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Discrete Time

Introduction

Content

- Definition of the discrete Fourier transform
- Leakage, window function and undersampling
- $z-$transform and time discrete LTI systems including canonical forms
- Relationships between different transforms

Study goals

- You shall be able to use a discrete Fourier transform including mapping between bin and frequency
- You shall be able to describe the effect of undersampling
- You shall be able to describe the consequences of changing sampling rate and number of samples

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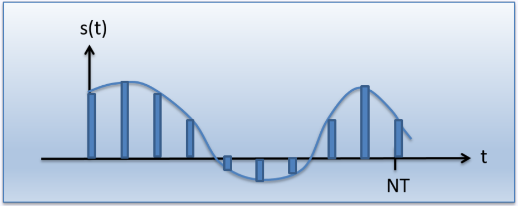
Discrete Time

Pulse modulated signal

Time domain

$$s_d(t) = \sum_{n=-\infty}^{\infty} s(nT)p(t-nT)$$



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Properties: Dirac comb

$$\mathcal{F}\left\{ \sum_{n=-\infty}^{\infty} \delta(t-nT) = \frac{1}{T}\sum_{-\infty}^{\infty} e^{j2\pi n \frac{t}{T}} \right\} = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

Frequency domain

$$\boxed{S_d(\omega) = \frac{1}{T}P(\omega)\sum_{n=-\infty}^{\infty} S\left(\omega - n\frac{2\pi}{T}\right)}$$

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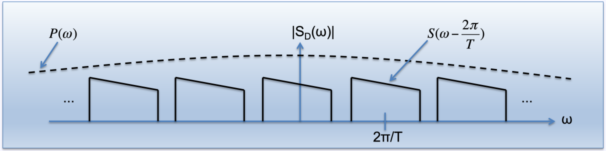

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Discrete Time

Discrete Fourier Transform

Discrete Fourier Transform

Motivation

The spectrum $S_d(\omega) = \frac{1}{T}P(\omega)\sum_{n=-\infty}^{\infty} S\left(\omega - n\frac{2\pi}{T}\right)$ of the signal

$$s_d(t) = \sum_{n=-\infty}^{\infty} s(nT)p(t-nT) = \left[s(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)\right] * p(t)$$

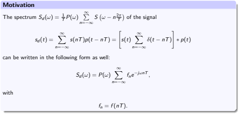
can be written in the following form as well:

$$S_d(\omega) = P(\omega)\sum_{n=-\infty}^{\infty} f_n e^{-j\omega nT},$$

with

$$f_n = f(nT).$$

Discrete Fourier Transform



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$$S_d(\omega) = \sum_{n=-\infty}^{\infty} f_n e^{-j\omega nT}.$$

motivates to define the discrete Fourier transform as follows:

$$\begin{aligned} F_k &= \left. \sum_{n=0}^{N-1} f_n e^{-\frac{j2\pi nk}{N}} \right|_{k=0\ldots N-1}, \\ f_n &= \left. \sum_{k=0}^{N-1} F_k e^{\frac{j2\pi nk}{N}} \right|_{n=0\ldots N-1} \end{aligned}$$

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Properties

Mapping to frequencies: Assuming

$$f_n = f(nT),$$

one can map

$$F_k = \sum_{n=0}^{N-1} f_n e^{-\frac{j2\pi nk}{N}}$$

to the frequency

$$\omega_k = \frac{2\pi}{NT}k.$$

Note: The result of the DFT is independent of T and depends only on the values f_n.

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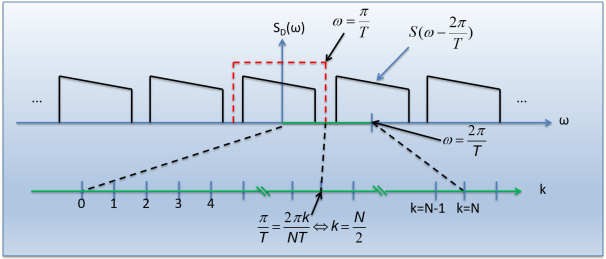
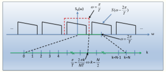


- ▷ Maximum frequency is given by sampling frequency $1/T$
- ▷ Frequency resolution is given by observation time: $df = 1/NT$

Discrete Fourier Transform

Properties



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Properties

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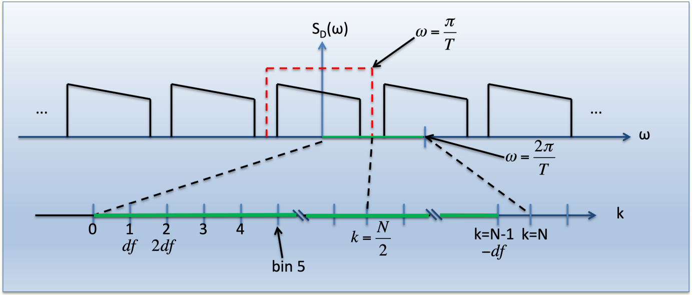
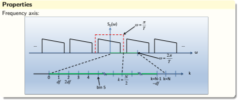
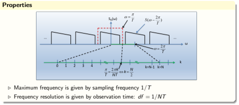
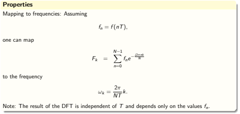


Nyquist–Shannon



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Nyquist–Shannon sampling theorem

Sufficient condition for the minimum sample rate so that a discrete sequence of samples contains all the information from a continuous-time signal of finite bandwidth f_{bw}:

$$f_s > 2f_{bw}$$

or

$$T < \frac{1}{2f_{bw}}.$$

Exercise

Nyquist–Shannon sampling theorem

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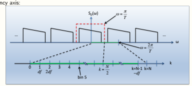

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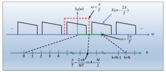

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Exercise (#6.1)

- Start NUMPY or MATLAB
- In the following you shall "sample" 5 periods of the signal $s(t) = \sin(2\pi t)$. Take 50 samples per period.
- Calculate the DFT by making use of `F=fft(s)`. Plot the result
- Determine the frequency axis to use `plot(freq,abs(F))`.
- "Sample" the sinus for 5.5 periods and compare the results. What do you observe?

