(\mathbf{r}', t) + \frac{\partial}{\partial t} \rho(\mathbf{r}', t) = 0.$$

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* In certain cases, such as here, aperiodic, magnetic currents are also considered.

^{**} See, for example, Blau, Theory of Electromagnetic Fields.

Fields of electrical currents

Thus, an integral representation for the electric field can be specified:

$$\mathbf{E}(\mathbf{r}, t) = -\nabla \frac{1}{4\pi\epsilon_0} \int_A \frac{J(\mathbf{r}', t')}{r'} d\mathbf{a}' - \frac{\partial}{\partial t} \frac{\mu}{4\pi} \int_A \frac{\hat{J}(\mathbf{r}', t')}{r'} d\mathbf{a}'.$$

The relationship between the electric current density J and the electric charge density ρ is described by the continuity equation:

$$\nabla \cdot \mathbf{J}(\mathbf{r}', t) + \frac{\partial}{\partial t} \rho(\mathbf{r}', t) = 0.$$

Potentials

$$\begin{aligned}\mathbf{A}(\mathbf{r}, t) &= \frac{\mu}{4\pi} \int_A \frac{J(\mathbf{r}', t')}{r} d\mathbf{a}', \\ \phi(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon_0} \int_A \frac{\rho(\mathbf{r}', t')}{r} d\mathbf{a}',\end{aligned}$$

with

$$r = |\mathbf{r} - \mathbf{r}'|$$

and

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Fields of Electrical CurrentsThe following considers electrically ideally conductive bodies in otherwise free space. The field radiated by currents* on the surface A of these bodies is given by^b:

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Fields of electrical currents

Integral representation of the electric charge density:

$$\rho(\mathbf{r}', t) = - \int_{-\infty}^t \nabla \cdot \mathbf{J}(\mathbf{r}', t') dt'.$$

And thus for $\mathbf{J}(\mathbf{r}, t) = 0$ for $t < 0$:

$$\begin{aligned}\mathbf{E}(\mathbf{r}, t) &= \nabla \frac{1}{4\pi\epsilon_0} \int_A \frac{\int_0^t \nabla \cdot \mathbf{J}(\mathbf{r}', t') dt'}{r} d\mathbf{a}' \\ &\quad - \frac{\partial}{\partial t} \frac{\mu}{4\pi} \int_A \frac{\mathbf{J}(\mathbf{r}', t')}{r} d\mathbf{a}'.\end{aligned}$$

Note that ∇ is applied to different coordinates (source (primed) and observation point (unprimed)).

Fields of electrical currents

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$$\begin{aligned} \mathbf{E}(r, t) &= -\nabla \frac{1}{4\pi\epsilon_0} \int_A \frac{\nabla \cdot \mathbf{J}(r', t')}{r'} da' \\ &\quad - \frac{\partial}{\partial t} \frac{\mu}{4\pi} \int_A \frac{\mathbf{J}(r', t')}{r'} da'. \end{aligned}$$

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$$\nabla \cdot \mathbf{J}(r', t) + \frac{\partial}{\partial t} \rho(r', t) = 0.$$

Field of electrical currents

Time harmonic case:

$$\mathbf{E}(r) = -\nabla \frac{1}{4\pi\epsilon_0} \int_A \frac{\nabla \cdot \mathbf{J}(r') e^{-jkr}}{j\omega r} da' - \frac{j\omega\mu}{4\pi} \int_A \mathbf{J}(r') \frac{e^{-jkr}}{r} da',$$

with

$$r = |\mathbf{r} - \mathbf{r}'|.$$

Potentials

$$\begin{aligned} \mathbf{A}(r, t) &= \frac{\mu}{4\pi} \int_A \frac{\mathbf{J}(r', t')}{r'} da', \\ \phi(r, t) &= \frac{1}{4\pi\epsilon_0} \int_A \frac{\rho(r', t')}{r'} da', \end{aligned}$$

with

$$r = |\mathbf{r} - \mathbf{r}'|$$

and

$$t' = t - \frac{r}{c}.$$

Fields of current distributions

2.4 Fields of current distributions

2.4.1 Hertzian dipole

2.4.2 General solution

2.4.3 Far field of antennas

Field of electrical currents

For large distances between antenna and observation point one can approximate

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= \nabla \frac{1}{4\pi\varepsilon} \int_A \frac{\nabla \cdot \mathbf{J}(\mathbf{r}')}{j\omega} \frac{e^{-jkr}}{r} d\mathbf{a}' - \frac{j\omega\mu}{4\pi} \int_A \mathbf{J}(\mathbf{r}') \frac{e^{-jkr}}{r} d\mathbf{a}' \\ &\approx \nabla \frac{1}{4\pi\varepsilon r_0} \int_A \frac{\nabla \cdot \mathbf{J}(\mathbf{r}')}{j\omega} e^{-jkr} d\mathbf{a}' - \frac{j\omega\mu}{4\pi r_0} \int_A \mathbf{J}(\mathbf{r}') e^{-jkr} d\mathbf{a}',\end{aligned}$$

with $r_0 \approx |\mathbf{r}|$ being the distance of the observation point to the center of gravity of the antenna.

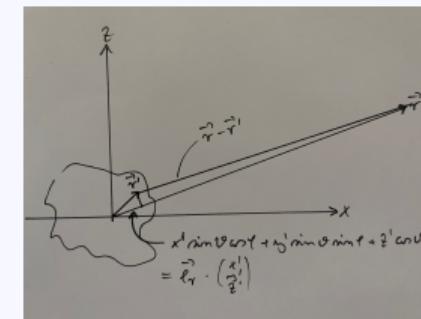
Field of electrical currents

For large distances between antenna and observation point one can approximate

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= \nabla \frac{1}{4\pi\epsilon_0} \int_A \frac{\nabla \cdot \mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} e^{-jk|\mathbf{r}'|} d\mathbf{r}' - \frac{jk\mu_0}{4\pi\epsilon_0} \int_A \mathbf{J}(\mathbf{r}') \frac{e^{-jk|\mathbf{r}'|}}{|\mathbf{r}'|} d\mathbf{r}' \\ &\approx \frac{1}{4\pi\epsilon_0\eta_0} \int_A \frac{\nabla \cdot \mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} e^{-jk|\mathbf{r}'|} d\mathbf{r}' - \frac{jk\mu_0}{4\pi\epsilon_0\eta_0} \int_A \mathbf{J}(\mathbf{r}') e^{-jk|\mathbf{r}'|} d\mathbf{r}', \end{aligned}$$

with $\eta_0 \approx |\mathbf{r}|$ being the distance of the observation point to the center of gravity of the antenna.**Field of electrical currents**The phase term can be approximated as follows^a:

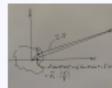
$$e^{-jk|\mathbf{r}-\mathbf{r}'|} \approx e^{-jk|\mathbf{r}|} e^{+jk(x' \sin \vartheta \cos \varphi + y' \sin \vartheta \sin \varphi + z' \cos \vartheta)}$$



$$^a \mathbf{e}_r = \sin \theta \cos \varphi \mathbf{e}_x + \sin \theta \sin \varphi \mathbf{e}_y + \cos \theta \mathbf{e}_z$$

Field of electrical currentsThe phase term can be approximated as follows^a:

$$e^{-jk(r-r')} \approx e^{-jk(r-r')}\left(1 + jk(r-r')\sin\vartheta\sin\varphi\right)$$



$$\mathbf{J}_0 = \sin\vartheta\sin\varphi\mathbf{e}_r + \sin\vartheta\cos\varphi\mathbf{e}_\theta + \cos\vartheta\mathbf{e}_\varphi$$

Field of electrical currents

For large distances between antenna and observation point one can approximate

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int_A \frac{\nabla \cdot \mathbf{J}(x', y', z')}{j\omega} e^{+jk(x' \sin\vartheta \cos\varphi + y' \sin\vartheta \sin\varphi + z' \cos\vartheta)} dx' dy' dz' \\ &\approx \frac{j\omega\mu}{4\pi |\mathbf{r}|} \int_A \mathbf{J}(x', y', z') e^{+jk(x' \sin\vartheta \cos\varphi + y' \sin\vartheta \sin\varphi + z' \cos\vartheta)} dx' dy' dz' \end{aligned}$$

$$\text{with } \eta_0 = |\mathbf{r}| \text{ being the distance of the observation point to the center of gravity of the antenna.}$$

Field of electrical currents

Thus, one gets^a:

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &\approx \nabla \frac{1}{4\pi\epsilon_0 |\mathbf{r}|} e^{-jk|\mathbf{r}|} \int_A \frac{\nabla \cdot \mathbf{J}(x', y', z')}{j\omega} e^{+jk(x' \sin\vartheta \cos\varphi + y' \sin\vartheta \sin\varphi + z' \cos\vartheta)} dx' dy' dz' \\ &\quad - \frac{j\omega\mu}{4\pi |\mathbf{r}|} e^{-jk|\mathbf{r}|} \int_A \mathbf{J}(x', y', z') e^{+jk(x' \sin\vartheta \cos\varphi + y' \sin\vartheta \sin\varphi + z' \cos\vartheta)} dx' dy' dz' \\ &= -\frac{jk}{4\pi} \mathbf{E}_0(\vartheta, \varphi) \frac{e^{-jk|\mathbf{r}|}}{|\mathbf{r}|} \end{aligned}$$

^aDue to the gradient in spherical coordinates given by

$$\text{grad } f = \frac{\partial f}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial f}{\partial \vartheta} \mathbf{e}_\vartheta + \frac{1}{r \sin \vartheta} \frac{\partial f}{\partial \varphi} \mathbf{e}_\varphi$$

the first anti-derivative can be neglected.

Field of electrical currentsThus, one gets^a:

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= \nabla \frac{1}{4\pi r_0 |\mathbf{r}|} e^{-jkr_0} \int_A \frac{\nabla \cdot \mathbf{J}(x', y', z')}{|z'|} j(k' \sin \vartheta \cos \varphi' \sin \vartheta' \sin \varphi' + k' \sin \vartheta \cos \varphi' \cos \vartheta' \sin \varphi' + k' \cos \vartheta \sin \varphi') dz' dy' dz' \\ &= -\frac{jk}{4\pi r_0} \int_A \mathbf{J}(x', y', z') e^{-j(k' \sin \vartheta \cos \varphi' \sin \vartheta' \sin \varphi' + k' \sin \vartheta \cos \varphi' \cos \theta' \sin \varphi' + k' \cos \vartheta \sin \varphi')} dz' dy' dz' \end{aligned}$$

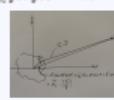
Due to the gradient in spherical coordinates follows by:

$$\text{grad } f = \frac{\partial f}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial f}{\partial \vartheta} \hat{e}_\vartheta + \frac{1}{r \sin \vartheta} \frac{\partial f}{\partial \varphi} \hat{e}_\varphi$$

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Field of electrical currentsThe phase term can be approximated as follows^a:

$$e^{j(k(r-r'))} \approx e^{-jkr'} e^{-jk(r-r') \cos \vartheta \cos \varphi' + (r-r') \sin \vartheta \sin \varphi' + k(r-r') \sin \vartheta \sin \varphi}$$

^a $\theta = \sin^{-1}(\sin \vartheta \cos \varphi + \sin \vartheta \sin \varphi \cos \theta) = \cos^{-1}(\cos \vartheta)$ **Field of electrical currents**

For large distances between antenna and observation point one can approximate

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= \nabla \frac{1}{4\pi r_0} \int_A \frac{\nabla \cdot \mathbf{J}(\mathbf{r}')}{|z'|} \frac{e^{-jkr'}}{|\mathbf{r}'|} dz' \\ &\approx \nabla \frac{1}{4\pi r_0} \int_A \frac{\nabla \cdot \mathbf{J}(\mathbf{r}')}{|z'|} e^{-jkr'} dz' - \frac{jkr_0}{4\pi} \int_A \mathbf{J}(\mathbf{r}') \frac{e^{-jkr'}}{|z'|^2} dz' \end{aligned}$$

with $r_0 = |\mathbf{r}'|$ being the distance of the observation point to the center of gravity of the antenna.**Field of electrical currents**

For large distances between antenna and observation point one can approximate

$$\mathbf{E}(\mathbf{r}) = -\frac{jk}{4\pi} \mathbf{E}_0(\vartheta, \varphi) \frac{e^{-jkr}}{|\mathbf{r}|},$$

where $\mathbf{E}_0(\vartheta, \varphi)$ contains anti-derivatives of the form

$$\int_A \mathbf{J}(x', y', z') e^{jk(x' \cos \vartheta \cos \varphi + y' \sin \vartheta \sin \varphi + z' \cos \vartheta)} dx' dy' dz'$$

Compare^a to **Fourier transform**: $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \mathcal{F}\{f(t)\}$.^aSee Klark, Antennen und Strahlungsfelder as well (available as ebook in library)

Field of electrical currents

For large distances between antenna and observation point one can approximate

$$\mathbf{E}(r) = -\frac{\rho}{4\pi} \mathbf{E}_d(\theta, \varphi) \frac{e^{jkr}}{|r|},$$

where $\mathbf{E}_d(\theta, \varphi)$ contains anti-derivatives of the form

$$\int J(x', y', z') j^k [x' \cos \theta \sin \varphi, y' \cos \theta \cos \varphi, z' \sin \theta] dx' dy' dz'$$

Compare* to Fourier transform: $F(w) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \mathcal{F}\{f(t)\}$.

*See Klark, Antennen und Strahlungsfelder as well (available as eBook in library)

Field of electrical currents

Thus, one gets*

$$\begin{aligned} \mathbf{E}(r) &\approx -\frac{1}{4\pi r^2} e^{-jkr} \int_{\text{ant}} \frac{\nabla \cdot J(x', y', z')}{j\omega} e^{-j[k(x' \cos \theta \sin \varphi + y' \cos \theta \cos \varphi + z' \sin \theta)]} dx' dy' dz' \\ &= -\frac{j\omega}{4\pi |r|^2} e^{-jkr} \int_{\text{ant}} J(x', y', z') j^k [x' \cos \theta \sin \varphi, y' \cos \theta \cos \varphi, z' \sin \theta] dx' dy' dz' \\ &= -\frac{\rho}{4\pi} \mathbf{E}_d(\theta, \varphi) \frac{e^{-jkr}}{|r|} \end{aligned}$$

*Due to the gradient in spherical coordinates given by

$$\operatorname{grad} f = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \varphi} \hat{\varphi}$$

the first anti-derivative can be neglected.

Summary

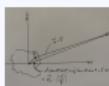
A more detailed examination is beyond the scope here. For further consideration, it is important to note:

- ▷ The use of potentials simplifies calculations (assumption: currents are known).
- ▷ An analysis of the time-harmonic case leads to the use of retarded potentials.
- ▷ The calculation of potentials leads to integrals of the type of Fourier transformation.
- ▷ Similarly to signals and systems, we will later examine exemplary current distributions.

Field of electrical currents

The phase term can be approximated as follows*.

$$e^{-j(kr - \tau)} \approx e^{-jkr} e^{-j\tau} [1 + jk(r \cos \theta \sin \varphi + r \cos \theta \cos \varphi + r \sin \theta)]$$

* $\mathbf{E}_d = \sin \theta \cos \varphi \mathbf{e}_x + \sin \theta \sin \varphi \mathbf{e}_y + \cos \theta \mathbf{e}_z$

Wave propagation

2.1 Introduction

2.2 Maxwell's Equations

2.3 Electromagnetic waves

2.4 Fields of current distributions

2.5 Reflection, diffraction and damping of plane waves

2.5.1 Reflection by a flat surface

2.5.2 Damping

2.5.3 Multipath

2.5.4 Diffraction & Physical optics

2.6 Micro Strip lines, (coplanar) waveguides

Reflection, diffraction and damping of plane waves

2.5 Reflection, diffraction and damping of plane waves

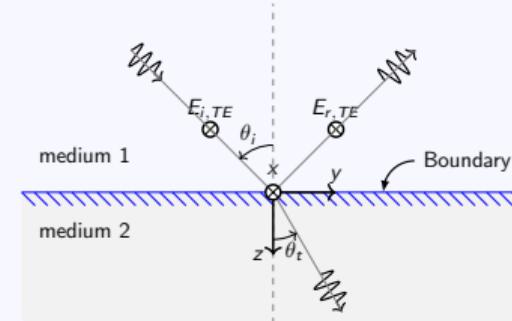
2.5.1 Reflection by a flat surface

2.5.2 Damping

2.5.3 Multipath

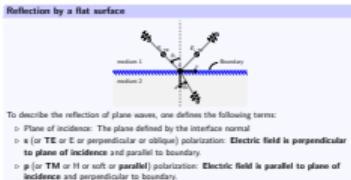
2.5.4 Diffraction & Physical optics

Reflection by a flat surface

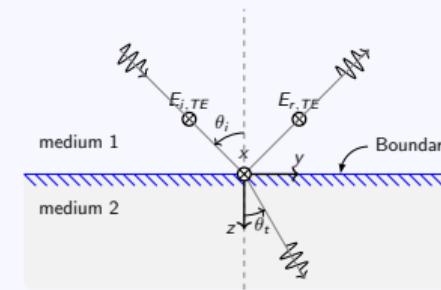


To describe the reflection of plane waves, one defines the following terms:

- ▷ Plane of incidence: The plane defined by the interface normal
- ▷ **s (or TE or E or perpendicular or oblique) polarization:** **Electric field is perpendicular to plane of incidence** and parallel to boundary.
- ▷ **p (or TM or H or soft or parallel) polarization:** **Electric field is parallel to plane of incidence** and perpendicular to boundary.



Snell's law

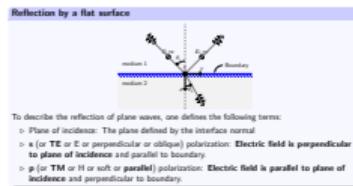


Snell's law is given by:

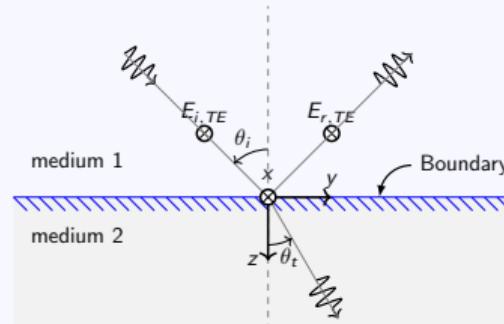
$$\frac{\sin \theta_i}{\sin \theta_t} = \frac{n_2}{n_1} \Rightarrow \sin \theta_t = \sin \theta_i \frac{n_1}{n_2},$$

with $n = \sqrt{\epsilon_r \mu_r}$ being the **refractive index** one gets

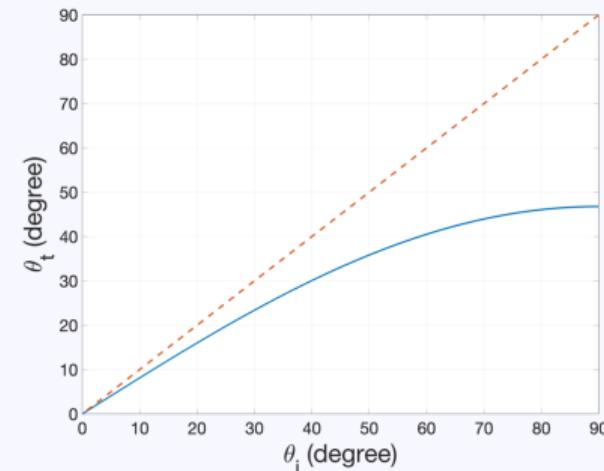
$$\boxed{\sin \theta_t = \sin \theta_i \frac{\sqrt{\epsilon_{r1}\mu_{r1}}}{\sqrt{\epsilon_{r2}\mu_{r2}}}}$$



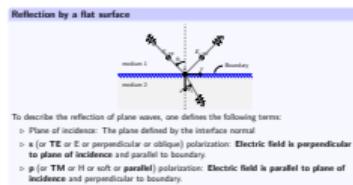
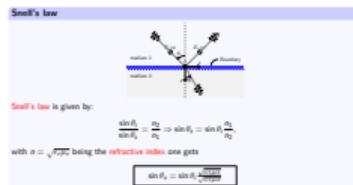
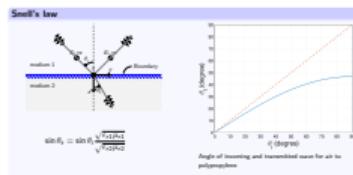
Snell's law



$$\sin \theta_t = \sin \theta_i \frac{\sqrt{\epsilon_r \mu_r}}{\sqrt{\epsilon_{r2} \mu_{r2}}}$$



Angle of incoming and transmitted wave for air to polypropylene



Reflection by a flat surface – TM

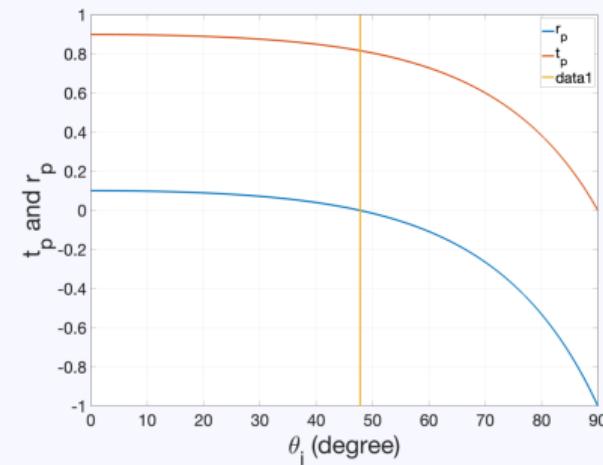
For p polarization (E-Field parallel to plane of incidence – **TM**):

$$r_p = r_{TM} = \frac{n_2 \cos \theta_i - n_1 \cos \theta_t}{n_2 \cos \theta_i + n_1 \cos \theta_t},$$

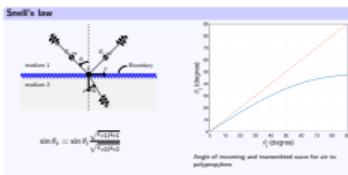
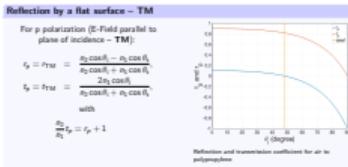
$$t_p = t_{TM} = \frac{2 n_1 \cos \theta_i}{n_2 \cos \theta_i + n_1 \cos \theta_t},$$

with

$$\frac{n_2}{n_1} t_p = r_p + 1$$



Reflection and transmission coefficient for air to polypropylene



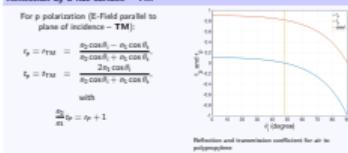
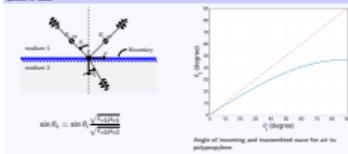
Brewster Angle

An electromagnetic wave with p polarization is completely transmitted in case of

$$\theta_B = \arctan \left(\frac{\nu_2}{\nu_1} \right).$$

Brewster AngleAn electromagnetic wave with p polarization is completely transmitted in case of

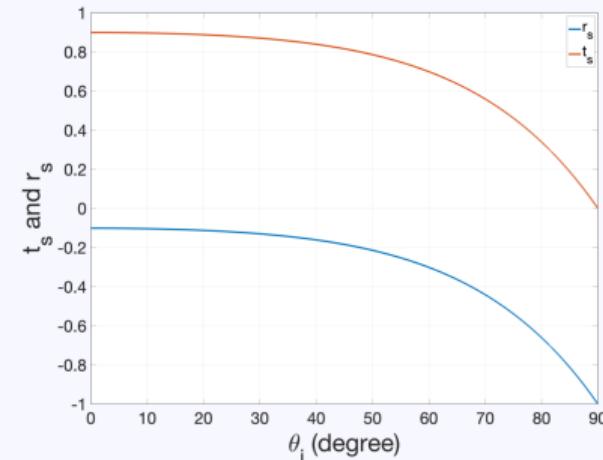
$$\theta_B = \arctan\left(\frac{n_2}{n_1}\right)$$

Reflection by a flat surface – TM**Snell's law****Reflection by a flat surface – TE**

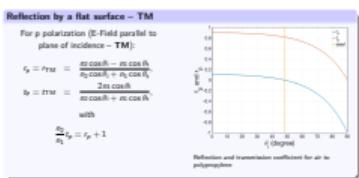
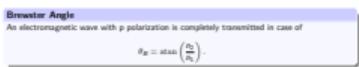
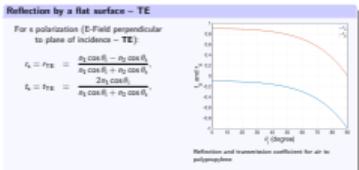
For s polarization (E-Field perpendicular to plane of incidence – TE):

$$r_s = r_{TE} = \frac{n_1 \cos \theta_i - n_2 \cos \theta_t}{n_1 \cos \theta_i + n_2 \cos \theta_t},$$

$$t_s = t_{TE} = \frac{2n_1 \cos \theta_i}{n_1 \cos \theta_i + n_2 \cos \theta_t},$$



Reflection and transmission coefficient for air to polypropylene



Reflection by a flat perfectly conducting surface

Using a complex permittivity

$$\varepsilon_{r,2} = \varepsilon'_{r,2} + j\varepsilon''_{r,2},$$

with $\varepsilon''_{r,2} \rightarrow \infty$ one gets

$$r_p = r_{TM} = 1,$$

$$t_p = t_{TM} = 0,$$

$$r_s = r_{TE} = -1,$$

$$t_s = t_{TE} = 0$$

Hint: Negative sign of r_{TE} can be explained by boundary condition (no tangential electrical field).