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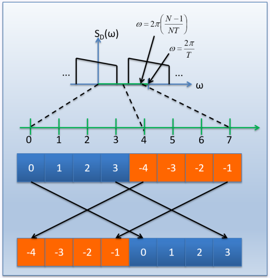
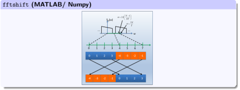
Discrete Fourier Transform

`fftshift` (MATLAB/ Numpy)



Exercise



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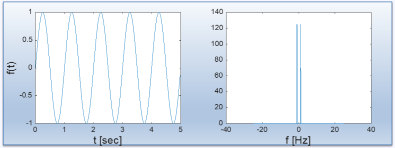
Discrete Fourier Transform

Leakage

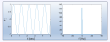


The spectrum is the one of the sinus multiplied with a rectangle:

$$g(t) = \sin(2\pi f_0 t)\mathrm{rect}\left(\frac{t - 0.5NT}{NT}\right)$$

In the example above, **exactly** 10 periods fit into NT.
In other words: $f_0 = \frac{n}{NT}$. In the next slide, we will look what happens, if n is not an integer.

Leakage



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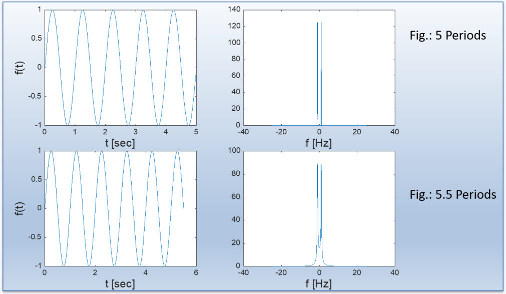
If f(t) does not (by chance) “fit” into the observation time, then an effect called leakage occurs (more correctly: Becomes visible)



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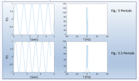

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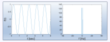

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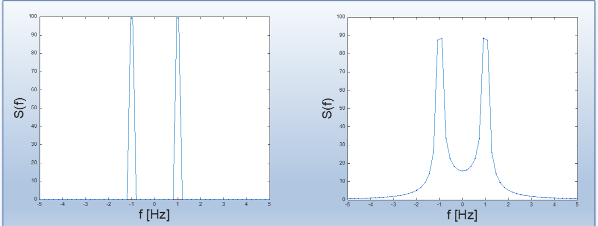
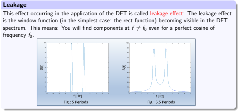


Fig.: 5 Periods
Fig.: 5.5 Periods

Zero of window function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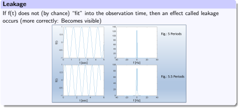
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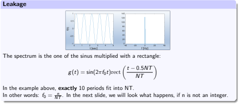
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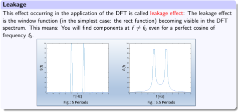
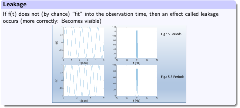
Zero of window function

Discrete Fourier Transform

Zero of window function





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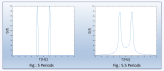

Fig.: 5 Periods   Fig.: 5.5 Periods

Leakage

For $p(t) = \delta(t)$ and

$$s(t) = \sin(\omega_0 t)\mathrm{rect}\left(\frac{t - 0.5TN}{NT}\right)$$

this leads to

$$S_d(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} S\left(\omega - n\frac{2\pi}{T}\right),$$

with

$$S(\omega) = NT\frac{\sin\left((\omega-\omega_0)\,0.5NT\right)}{(\omega-\omega_0)\,0.5NT} = NT\mathrm{si}\left((\omega-\omega_0)\,0.5NT\right)$$

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Leakage

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With

$$f_0 = ndf = \frac{n}{NT}$$

we get

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$$s_d(t) = \sum_{n=-\infty}^{\infty} s(nT)p(t-nT) = \left[s(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)\right] * p(t)$$

has the spectrum

$$S_d(\omega) = \frac{1}{T}P(\omega)\sum_{n=-\infty}^{\infty} S\left(\omega - n\frac{2\pi}{T}\right)$$

Leakage

$$S(\omega) = NT\,\mathrm{si}\left(\left(\omega - 2\pi\frac{n}{NT}\right)0.5NT\right)$$

The first 0 of the si-function is located at

$$\frac{\omega - \frac{2\pi n}{NT}}{2}NT = \pm\pi \Rightarrow \omega = \frac{2\pi}{NT}(n\pm 1) = (n\pm 1)2\pi df.$$

This means that the maximum appears exactly at one bin of the DFT:

$$0 = \frac{\omega_0 - \frac{2\pi n}{NT}}{2}NT \Rightarrow \omega_0 = \frac{2\pi n}{NT} = n2\pi df$$

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$$S(\omega) = NT\,\mathrm{si}\left((\omega - \omega_0)\,0.5NT\right)$$

With

$$f_0 = ndf = \frac{n}{NT}$$

we get

$$S(\omega) = NT\,\mathrm{si}\left(\left(\omega - 2\pi\frac{n}{nT}\right)0.5NT\right)$$

Leakage

For $p(t) = \delta(t)$ and

$$s(t) = \sin(\omega_0 t)\,\mathrm{rect}\left(\frac{t - 0.5TN}{NT}\right)$$

this leads to

$$S_d(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} S\left(\omega - n\frac{2\pi}{T}\right),$$

with

$$S(\omega) = NT\frac{\sin\left((\omega - \omega_0)\,0.5NT\right)}{(\omega - \omega_0)\,0.5NT} = NT\,\mathrm{si}\left((\omega - \omega_0)\,0.5NT\right)$$

Spectrum of periodic function

Spectrum of periodic function

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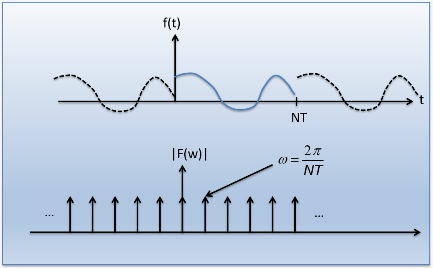
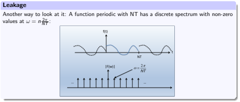
Leakage

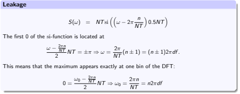
Another way to look at it: A function periodic with NT has a discrete spectrum with non-zero values at $\omega = n\frac{2\pi}{NT}$:



Discrete Fourier Transform



Spectrum of periodic function



Leakage

This can be shown by using the Fourier series: Each function periodic with NT can be expanded as a Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\frac{2\pi}{NT}},$$

$$F_n = \int_0^T f(t)e^{jn\frac{2\pi}{NT}}\,dt.$$

The corresponding Fourier transform is given by:

$$F(\omega) = \sum_{n=-\infty}^{\infty} F_n 2\pi\delta\left(\omega - n\frac{2\pi}{NT}\right)$$

In words: The spectrum of the DFT is equivalent to the Fourier series of the periodic repetition of the same signal

Periodicity of inverse transform

Leakage

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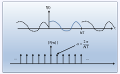
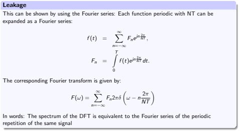
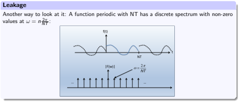


Spectrum of periodic function

Periodicity of inverse transform

Periodicity of inverse transform





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Another way to look at it:
One can easily show that the signal calculated by the inverse transform is periodic:

$$f_{n+N} = \sum_{k=0}^{N-1} F_k e^{\frac{j2\pi(n+N)k}{N}} \Bigg|_{n=0\ldots N-1} = f_n,$$

This means the DFT spectrum is equivalent to the spectrum of periodic repetition of the signal

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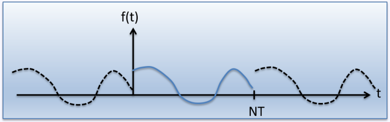
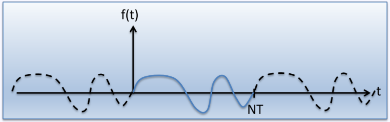


Strong leakage effect: “Jumps” of signal are seen in the spectrum



Leakage effect not visible

Leakage

This can be shown by using the Fourier series: Each function periodic with NT can be expanded as a Fourier series:

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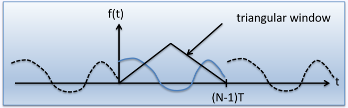
In words: The spectrum of the DFT is equivalent to the Fourier series of the periodic repetition of the same signal

Discrete Fourier Transform

Window functions

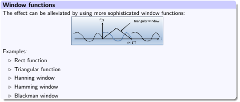
The effect can be alleviated by using more sophisticated window functions:



Examples:

- ▷ Rect function
- ▷ Triangular function
- ▷ Hanning window
- ▷ Hamming window
- ▷ Blackman window

Window functions



Properties

Applying window functions does not remove the leakage effect but windows can be chosen to either to

- ▷ get a sharp peak (resolve signals with similar strength and frequency)

or

- ▷ low sidelobes (resolve signals with dissimilar strength and frequency)

Window functions

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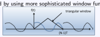

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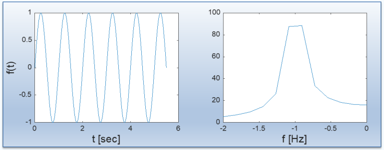
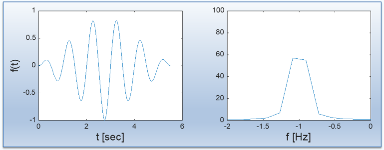


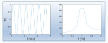
Rectangular window



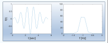
Triangular window

Window functions

Example



Rectangular window



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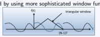

Examples:

- Rect function
- Triangular function
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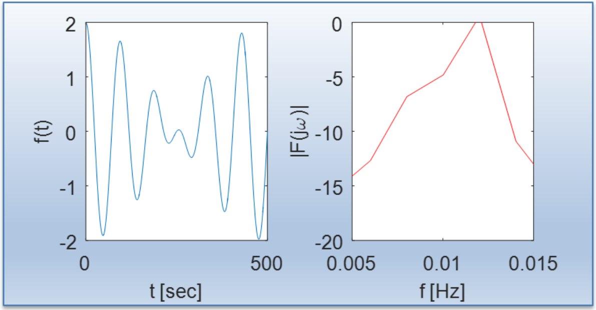
- Blackman window

Example

Rectangular window and signal with two sinus-functions:



Window functions

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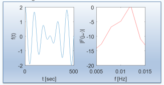

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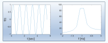

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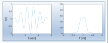

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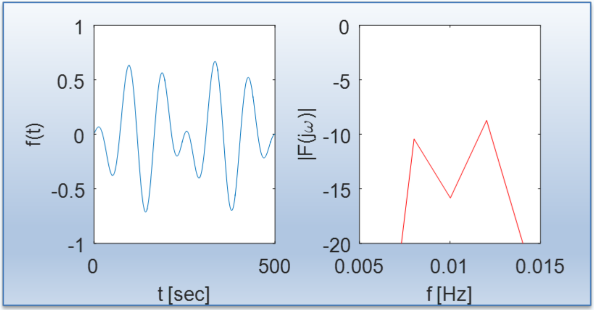
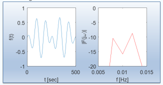


Exercise

Example

Triangular window and signal with two sinus-functions:



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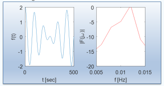

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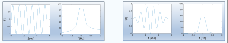

Rectangular window   Triangular window

Exercise (#6.3)

Consider the function

$$s(nT) \quad = \quad \sin\left(2\pi\frac{100}{512}nT\right) + \sin\left(2\pi\frac{110}{512}nT\right)$$

Plot the DFT-spectrum for $N = 512$, $T = 0.1\,\mathrm{sec}$ and

- ▷ using a triangular window function
- ▷ using a rectangular window function.

Use a meaningful frequency axis.

Exercise

Exercise (#6.3)

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Use a meaningful frequency axis.

Discrete Fourier Transform

Discrete Fourier Transform

Exercise

Exercise (#6.4)

- Setup a time vector `t = [0:0.1:10]`
- Calculate $s(t) = sin(2\pi ft)$ and plot the energy spectral density
- Start with f $= 1$ Hz and increase the frequency step by step. What happens?

Exercise

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- ▷ Setup a time vector t = [0:0.1:10]
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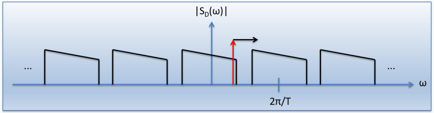
Example

What happens, if the signal contains a frequency above $2\pi/T$?



If the frequency of the signal becomes larger than π/T, then it is interpreted as a negative frequency!