Reflection, diffraction and damping of plane waves

2.5 Reflection, diffraction and damping of plane waves

2.5.1 Reflection by a flat surface

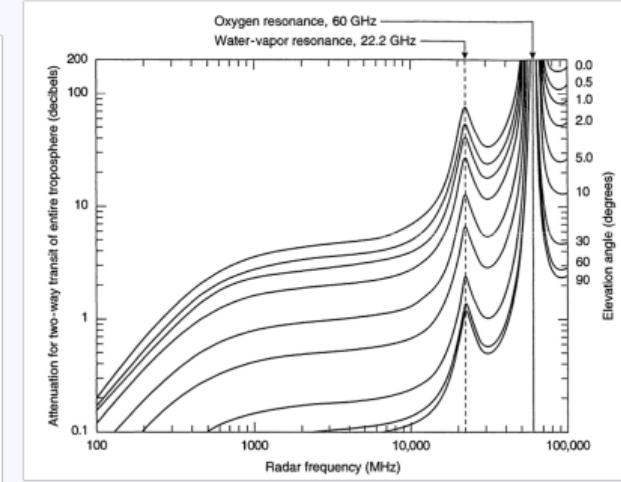
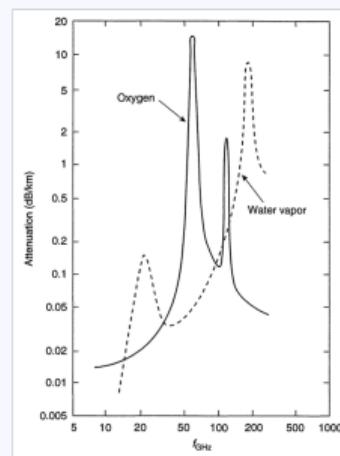
2.5.2 Damping

2.5.3 Multipath

2.5.4 Diffraction & Physical optics

Damping of plane waves

Road Surface



Theoretical attenuation for oxygen and water vapor and wave attenuation in standard atmosphere for target outside the troposphere (source: Peebles, radar principles)

Reflection, diffraction and damping of plane waves

2.5 Reflection, diffraction and damping of plane waves

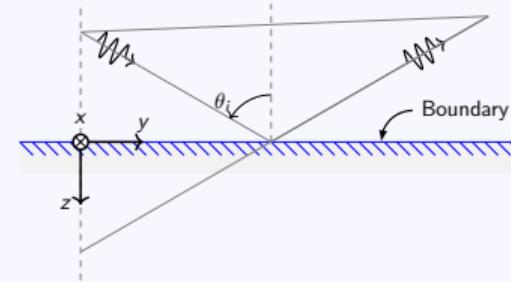
2.5.1 Reflection by a flat surface

2.5.2 Damping

2.5.3 Multipath

2.5.4 Diffraction & Physical optics

Multipath



Considered is a reflector at $(0, d, -h_2)$, which is illuminated by source at $(0, 0, -h_1)$. The following paths exist:

- ▷ Source - reflector - source (SRS): r_1
- ▷ Source - reflector - boundary - source (SRBS): r_2
- ▷ Source - boundary - reflector - source (SBRS): r_3
- ▷ Source - boundary - reflector - boundary - source (SBRBS): r_4

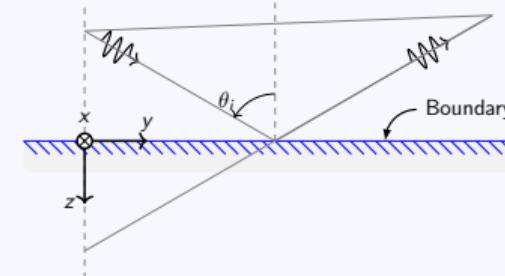
└ Wave propagation

└ Reflection, diffraction and damping of plane waves

Multipath

Considered is a reflector at $(0, d, -h_2)$, which is illuminated by source at $(0, 0, -h_1)$. The following paths exist:

- ▷ Source - reflector - source (SRS): r_1
- ▷ Source - reflector - boundary - source (SRBS): r_2
- ▷ Source - boundary - reflector - source (SBRIS): r_3
- ▷ Source - boundary - reflector - boundary - source (SBRSBIS): r_4

Multipath

Thus, the overall signal (assuming that the reflector is isotropic):

$$R_x \sim \frac{e^{-jkr_1}}{r_1} + r_{p,s} \frac{e^{-jkr_2}}{r_2} + r_{p,s} \frac{e^{-jkr_3}}{r_3} + r_{p,s}^2 \frac{e^{-jkr_4}}{r_4},$$

with $r_1 = 2\sqrt{(h_2 - h_1)^2 + d^2}$, $r_2 = r_3 = \sqrt{(h_2 + h_1)^2 + d^2}$ and $r_4 = 2\sqrt{(h_2 + h_1)^2 + d^2}$, for $h_2 > h_1$.

- └ Wave propagation

- └ Reflection, diffraction and damping of plane waves

Multipath

Thus, the overall signal (assuming that the reflector is isotropic)

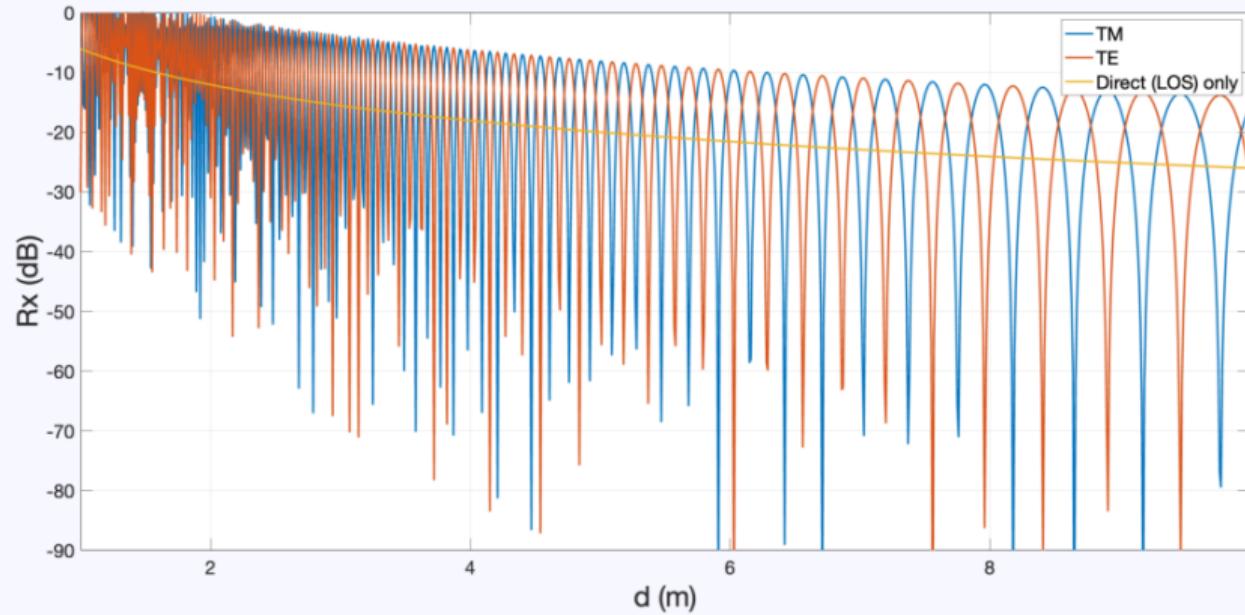
$$R_1 = \frac{e^{-j2\pi f t}}{r_1} + r_{pA} \frac{e^{-j2\pi f t}}{r_2} + r_{pB} \frac{e^{-j2\pi f t}}{r_1} + r'_{pA} \frac{e^{-j2\pi f t}}{r_2}$$

with $n = 2\sqrt{(h_2 - h_1)^2 + d^2}$, $\alpha = \sqrt{(h_2 + h_1)^2 + d^2}$ and $n = 2\sqrt{(h_2 + h_1)^2 + d^2}$, for $h_2 > h_1$.

Multipath

Considered is a reflector at $(0, d, -h_2)$, which is illuminated by source at $(0, 0, -h_1)$. The following paths exist:

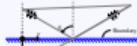
- ▷ Source - reflector - source (SRS); r_1
- ▷ Source - reflector - boundary - source (SRBS); r_2
- ▷ Source - boundary - reflector - source (BRSR); r_3
- ▷ Source - boundary - reflector - boundary - source (BRBSR); r_4

Perfectly conducting surface

$$h_2 = 0.6 \text{ m}, h_1 = 0.5 \text{ m}, f = 76 \text{ GHz}$$

└ Wave propagation

└ Reflection, diffraction and damping of plane waves

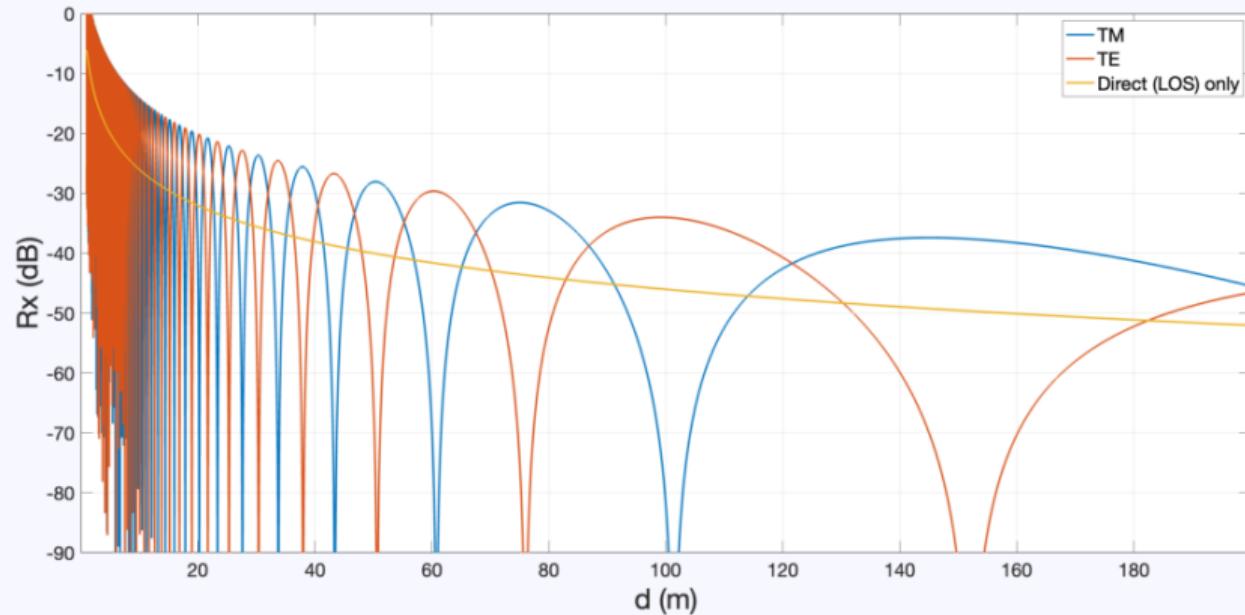
Multipath

Thus, the overall signal (assuming that the reflector is isotropic)

$$R_s = \frac{e^{-j2\pi f t_1}}{t_1} + r_{pA} \frac{e^{-j2\pi f t_2}}{t_2} + r_{pB} \frac{e^{-j2\pi f t_3}}{t_3} + r'_{pA} \frac{e^{-j2\pi f t_4}}{t_4}$$

with $t_1 = 2\sqrt{(ls - h_1)^2 + d^2}$, $t_2 = \sqrt{(ls + h_1)^2 + d^2}$ and $t_3 = 2\sqrt{(ls + h_2)^2 + d^2}$, for $ls \geq h_2$.**Multipath**Considered is a reflector at $(0, d, -h_2)$, which is illuminated by source at $(0, 0, -h_1)$. The following paths exist:

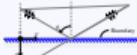
- ▷ Source - reflector - source (SRS); t_1
- ▷ Source - reflector - boundary - source (SRBS); t_2
- ▷ Source - boundary - reflector - source (BRSB); t_3
- ▷ Source - boundary - reflector - boundary - source (BRBSB); t_4

Perfectly conducting surface

$$h_2 = 0.6 \text{ m}, h_1 = 0.5 \text{ m}, f = 76 \text{ GHz}$$

└ Wave propagation

└ Reflection, diffraction and damping of plane waves

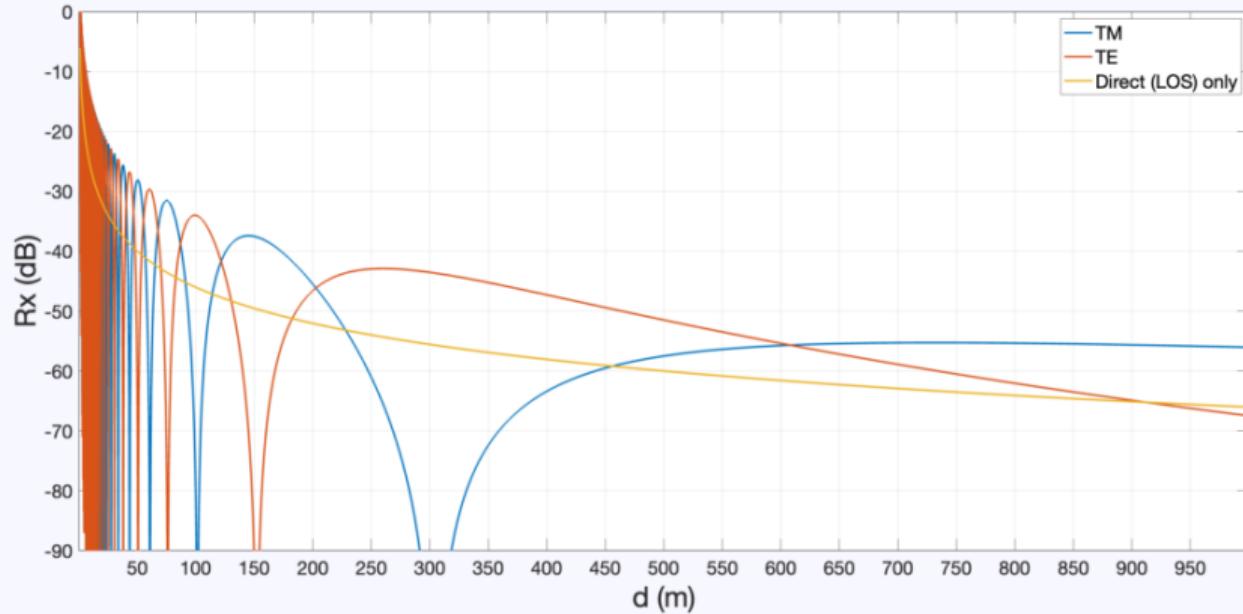
Multipath

Thus, the overall signal (assuming that the reflector is isotropic):

$$R_s = \frac{e^{-j2\pi f t_1}}{t_1} + r_p e^{-j2\pi f t_2} + r_p e^{-j2\pi f t_1} + r_p' e^{-j2\pi f t_2}$$

with $n = 2\sqrt{(h_2 - h_1)^2 + d^2}$, $\alpha = \sqrt{(h_2 + h_1)^2 + d^2}$ and $\beta = 2\sqrt{(h_2 + h_1)^2 + d^2}$, for $h_2 > h_1$.**Multipath**Considered is a reflector at $(0, d, -h_2)$, which is illuminated by source at $(0, 0, -h_1)$. The following paths exist:

- ▷ Source - reflector - source (SRS); t_1
- ▷ Source - reflector - boundary - source (SRBS); t_2
- ▷ Source - boundary - reflector - source (BRS); t_3
- ▷ Source - boundary - reflector - boundary - source (BRBS); t_4

Perfectly conducting surface

$$h_2 = 0.6 \text{ m}, h_1 = 0.5 \text{ m}, f = 76 \text{ GHz}$$

└ Wave propagation

└ Reflection, diffraction and damping of plane waves

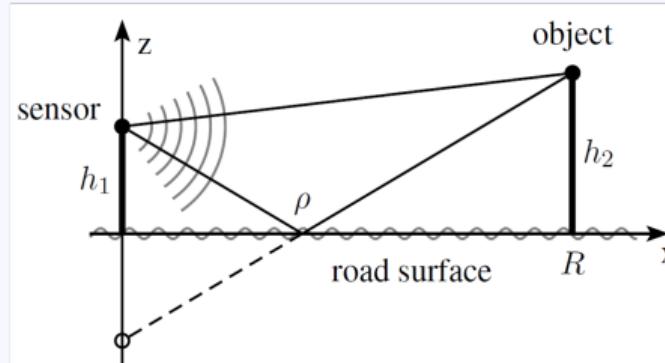
Multipath

Thus, the overall signal (assuming that the reflector is isotropic)

$$R_s = \frac{e^{-j2\pi f t}}{t_1} + r_{\text{ref}} \frac{e^{-j2\pi f t}}{t_2} + r_p e^{-j2\pi f t} + r'_p \frac{e^{-j2\pi f t}}{t_4}$$

with $t_1 = 2\sqrt{(R_s - h_1)^2 + d^2}$, $t_2 = \sqrt{(R_s + h_1)^2 + d^2}$ and $t_4 = 2\sqrt{(R_s + h_2)^2 + d^2}$, for $h_2 > h_1$.**Multipath**Considered is a reflector at $(0, d, -h_2)$, which is illuminated by source at $(0, 0, -h_1)$. The following paths exist:

- ▷ Source - reflector - source (SRS); r_1
- ▷ Source - reflector - boundary - source (SRBS); r_2
- ▷ Source - boundary - reflector - source (SBRBS); r_3
- ▷ Source - boundary - reflector - boundary - source (SBRBBS); r_4

Road Surface

Source: Büren, Yang, Characterization of Automotive Radar Targets from 22 to 29 GHz

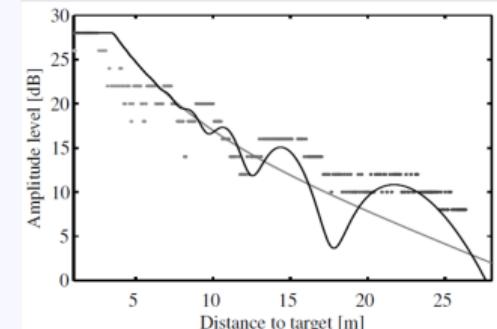


Fig. 4. Measured (dots) and simulated (line) radar amplitude of measurement with Opel Vectra as target

Reflection, diffraction and damping of plane waves

2.5 Reflection, diffraction and damping of plane waves

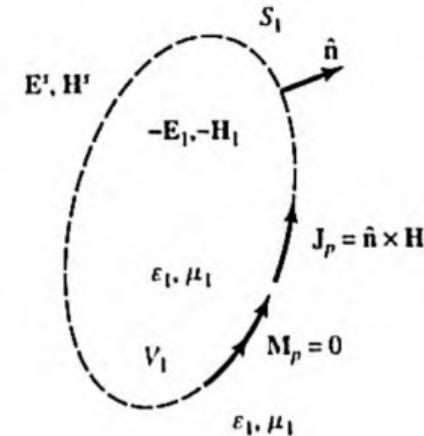
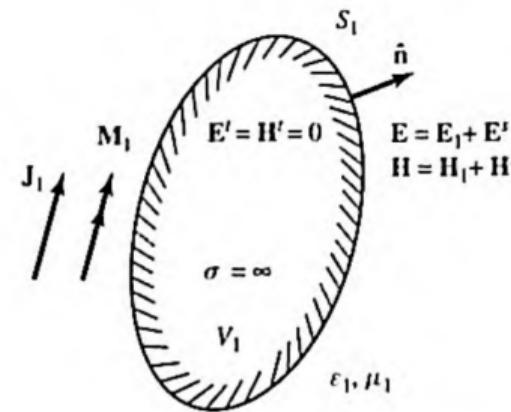
2.5.1 Reflection by a flat surface

2.5.2 Damping

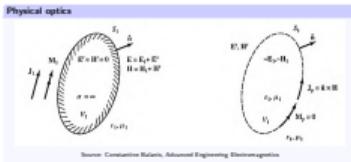
2.5.3 Multipath

2.5.4 Diffraction & Physical optics

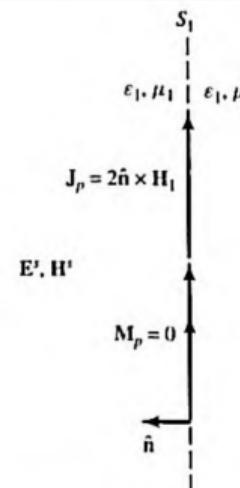
Physical optics



Source: Constantine Balanis, Advanced Engineering Electromagnetics



Physical optics



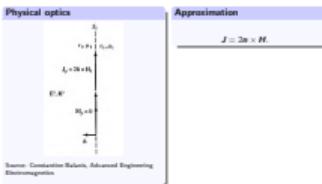
Source: Constantine Balanis, Advanced Engineering Electromagnetics

Approximation

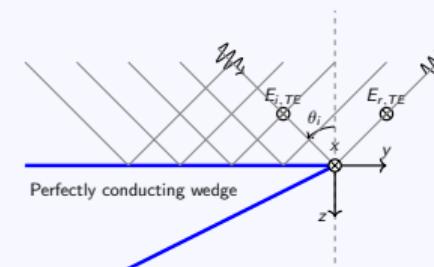
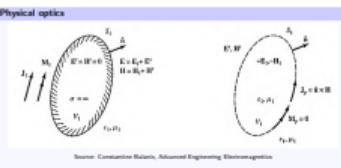
$$\mathbf{J} = 2\mathbf{n} \times \mathbf{H}$$

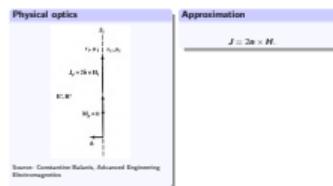
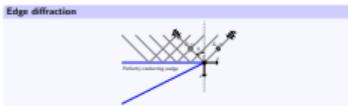
└ Wave propagation

└ Reflection, diffraction and damping of plane waves

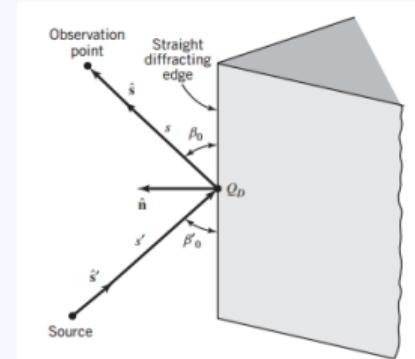
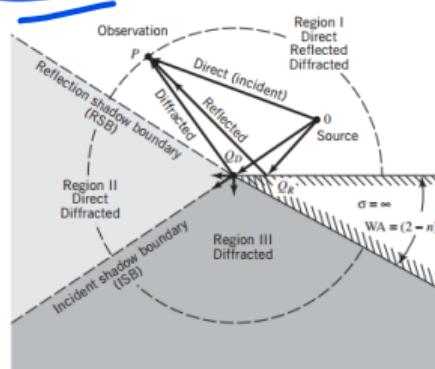


Edge diffraction

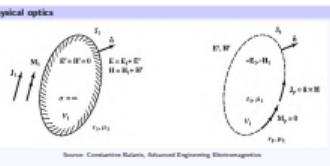




Edge diffraction



Source: Constantine Balanis, Advanced Engineering Electromagnetics

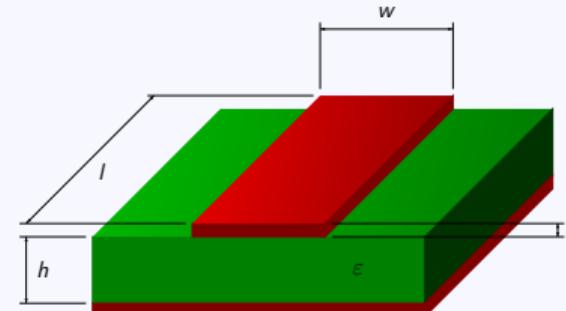


Source: Constantine Balanis, Advanced Engineering Electromagnetics

Wave propagation

- 2.1 Introduction
- 2.2 Maxwell's Equations
- 2.3 Electromagnetic waves
- 2.4 Fields of current distributions
- 2.5 Reflection, diffraction and damping of plane waves
- 2.6 Micro Strip lines, (coplanar) waveguides

Micro Strip lines



Figure



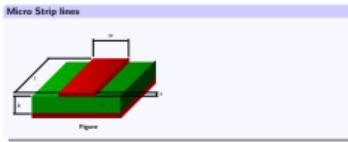
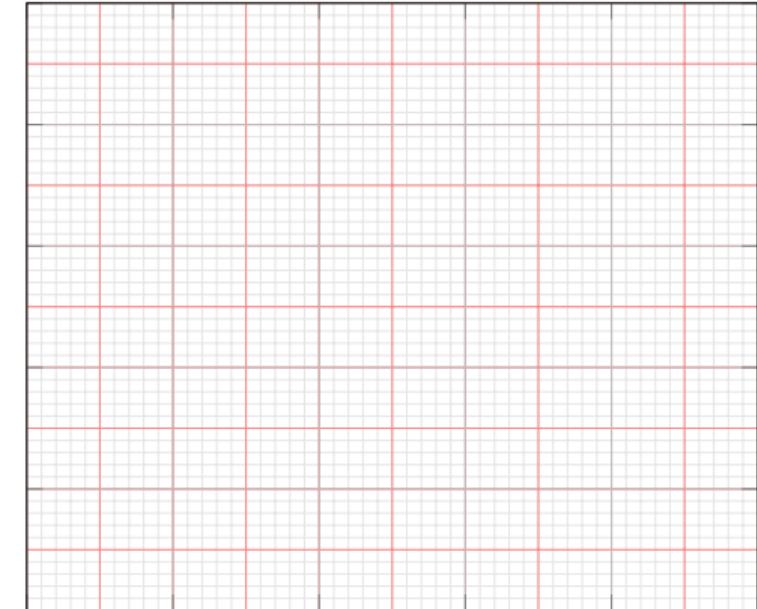
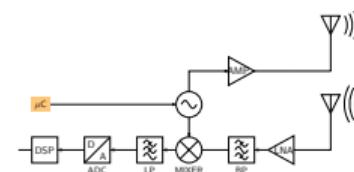
Coplanar waveguide



Figure

└ Wave propagation

└ Micro Strip lines, (coplanar) waveguides



Introduction

1. Introduction
2. Wave propagation
3. Block diagram

Block diagram

3.1 Introduction

3.2 Voltage controlled oscillator (VCO)

3.3 PLL

3.4 Amplifier

3.5 Antennas

3.6 Radar equation

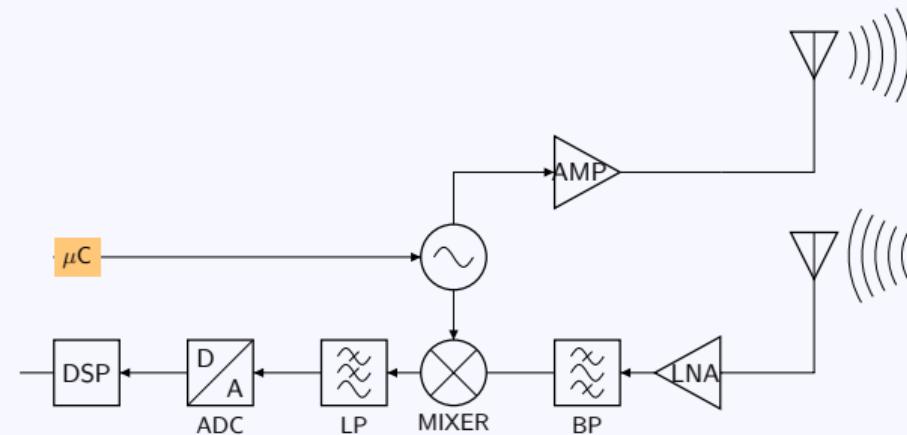
3.7 Low noise amplifier

3.8 Filter

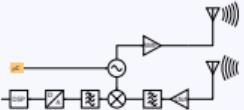
3.9 Mixer

3.10 ADC

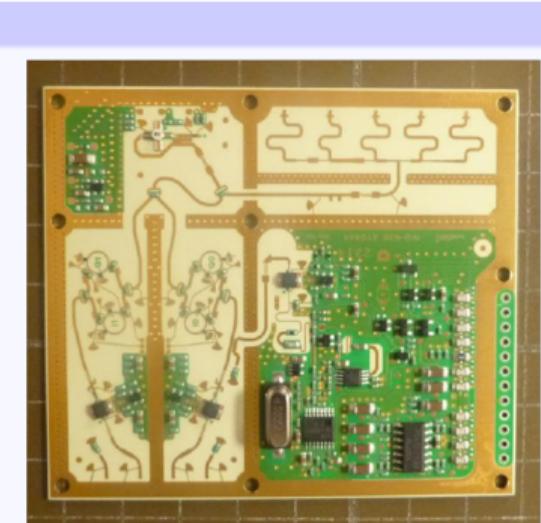
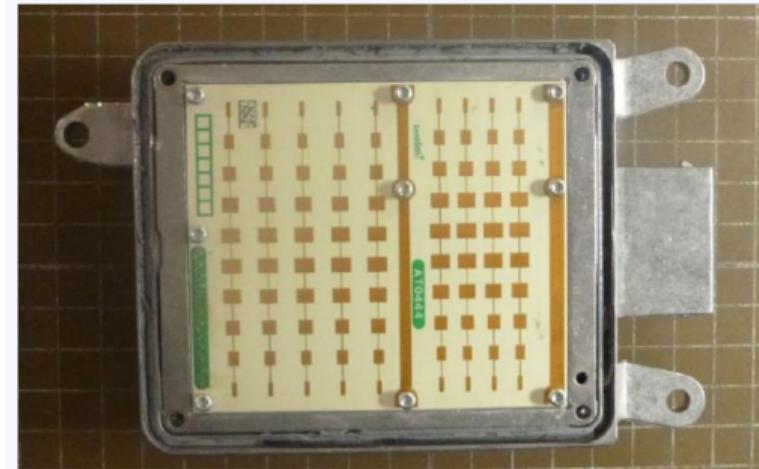
Blockdiagramm



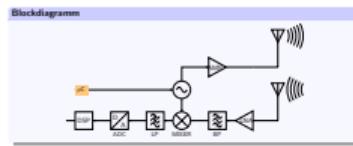
Blockdiagramm



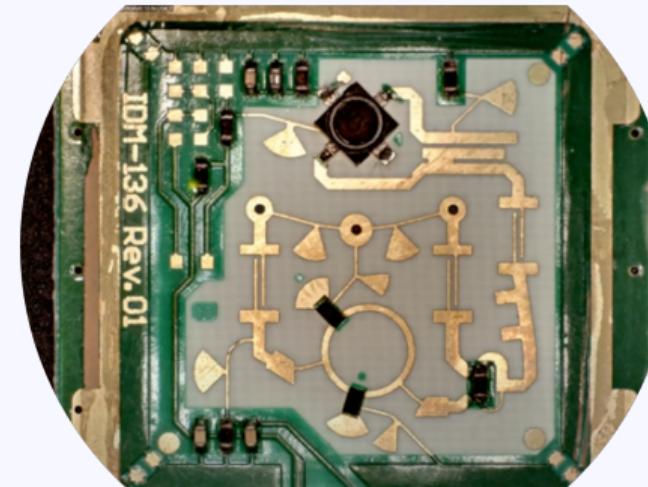
Hella SWA 1.5



Source: FCC



CDM 324



Source: [The Signal Path \(Youtube\)](#)

Block diagram

3.1 Introduction

3.2 Voltage controlled oscillator (VCO)

3.3 PLL

3.4 Amplifier

3.5 Antennas

3.6 Radar equation

3.7 Low noise amplifier

3.8 Filter

3.9 Mixer

3.10 ADC

Key performance indicators

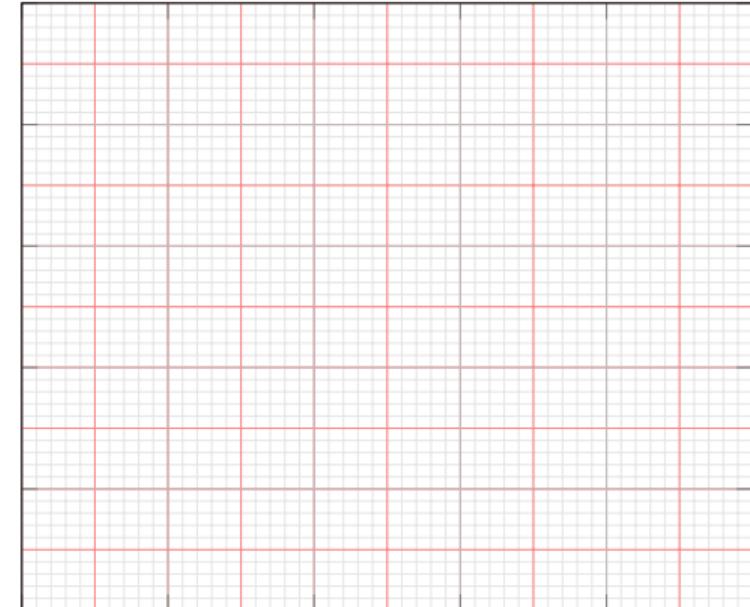
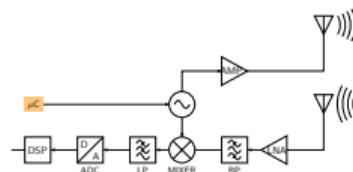
- ▷ Phase stability
- ▷ Large electrical tuning range
- ▷ Linearity of frequency versus control voltage
- ▷ Large gain factor
- ▷ Capability for accepting wideband modulation
- ▷ Low cost
- ▷ Load pulling

└ Block diagram

└ Voltage controlled oscillator (VCO)

Key performance indicators

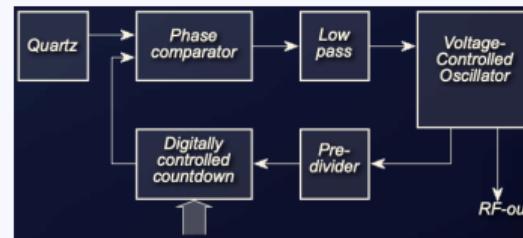
- ▷ Phase stability
- ▷ Large electrical tuning range
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- ▷ Large gain factor
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- ▷ Low cost
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Block diagram

- 3.1 Introduction
- 3.2 Voltage controlled oscillator (VCO)
- 3.3 PLL**
- 3.4 Amplifier
- 3.5 Antennas
- 3.6 Radar equation
- 3.7 Low noise amplifier
- 3.8 Filter
- 3.9 Mixer
- 3.10 ADC

PLL



Source: [Radartutorial](#)

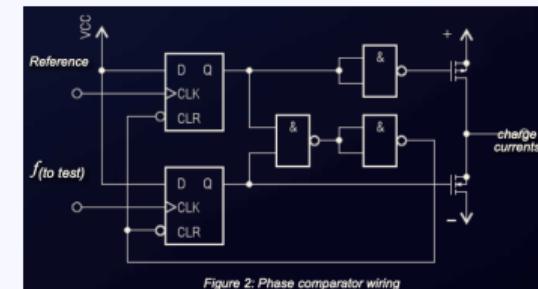
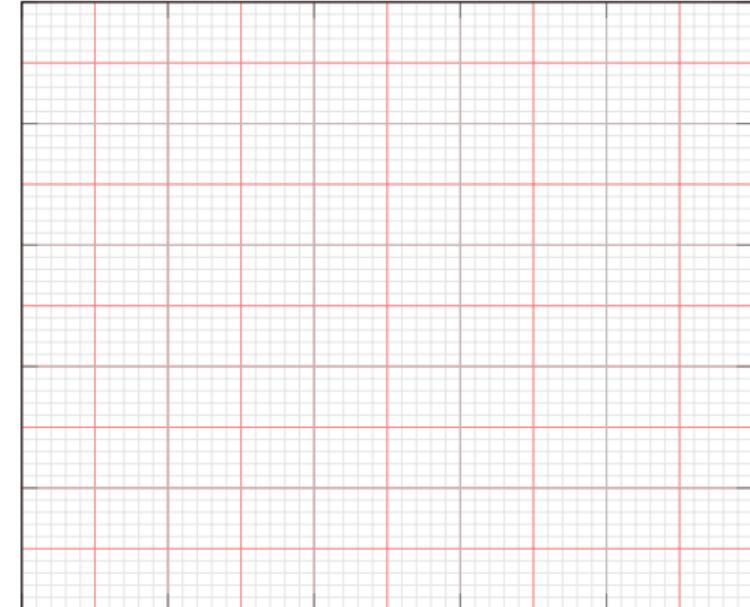
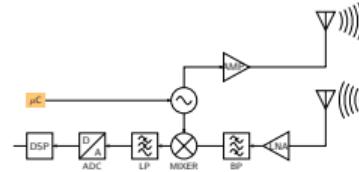


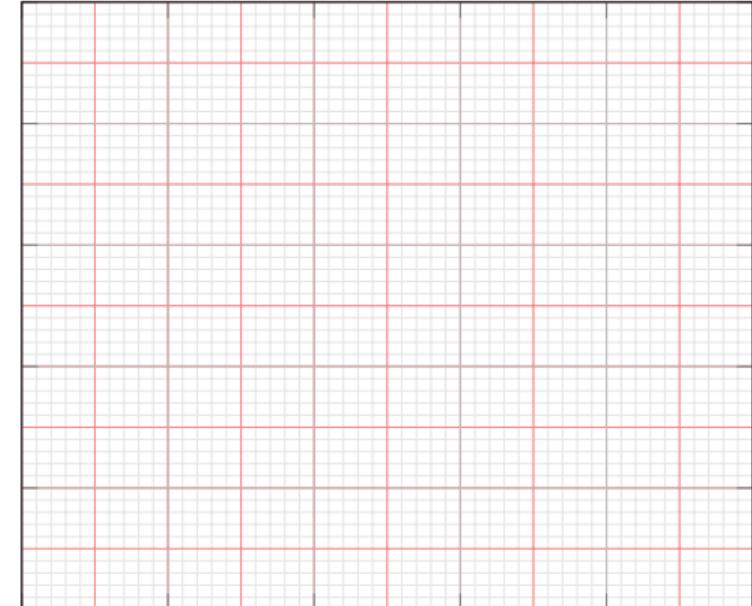
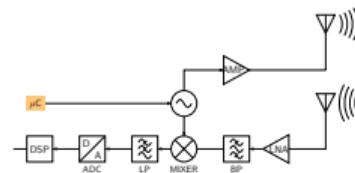
Figure 2: Phase comparator wiring

Source: [Radartutorial](#)



Block diagram

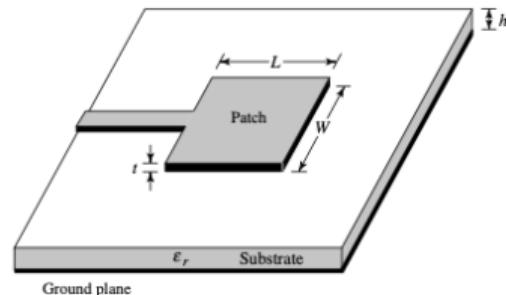
- 3.1 Introduction
- 3.2 Voltage controlled oscillator (VCO)
- 3.3 PLL
- 3.4 Amplifier**
- 3.5 Antennas
- 3.6 Radar equation
- 3.7 Low noise amplifier
- 3.8 Filter
- 3.9 Mixer
- 3.10 ADC



Block diagram

- 3.1 Introduction
- 3.2 Voltage controlled oscillator (VCO)
- 3.3 PLL
- 3.4 Amplifier
- 3.5 Antennas**
 - 3.5.1 Radar Cross Section
- 3.6 Radar equation
- 3.7 Low noise amplifier
- 3.8 Filter
- 3.9 Mixer
- 3.10 ADC

Example

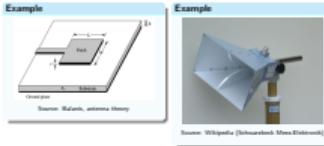


Source: Balanis, antenna theory

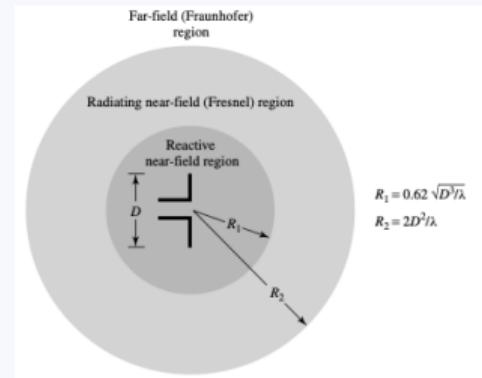
Example



Source: Wikipedia (Schwarzbeck Mess-Elektronik)

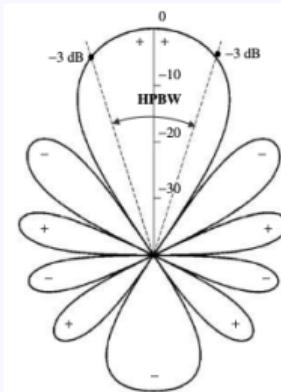


Far field and near field

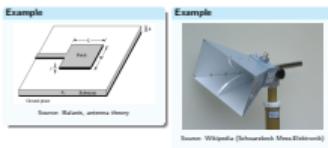
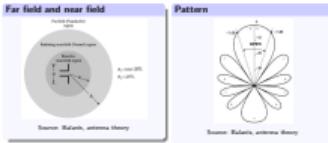


Source: Balanis, antenna theory

Pattern



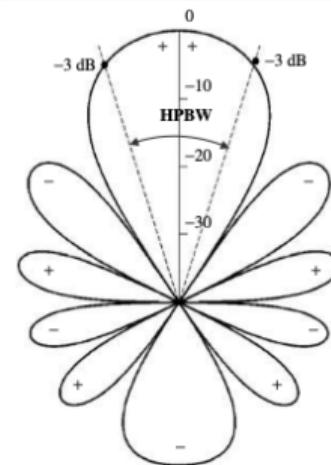
Source: Balanis, antenna theory



Properties

- ▷ Directivity: Ratio of radiation density in a given direction over that of an isotropic source
- ▷ Gain: Taking efficiency into account
- ▷ Polarization
- ▷ Bandwidth / Input resistance

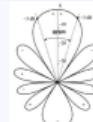
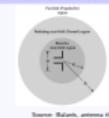
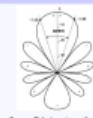
Pattern



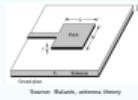
Source: Balanis, antenna theory

Properties

- ▷ Directivity: Ratio of radiation density in a given direction over that of an isotropic source
- ▷ Gain: Taking efficiency into account
- ▷ Polarization
- ▷ Bandwidth / Input resistance

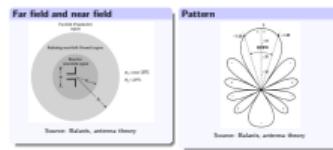
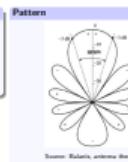
Pattern**Far field and near field****Pattern****Types**

- ▷ Micro strip (e.g. Bosch MRR)
- ▷ Surface integrated waveguides (e.g. Aptiv SRR 5)
- ▷ Air-Waveguides (e.g. Aptiv SRR 7)

Example**Example**

- Types**
- ▷ Micro strip (e.g. Bosch MBF)
 - ▷ Surface integrated waveguides (e.g. Aptiv SRR 5)
 - ▷ Air-Waveguides (e.g. Aptiv SRR 7)

- Properties**
- ▷ Directivity: Ratio of radiation density in a given direction over that of an isotropic radiator.
 - ▷ Gain: Taking efficiency into account
 - ▷ Polarization
 - ▷ Bandwidth / Input resistance



Directivity

Antenna	Directivity (logarithmic scale)	Directivity (linear scale)
Isotropic radiator	0 dBi	1
Hertzian dipole	1.76 dBi	1.5
$\lambda/2$ -dipole	2.15 dBi	1.64
$\lambda/4$ -monopole	5.15 dBi	3.28
Patch antenna	6 dBi	4
Logarithmic periodic dipole antenna	7 dBi	5
Horn antenna	20 dBi	100
Parabolic dish antenna	> 30 dBi	> 1000

Source: Gustrau, F. (2012). Rf and microwave engineering : Fundamentals of wireless communications. John Wiley & Sons, Incorporated.

Directivity		
Antenna	Directivity (directive gain)	Directivity (directivity)
Isotropic source	1.00	1.00
Horizontal dipole	1.176 dB	1.5
Vertical dipole	-1.176 dB	1.5
λ/2 monopole	3.156 dB	3.28
Patch antenna	9.0 dB	4
Log-periodic dipole array	2.0 dB	1
dipole antenna	20.0 dB	100
Horizontal horn	>30 dB	>1000
Parabolic dish antenna		

Source: Gauthier, P. (2012). RF and microwave engineering - Fundamentals of wireless communications. John Wiley & Sons, Incorporated.