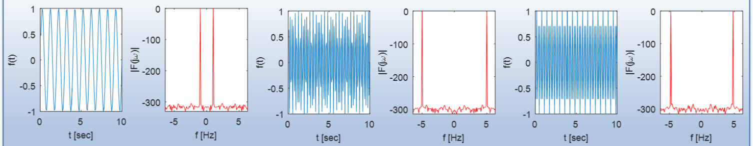


Fig.: f =1 Hz / T=78.125msec
Fig.: f =5 Hz / T=78.125msec
Fig.: f =8 Hz / T=78.125msec

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Discrete Fourier Transform

Exercise

Exercise (#6.5)

Given is $n = [1 \ldots 512]$ and $s_n = \sin\left(2\pi \frac{n}{512} 10\right)$.

- ▷ Add some noise to the signal (use `randn`)
- ▷ Calculate the FFT in NUMPY or MATLAB.
- ▷ Compare the spectrum for positive and negative frequencies (compare $S(-\omega)$ and $S(\omega)$).

Exercise

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Discrete Fourier Transform

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- Plot the magnitude of the spectrum in dB.
- Compare signal to noise ratio for different noise values.

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FFT Gain

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- Compare signal to noise ratio for different noise values.

Properties

- The FFT offers some sort of *gain*. This effect is often called FFT-gain.
- Rule of thumb: In case of increasing the number of samples by two, then the amplitude of the FFT increases by 6 dB (signal) and 3 dB (noise), respectively.

Discrete Fourier Transform

Exercise

Exercise (#6.7)

Given is $n = [1 \ldots 512]$ and $s_n = \sin\left(\frac{n}{512}10\right)$.

- ▷ Simulate the effect of discretization by making use of `round`[a] and different accuracies.
- ▷ Calculate the FFT in NumPy or MATLAB.
- ▷ Plot the magnitude of the spectrum in dB.
- ▷ Change the frequency of the signal.

[a] e.g. `x = round(x*10)/10`

Exercise

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Discretization

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Example

Numerical example using MATLAB.

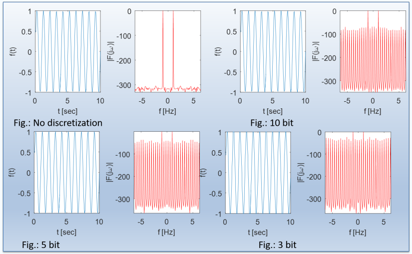

Fig.: No discretization

Fig.: 10 bit

Fig.: 5 bit

Fig.: 3 bit

Discrete Fourier Transform

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Discrete Time

Z-Transform

Discrete Fourier transform

Discrete Fourier Transform

$$F_k = \sum_{n=0}^{N-1} f_n e^{-\frac{j2\pi nk}{N}},$$

with a mapping to *real* frequencies as follows: $\omega_k = \frac{2\pi k}{NT}$.

Inverse transform

$$f_n = \sum_{k=0}^{N-1} F_k e^{\frac{j2\pi nk}{N}}$$

Z transform

Discrete Fourier Transform

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Discrete Fourier Transform

$$F_k = \sum_{n=0}^{N-1} f_n e^{-\frac{j2\pi nk}{N}}$$

Definition

The z transform is defined as follows:

$$\mathcal{Z}\left\{x_n\right\} = X(z) = \sum_{n=-\infty}^{\infty} x_n z^{-n}$$

Discrete Fourier Transform

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Properties

$$\mathcal{Z}\{x_n\} = X(z) = \sum_{n=-\infty}^{\infty} x_n z^{-n}$$

- z is a complex number!
- This series above does not necessarily converge for all z (see next slides)
- The (inverse) transform is only unique when specifying the region of convergence (ROC)

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Z-Transform

Convergence

The following behaviours can be observed:

1. $X(z)$ converges for all z with $|z| < \infty$
2. $X(z)$ converges for all z with $|z| < R_0$
3. **$X(z)$ converges for all** z with $|z| > R_0$
4. $X(z)$ converges for all z with $R_0 < |z| < R_1$
5. $X(z)$ converges only for $z = 0$

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Exercise (#6.8)

Consider the two transforms

$$\mathcal{Z}\left\{f_n = a^{n-1}u_{n-1}\right\}$$

and

$$\mathcal{Z}\left\{g_n = -a^{n-1}u_{-n}\right\}$$

- ▷ Sketch both series
- ▷ Calculate the Z transform

Note: Use $\sum\limits_{k=0}^{\infty} q^k = \frac{1}{1-q}$

Exercise

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Z transform

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Convergence

The following behaviours can be observed:

1. $X(z)$ converges for all z with $|z| < \infty$
2. $X(z)$ converges for all z with $|z| < R_0$
3. **$X(z)$ converges for all z with $|z| > R_0$**
4. $X(z)$ converges for all z with $R_0 < |z| < R_1$
5. $X(z)$ converges only for $z = 0$

Importance of the region of convergence

Both sequences have the same Z transform:

$$\mathcal{Z}\left\{f_n = a^{n-1}u_{n-1}\right\} = \frac{1}{z-a}, \text{ with ROC: } |z| > |a|$$

and

$$\mathcal{Z}\left\{g_n = -a^{n-1}u_{-n}\right\} = \frac{1}{z-a}, \text{ with ROC: } |z| < |a|$$

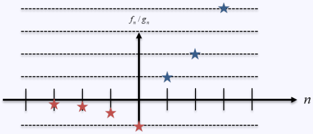

Figure 24: Both series for $a = 2$

Z transform

Importance of the region of convergence

Both sequences have the same Z transform:

$$\mathcal{Z}\left\{f_n = a^{n-1}u_{n-1}\right\} = \frac{1}{z-a}, \text{ with ROC: } |z| > |a|$$

and

$$\mathcal{Z}\left\{g_n = -a^{n-1}u_{-n}\right\} = \frac{1}{z-a}, \text{ with ROC: } |z| < |a|$$

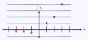

Figure 24: Both series for $a = 2$

Exercise (#6.8)

Consider the two transforms

$$\mathcal{Z}\left\{f_n = a^{n-1}u_{n-1}\right\}$$

and

$$\mathcal{Z}\left\{g_n = -a^{n-1}u_{-n}\right\}$$

- Sketch both series
- Calculate the Z transform

Note: Use $\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$

Convergence

The following behaviours can be observed:

1. $X(z)$ converges for all z with $|z| < \infty$
2. $X(z)$ converges for all z with $|z| < R_0$
3. $X(z)$ **converges for all** z with $|z| > R_0$
4. $X(z)$ converges for all z with $R_0 < |z| < R_1$
5. $X(z)$ converges only for $z = 0$

Convergence

In the following we will primarily consider causal signals (and systems): $x_n = 0 \;\forall\; n < 0$

$$\mathcal{Z}\left\{x_n\right\} = X(z) = \sum_{n=0}^{\infty} x_n z^{-n}.$$

For such signals: If the series converges for any radius r_0, then the series will converge for $r_1 > r_0$ as well.