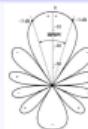
Types

- ▷ Micro strip (e.g. Bosch MRR)
- ▷ Surface integrated waveguides (e.g. Aptiv SRR 5)
- ▷ Air-Waveguides (e.g. Aptiv SRR 7)

Properties

- ▷ Directivity: Ratio of radiation density in a given direction over that of an isotropic source
- ▷ Gain: Taking efficiency into account
- ▷ Polarization
- ▷ Bandwidth / Input resistance

Pattern



Source: Balanis, antenna theory

Far field

For a nonisotropic transmitting antenna, the power density in the direction ϑ_t, φ_t can be written as^a:

$$W_t = \frac{P_t \mathbf{G}_t(\vartheta_t, \varphi_t)}{4\pi R^2} = \mathbf{e}_t \frac{P_t D_t(\vartheta_t, \varphi_t)}{4\pi R^2}$$

^aBalanis, Antenna Theory, page 95

Far field

For a nonisotropic transmitting antenna, the power density in the direction ϑ_t, φ_t can be written as^a:

$$W_t = \frac{P_t G_t(\vartheta_t, \varphi_t)}{4\pi R^2} = \epsilon_t \frac{P_t D_t(\vartheta_t, \varphi_t)}{4\pi R^2}$$

(Balanis, Antenna Theory, page 95)

Directivity

Antenna	Directive (isotropic scale)	Directive (dBi scale)
Isoelectric radiator	1.00	1
Horizontal dipole	0.90	-0.10
Vertical dipole	2.15 (D)	1.00
z-isotropic	3.15 (D)	3.28
Plane wave	4.00	4
Logarithmic periodic	7.00	7
Aperture		
Hour antenna	20 dBi	100
Parabolic dish	> 30 dBi	> 1000
Planar horn		

Source: Ghorbani, F. (2012). RF and microwave engineering - Fundamentals of wireless communications. John Wiley & Sons, Incorporated.

Far field

For a nonisotropic receiving antenna, the from direction ϑ_r, φ_r received power can be written as^a:

$$W_r = \epsilon_r D_r(\vartheta_r, \varphi_r) \frac{\lambda^2}{4\pi} W_t$$

For reflection and polarization-matched antennas aligned for maximum directional radiation and reception, one gets

$$\frac{P_r}{P_t} = \left(\frac{\lambda}{4\pi R} \right)^2 G_{\max,t} G_{\max,r}$$

^aBalanis, Antenna Theory, page 95

Types

- ▷ Micro strip (e.g. Bosch MMW)
- ▷ Surface integrated waveguides (e.g. Aptiv SRR 5)
- ▷ Air-Waveguides (e.g. Aptiv SRR 7)

Far field

For a nonisotropic receiving antenna, the front direction θ_0, φ_0 , received power can be written as⁴⁶

$$W_r = \epsilon_0 D_r(\theta_0, \varphi_0) \frac{\lambda^2}{4\pi R} W_t$$

For reflection and polarization-matched antennas aligned for maximum directional radiation and reception, one gets

$$\frac{P_r}{P_t} = \left(\frac{\lambda}{4\pi R} \right)^2 G_{max,t} G_{max,r}$$

⁴⁶Stalans, Antenna Theory, page 95

Far field

For a nonisotropic transmitting antenna, the power density in the direction θ_0, φ_0 , can be written as⁴⁷

$$W_t = \frac{P_t G_t(\theta_0, \varphi_0)}{4\pi R^2} = \epsilon_0 \frac{P_t D_t(\theta_0, \varphi_0)}{4\pi R^2}$$

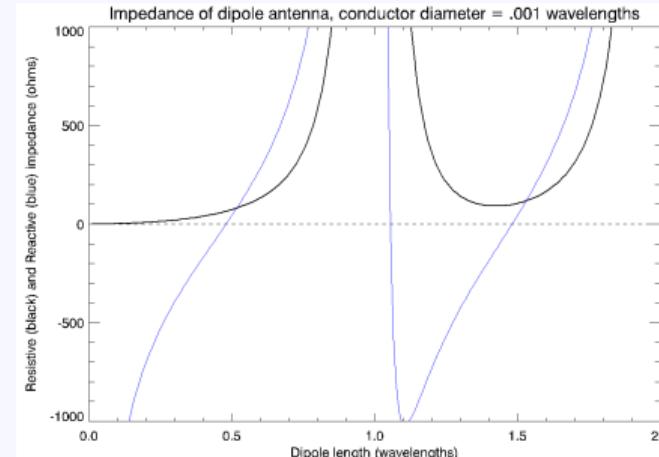
⁴⁷Stalans, Antenna Theory, page 95

Directivity

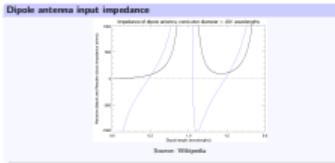
Antennas	Directive (directive scale)	Directive (direct scale)
Isotropic radiator	0.00	-
Half-wave dipole	1.00	1.5
1/2-dipole	2.15x10 ⁻³	1.00
1/4-dipole	3.15x10 ⁻³	1.00
Point antenna	0.00	4
Lognorm. parabolic	7.00	7
Log. spiral antenna	20.00	100
Half-wavelength dipole	> 10 ³	> 1000
Parabolic horn antenna	-	-

Source: Gauthier, P. (2012). RF and microwave engineering - Fundamentals of wireless communications. John Wiley & Sons, Incorporated.

Dipole antenna input impedance



Source: Wikipedia



Far field

For a nonisotropic receiving antenna, the from direction θ_1, φ_1 received power can be written as¹⁴:

$$W_r = \epsilon_0 D_1(\theta_1, \varphi_1) \frac{\lambda^2}{4\pi} \mathbf{W}_r$$

For reflection and polarization-matched antennas aligned for maximum directional radiation and reception, one gets

$$\frac{P_r}{P_t} = \left(\frac{\lambda}{4\pi R} \right)^2 G_{max,r} G_{max,t}$$

¹⁴Balanis, Antenna Theory, page 95

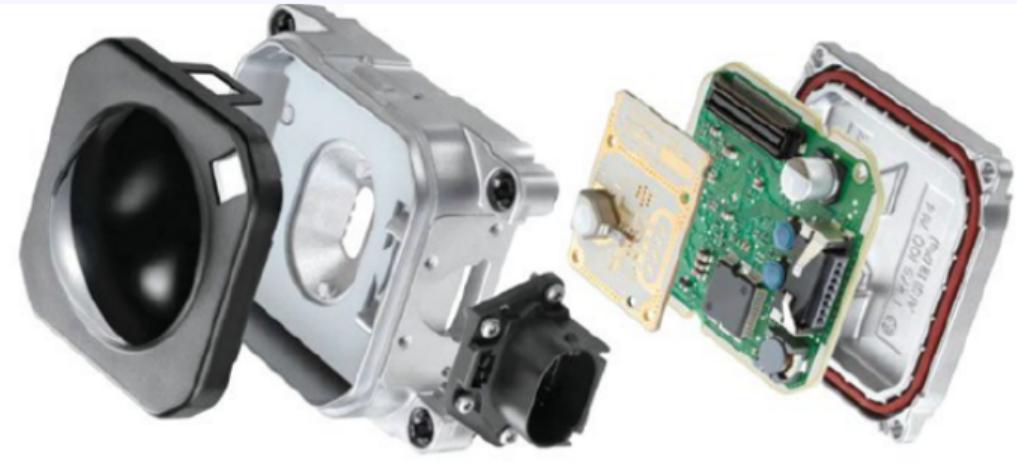
Far field

For a nonisotropic transmitting antenna, the power density in the direction θ_1, φ_1 can be written as¹⁴:

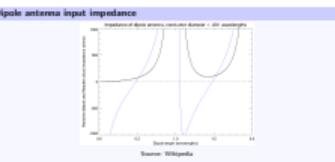
$$W_t = \frac{P_t G_t(\theta_1, \varphi_1)}{4\pi R^2} = \kappa \frac{P_t D_t(\theta_1, \varphi_1)}{4\pi R^2}$$

¹⁴Balanis, Antenna Theory, page 95

Bosch MRR



Source: Handbuch Fahrerassistenzsysteme



Far field

For a nonisotropic receiving antenna, the from direction θ_0, φ_0 received power can be written as:

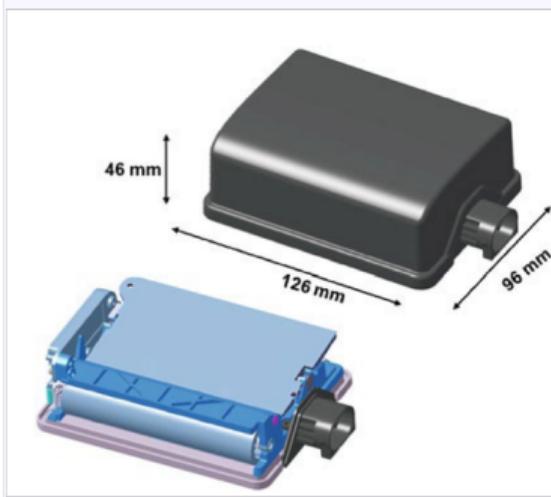
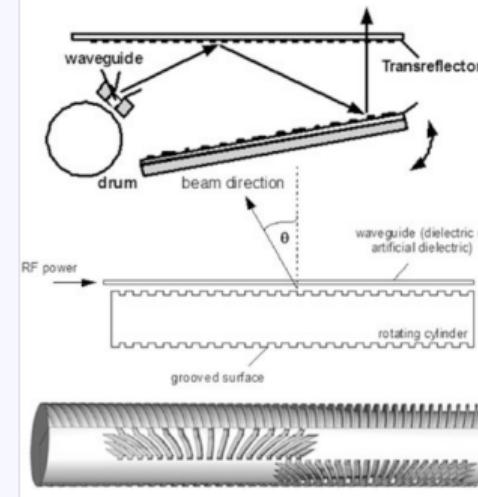
$$W_r = \epsilon_0 D_0(\theta_0, \varphi_0) \frac{\lambda^2}{4\pi} \mathbf{W}_r$$

For reflection and polarization-matched antennas aligned for maximum directional radiation and reception, one gets

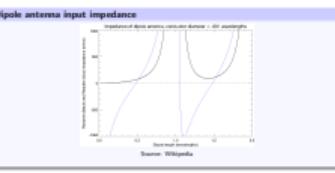
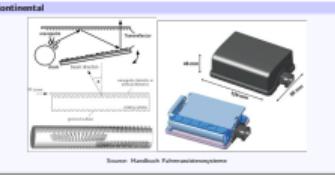
$$\frac{P_r}{P_t} = \left(\frac{\lambda}{4\pi R} \right)^2 G_{max,r} G_{max,t}$$

*Stiles, Antenna Theory, page 95

Continental



Source: Handbuch Fahrerassistenzsysteme



Continental ARS 4B



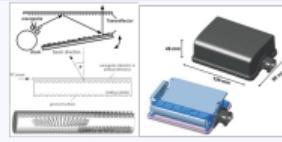
E.g. used in Tesla 3. Source: FCC

Continental ARS 4B



E.g. used in Tesla S. Source: FCC

Continental



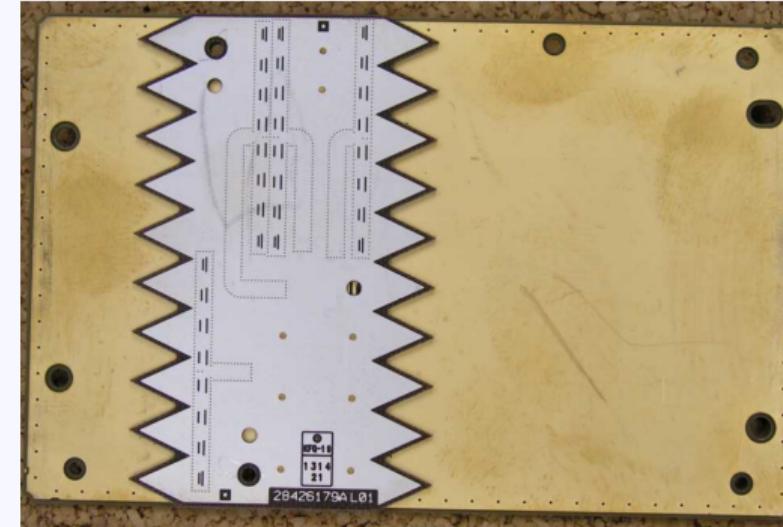
Source: Handbuch Fahrerassistenzsysteme

Bosch MRR



Source: Handbuch Fahrerassistenzsysteme

Delphi (APTIV) SRR2



SRR 5 (Source: FCC)

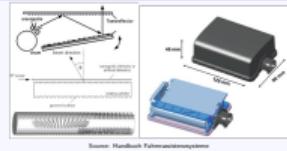
Delphi (APTV) SRR2



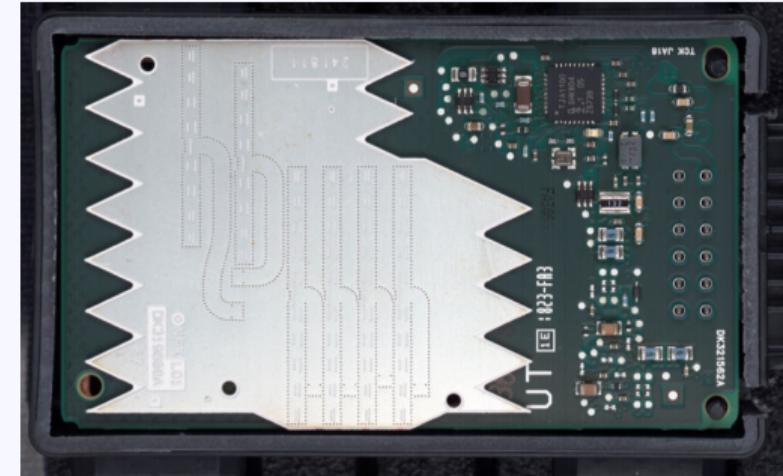
Continental ARS 4B



Continental



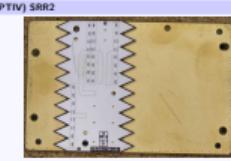
Aptiv SRR



SRR 5 (Source: FCC)



ISRR 5 [Source: FCC]

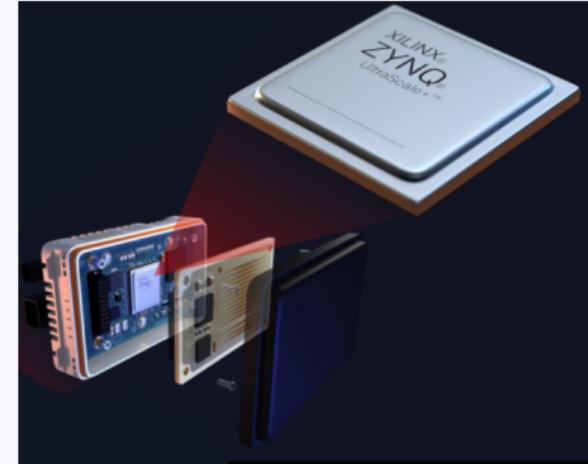


ISRR 5 [Source: FCC]



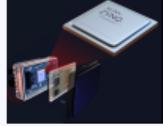
Rg used in Theta 5. Source: FCC

Continental ARS 540



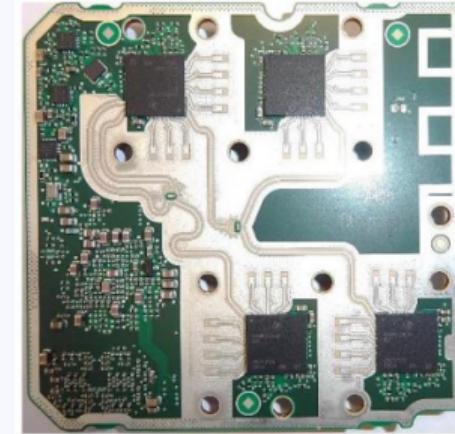
ARS 540 with 12x TX + 16x RX antennas. Source: Xilinx (modifiziert)

Continental ARS 540



ARS 540 with 12x TX + 16x RX antennas. Source: Xlens (modified)

Continental



Aptiv SRR



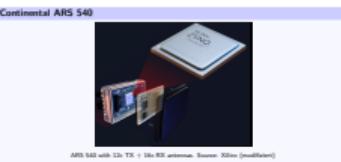
SRR 1 [Source: FCC]

Delphi (APTV) SRR2

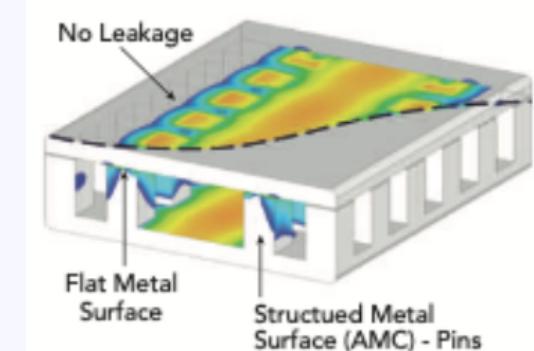
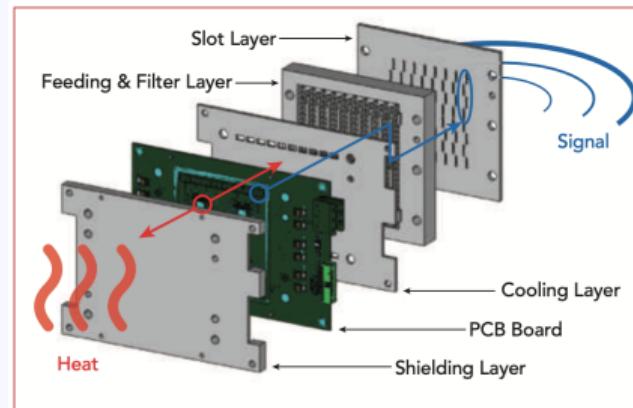


SRR 2 [Source: FCC]

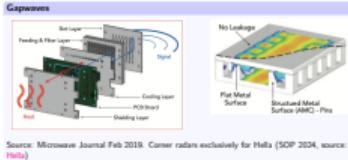
Source: FCC. See Huber+Suhner as well.



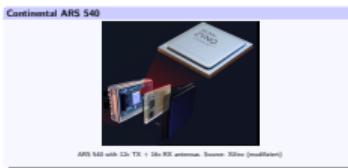
Gapwaves



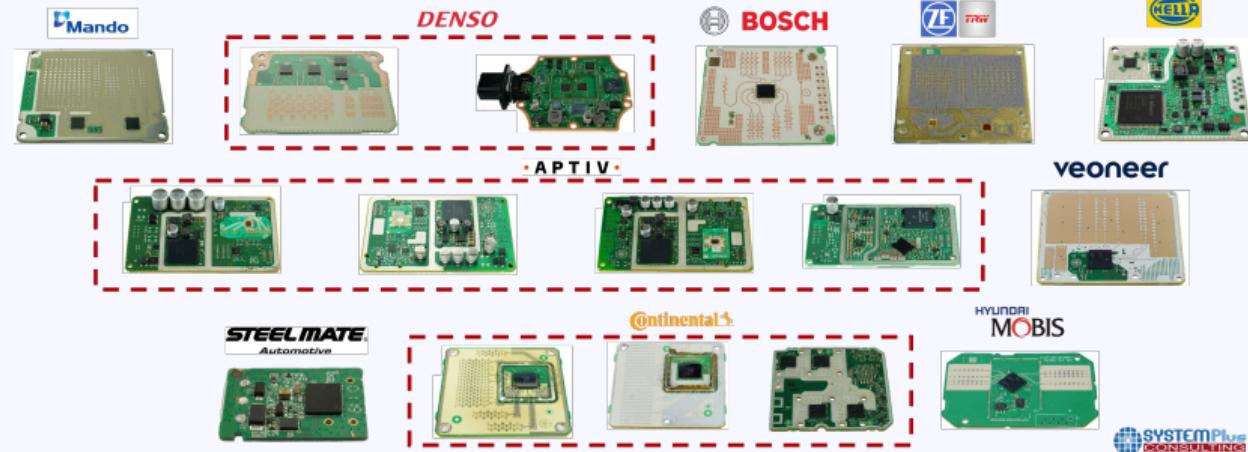
Source: Microwave Journal Feb 2019. Corner radars exclusively for Hella (SOP 2024, source: [Hella](#))

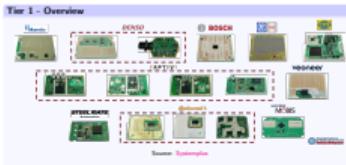


Source: FCC. See Huber+Suhner as well.

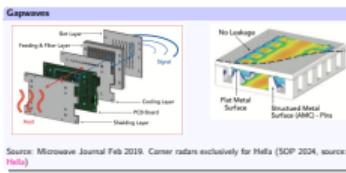


Tier 1 - Overview





EIRP



Source: Microwave Journal Feb 2019. Corner radars exclusively for Hella (SDP 2014, source: Hella.)

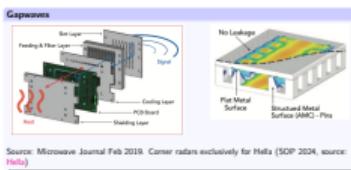


Source: FCC. See Hella + Schenker as well.

EIRP

Reports & Links

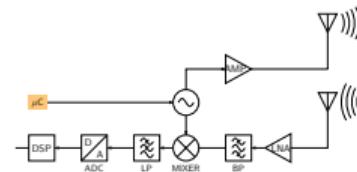
Comprehensive Survey of 77, 79 GHz Automotive Radar Companies – Sensors and ICs



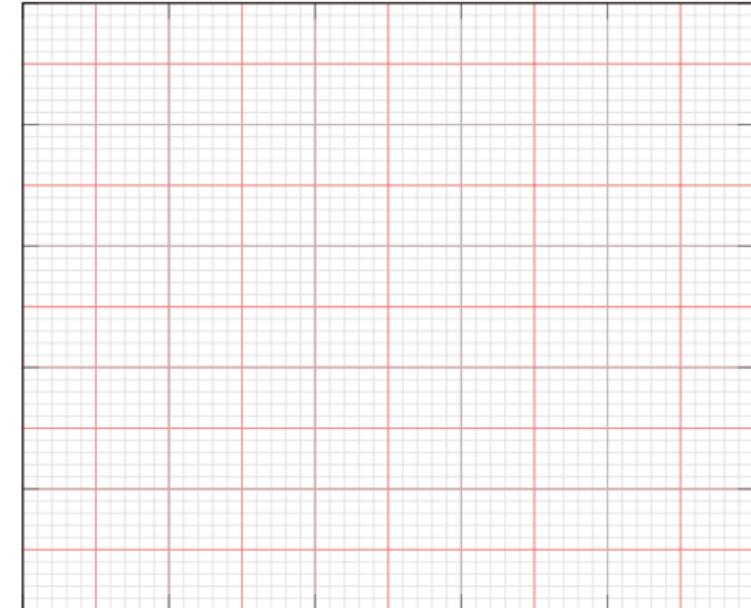
Source: Microwave Journal Feb 2019. Corner radars exclusively for Hella (SOP 2024, source: Hella)

Reports & Links

Comprehensive Survey of 77+ 79 GHz Automotive Radar Companies – Sensors and ICs



EIRP



Tier 1 - Overview



Antennas

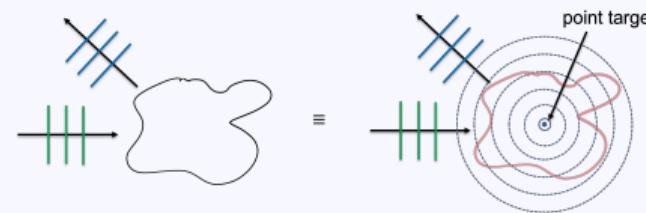
3.5 Antennas

3.5.1 Radar Cross Section

RCS

The RCS is the hypothetical area required to intercept the transmitted power density at the target such that, if the total intercepted power were re-radiated isotropically, the power density actually observed at the receiver would be produced:

$$\sigma = \lim_{R \rightarrow \infty} 4\pi R^2 \frac{P_s}{P_i} = \lim_{R \rightarrow \infty} 4\pi R^2 \frac{|\mathbf{E}_s|}{|\mathbf{E}_i|}.$$



Unit: Area (m^2). Usually given in dBsm (power: Use $10 \log_{10} x$).

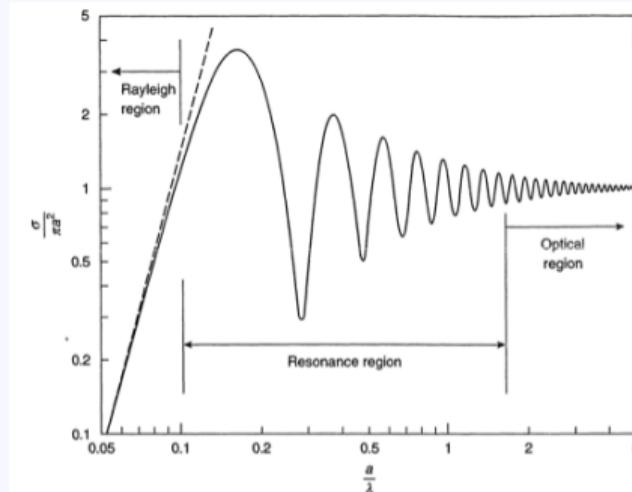
RCS
The RCS is the hypothetical area required to intercept the transmitted power density at the target such that, if the total intercepted power were re-radiated isotropically, the power density actually observed at the receiver would be produced produced:

$$\sigma = \lim_{R \rightarrow \infty} 4\pi R^2 \frac{P_r}{P_t} = \lim_{R \rightarrow \infty} 4\pi R^2 \frac{|E_r|}{|E_t|}$$



Unit: Area (m^2). Usually given in dBsm (power: use $10 \log_{10} \sigma$).

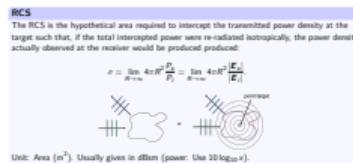
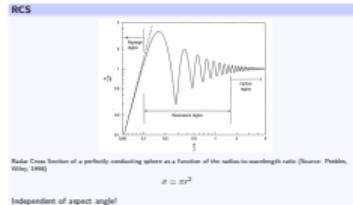
RCS



Radar Cross Section of a perfectly conducting sphere as a function of the radius-to-wavelength ratio (Source: Peebles, Wiley, 1998)

$$\sigma = \pi r^2$$

Independent of aspect angle!



RCS

The maximum of the RCS of a perfectly conducting flat plate:

$$\sigma = 4\pi w^2 h^2 / \lambda^2$$

This is only the maximum (perpendicular). Rule of thumb: License plate up to 30 dBsm at 76 GHz (far field, near field: reflect energy away) Example:

$$\sigma = 4\pi 0.3^2 0.1^2 / 0.0039^2 \approx 743$$

and thus

$$dBsm = 10 \log_{10} 743 = 28.7 \text{ dB}$$

RCS

The maximum of the RCS of a perfectly conducting flat plate:

$$\sigma = 4\pi a^2 \lambda^2 / \lambda^2$$

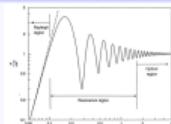
This is only the maximum (perpendicular) Rule of thumb: License plate up to 30 dBsm at 75 GHz (far field, near field: reflect energy away) Example:

$$\sigma = 4\pi(0.17)^2 / 0.0039^2 \approx 743$$

and thus

$$\text{dBsm} = 10\log_{10}(743) \approx 28.7 \text{ dB}$$

RCS



Radar Cross Section of a perfectly conducting sphere as a function of the radius-to-wavelength ratio [Source: Pergamon, May 1960]

$$\sigma = \pi r^2$$

Independent of aspect angle!

RCS

The RCS is the hypothetical area required to intercept the transmitted power density at the target such that, if the total intercepted power were re-radiated isotropically, the power density actually observed at the receiver would be produced:

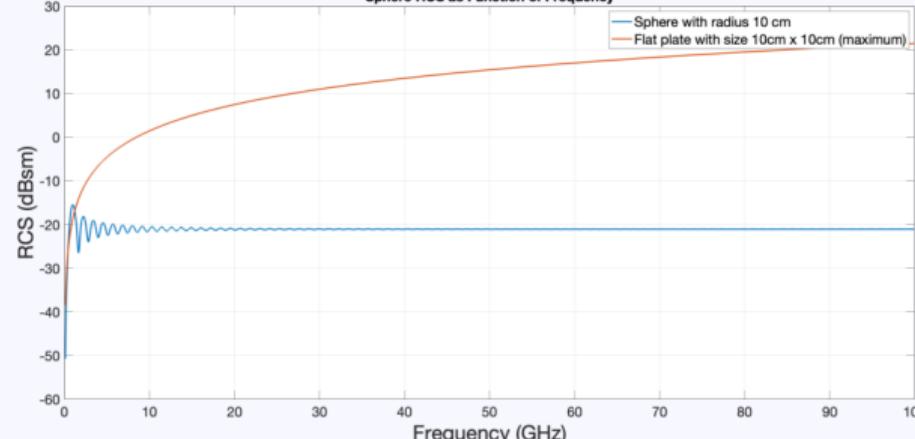
$$\sigma = \lim_{\theta \rightarrow \infty} 4\pi R^2 \frac{P_t}{P_r} = \lim_{\theta \rightarrow \infty} 4\pi R^2 \frac{|E_r|}{|E_t|}$$



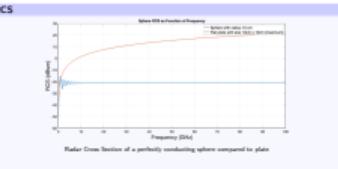
Unit: Area (m^2). Usually given in dBsm (power: use $10\log_{10}(x)$).

RCS

Sphere RCS as Function of Frequency



Radar Cross Section of a perfectly conducting sphere compared to plate



RCS

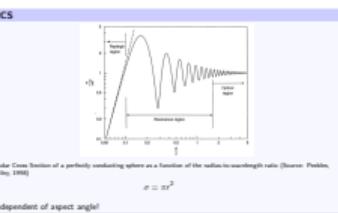
The maximum of the RCS of a perfectly conducting flat plate:

$$\sigma = 4\pi w^2 \lambda^2 / \lambda^2$$

This is only the maximum (perpendicular). Rule of thumb: License plate up to 30 dBsm at 76 GHz (far field, near field: reflect energy away) Example:

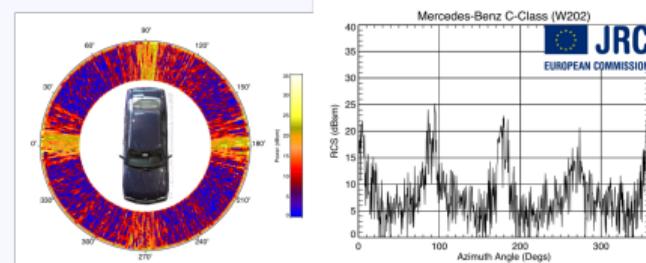
$$\sigma = 4\pi(0.3)^2 / 0.003^2 \approx 743$$

and thus

$$dBsm = 10 \log_{10} 743 = 20.7 dB$$


RCS

The received power usually fluctuates. The figure is for a single frequency (CW) and performed at 76 GHz for a Mercedes C Class. Note: Pedestrian approx. -8 dBsm with strong fluctuation



Block diagram

- 3.1 Introduction
- 3.2 Voltage controlled oscillator (VCO)
- 3.3 PLL
- 3.4 Amplifier
- 3.5 Antennas
- 3.6 Radar equation**
- 3.7 Low noise amplifier
- 3.8 Filter
- 3.9 Mixer
- 3.10 ADC

Radar equation

The received power P_r is given by the equation

$$P_r = \frac{P_t G_t G_r \lambda^2 \sigma F^4}{(4\pi)^3 R^4},$$

where P_t is the transmitter power, G_t is the antenna gain of the transmitting antenna, λ is the transmitted wavelength, G_r is the gain of receiving antenna, σ is the radar cross section of the target, F is the pattern propagation factor and R is distance from the radar to the target (transmitter and receiver at the same location).

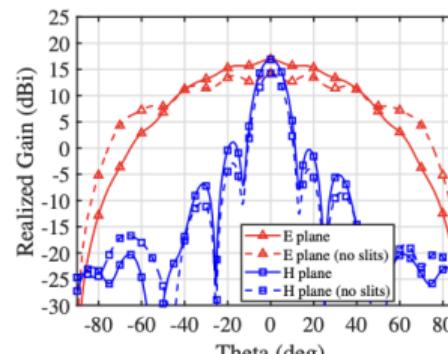
Radar equationThe received power P_r is given by the equation

$$P_r = \frac{P_t G_t G_r \lambda^2 F^2}{(4\pi)^2 R^4}$$

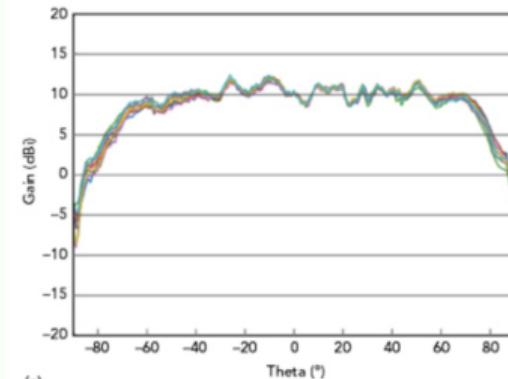
where P_t is the transmitter power, G_t is the antenna gain of the transmitting antenna, λ is the operating wavelength, G_r is the gain of receiving antenna, r is the radar cross section of the target, F is the pattern proportion factor and R is distance from the radar to the target (transmitter and receiver at the same location).

Exercise (#3.1)

Assume that a radar has a maximum range of 100 m at boresight for a 20 dBsm target and the patterns shown below. Plot the coverage for a 20 dBsm and a 0 dBsm target.



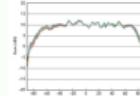
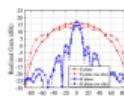
Pattern. Source: Ren et al. Gapwaveguide Automotive Imaging Radar Antenna With Launcher in Package Technology



Pattern. Source: [Microwave Journal](#)

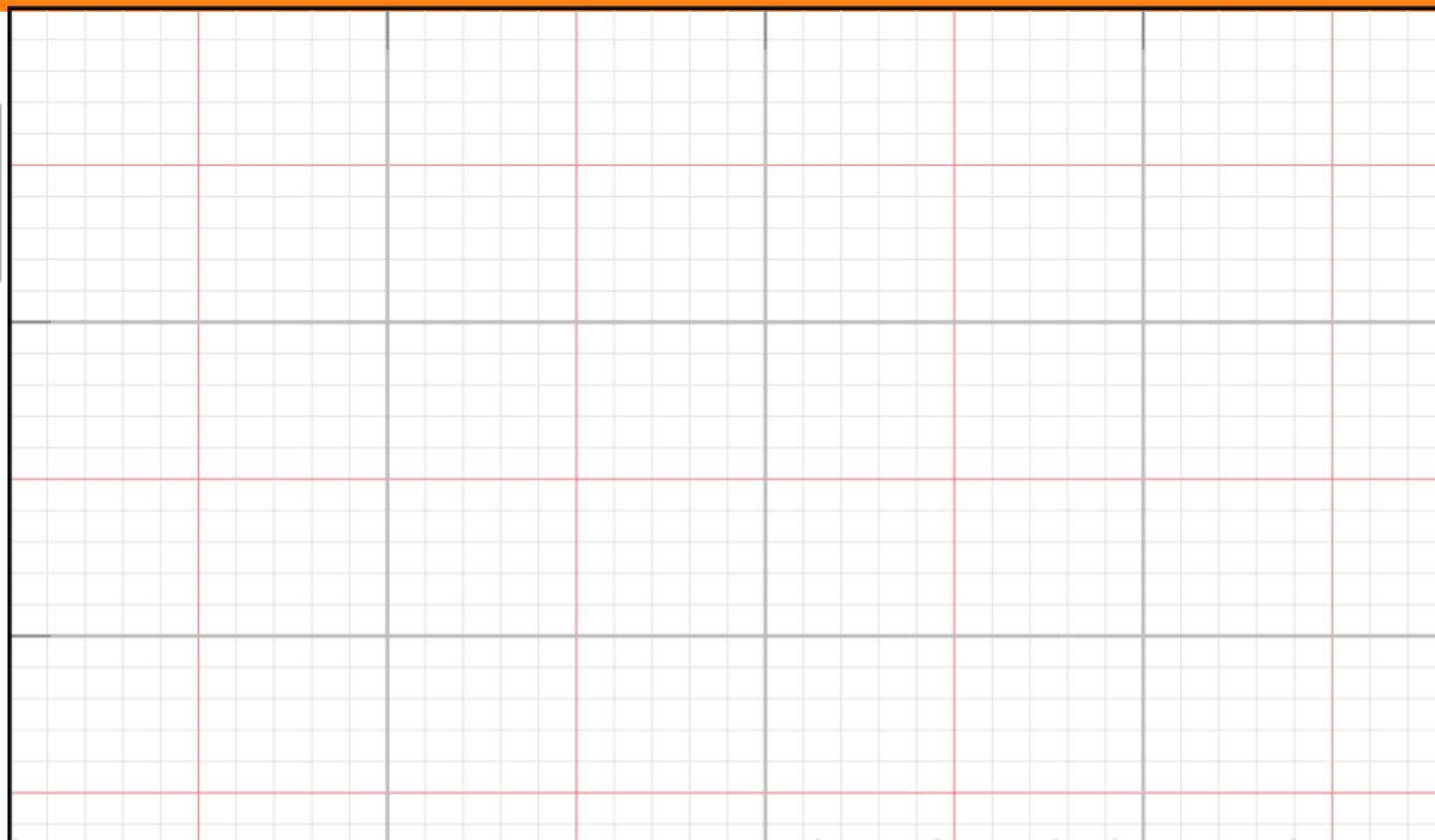
Exercise (#3.1)

Assume that a radar has a maximum range of 100m at boresight for a 20 dBm target and the patterns shown below. Plot the coverage for a 20 dBm and a 0 dBm target.



Patent Source: Ren et al. Capillarylike
Automotive Imaging Radar Antennas With
Lambertian-to-Fresnel Technology

Patent Source: Microwave Journal



Block diagram

- 3.1 Introduction
- 3.2 Voltage controlled oscillator (VCO)
- 3.3 PLL
- 3.4 Amplifier
- 3.5 Antennas
- 3.6 Radar equation
- 3.7 Low noise amplifier**
- 3.8 Filter
- 3.9 Mixer
- 3.10 ADC

Noise figure

The noise figure F is given by

$$F = \frac{\text{SNR}_{\text{in}}}{\text{SNR}_{\text{out}}} = \frac{1}{G} \cdot \frac{N_2}{N_1},$$

with gain factor G of the amplifier..

Cascading

If several two-ports are connected in series, the following applies according to the so-called **Friis formula**:

$$F_{\text{total}} = \frac{\text{SNR}_{\text{in}}}{\text{SNR}_{\text{out}}} = 1 + (F_1 - 1) + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1 \cdot G_2} + \cdots + \frac{F_n - 1}{G_1 G_2 G_3 \cdots G_{n-1}}$$

Placing

- ▷ Placing a filter, like any a device, before the first LNA negatively impacts system noise figure. Ideal position for a filter is after the LNA to preserve noise figure integrity
 - ▷ If filter is placed after the LNA, risk exist that strong out-of-band signals may cause LNA compression
 - ▷ See [TI application guide](#) as well.

- └ Block diagram
- └ Low noise amplifier

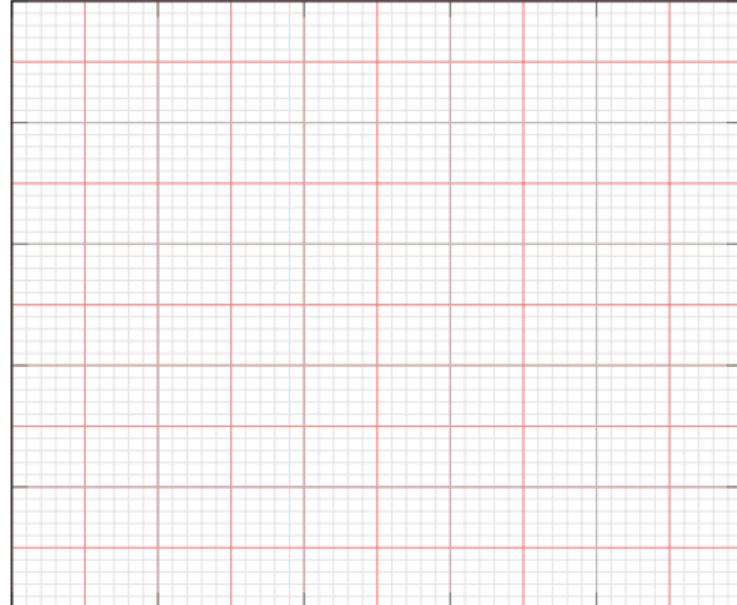
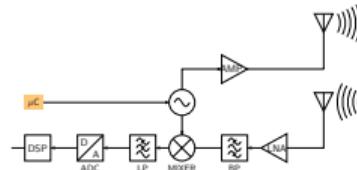
Cascading

If several two-ports are connected in series, the following applies according to the so-called Friis formula:

$$F_{\text{total}} = \frac{\text{SNR}_{\text{out}}}{\text{SNR}_{\text{in}}} \approx 1 + (F_1 - 1) + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1 G_2} + \dots + \frac{F_n - 1}{G_1 G_2 G_3 \dots G_{n-1}}$$

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**Noise figure**

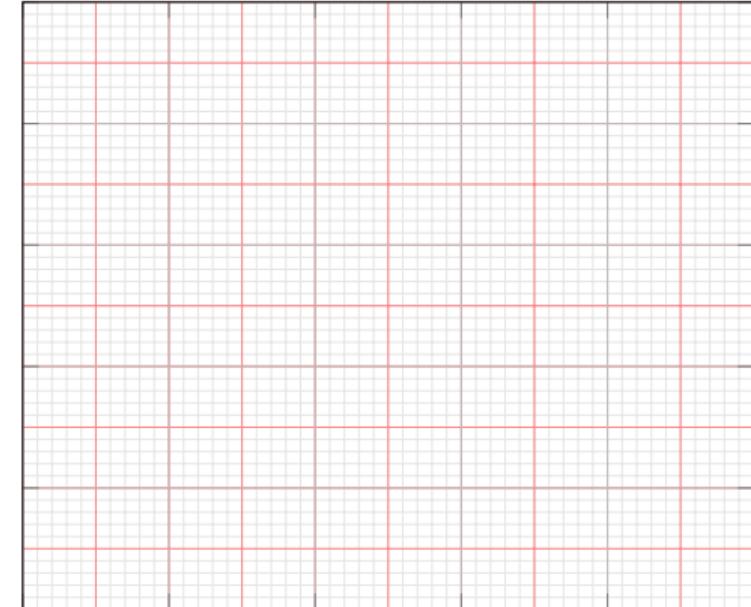
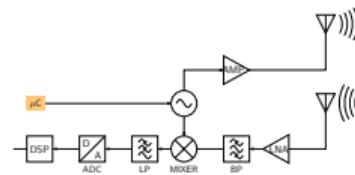
The noise figure F is given by

$$F = \frac{\text{SNR}_{\text{out}}}{\text{SNR}_{\text{in}}} = \frac{1}{G} \frac{N_0}{N}$$

with gain factor G of the amplifier...

Block diagram

- 3.1 Introduction
- 3.2 Voltage controlled oscillator (VCO)
- 3.3 PLL
- 3.4 Amplifier
- 3.5 Antennas
- 3.6 Radar equation
- 3.7 Low noise amplifier
- 3.8 Filter**
- 3.9 Mixer
- 3.10 ADC



Block diagram

- 3.1 Introduction
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Frequency conversion

$$\cos(\omega_1 t) \cos(\omega_2 t) = \frac{1}{2} (\cos(\omega_1 t + \omega_2 t) + \cos(\omega_1 t - \omega_2 t))$$

Frequency conversion

$$\cos(\omega_1 t) \cos(\omega_2 t) = \frac{1}{2} (\cos(\omega_1 t + \omega_2 t) + \cos(\omega_1 t - \omega_2 t))$$

Diode

Shockley-Equation^a (see e.g. [Wikipedia](#)):

$$\begin{aligned} I_D &= I_S \left(e^{\frac{U_D}{n U_T}} - 1 \right) \\ &\approx I_S \left(1 + \frac{U_D}{n U_T} + \frac{1}{2} \left(\frac{U_D}{n U_T} \right)^2 + \dots \right) \end{aligned}$$

^a with $I_S \approx 10^{-12} \dots 10^{-6} \text{ A}$ dem Sättigungsstrom, n dem Emissionskoeffizient ($n \approx 1 \dots 2$), $U_T = \frac{k \cdot T}{q} \approx 25 \text{ mV}$ (bei Raumtemperatur) der Temperaturspannung, der absoluten Temperatur T und der Boltzmannkonstante $k = 1,381 \cdot 10^{-23} \text{ Ws/K}$ sowie der Elementarladung $q = 1,602 \cdot 10^{-19} \text{ As}$

- └ Block diagram
- └ Mixer

Diode

Shockley-Equation* (see e.g. Wikipedia):

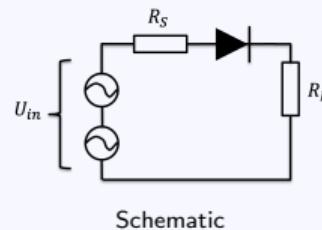
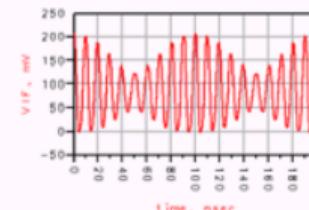
$$i_d = i_0 \left(e^{\frac{qU_D}{kT}} - 1 \right)$$

$$\approx i_0 \left(1 + \frac{qU_D}{kT} + \frac{1}{2} \left(\frac{qU_D}{kT} \right)^2 + \dots \right)$$

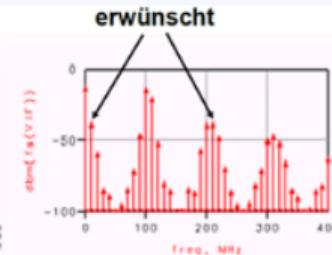
* with $i_0 = 10^{-17} \dots 10^{-15} A$, the saturation current, the reverse saturation current ($i_0 < 1 \dots 20$), $kT = \frac{kT}{e} = 25 \text{ mV}$ (at $T = 300 \text{ K}$), the temperature factor, the absolute Temperature T and the Boltzmann constant $k = 1.381 \cdot 10^{-23} \text{ J/K}$ and the elementary charge $e = 1.602 \cdot 10^{-19} \text{ As}$

Frequency conversion

$$\cos(\omega_1 t) \cos(\omega_2 t) = \frac{1}{2} \{ \cos(\omega_1 t + \omega_2 t) + \cos(\omega_1 t - \omega_2 t) \}$$

Schematic**Measurement**

$$\begin{aligned} F_{RF} &= 110 \text{ MHz} & |V_{RF}| &= 0.1 \text{ V} \\ F_{LO} &= 100 \text{ MHz} & |V_{LO}| &= 0.2 \text{ V} \end{aligned}$$

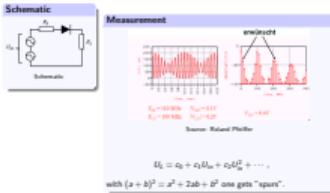


$$V_{DC} = 0.6 \text{ V}$$

Source: Roland Pfeiffer

$$U_L = c_0 + c_1 U_{in} + c_2 U_{in}^2 + \dots ,$$

with $(a + b)^2 = a^2 + 2ab + b^2$ one gets "spurs".