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Z-Transform

Properties

$$
\begin{aligned}
X(z) &= \mathcal{Z}\{x_n\} = \sum_{n=-\infty}^{\infty} x_n z^{-n} \\
\mathcal{Z}\{af_n + bg_n\} &= a\mathcal{Z}\{f_n\} + b\mathcal{Z}\{g_n\} \\
\mathcal{Z}\{a^n f_n\} &= F\left(\frac{z}{a}\right) \\
\mathcal{Z}\{f_{n-1}\} &= \mathcal{Z}\{f_n\} z^{-1} \\
\mathcal{Z}\{nf_n\} &= -z\frac{\partial F(z)}{\partial z}\mathcal{Z}\{f_n\} \\
\mathcal{Z}\{f_{-n}\} &= F\left(z^{-1}\right) \\
\mathcal{Z}\{f_n^*\} &= F^*\left(z^*\right)
\end{aligned}
$$

Z transform

Properties

$$
\begin{aligned}
X(z) &= \mathcal{Z}\{x_n\} = \sum_{n=-\infty}^{\infty} x_n z^{-n} \\
\mathcal{Z}\{af_n + bg_n\} &= a\mathcal{Z}\{f_n\} + b\mathcal{Z}\{g_n\} \\
\mathcal{Z}\{a^n f_n\} &= F\left(\frac{z}{a}\right) \\
\mathcal{Z}\{f_{n-1}\} &= \mathcal{Z}\{f_n\} z^{-1} \\
\mathcal{Z}\{nf_n\} &= -z\frac{\partial F(z)}{\partial z}\mathcal{Z}\{f_n\} \\
\mathcal{Z}\{f_{-n}\} &= F\left(z^{-1}\right) \\
\mathcal{Z}\{f_n^*\} &= F^*\left(z^*\right)
\end{aligned}
$$

Properties

$$\mathcal{Z}\{f_n * g_n\} = \mathcal{Z}\left\{\sum_{l=-\infty}^{\infty} f_l g_{n-l}\right\} = F(z)G(z)$$

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Z-Transform

Unilateral transform

Definition

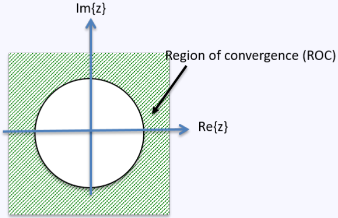
The z-Transform of an infinite series f_n (with $n \geq 0$) is given as follows:

$$\mathcal{Z}\left\{f_n\right\} = \sum_{k=0}^{\infty} f_k z^{-k}$$

Z transform

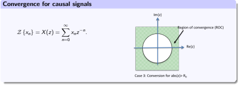
Convergence for causal signals

$$\mathcal{Z}\left\{x_n\right\} = X(z) = \sum_{n=0}^{\infty} x_n z^{-n}.$$



Case 3: Conversion for abs(z)> R_0

Z transform



Transform pairs

Time domain	$z-$domain
u_n	$\frac{z}{z-1}$
$u_n a^n$	$\frac{z}{z-a}$

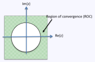
Z transform

Transform pairs

Time domain	$z-$domain
u_n	$\frac{z}{z-1}$
$u_n a^n$	$\frac{z}{z-a}$

Convergence for causal signals

$$\mathcal{Z}\{x_n\} = X(z) = \sum_{n=0}^{\infty} x_n z^{-n}.$$



Case 3: Conversion for abs(z)> R_0

Transform pairs

Time domain	Laplace Domain	$z-$domain
$\delta(t)$	1	
Unit step	$\frac{1}{s}$	$\frac{z}{z-1}$
t	$\frac{1}{s^2}$	$\frac{T\cdot z}{(z-1)^2}$

Properties

Table 13.2 Properties of the z-Transform

	$x(t)$	$X(z)$
1.	$kx(t)$	$kX(z)$
2.	$x_1(t) + x_2(t)$	$X_1(z) + X_2(z)$
3.	$x(t+T)$	$zX(z) - zx(0)$
4.	$tx(t)$	$-Tz\dfrac{dX(z)}{dz}$
5.	$e^{-at}x(t)$	$X(ze^{aT})$
6.	$x(0)$, initial value	$\lim_{z\to\infty} X(z)$ if the limit exists
7.	$x(\infty)$, final value	$\lim_{z\to 1}(z-1)X(z)$ if the limit exists and the system is stable; that is, if all poles of $(z-1)X(z)$ are inside the unit circle $\lvert z\rvert = 1$ on z-plane.

Unilateral transform

Properties

Table 13.1 z-Transforms

$x(t)$	$X(s)$	$X(z)$
$\delta(t) = \begin{cases} \frac{1}{\epsilon}, & t < \epsilon, \epsilon \to 0 \\ 0 & \text{otherwise} \end{cases}$	1	—
$\delta(t-a) = \begin{cases} \frac{1}{\epsilon}, & a < t < a + \epsilon, \epsilon \to 0 \\ 0 & \text{otherwise} \end{cases}$	e^{-as}	—
$\delta_o(t) = \begin{cases} 1 & t = 0, \\ 0 & t = kT, k \neq 0 \end{cases}$	—	1
$\delta_o(t-kT) = \begin{cases} 1 & t = kT, \\ 0 & t \neq kT \end{cases}$	—	z^{-k}

Unilateral transform

Properties

Table 13.1 z-Transforms

$x(t)$	$X(s)$	$X(z)$
$u(t)$, unit step	$1/s$	$\frac{z}{z-1}$
t	$1/s^2$	$\frac{Tz}{(z-1)^2}$
e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$
$1-e^{-at}$	$\frac{1}{s(s+a)}$	$\frac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})}$