Diode
Shockley-Equation* (see e.g. Wikipedia):

$$i_v = A_0 \left(e^{\frac{qU_v}{kT}} - 1 \right)$$

$$\approx A_0 \left(1 + \frac{qU_v}{kT} + \frac{1}{2} \left(\frac{qU_v}{kT} \right)^2 + \dots \right)$$

* mit $A_0 = 10^{14} \dots 10^{16}$ A, dem Spätstromgitter, q den Elementladungswerten ($q = 1 \dots 2$), $kT = \frac{qV}{e}$ = 25 mV (Bei Raumtemperatur) die Temperaturspannung, der absolute Temperatur T und der Boltzmannkonstante $k = 1,381 \cdot 10^{-23}\text{ J/K}$; wenn die Diemittelspannung $q = 1,602 \cdot 10^{-19}\text{ C}$

Key performance indicators

- ▷ Linearity
- ▷ Conversion gain
- ▷ Noise figure
- ▷ Power consumption
- ▷ Bias

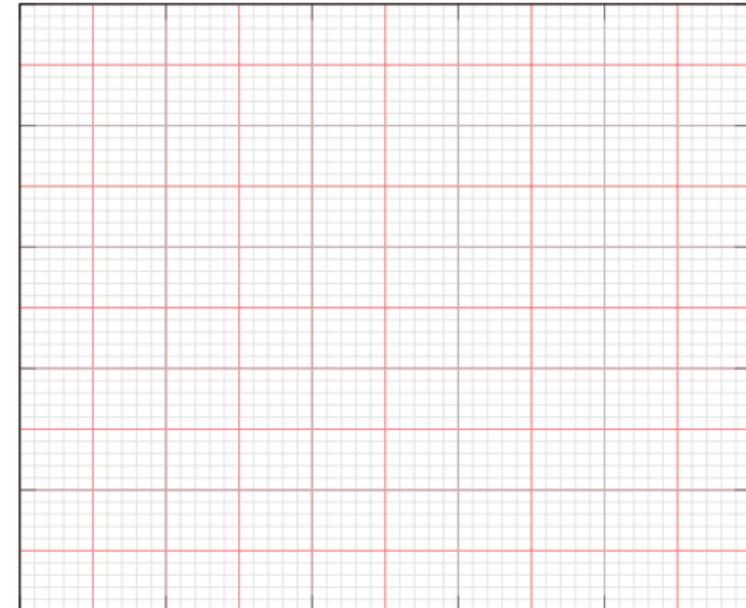
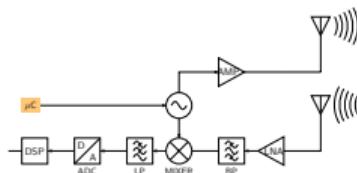
Frequency conversion

$$\cos(\omega_1 t) \cos(\omega_2 t) = \frac{1}{2} (\cos(\omega_1 t + \omega_2 t) + \cos(\omega_1 t - \omega_2 t))$$

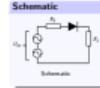
Notes

Key performance indicators

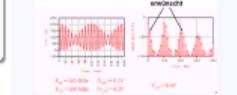
- ▷ Linearity
 - ▷ Conversion gain
 - ▷ Noise figure
 - ▷ Power consumption



Schematic



Measurement



with $(x + h)^2 = x^2 + 2xh + h^2$ one gets "square"

Blanks

Shockley-Equation^a (siehe z.B. Wikipedia)

$$k_2 = k_0 \left(e^{\frac{2\pi i}{\lambda}} - 1 \right)$$

^a with $k_b = 10^{-23} \dots 10^{-24}$ A in the Röntgenquatum, α the Emissivitätskoeffizient ($\alpha = 1 \dots 20$), $D_0 = \frac{h \cdot T}{4}$ or 25 meV [bei Raumtemperatur] der Temperaturspannung, der absolute Temperatur T und der Emissivitätskonstante.

Block diagram

- 3.1 Introduction
- 3.2 Voltage controlled oscillator (VCO)
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TI AWR 1843

- ▷ 25 MSPS
- ▷ 4 ADC on board (for receive antennas, further ADC on board)
- ▷ 12 Bit resolution
- ▷ Source: [TI](#)

TI AWR 1843

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TI AWR 2243

- ▷ up to 45 MSPS (22.5 complex)
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Receiver dynamics

Receiver dynamics:

$$D = \frac{P_{\max}}{P_{\min}}$$

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Receiver dynamics

$$D = \frac{P_{\text{max}}}{P_{\text{min}}}$$

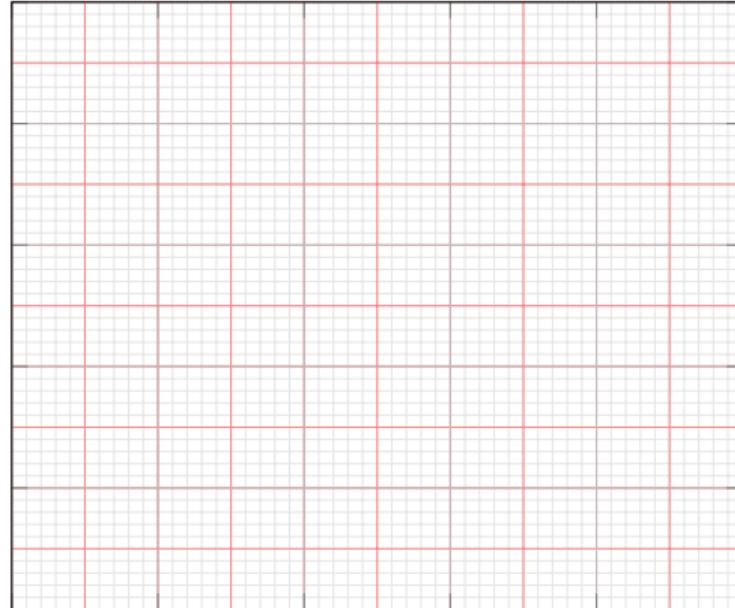
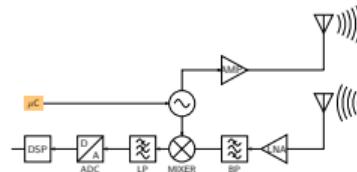
RCS in dBsm	RCS	R	$\frac{1}{R^4}$	$\frac{\sigma}{R^4}$
0	1	1	1	1
-8	0.1585	1	1	0.1585
30	1000	10	0.0001	0.1
30	1000	80	2.44e-08	2.44e-05

Compare to $1/2^{12} = 2.44\text{e-}04$

└ Block diagram

└ ADC

Receiver dynamics			
RCS in dBm	RCS	$\frac{1}{P}$	$\frac{1}{P}$
0	1	1	1
-8	0.1585	1	0.1585
30	1000	10	0.0001
30	1000	80	2.44e-08
Compare to $1/2^{12} \approx 2.44e-08$			



Receiver dynamics

Receiver dynamics:

$$D = \frac{P_{\text{avg}}}{P_{\text{min}}}$$

TI AWR 2243

- ▷ up to 45 MSIPS (22.5 complex)
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Signal Processing

- 4. Spectral analysis
- 5. Index

Spectral analysis

- 4.1 Introduction
- 4.2 Modulation Schemes
- 4.3 Range and Range Rate
- 4.4 Angle of Arrival
- 4.5 Related concepts

Content

- ▷ Modulation schemes (CW, FMCW, Chirp sequence, Tx Phase coding)
- ▷ IQ mixer
- ▷ CFAR
- ▷ Range and range rate estimation

Study goals

- ▷ Implement simple simulators
- ▷ Implement simple CFAR
- ▷ Implement range and range rate estimators
- ▷ Implement simple target simulators

Spectral analysis

4.1 Introduction

4.2 Modulation Schemes

4.2.1 CW

4.2.2 FSK

4.2.3 FMCW

4.2.4 Chirp Sequence

4.2.5 Phase Coded Chirp Sequence

4.2.6 Other modulation schemes

4.3 Range and Range Rate

4.4 Angle of Arrival

4.5 Related concepts

Modulation Schemes

4.2 Modulation Schemes

4.2.1 CW

4.2.2 FSK

4.2.3 FMCW

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4.2.6 Other modulation schemes

Exercise (#4.1)

Implement a simulator in Matlab that simulates the receive signal for a CW radar in case of a single target.

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Implement a simulator in Matlab that simulates the receive signal for a CW radar in case of a single target.



Modulation Schemes

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Modulation Schemes

4.2 Modulation Schemes

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4.2.2 FSK

4.2.3 FMCW

4.2.4 Chirp Sequence

4.2.5 Phase Coded Chirp Sequence

4.2.6 Other modulation schemes

Phase and frequency

Phase is the integral of the frequency. I.e. if

$$\omega = \omega_0 + \Delta\omega t,$$

then

$$\varphi = \omega_0 t + 0.5 \Delta\omega t^2.$$

Model

Given is a target with radial distance $R(t)$ and a transmit signal of the form

$$\omega_{\text{rx}}(t) = \omega_0 + \Delta\omega \cdot t$$

and thus

$$s_{\text{tx}} = A_{\text{tx}} \sin(\omega_0 t + 0.5 \Delta\omega t^2)$$

and for the down-mixed and filtered signal:

$$s_{\text{rx}}(t) = A_{\text{rx}} \sin \left(\omega_0 t + 0.5 \Delta\omega t^2 - \left(\omega_0 \left(t - \frac{2 R(t)}{c} \right) + 0.5 \Delta\omega \left(t - \frac{2 R(t)}{c} \right) \right)^2 \right),$$

respectively.

- └ Spectral analysis
- └ Modulation Schemes

Model

Given is a target with radial distance $R(t)$ and a transmit signal of the form

$$\omega_0 t = \omega_0 t + \Delta\omega \cdot t$$

and thus

$$a_{tx} = A_{tx} \sin(\omega_0 t + 0.5 \Delta\omega t^2)$$

and for the down-mixed and filtered signal:

$$a_{rx}(t) = A_{rx} \sin \left(\omega_0 t + 0.5 \Delta\omega t^2 - \left(\omega_0 \left(t - \frac{2 R(t)}{c} \right) + 0.5 \Delta\omega \left(t - \frac{2 R(t)}{c} \right)^2 \right) \right),$$

respectively.

Receive signal

Thus it follows that

$$\begin{aligned} s_{rx}(t) &= A_{rx} \sin \left(0.5 \Delta\omega t^2 + \omega_0 \frac{2 R(t)}{c} - 0.5 \Delta\omega t^2 + \Delta\omega \frac{2 R(t)}{c} t - \Delta\omega \frac{2 R^2(t)}{c^2} \right). \\ &= A_{rx} \sin \left(\frac{2 \omega_0}{c} R(t) + \frac{2 \Delta\omega}{c} R(t) t - \frac{2 \Delta\omega}{c^2} R^2(t) \right) \end{aligned}$$

Phase and frequency

Phase is the integral of the frequency, i.e. if

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- └ Spectral analysis
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ModelGiven is a target with radial distance $R(t)$ and a transmit signal of the form

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and thus

$$\omega_0 = A_{rx} \sin(\omega_0 t + 0.5 \Delta\omega t^2)$$

and for the down-mixed and filtered signal:

$$s_{rx}(t) = A_{rx} \sin \left(\omega_0 t + 0.5 \Delta\omega t^2 - \left(\omega_0 \left(t - \frac{2R(t)}{c} \right) + 0.5 \Delta\omega \left(t - \frac{2R(t)}{c} \right)^2 \right) \right),$$

respectively.

Phase and frequency

Phase is the integral of the frequency, i.e. if

$$\omega = \omega_0 + \Delta\omega t,$$

then

$$\varphi = \omega_0 t + 0.5 \Delta\omega t^2.$$

Receive signal for a moving target

Expanding

$$R(t) = R_0 + \dot{R}t$$

into

$$s_{rx}(t) = A_{rx} \sin \left(\frac{2\omega_0}{c} R(t) + \frac{2\Delta\omega}{c} R(t)t - \frac{2\Delta\omega}{c^2} R^2(t) \right)$$

leads to

$$s_{rx}(t) = A_{rx} \sin \left(\frac{2\omega_0}{c} (R_0 + \dot{R}t) + \frac{2\Delta\omega}{c} (R_0 + \dot{R}t)t - \frac{2\Delta\omega}{c^2} (R_0 + \dot{R}t)^2 \right)$$

Neglecting constant terms (A_{rx} might be complex) and expanding:

$$s_{rx}(t) = A_{rx} \sin \left(\frac{2\omega_0}{c} \dot{R}t + \frac{2\Delta\omega}{c} R_0 t + \frac{2\Delta\omega}{c} \dot{R}t^2 - \frac{2\Delta\omega}{c^2} 2R_0 \dot{R}t - \frac{2\Delta\omega}{c^2} 2\dot{R}^2 t^2 \right)$$

Receive signal for a moving target

Assume

$$R(t) = R_0 + \dot{R}t$$

into

$$s_{rx}(t) = A_{rx} \sin \left(\frac{2\Delta\omega}{c} R(t) + \frac{2\Delta\omega}{c} R(t)t - \frac{2\Delta\omega}{c^2} R^2(t) \right)$$

leads to

$$s_{rx}(t) = A_{rx} \sin \left(\frac{2\Delta\omega}{c} (R_0 + \dot{R}t) + \frac{2\Delta\omega}{c} (R_0 + \dot{R}t)t - \frac{2\Delta\omega}{c^2} (R_0 + \dot{R}t)^2 \right)$$

Neglecting constant terms (A_{rx} might be complex) and expanding:

$$s_{rx}(t) = A_{rx} \sin \left(\frac{2\Delta\omega}{c} \dot{R}t + \frac{2\Delta\omega}{c} R_0 t + \frac{2\Delta\omega}{c} \dot{R}t^2 - \frac{2\Delta\omega}{c^2} 2R_0 \dot{R}t - \frac{2\Delta\omega}{c^2} 2\dot{R}^2 t^2 \right)$$

Receive signal

Thus it follows that:

$$\begin{aligned} s_{rx}(t) &= A_{rx} \sin \left(0.5\Delta\omega t^2 + \omega_0 \frac{2R(t)}{c} - 0.5\Delta\omega t^2 + \Delta\omega \frac{2R(t)}{c} t - \Delta\omega \frac{2R^2(t)}{c^2} \right) \\ &= A_{rx} \sin \left(\frac{2\Delta\omega}{c} R(t) + \frac{2\Delta\omega}{c} R(t)t - \frac{2\Delta\omega}{c^2} R^2(t) \right) \end{aligned}$$

ModelGiven is a target with radial distance $R(t)$ and a transmit signal of the form

$$\omega_0(t) = \omega_0 + \Delta\omega \cdot t$$

and thus

$$s_{rx} = A_{rx} \sin[\omega_0 t + 0.5\Delta\omega t^2]$$

and for the down-mixed and filtered signal:

$$s_{rx}(t) = A_{rx} \sin \left(\omega_0 t + 0.5\Delta\omega t^2 - \left(\omega_0 \left(t - \frac{2R(t)}{c} \right) + 0.5\Delta\omega \left(t - \frac{2R(t)}{c} \right)^2 \right) \right),$$

respectively

Receive signal for a moving targetSorting, and neglecting terms with c^2 :

$$s_{rx}(t) = A_{rx} \sin \left(\frac{2\Delta\omega}{c} R_0 t + \frac{2\omega_0}{c} \dot{R}t + \frac{2\Delta\omega}{c} \dot{R}t^2 \right)$$

And thus

$$s_{rx}(t) = A_{rx} \sin \left(2\pi C_1 R_0 t + 2\pi C_2 \dot{R}t + \frac{2\Delta\omega}{c} \dot{R}t^2 \right),$$

with

$$2\pi C_1 = \frac{2\Delta\omega}{c} \Rightarrow C_1 = \frac{\Delta\omega}{\pi c} = \frac{2\Delta f}{c} = \frac{2}{\Delta\lambda}$$

and

$$2\pi C_2 = \frac{2\omega_0}{c} \Rightarrow C_2 = \frac{\omega_0}{\pi c} = \frac{2}{\lambda_0},$$

respectively (Note: Doppler shift $f_d = \frac{2\dot{R}}{\lambda_0}$)

Receive signal for a moving target

Starting, and neglecting terms with c^2 :

$$s_{\text{rx}}(t) = A_{\text{rx}} \sin \left(\frac{2\Delta\omega}{c} R_0 t + \frac{2\omega_0}{c} R t + \frac{2\Delta\omega}{c} R t^2 \right)$$

And thus

$$s_{\text{rx}}(t) = A_{\text{rx}} \sin \left(2\pi C_1 R_0 t + 2\pi C_2 R t + \frac{2\Delta\omega}{c} R t^2 \right),$$

with

$$2\pi C_1 \approx \frac{2\Delta\omega}{c} \Rightarrow C_1 = \frac{\Delta\omega}{\pi c} = \frac{2\Delta f}{c} = \frac{2}{\lambda_0}$$

and

$$2\pi C_2 \approx \frac{2\omega_0}{c} \Rightarrow C_2 = \frac{\omega_0}{\pi c} = \frac{2}{\lambda_0}$$

respectively (Note: Doppler shift: $\dot{f}_D = \frac{2\Delta\omega}{c}$)**Receive signal for a moving target**

Expanding

$$R(t) = R_0 + \dot{R}t$$

into

$$s_{\text{rx}}(t) = A_{\text{rx}} \sin \left(\frac{2\omega_0}{c} R(t) + \frac{2\Delta\omega}{c} R(t) t - \frac{2\Delta\omega}{c^2} R^2(t) \right)$$

leads to

$$s_{\text{rx}}(t) = A_{\text{rx}} \sin \left(\frac{2\omega_0}{c} (R_0 + \dot{R}t) + \frac{2\Delta\omega}{c} (R_0 + \dot{R}t) t - \frac{2\Delta\omega}{c^2} (R_0 + \dot{R}t)^2 \right)$$

Neglecting constant terms (A_{rx} might be complex) and expanding:

$$s_{\text{rx}}(t) = A_{\text{rx}} \sin \left(\frac{2\omega_0}{c} R_0 t + \frac{2\Delta\omega}{c} R_0 t + \frac{2\Delta\omega}{c^2} R t^2 - \frac{2\Delta\omega}{c^2} 2 R_0 \dot{R} t - \frac{2\Delta\omega}{c^2} R^2 t^2 \right)$$

Receive signal

Thus it follows that

$$\begin{aligned} s_{\text{rx}}(t) &= A_{\text{rx}} \sin \left(0.5 \Delta\omega t^2 + \omega_0 \frac{2 R(t)}{c} - 0.5 \Delta\omega t^2 + \Delta\omega \frac{2 R(t)}{c} t - \Delta\omega \frac{2 R^2(t)}{c^2} \right) \\ &= A_{\text{rx}} \sin \left(\frac{2\omega_0}{c} R(t) + \frac{2\Delta\omega}{c} R(t) t - \frac{2\Delta\omega}{c^2} R^2(t) \right) \end{aligned}$$

Resolution

Considering

$$s_{\text{rx}}(t) \approx A_{\text{rx}} \sin \left(2\pi C_1 R_0 t + 2\pi C_2 \dot{R} \right),$$

with $C_2 = \frac{2}{\lambda_0}$, an observation interval of T and

$$\Delta f = \frac{\Delta f_{\text{BW}}}{T} \Rightarrow C_1 = \frac{2\Delta f}{c} = \frac{2\Delta f_{\text{BW}}}{Tc} = \frac{2}{T\lambda_{\text{BW}}}$$

leads to a frequency resolution of $\Delta f = \frac{1}{T}$ and thus

$$\begin{aligned} \frac{1}{T} &= \Delta R \frac{2}{T\lambda_{\text{BW}}} \Rightarrow \Delta R = \frac{\lambda_{\text{BW}}}{2} \\ \frac{1}{T} &= \Delta \dot{R} \frac{2}{\lambda_0} \Rightarrow \Delta \dot{R} = \frac{\lambda_0}{2T} \end{aligned}$$

Resolution

Considering

$$s_R(t) \approx A_{R0} \sin\left(2\pi C_1 R_0 t + 2\pi C_2 R\right),$$

with $C_1 = \frac{2}{c}$, an observation interval of T and

$$\Delta f = \frac{\Delta f_{RF}}{T} \Rightarrow C_1 = \frac{2\Delta f}{c} = \frac{2\Delta f_{RF}}{f_C} = \frac{2}{T \lambda_{RF}}$$

leads to a frequency resolution of $\Delta f \approx \frac{1}{T}$ and thus

$$\begin{aligned}\frac{1}{T} &= \Delta R \frac{2}{\lambda_{RF}} \Rightarrow \Delta R = \frac{\lambda_{RF}}{2} \\ \frac{1}{T} &= \Delta R \frac{2}{\lambda_0} \Rightarrow \Delta R = \frac{\lambda_0}{2T}\end{aligned}$$

Receive signal for a moving targetSorting, and neglecting terms with c^2 :

$$s_R(t) \approx A_{R0} \sin\left(\frac{2\Delta f}{c} R_0 t + \frac{2\omega_0}{c} R t + \frac{2\Delta f}{c} R t^2\right)$$

And thus

$$s_R(t) \approx A_{R0} \sin\left(2\pi C_1 R_0 t + 2\pi C_2 R t + \frac{2\Delta f}{c} R t^2\right),$$

with

$$2\pi C_1 = \frac{2\Delta f}{c} \Rightarrow C_1 = \frac{\Delta f}{c} = \frac{2\Delta f}{v_C} = \frac{2}{\Delta \lambda}$$

and

$$2\pi C_2 = \frac{2\omega_0}{c} \Rightarrow C_2 = \frac{\omega_0}{\frac{c}{2}} = \frac{2}{\lambda_0},$$

respectively (Note: Doppler shift: $\delta_f = \frac{2\omega_0}{c}$)**Receive signal for a moving target**