Unilateral transform

Properties

Table 13.1 z-Transforms

$x(t)$	$X(s)$	$X(z)$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$	$\frac{z\sin(\omega T)}{z^2 - 2z\cos(\omega T) + 1}$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$	$\frac{z(z - \cos(\omega T))}{z^2 - 2z\cos(\omega T) + 1}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s + a)^2 + \omega^2}$	$\frac{(ze^{-aT}\sin(\omega T))}{z^2 - 2ze^{-aT}\cos(\omega T) + e^{-2aT}}$
$e^{-at}\cos(\omega t)$	$\frac{s + a}{(s + a)^2 + \omega^2}$	$\frac{z^2 - ze^{-aT}\cos(\omega T)}{z^2 - 2ze^{-aT}\cos(\omega T) + e^{-2aT}}$

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Z-Transform

Mapping to a frequency

$$\mathcal{Z}\left\{x_n\right\} = X(z) = \sum_{n=-\infty}^{\infty} x_n z^{-n}$$

The following subtitution maps to a *normalized* frequency:

$$z = e^{j\omega}$$

The following subtitution maps to a frequency (sampling period T):

$$z = e^{j\omega T}$$

Mapping to a frequency

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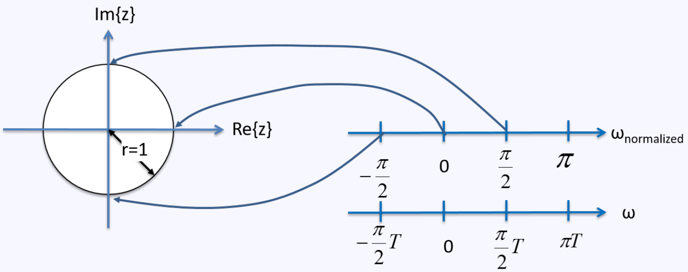
Mapping to a frequency



Note: In case of undersampling, signal components with different frequencies map to the same position on the circle.

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Z-Transform

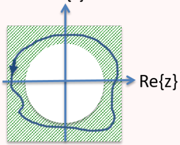
Inverse Z transform

Definition

The inverse transform is defined as follows:

$$f_n = \frac{1}{2\pi j} \oint F(z) z^{n-1} dz.$$

This is a counterclockwise integral with a path inside the region of convergence (independent of the integration path, can be solved with Cauchy's residue theorem). This means that the inverse transform only depends on the poles of $F(z)$.



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Properties

- We will use the Z-transform for the analysis of discrete-time LTI systems.
- Thus, we will use
  - Z-transform tables,
  - partial fraction decomposition,
  - polynominal long division,

  and other techniques of signal decomposition for transformations
- Using well-known structures (see next block) will give you the ability to implement e.g. filters without applying the inverse transformation.

Inverse Z transform

Properties

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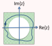

Polynomial long division

Bring

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots}{a_0 + a_1 z^{-1} + \cdots}$$

into the form

$$\begin{aligned} H(z) &= c_0 + c_1 z^{-1} + c_2 z^{-2} + \cdots \\ y_n &= c_0 x_n + c_1 x_{n-1} + c_2 x_{n-2} + \cdots = x_n * h_n \\ &= \sum_{l=0}^{\infty} h_l x_{n-l} \end{aligned}$$

Inverse Z transform

Polynomial long division

Bring

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots}{a_0 + a_1 z^{-1} + \cdots}$$

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Properties

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Definition

The inverse transform is defined as follows:

$$f_n = \frac{1}{2\pi j} \oint F(z) z^{n-1} dz.$$

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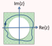

Example

$$\begin{aligned} H(z) &= \frac{3z^{-1} + z^{-2} + 2}{z^{-1} + 1} \\ &= 2 + z^{-1} \Rightarrow h_0 = 2, h_1 = 1 \end{aligned}$$

Note: The polyomial division gives an infinite result in case of an IIR filter (see next chapter)

Z-Transform

Inverse Z transform

Properties

If the unit circle belongs to the region of convergence, then one can integrate on the unit circle (substitution):

$$\begin{aligned} f_n &= \frac{1}{2\pi j}\oint F(z) z^{n-1} dz \\ &= \frac{1}{2\pi j}\int_{-\pi}^{\pi} F\left(z = e^{j\omega}\right) e^{j\omega(n-1)} j e^{j\omega} d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} F\left(z = e^{j\omega}\right) e^{j\omega n} d\omega \end{aligned}$$

Inverse Z transform

Properties

If the unit circle belongs to the region of convergence, then one can integrate on the unit circle (substitution):

$$\begin{aligned} f_n &= \frac{1}{2\pi j}\oint F(z)z^{n-1}dz \\ &= \frac{1}{2\pi j}\int_{-\pi}^{\pi} F\left(z=e^{j\omega}\right) e^{j\omega(n-1)} j e^{j\omega} d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} F\left(z=e^{j\omega}\right) e^{j\omega n} d\omega \end{aligned}$$

Properties

Unit circle belonging to the region of convergence:

$$f_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} F\left(z=e^{j\omega}\right) e^{j\omega n} d\omega$$

This is equivalent to the inverse Fourier transform:

$$\mathcal{F}^{-1}\left\{F_d(\omega)\mathrm{rect}\left(\frac{\omega}{2\pi}\right)\right\}(t=nT) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_d(\omega)e^{j\omega nT}d\omega,$$

with $F_d(\omega)$ being the Fourier transform of the signal $f_d(t) = \sum_{n=-\infty}^{\infty} f_n\delta(t-nT)$.

Inverse Z transform

Properties

Unit circle belonging to the region of convergence:

$$f_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} F\left(z = e^{j\omega}\right) e^{j\omega n} d\omega$$

This is equivalent to the inverse Fourier transform:

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Properties

If the unit circle belongs to the region of convergence, then one can integrate on the unit circle (substitution):

$$\begin{aligned} f_n &= \frac{1}{2\pi j}\oint F(z)z^{n-1}dz \\ &= \frac{1}{2\pi j}\int_{-\pi}^{\pi} F\left(z = e^{j\omega}\right) e^{j\omega(n-1)} j e^{j\omega} d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} F\left(z = e^{j\omega}\right) e^{j\omega n} d\omega \end{aligned}$$

Properties

$$\begin{aligned} f_d(t) &= \sum_{n=-\infty}^{\infty} f_n\delta(t-nT) \\ F_d(\omega) &= \int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f_n\delta(t-nT)e^{-j\omega t}dt \\ &= \sum_{n=-\infty}^{\infty} f_n e^{-j\omega nT}\int_{-\infty}^{\infty}\delta(t-nT)dt = \underbrace{\sum_{n=-\infty}^{\infty} f_n e^{-j\omega nT}}_{\text{discrete Fourier transform}} \end{aligned}$$

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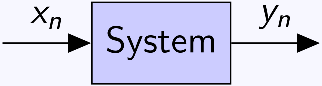
Discrete Time

Inverse transform using Fourier

LTI

We now consider the LTI system shown below:



With the transfer function $H(z)$ and impulse response h_n the following relations hold true:

$$
\begin{aligned}
y_n &= x_n * h_n = h_n * x_n = \sum_{n=-\infty}^{\infty} x_l h_{n-l} \\
H(z) &= \mathcal{Z}\left\{h_n\right\} \\
Y(z) &= X(z)H(z)
\end{aligned}
$$

Frequency response

$$
\begin{aligned}
H(z) &= \mathcal{Z}\{h_n\} \\
Y(z) &= X(z)H(z)
\end{aligned}
$$

The frequency response can be calculated by letting

$$z = e^{j\omega T}.$$

Note: In the literature T is most times omitted (set to 1) for a “normalized” frequency:

$$z = e^{j\omega}.$$

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Time discrete LTI systems

Stability

The system is BIBO stable (bounded input, bounded output) if and only if

$$\sum_{n=0}^{\infty} |h_n| < \infty$$

Stability

Stability can also be checked in the z-domain. Usually H(Z) can be written in the following form:

$$H(z) = \frac{N(z)}{D(z)}.$$

The system is stable, if the poles of H(z) lie inside the unity circle:

$$D(z) \neq 0 \text{ for } |z| \leq 1$$

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Time discrete LTI systems

Classification

Finite Impulse Response (FIR)

- h_n has only an endless number of values unequal to 0
- non-recursive
- always stable

Infinite Impulse Response (IIR)

- h_n has an unlimited number of values unequal to 0
- recursive

Classification

Finite Impulse Response (FIR)

- ▷ h_n has only an endless number of values unequal to 0
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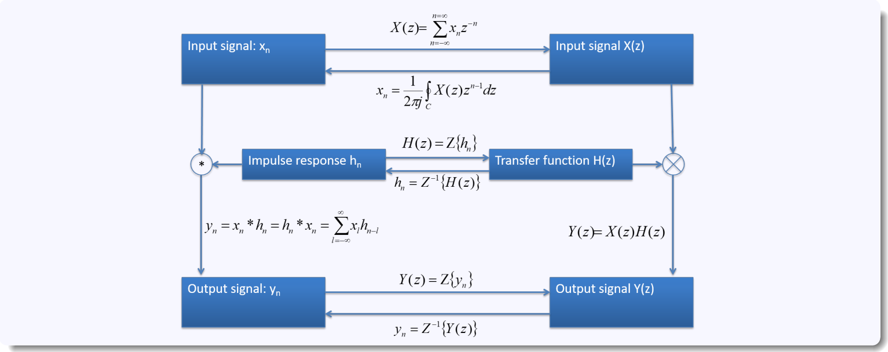
- ▷ always stable

Infinite Impulse Response (IIR)

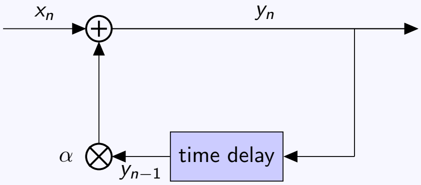
- ▷ h_n has an unlimited number of values unequal to 0
- ▷ recursive



Time discrete LTI systems

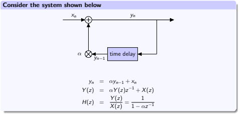
Example system

Consider the system shown below

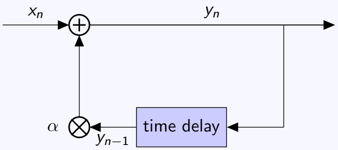


$$
\begin{aligned}
y_n &= \alpha y_{n-1} + x_n \\
Y(z) &= \alpha Y(z) z^{-1} + X(z) \\
H(z) &= \frac{Y(z)}{X(z)} = \frac{1}{1-\alpha z^{-1}}
\end{aligned}
$$

Example system



Finding the poles



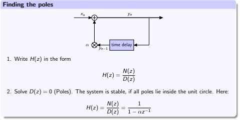
1. Write $H(z)$ in the form

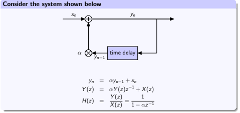
$$H(z) = \frac{N(z)}{D(z)}$$

2. Solve $D(z) = 0$ (Poles). The system is stable, if all poles lie inside the unit circle. Here:

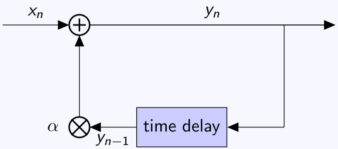
$$H(z) = \frac{N(z)}{D(z)} = \frac{1}{1 - \alpha z^{-1}}$$

Example system





Finding the poles



2. Solve $D(z) = 0$ (Poles).

$$
\begin{aligned}
0 &= 1 - \alpha z^{-1} \\
z &= \alpha z
\end{aligned}
$$

This system is stable if

$$|\alpha| < 1$$

Time discrete LTI systems

Exercise (#6.9)

Given is a system with the transfer function

$$H(z) = \frac{z}{z^2 - \frac{1}{4}z - \frac{3}{8}}.$$

Calculate the impulse response.

Exercise

Exercise (#6.9)

Given is a system with the transfer function

$$H(z) = \frac{z}{z^2 - \frac{1}{4}z - \frac{3}{8}}.$$

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Exercise (#6.9)

Given is a system with the transfer function

$$H(z) = \frac{z}{z^2 - \frac{1}{4}z - \frac{1}{8}}.$$

Calculate the impulse response.

Exercise (#6.10)

Given is a system with the transfer function

$$H(z) = \frac{z-1}{z+\frac{1}{2}}.$$

Sketch the impulse response.

Exercise

Exercise (#6.10)

Given is a system with the transfer function

$$H(z) = \frac{z-1}{z+\frac{1}{2}}.$$

Sketch the impulse response.