Expanding

$$R(t) = R_0 + R t$$

into

$$s_R(t) \approx A_{R0} \sin\left(\frac{2\omega_0}{c} R(t) + \frac{2\Delta f}{c} R(t)t - \frac{2\Delta f}{c^2} R^2(t)\right)$$

leads to

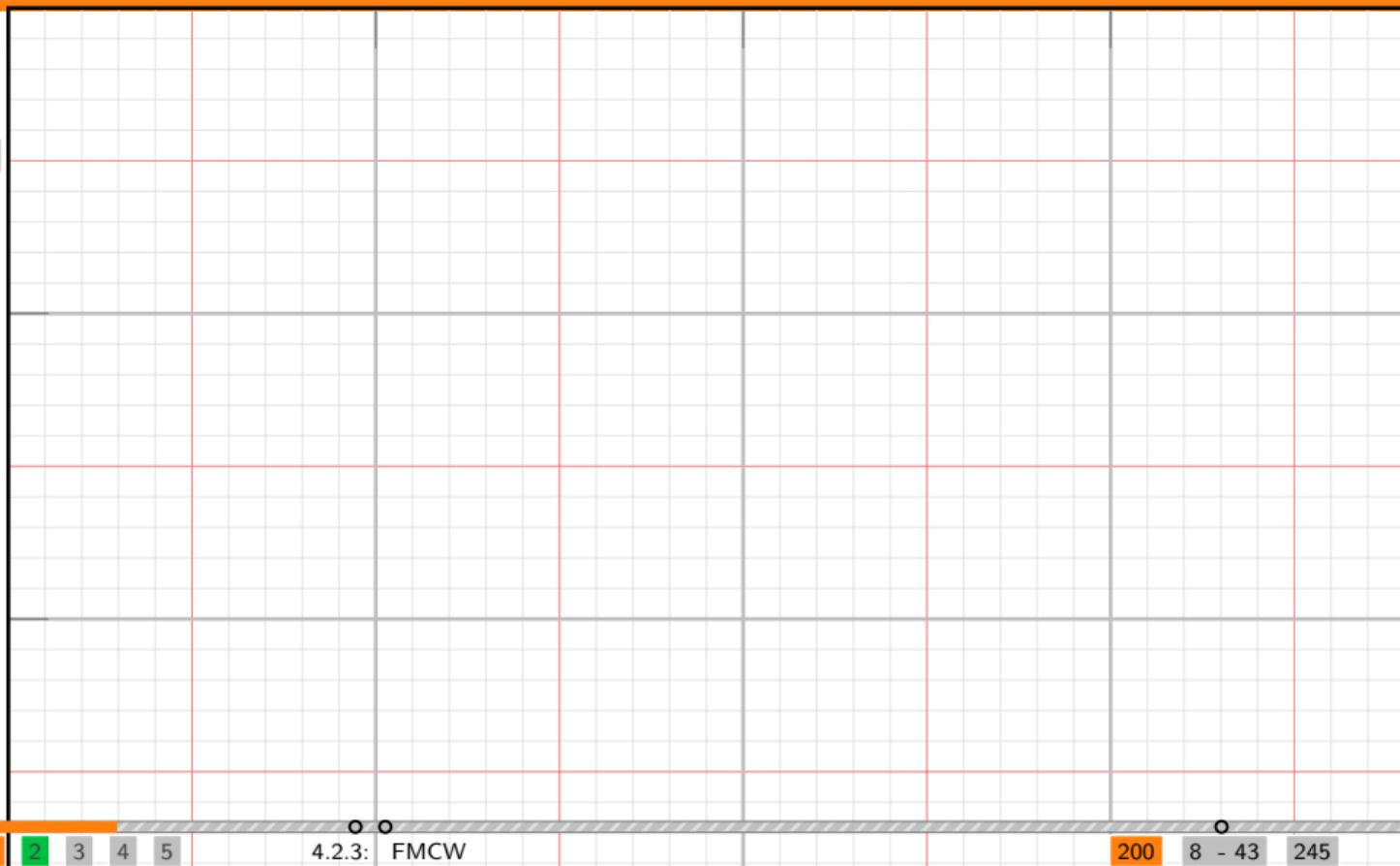
$$s_R(t) \approx A_{R0} \sin\left(\frac{2\omega_0}{c} (R_0 + R t) + \frac{2\Delta f}{c} (R_0 + R t)t - \frac{2\Delta f}{c^2} (R_0 + R t)^2\right)$$

Neglecting constant terms (A_{R0} might be complex) and expanding:

$$s_R(t) \approx A_{R0} \sin\left(\frac{2\omega_0}{c} R_0 t + \frac{2\Delta f}{c} R_0 t + \frac{2\omega_0}{c} R t^2 - \frac{2\Delta f}{c^2} 2 R_0 R t - \frac{2\Delta f}{c^2} 2 R^2 t^2\right)$$

Exercise (#4.2)

Implement a simulator in Matlab that simulates the receive signal for a FMCW radar in case of a single target (w/o IQ-mixer)



Exercise (#4.2)

Implement a simulator in Matlab that simulates the receive signal for a FMCW radar in case of a single target (w/wo IQ-mixer).

Exercise (#4.3)

Implement a simulator in Matlab that simulates the receive signal for a FMCW radar in case of a single target (w/wo IQ-mixer) plus a distributed target (guardrail and/or walking person).

Resolution

Considering

$$a_R(t) \approx A_{R0} \sin \left(2\pi C_0 R_0 t + 2\pi C_2 R \right),$$

with $C_0 = \frac{2\Delta f}{c}$ an observation interval of T and

$$\Delta R = \frac{\Delta f_{\text{RF}}}{c} \Rightarrow C_1 = \frac{2\Delta f}{c} = \frac{2\Delta f_{\text{RF}}}{f_c} = \frac{2}{T \Delta f_{\text{RF}}},$$

leads to a frequency resolution of $\Delta f = \frac{1}{T}$ and thus

$$\begin{aligned} \frac{1}{T} &= \Delta R \frac{2}{T \Delta f_{\text{RF}}} \Rightarrow \Delta R = \frac{\Delta f_{\text{RF}}}{2} \\ \frac{1}{T} &= \Delta R \frac{2}{\lambda_0} \Rightarrow \Delta R = \frac{\lambda_0}{2T} \end{aligned}$$

Receive signal for a moving target

Sorting, and neglecting terms with c^2 :

$$a_R(t) \approx A_{R0} \sin \left(\frac{2\Delta f}{c} R_0 t + \frac{2\omega_0}{c} R t + \frac{2\Delta f}{c} R t^2 \right)$$

And thus

$$a_R(t) \approx A_{R0} \sin \left(2\pi C_0 R_0 t + 2\pi C_2 R t + \frac{2\Delta f}{c} R t^2 \right),$$

with

$$2\pi C_0 = \frac{2\Delta f}{c} \Rightarrow C_1 = \frac{\Delta f}{\pi c} = \frac{2\Delta f}{c} = \frac{2}{\Delta \lambda}$$

and

$$2\pi C_2 = \frac{2\omega_0}{c} \Rightarrow C_2 = \frac{\omega_0}{\pi c} = \frac{2}{\lambda_0}$$

respectively (Note: Doppler shift: $f_d = \frac{\Delta f}{\lambda_0}$)

Exercise (#4.3)

Implement a simulator in Matlab that simulates the receive signal for a FMCW radar in case of a single target (w/wo IQ-mixer) plus a distributed target (guardrail and/or walking person).



Modulation Schemes

4.2 Modulation Schemes

4.2.1 CW

4.2.2 FSK

4.2.3 FMCW

4.2.4 Chirp Sequence

4.2.5 Phase Coded Chirp Sequence

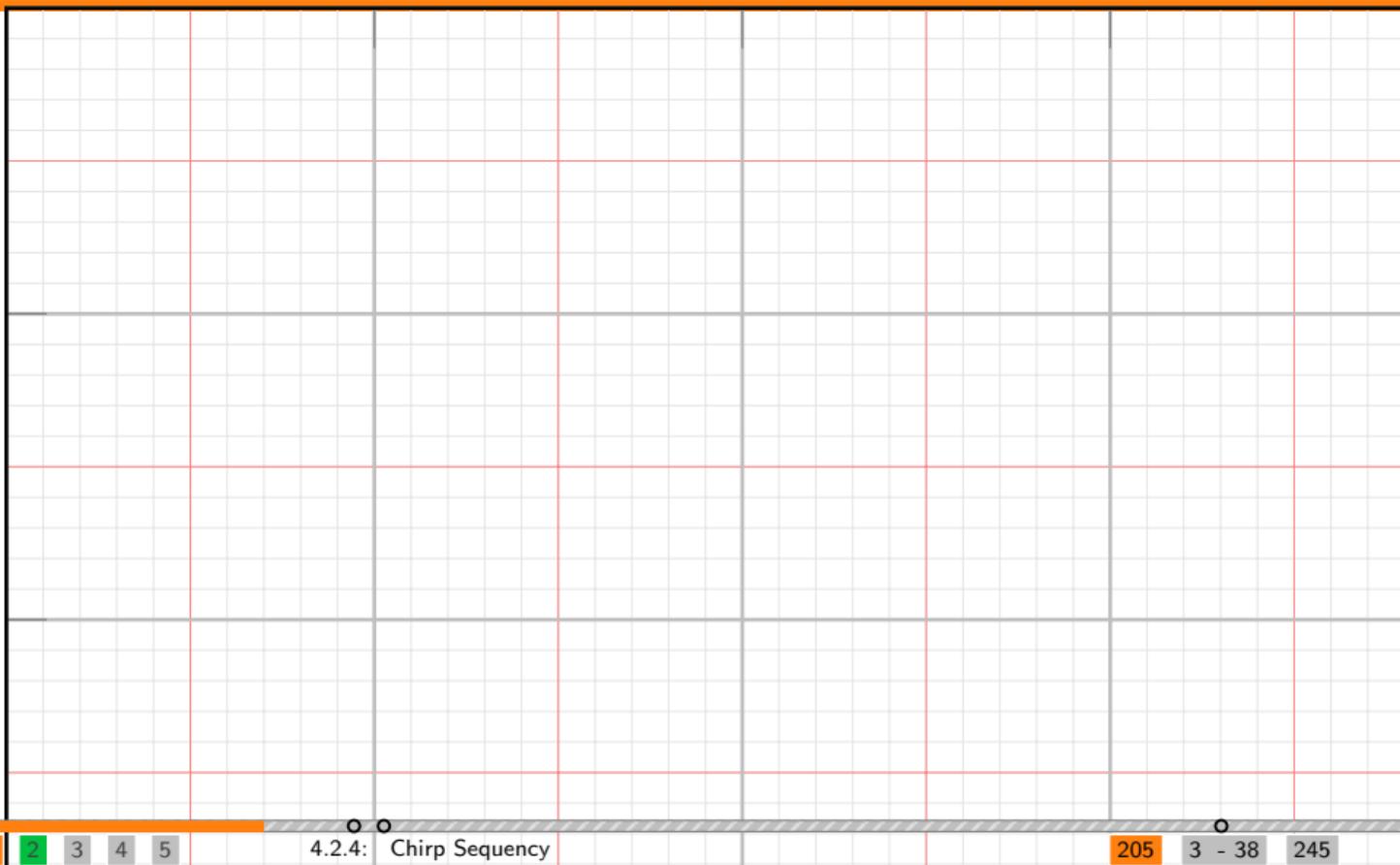
4.2.6 Other modulation schemes

Exercise (#4.4)

Implement a simulator in Matlab that simulates the receive signal for a chirp sequence radar in case of a single target.

Exercise (#4.4)

Implement a simulator in Matlab that simulates the receive signal for a chirp sequence radar in case of a single target.



Modulation Schemes

4.2 Modulation Schemes

4.2.1 CW

4.2.2 FSK

4.2.3 FMCW

4.2.4 Chirp Sequence

4.2.5 Phase Coded Chirp Sequence

4.2.6 Other modulation schemes

Modulation Schemes

4.2 Modulation Schemes

4.2.1 CW

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4.2.6 Other modulation schemes

Spectral analysis

4.1 Introduction

4.2 Modulation Schemes

4.3 Range and Range Rate

4.4 Angle of Arrival

4.5 Related concepts

Spectral analysis

4.1 Introduction

4.2 Modulation Schemes

4.3 Range and Range Rate

4.4 Angle of Arrival

4.4.1 Steering vector of ULA

4.4.2 Monopulse phase comparision

4.4.3 Spectral-Based algorithms

4.4.4 Other techniques

4.4.5 OMP

4.5 Related concepts

Angle of Arrival

4.4 Angle of Arrival

4.4.1 Steering vector of ULA

- 4.4.2 Monopulse phase comparision
- 4.4.3 Spectral-Based algorithms
- 4.4.4 Other techniques
- 4.4.5 OMP

Uniform linear array

Assuming N -antennas with identical directivity on the z – axis with positions $z_n = n \cdot d$ and an incoming plane wave, the steering vector can be described as follows:

$$\mathbf{a}(\vartheta) = [1 \quad e^{jkd \cos \vartheta} \quad e^{j2kd \cos \vartheta} \quad \dots \quad e^{j(N-1)d \cos \vartheta}]^T.$$

Adding all receive signals and using

$$\sum_{k=0}^{N-1} q^k = \frac{1 - q^N}{1 - q}$$

leads to

$$|\mathbf{a}(\vartheta)| = \left| \frac{1 - e^{jkNd \cos \vartheta}}{1 - e^{jkd \cos \vartheta}} \right| = \frac{|\sin(\frac{1}{2}Nkd \cos \vartheta)|}{|\sin(\frac{1}{2}kd \cos \vartheta)|}$$

Uniform linear array

Assuming N -antennas with identical directivity on the x -axis with positions $z_n = n \cdot d$ and an incoming plane wave, the steering vector can be described as follows:

$$\mathbf{a}(\vartheta) = [1 \quad e^{j2\pi d \cos \vartheta} \quad e^{j4\pi d \cos \vartheta} \quad \dots \quad e^{j2(N-1)\pi d \cos \vartheta}]^T.$$

Adding all receive signals and using

$$\sum_{k=0}^{N-1} q^k = \frac{1-q^N}{1-q}$$

leads to

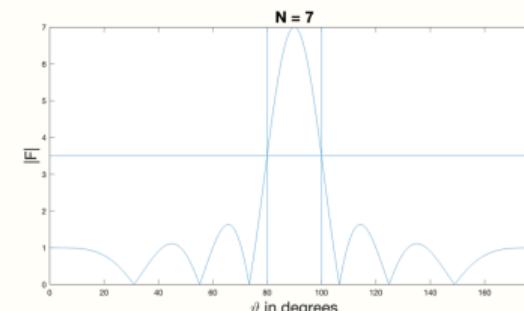
$$|\mathbf{a}(\vartheta)| = \left| \frac{1 - e^{j2Nd \cos \vartheta}}{1 - e^{j2d \cos \vartheta}} \right| = \frac{|\sin(\frac{1}{2}Nd \cos \vartheta)|}{|\sin(\frac{1}{2}d \cos \vartheta)|}$$

Properties

- ▷ Maximum of N .
- ▷ For $d = \lambda/2$: First zero at

$$\begin{aligned} \frac{1}{2}\pi N \cos_0 \vartheta &= \pi \\ \Leftrightarrow \cos(\vartheta_0) &= 2/N. \end{aligned}$$

- ▷ Thus^a



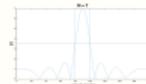
^aLinearization around $\pi/2$: $\Delta\vartheta \approx 2/N$

Properties

- Maximum of N .
- For $d = \lambda/2$: First zero at

$$\frac{1}{2}N\cos(\vartheta) = \pi \\ \cos(\vartheta) = 2/N.$$

⇒ Thus

Generalization around $n=0$: $\Delta d = \lambda/2$

Uniform linear array

Assuming N -antennas with identical directivity on the x -axis with positions $z_n = n \cdot d$ and an incoming plane wave, the steering vector can be described as follows:

$$\mathbf{a}(\vartheta) = [1, e^{j2\pi d \cos \vartheta}, e^{j4\pi d \cos \vartheta}, \dots, e^{j(2N-1)d \cos \vartheta}]^T.$$

Adding all receive signals and using

$$\sum_{k=0}^{N-1} q^k = \frac{1-q^N}{1-q}$$

leads to

$$|\mathbf{a}(\vartheta)| = \left| \frac{1 - e^{j2Nd \cos \vartheta}}{1 - e^{j2d \cos \vartheta}} \right| = \frac{\left| \sin \left(\frac{1}{2}Nd \cos \vartheta \right) \right|}{\left| \sin \left(\frac{1}{2}d \cos \vartheta \right) \right|}$$

Uniform linear array: $\lambda/2$ -spacing

Uniform linear array with array spacing $\lambda/2$ located on z-axis. Phase of element n :

$$e^{jkn\frac{\lambda}{2} \cos(\vartheta)} = e^{j\pi \cos(\vartheta)n}.$$

Compare to sampled signal (nT) of frequency f_0 :

$$f_n = e^{j2\pi f_0 n T}.$$

This leads to a spacial frequency

$$f_{\text{spat}} = 0.5 \cos(\vartheta).$$

Uniform linear array: $\lambda/2$ -spacingUniform linear array with array spacing $\lambda/2$ located on z-axis. Phase of element n :

$$e^{j\theta + \frac{\pi}{2}\cos(\vartheta)} = e^{j\theta}\cos(\vartheta)$$

Compare to sampled signal (πT) of frequency f_0 :

$$f_0 = e^{j2\pi f_0 T}$$

This leads to a spatial frequency

$$f_{\text{spat}} = 0.5 \cos(\vartheta)$$

Properties

- ▷ Maximum of N .
- ▷ For $d = \lambda/2$: First zero at

$$\begin{aligned} \frac{1}{2} \pi N \cos \vartheta &= \pi \\ \Leftrightarrow \cos(\vartheta_0) &= 2/N. \end{aligned}$$

- ▷ Thus^a



Uniform linear array

Assuming N -antennas with identical directivity on the z -axis with positions $x_n = n \cdot d$ and an incoming plane wave, the steering vector can be described as follows:

$$\mathbf{a}(\vartheta) = [1 \quad e^{j2\pi d \cos \vartheta} \quad e^{j4\pi d \cos \vartheta} \quad \dots \quad e^{j(2N-1)d \cos \vartheta}]^T.$$

Adding all receive signals and using

$$\sum_{k=0}^{N-1} q^k = \frac{1-q^N}{1-q}$$

leads to

$$|\mathbf{a}(\vartheta)| = \left| \frac{1 - e^{j2N\pi d \cos \vartheta}}{1 - e^{j2\pi d \cos \vartheta}} \right| = \frac{\sin\left(\frac{1}{2}N\pi d \cos \vartheta\right)}{\sin\left(\frac{1}{2}\pi d \cos \vartheta\right)}$$

Uniform linear array: $\lambda/2$ -spacing

Using a DFT with N samples (antennas) yields a frequency resolution^a of $\Delta f = \frac{1}{N}$. For $\vartheta \approx \Delta\vartheta + \frac{\pi}{2}$:

$$f_{\text{spat}} \approx 0.5 \Delta\vartheta.$$

Thus:

$$\begin{aligned} \frac{1}{N} &= 0.5 \Delta\vartheta \\ \Leftrightarrow \Delta\vartheta &= \frac{2}{N} \end{aligned}$$

$$^a T = 1$$

Uniform linear array: $\lambda/2$ -spacing

Using a DFT with N samples (antennas) yields a frequency resolution* of $\Delta f = \frac{1}{T}$. For $\theta \approx \Delta\theta + \frac{\pi}{2}$:

$$f_{\text{peak}} \approx 0.5\Delta f.$$

Thus:

$$\begin{aligned} \frac{1}{N} &= 0.5\Delta\theta \\ \Leftrightarrow \Delta\theta &= \frac{2}{N} \end{aligned}$$

 $\star T = 1$ **Uniform linear array: $\lambda/2$ -spacing**Uniform linear array with array spacing $\lambda/2$ located on z-axis. Phase of element n :

$$e^{j\frac{\pi}{\lambda} n z \cos(\theta)} = e^{j2\pi n \cos(\theta)}.$$

Compare to sampled signal ($e^{j\omega t}$) of frequency f_0 :

$$f_0 = e^{j2\pi n \Delta t}.$$

This leads to a spatial frequency

$$f_{\text{spat}} = 0.5 \cos(\theta).$$

Exercise (#4.5)

Implement a ULA in MATLAB and numerically determine the 3 dB–beamwidth.

Properties

- ▷ Maximum of N .
- ▷ For $d = \lambda/2$: First zero at

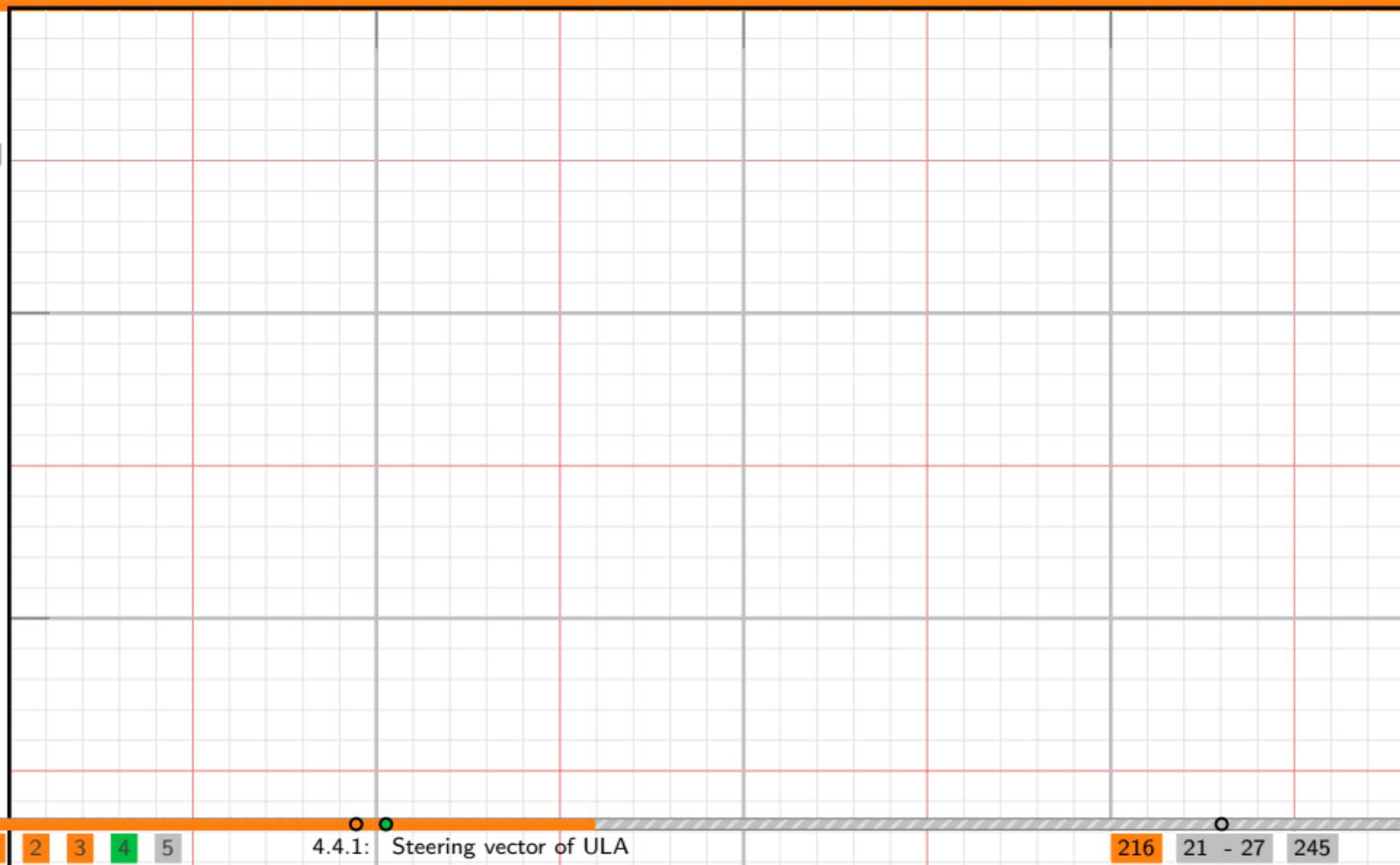
$$\begin{aligned} \frac{1}{2}N\cos_0\theta &= \pi \\ \cos(\theta_0) &= 2/N. \end{aligned}$$

- ▷ That*

* Localization around $\pi/2$: $\Delta\theta \approx 2/N$

Exercise (#4.5)

Implement a ULA in MATLAB and numerically determine the 3dB-beamwidth.