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Discrete Time

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Special classes of time discrete systems

Standard form

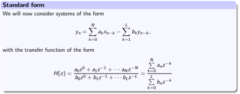
We will now consider systems of the form

$$y_n = \sum_{k=0}^{N} a_k x_{n-k} - \sum_{k=1}^{L} b_k y_{n-k},$$

with the transfer function of the form

$$H(z) = \frac{a_0 z^0 + a_1 z^{-1} + \cdots a_N z^{-N}}{b_0 z^0 + b_1 z^{-1} + \cdots b_L z^{-L}} = \frac{\sum\limits_{k=0}^{N} a_n z^{-k}}{\sum\limits_{k=0}^{L} b_n z^{-k}}$$

Poles and Zeros (holes)


Standard form

We will now consider systems of the form

$$y_n = \sum_{k=0}^{N} a_k x_{n-k} - \sum_{k=1}^{L} b_k y_{n-k},$$

with the transfer function of the form

$$H(z) = \frac{a_0 z^0 + a_1 z^{-1} + \cdots a_N z^{-N}}{b_0 z^0 + b_1 z^{-1} + \cdots b_L z^{-L}} = \frac{\sum_{k=0}^{N} a_n z^{-k}}{\sum_{k=0}^{L} b_n z^{-k}}$$

The transfer function of the form

$$H(z) = \frac{\sum_{k=0}^{N} a_n z^{-k}}{\sum_{k=0}^{L} b_n z^{-k}} = \frac{N(z)}{D(z)}$$

can be modified using the fundamental theorem of algebra as follows:

$$H(z) = \frac{a_N \prod_{k=0}^{N} (z - h_k)}{b_L \prod_{k=0}^{L} (z - p_k)}$$

Partial fraction decomposition

The transfer function of the form

$$H(z) = \frac{\sum_{k=0}^{N} a_n z^{-k}}{\sum_{k=0}^{L} b_n z^{-k}} = \frac{N(z)}{D(z)}$$

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We will now consider systems of the form

$$y_n = \sum_{k=0}^{N} a_k x_{n-k} - \sum_{k=1}^{L} b_k y_{n-k},$$

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$$H(z) = \frac{a_0 z^0 + a_1 z^{-1} + \cdots a_N z^{-N}}{b_0 z^0 + b_1 z^{-1} + \cdots b_L z^{-L}} = \frac{\sum_{k=0}^{N} a_n z^{-k}}{\sum_{k=0}^{L} b_n z^{-k}}$$

Properties

In case of $N(z)$ being a polynom with smaller grade than $D(z)$ and O_k the order of the pole k one can use the partial fraction composition to derive

$$H(z) = \sum_{k=1}^{N} \sum_{l=1}^{O_k} \frac{a_{k,l}}{(z - p_k)^l}.$$

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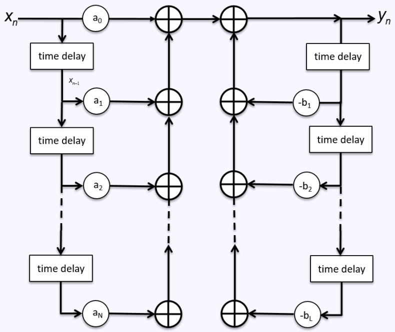
Special classes of time discrete systems

Direct form I

Direct form I



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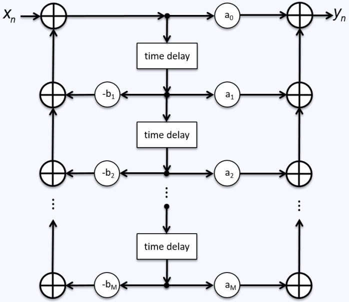
Special classes of time discrete systems

Direct form II



$$m = \max(N, L)$$

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Discrete Time

Relationship between different transforms

Laplace and Fourier

Properties

If

$f(t < 0) = 0$

and if

the imaginary axis belongs to the ROC,

then

$$\mathcal{L}\left\{f(t)\right\}(s = j\omega) = \mathcal{F}\left\{f(t)\right\}$$

Note: Frequently, the Fourier transform is written as follows:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}$$

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Relationship between different transforms

Laplace and Z-transform

	continuous time	**discrete time**
Variable	t	n
Transformation	One-sided Laplace	Unilateral Z-transform[a]
Frequency domain variable	s	z
Area of convergence	Half spaces	$r > r_0$
Frequency response	$s = j\omega$	$z = e^{j\omega T}$
Stability condition for poles	$Re\{s_p\} < 0$	$\|z_p\| < 1$
Stability condition in time domain	$\int\limits_0^\infty \|h(t)\|dt < \infty$	$\sum\limits_{n=0}^{\infty} \|h_n\| < \infty$

[a] In the following we will only consider the unilateral transform. For properties of the bilateral Z-transform: See the according chapter

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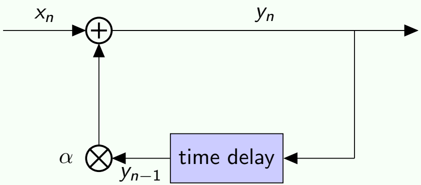
Discrete Time

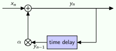
Exercise

Exercise (#6.11)

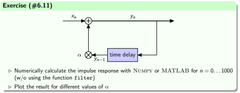


- Numerically calculate the impulse response with NUMPY or MATLAB for $n = 0 \ldots 1000$ (w/o using the function `filter`)
- Plot the result for different values of α

Exercise (#6.11)



- Numerically calculate the impulse response with NumPy or MATLAB for $n = 0 \ldots 1000$ (w/o using the function `filter`)
- Plot the result for different values of α



Exercise (#6.12)

Given is a system with the transfer function

$$H(z) = 1 - z^{-1}$$

- ▷ Find the impulse response h_n
- ▷ Write y_n as a function of x_n
- ▷ Sketch the magnitude of the transfer function as a function of the frequency ω.
- ▷ Find an analog circuit with a *similar* transfer function.

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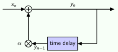

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- Plot the result for different values of α

Exercise (#6.13)

Given is a system with the transfer function

$$H(z) = \frac{z^2 + z}{z^3 - 2z^2 + z - 0.5}$$

- Find the impulse response h_n
- Find the impulse response making use of NUMPY or MATLAB and the function `impz`.

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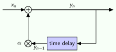

- ▷ Numerically calculate the impulse response with NUMPY or MATLAB for $n = 0 \ldots 1000$ (w/o using the function `filter`)
- ▷ Plot the result for different values of α

Exercise (#6.14)

Given is the file `B6E14.csv` which contains a heartbeat signal (first column: time in m sec).

- ▷ Plot the spectrum of the signal using meaningful axis.
- ▷ Interpret the result.

Exercise

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