Angle of Arrival

4.4 Angle of Arrival

4.4.1 Steering vector of ULA

4.4.2 Monopulse phase comparision

4.4.3 Spectral-Based algorithms

4.4.4 Other techniques

4.4.5 OMP

Angle of Arrival

4.4 Angle of Arrival

4.4.1 Steering vector of ULA

4.4.2 Monopulse phase comparision

4.4.3 Spectral-Based algorithms

4.4.4 Other techniques

4.4.5 OMP

Signal model

Signal model

Signal model

$$\mathbf{x}(t) = \mathbf{A}(\vartheta)\mathbf{s}(t) + \mathbf{n}(t),$$

where \mathbf{A} is a $N_a \times N_s$ matrix, with

N_a : number of antennas

and

N_s : number of signals

\mathbf{A} contains N_s steering vectors. $\mathbf{s}(t)$ contains N_s – impinging signals (unknown!).

Signal model $x(t) = \mathbf{A}(t)\mathbf{s}(t) + \mathbf{n}(t)$,where \mathbf{A} is a $N_a \times N_s$ matrix, with N_a : number of antennas

and

 N_s : number of signals \mathbf{A} contains N_a steering vectors. $\mathbf{s}(t)$ contains N_s -impinging signals ('unknown')

Weighting vector

Signal model

Weighting vector

Weighting vector

Use a weighting vector

$$\mathbf{w}(\vartheta)$$

and calculate the power:

$$\begin{aligned} P(\vartheta) &= \left[\mathbf{w}^H(\vartheta) \mathbf{s} \right] \left[\mathbf{w}^H(\vartheta) \mathbf{s} \right]^T \\ &= \mathbf{w}^H(\vartheta) \mathbf{s} \mathbf{s}^T \mathbf{w}. \end{aligned}$$

Task: Maximize the receive power:

$$\arg \max_{\vartheta} \{P(\vartheta)\}$$

Signal model

where \mathbf{A} is a $N_r \times N_t$ matrix, with N_r : number of antennas

and

 N_t : number of signals \mathbf{A} contains N_t steering vectors. $\mathbf{s}(t)$ contains N_r -impinging signals (unknown!)

Weighting vector

Use a weighting vector

 $w(t)$

and calculate the power:

$$\begin{aligned} P(t) &= \left[w^H(t)x(t) \right] \left[w^H(t)x(t) \right]^T \\ &= w^H(t)xx^T w. \end{aligned}$$

Task: Maximize the receive power:

$$\arg \max_w [P(t)]$$

Weighting vector

Spatial covariance matrix

Signal model

$$x(t) = \mathbf{A}(t)s(t) + n(t),$$

where \mathbf{A} is a $N_r \times N_t$ matrix, with N_r : number of antennas

and

 N_t : number of signals \mathbf{A} contains N_t steering vectors. $s(t)$ contains N_r -impinging signals (unknown).

Spatial covariance matrix

Spatial covariance matrix

The spatial covariance matrix is then given by:

$$\begin{aligned}\mathbf{R} &= E \left\{ \mathbf{x}(t) \mathbf{x}^H(t) \right\} \\ &= \mathbf{A} \mathbf{P} \mathbf{A}^H + \sigma^2 \mathbf{I},\end{aligned}$$

with

$$\mathbf{P} = E \left\{ \mathbf{s}(t) \mathbf{s}(t)^H \right\}$$

being the source covariance matrix. Thus, the receive power becomes:

Weighting vector

$$P(\vartheta) = \mathbf{w}^H(\vartheta) \mathbf{s} \mathbf{s}^T \mathbf{w} = \mathbf{w}^H(\vartheta) \mathbf{R} \mathbf{w}.$$

Spatial covariance matrix

The spatial covariance matrix is then given by:

$$\mathbf{R} = \mathbb{E} \left[\mathbf{x}(t) \mathbf{x}^H(t) \right] \\ = \mathbf{A} \mathbf{P} \mathbf{A}^H + \sigma^2 \mathbf{I}_N$$

with

$$\mathbf{P} = \mathbb{E} \left[\mathbf{x}(t) \mathbf{x}(t)^H \right]$$

being the source covariance metric. Thus, the receive power becomes:

$$P(\theta) = \mathbf{w}^H(\theta) \mathbf{x}^T \mathbf{x} = \mathbf{w}^H(\theta) \mathbf{R} \mathbf{w}.$$

Spatial covariance matrix

Classical beamforming (Bartellt)

Weighting vector

Use a weighting vector

$$\mathbf{w}(\theta)$$

and calculate the power:

$$P(\theta) = \left[\mathbf{w}^H(\theta) \mathbf{x} \right] \left[\mathbf{w}^H(\theta) \mathbf{x} \right]^T \\ = \mathbf{w}^H(\theta) \mathbf{x} \mathbf{x}^T \mathbf{w}.$$

Task: Maximize the receive power:

$$\arg \max_{\theta} \{ P(\theta) \}$$

Classical beamforming (Barteltt)

Classical beamformer

Use a weighting vector

$$\mathbf{w}(\vartheta) = \mathbf{a}$$

and normalizing $\mathbf{w}(\vartheta)$ using

$$\mathbf{w}(\vartheta) = \frac{\mathbf{a}(\vartheta)}{\sqrt{\mathbf{a}^H(\vartheta)\mathbf{a}(\vartheta)}}$$

leads to

$$P(\vartheta) = \mathbf{w}^H(\vartheta)\mathbf{R}\mathbf{w} = \frac{\mathbf{a}^H(\vartheta)\mathbf{R}\mathbf{a}}{\mathbf{a}^H(\vartheta)\mathbf{a}(\vartheta)}.$$

Spatial covariance matrix

Target separation: Approx.

$$\Delta\varphi = \frac{2}{N},$$

assuming ULA with $\lambda/2$ spacing.

See e.g. *Two decades of array signal processing research: The parametric approach*, page 73.

- └ Spectral analysis
- └ Angle of Arrival

Classical beamformer

Use a weighting vector

$$\mathbf{w}(\vartheta) = \mathbf{a}$$

and normalizing $\mathbf{w}(\vartheta)$ using

$$\mathbf{w}(\vartheta) = \frac{\mathbf{a}(\vartheta)}{\sqrt{\mathbf{a}^H(\vartheta)\mathbf{a}(\vartheta)}}$$

leads to

$$P(\vartheta) = \mathbf{w}^H(\vartheta) R \mathbf{w} = \frac{\mathbf{a}^H(\vartheta) R \mathbf{a}}{\mathbf{a}^H(\vartheta) \mathbf{a}(\vartheta)}$$

Target separation: Approx.

$$\Delta\varphi = \frac{\lambda}{N}$$

assuming ULA with $\lambda/2$ spacing.

See e.g. Two decades of array signal processing research: The parametric approach, page 73

Classical beamforming (Bartelt)

Exercise (#4.6)

Implement a ULA in MATLAB with $d = \frac{\lambda}{2}$ and $N = 6$. Simulate three targets at 30° , 90° and 110° , respectively. Implement a delay-and-sum beamformer and an FFT based approach. Change N to be 5,

Spatial covariance matrix

The spatial covariance matrix is then given by:

$$\mathbf{R} = E \left[\mathbf{x}(t) \mathbf{x}^H(t) \right] \\ = \mathbf{A} \mathbf{P} \mathbf{A}^H + \sigma^2 \mathbf{I},$$

with

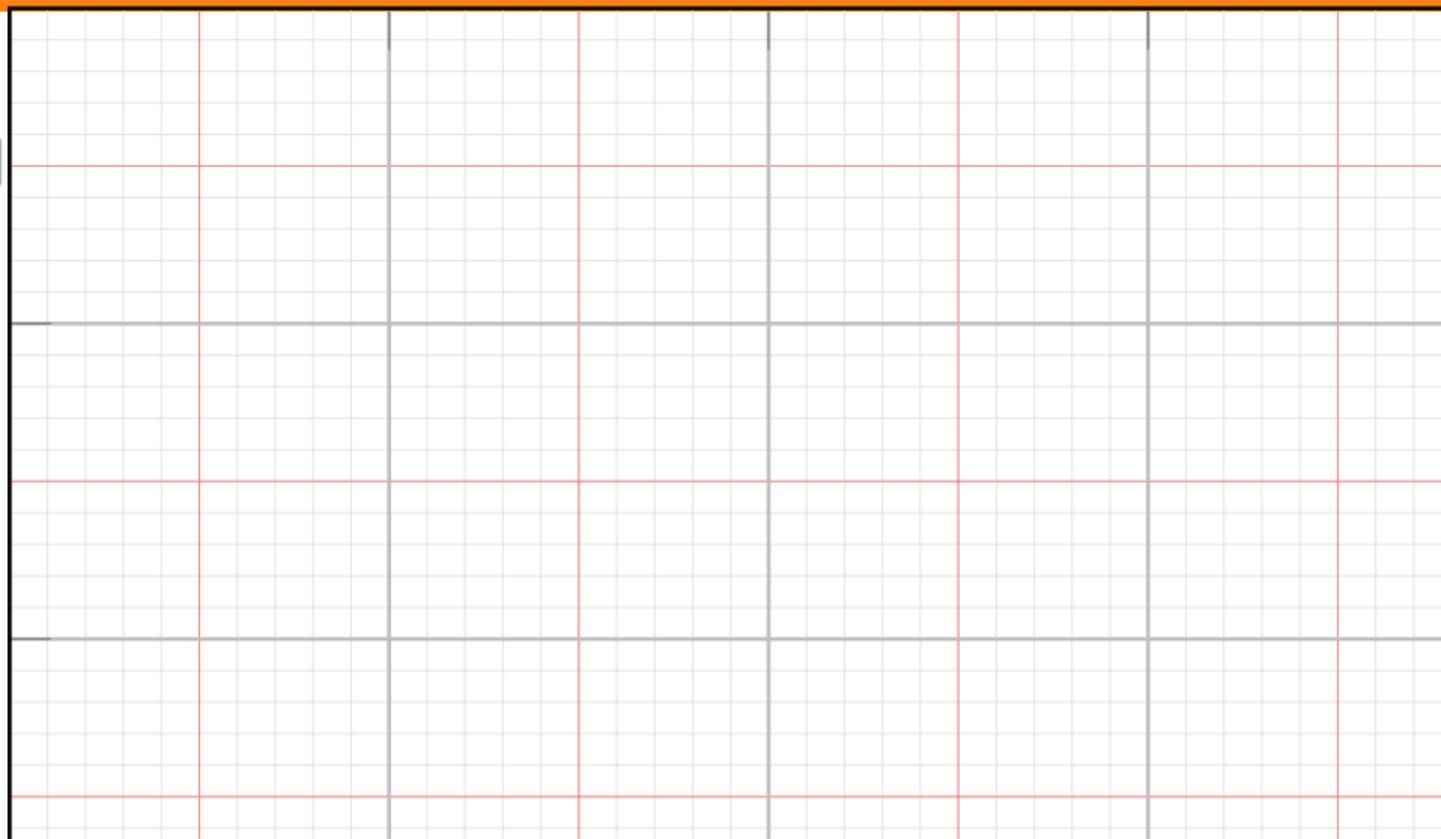
$$\mathbf{P} = E \left[\mathbf{a}(\vartheta) \mathbf{a}(\vartheta)^H \right]$$

being the source covariance matrix. Thus, the receive power becomes:

$$P(\vartheta) = \mathbf{w}^H(\vartheta) \mathbf{R} \mathbf{w} = \mathbf{w}^H(\vartheta) \mathbf{R} \mathbf{w}$$

Exercise (#4.6)

Implement a ULA in MATLAB with $d = \frac{\lambda}{2}$ and $N = 6$. Simulate three targets at 30° , 90° and 110° , respectively. Implement a delay-and-sum beamformer and an FFT based approach. Change N to be 5.



Exercise (#4.6)

Implement a ULA in MATLAB with $d = \frac{\lambda}{2}$ and $N = 6$. Simulate three targets at 30° , 90° and 110° , respectively. Implement a delay-and-sum beamformer and an FFT based approach. Change N to be 5.

Classical beamformer

Use a weighting vector $\mathbf{w}(\vartheta) = \mathbf{a}$

and normalizing $\mathbf{w}(\vartheta)$ using

$$\mathbf{w}(\vartheta) = \frac{\mathbf{a}(\vartheta)}{\sqrt{\mathbf{a}^H(\vartheta)\mathbf{a}(\vartheta)}}$$

leads to

$$P(\vartheta) = \mathbf{w}^H(\vartheta)\mathbf{R}\mathbf{w} = \frac{\mathbf{a}^H(\vartheta)\mathbf{R}\mathbf{a}}{\mathbf{a}^H(\vartheta)\mathbf{a}(\vartheta)}$$

Target separation: Approx.

$$\Delta\varphi \approx \frac{2}{N}$$

assuming ULA with $\lambda/2$ spacing

See e.g. *Two decades of array signal processing research: The parametric approach*, page 72

Capon beamforming

Classical beamforming (Barteltt)

Capon beamforming

Capon beamforming

Spatial spectrum:

$$P(\vartheta) = \frac{1}{\mathbf{a}^H(\vartheta)\mathbf{R}^{-1}\mathbf{a}(\vartheta)}$$

Exercise (#4.6)

Implement a ULA in MATLAB with $d = \frac{\lambda}{2}$ and $N = 6$. Simulate three targets at 30° , 90° and 110° , respectively. Implement a delay-and-sum beamformer and an FFT based approach. Change N to be 5.

Classical beamformer

Use a weighting vector

$$\mathbf{w}(\vartheta) = \mathbf{a}$$

and normalizing $\mathbf{w}(\vartheta)$ using

$$\mathbf{w}(\vartheta) = \frac{\mathbf{a}(\vartheta)}{\sqrt{\mathbf{a}^H(\vartheta)\mathbf{a}(\vartheta)}}$$

leads to

$$P(\vartheta) = \mathbf{w}^H(\vartheta)\mathbf{R}\mathbf{w} = \frac{\mathbf{a}^H(\vartheta)\mathbf{R}\mathbf{a}}{\mathbf{a}^H(\vartheta)\mathbf{a}(\vartheta)}$$

Target separation: Approx.

$$\Delta\varphi = \frac{2}{N}$$

assuming ULA with $\lambda/2$ spacing.

See e.g. "Two decades of array signal processing research: The parametric approach, page 73"

Capon beamforming

Spatial spectrum:

$$P(\theta) \triangleq \frac{1}{\mathbf{a}^H(\theta) R^{-1} \mathbf{a}(\theta)}$$

Capon beamforming

Music beamforming

Exercise (#4.6)

Implement a ULA in MATLAB with $d = \frac{\lambda}{2}$ and $N = 6$. Simulate three targets at 30° , 80° and 110° , respectively. Implement a delay-and-sum beamformer and an FFT based approach. Change N to be 5.

Eigendecomposition of spatial covariance matrix

Music beamforming

Eigendecomposition of spatial covariance matrix

The spatial covariance matrix is given by:

$$\mathbf{R} = \mathbf{A}\mathbf{P}\mathbf{A}^H + \sigma^2\mathbf{I},$$

with

$$\mathbf{P} = E \left\{ \mathbf{s}(t)\mathbf{s}(t)^H \right\}$$

being the source covariance matrix. \mathbf{R} can be Eigen-decomposed as follows:

$$\mathbf{R} = \mathbf{U}_s \boldsymbol{\lambda}_s \mathbf{U}_s^H + \sigma^2 \mathbf{U}_n \boldsymbol{\lambda}_n \mathbf{U}_n^H,$$

Capon beamforming

where $\boldsymbol{\lambda}_s$ contains the N_s largest Eigenvalues, \mathbf{U}_s contains the corresponding Eigenvectors, \mathbf{U}_n contains the Eigenvectors related to noise and $\boldsymbol{\lambda}_n$ the Eigenvalues related to noise.

Eigendecomposition of spatial covariance matrix

The spatial covariance matrix is given by:

$$\mathbf{R} = \mathbf{A}\mathbf{P}\mathbf{A}^H + \sigma^2\mathbf{I},$$

with

$$\mathbf{P} = \mathbb{E} \left[\mathbf{s}(z)\mathbf{s}(z)^H \right]$$

being the source covariance matrix. \mathbf{R} can be Eigen-decomposed as follows:

$$\mathbf{R} = \mathbf{U}_s \mathbf{\Lambda}_s \mathbf{U}_s^H + \sigma^2 \mathbf{U}_n \mathbf{\Lambda}_n \mathbf{U}_n^H,$$

where $\mathbf{\Lambda}_s$ contains the N_s largest Eigenvalues, \mathbf{U}_s contains the corresponding Eigenvectors, \mathbf{U}_n contains the Eigenvectors related to noise and $\mathbf{\Lambda}_n$ the Eigenvalues related to noise.

Music beamforming

Music beamforming

The receiver power becomes

$$P(\vartheta) = \mathbf{w}^H(\vartheta) \mathbf{s} \mathbf{s}^T \mathbf{w}.$$

This becomes minimal in case of a steering vector pointing into the direction of the noise subspace (orthogonal to signal subspace). Taking the reciprocal and normalizing:

$$P(\vartheta) = \frac{1}{\mathbf{a}^H(\vartheta) \mathbf{U}_n \mathbf{U}_n^H \mathbf{a}(\vartheta)} \Rightarrow P_{\text{norm}}(\vartheta) = \frac{\mathbf{a}^H(\vartheta) \mathbf{a}}{\mathbf{a}^H(\vartheta) \mathbf{U}_n \mathbf{U}_n^H \mathbf{a}(\vartheta)}.$$

Notes:

- ▷ Does not work in case of correlated signals. Signals can be de-correlated e.g. by making use of spatial smoothing.

Capon beamforming

Spatial spectrum:

$$P(\vartheta) = \frac{1}{\mathbf{a}^H(\vartheta) \mathbf{R}^{-1} \mathbf{a}(\vartheta)}$$

Angle of Arrival

4.4 Angle of Arrival

- 4.4.1 Steering vector of ULA
- 4.4.2 Monopulse phase comparision
- 4.4.3 Spectral-Based algorithms
- 4.4.4 Other techniques**
- 4.4.5 OMP

Angle of Arrival

4.4 Angle of Arrival

- 4.4.1 Steering vector of ULA
- 4.4.2 Monopulse phase comparision
- 4.4.3 Spectral-Based algorithms
- 4.4.4 Other techniques
- 4.4.5 OMP

OMP

Signal decomposition

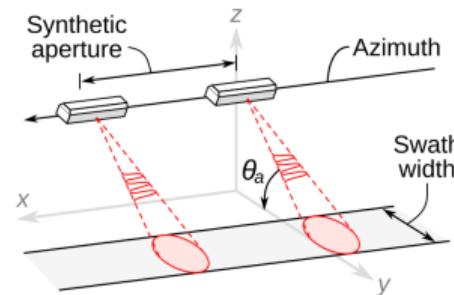
- ▷ use a large, highly redundant dictionary that spans the signal space
- ▷ choose the smallest subset

Spectral analysis

- 4.1 Introduction
- 4.2 Modulation Schemes
- 4.3 Range and Range Rate
- 4.4 Angle of Arrival
- 4.5 Related concepts**
 - 4.5.1 Wiener Filter

SAR

Synthetic aperture radar



Source: Wikipedia

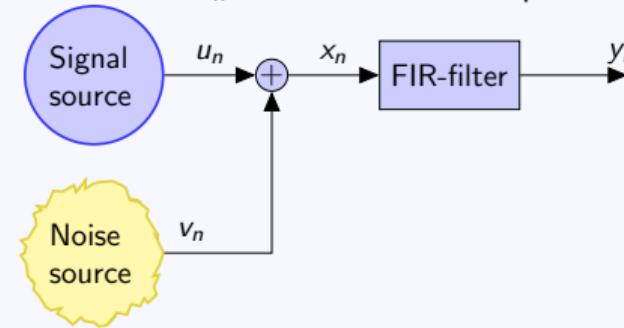
Related concepts

4.5 Related concepts

4.5.1 Wiener Filter

Problem definition

Given is the system as shown below. u_n is a WSS stochastic process.



Goal: Find the filter taps w to maximize the signal-to-noise ratio.

Problem definitionGiven is the system as shown below. u_n is a WSS stochastic process.Goal: Find the filter taps w to maximize the signal-to-noise ratio.**Corollary**

The average power of the signal output:

$$P_y = \mathbf{w}^H \mathbf{R} \mathbf{w},$$

with \mathbf{R} being the autocorrelation matrix of the input signal u_n . The average power of the noise component:

$$N_y = \sigma^2 \mathbf{w}^H \mathbf{w}.$$

Signal to noise ratio:

$$(SNR)_y = \frac{\mathbf{w}^H \mathbf{R} \mathbf{w}}{\sigma^2 \mathbf{w}^H \mathbf{w}}$$

Corollary

The average power of the signal output:

$$P_y = \mathbf{w}^H \mathbf{R} \mathbf{w},$$

with \mathbf{R} beeing the autocorrelation matrix of the input signal u_0 . The average power of the noise component:

$$N_y = \sigma^2 \mathbf{w}^H \mathbf{w}.$$

Signal to noise ratio:

$$(SNR)_y = \frac{\mathbf{w}^H \mathbf{R} \mathbf{w}}{\sigma^2 \mathbf{w}^H \mathbf{w}}$$

Problem definitionGiven is the system as shown below. u_0 is a WSS stochastic process.Goal: Find the filter taps \mathbf{w} to maximize the signal-to-noise ratio.**Corollary**

The filter values w_n of the FIR-filter for maximum signal to noise ratio subject to the constraint $\mathbf{w}^H \mathbf{w} = 1$ can be calculated as follows:

1. Perform an eigendecomposition of \mathbf{R}
2. Find the largest eigenvalue λ_m and the corresponding eigenvector \mathbf{q}_m .
3. \mathbf{q}_m defines the impulse response of the optimum filter.

And thus:

$$(SNR)_{y,max} = \frac{\lambda_m}{\sigma^2},$$

with σ^2 beeing the noise covariance.

Corollary

The filter values w_r of the FIR-filter for maximum signal to noise ratio subject to the constraint $w^H w = 1$ can be calculated as follows:

1. Perform an eigendecomposition of R .
2. Find the largest **eigenvalue** λ_m and the corresponding **eigenvector** q_m .
3. q_m defines the impulse response of the optimum filter.

And thus:

$$(SNR)_{r,\max} = \frac{\lambda_m}{\sigma^2},$$

with σ^2 being the noise covariance.

Corollary

The average power of the signal output:

$$P_r := w^H R w,$$

with R being the autocorrelation matrix of the input signal x_r . The average power of the noise component:

$$N_r := \sigma^2 w^H w.$$

Signal to noise ratio:

$$(SNR)_r = \frac{w^H R w}{\sigma^2 w^H w}$$

Corollary

This type of filter can be viewed as the *stochastic* counterpart of the so-called **matched filter**. The latter maximizes the output signal-to-noise ratio for a known signal subject to additive white noise. Furthermore, it can be understood as an example for **principal component analysis**.

Problem definition

Given is the system as shown below. x_r is a WSS stochastic process.



Goal: Find the filter taps w to maximize the signal-to-noise ratio.

- 4. Spectral analysis
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