

LTI systems

3.1 Introduction

3.2 Mathematical foundations: Differential equations

3.3 LTI systems

3.4 Transfer function

3.5 Impulse response

3.6 Causality

3.7 Stability

3.8 Network of systems

3.9 Graphical representation

Content

- ▷ Differential equations
- ▷ LTI systems
- ▷ Frequency-domain representations

Study goals

- ▷ Describe LTI systems in time and frequency domain
- ▷ Use the concepts of transfer functions and impulse response
- ▷ Discuss stability of LTI systems
- ▷ Make use of Bode plots

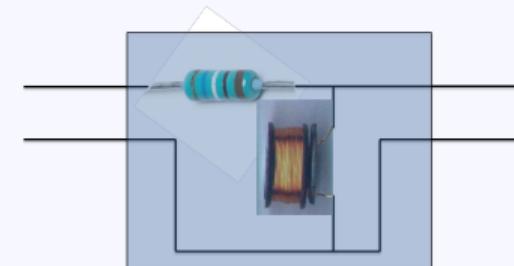
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What is a system?



- ▷ A system is “something” with input and output
- ▷ The system “reacts” to its environment
- ▷ Usually, models are used for the description of systems. I.e. real-world behavior is simplified

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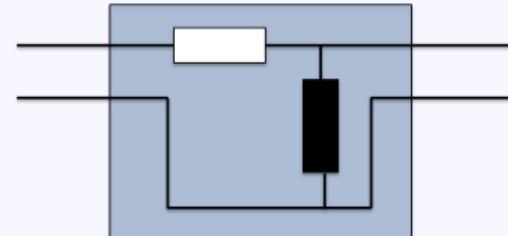
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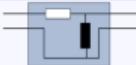
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- ▷ Here we can define voltages as input and output
- ▷ A simple model for this circuit is shown above
- ▷ The output voltage can be calculated as follows (for open terminals):

$$U_{out} = U_{in} \frac{j\omega L}{R + j\omega L}.$$

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Physical systems...

- ... frequently have a dynamic behaviour that can be described by differential equations
- ... might be non-linear
- ... can often be linearized by making use of the Taylor series / Jacobi-matrix
- ... can usually be subdivided into sub-systems with corresponding block diagrams

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Network analysis

- ▷ In network analysis, we normally make the following assumptions:
 - ▷ Networks are linear
 - ▷ Networks are time invariant
- ▷ Analysis is usually done in the frequency domain:

$$U_{out} = U_{in} \frac{j\omega L}{R + j\omega L}.$$

- ▷ The fractional term can be interpreted as a transfer function $H(\omega)$: The frequency dependent reaction of the system to the input.

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3.2.1 Definitions

3.2.2 Second-order differential equation

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Mathematical foundations: Differential equations

3.2 Mathematical foundations: Differential equations

3.2.1 Definitions

3.2.2 Second-order differential equation

Definition

A *ordinary differential equation* is an equality involving one or more dependent variables, one independent variable, and one or more derivatives of the dependent variables, with respect to the independent variable.

Williams et al., Feedback and Control Systems

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Definition

An equation of the form

$$\sum_{i=0}^n a_i \frac{d^i}{dt^i} y(t) = \sum_{i=0}^m b_i \frac{d^i}{dt^i} u(t)$$

is called a **linear constant-coefficient ordinary differential equation**.

Definition

The **initial condition** is a set of equations of the form

$$y(0) = A_0, \frac{d}{dt}y(t=0) = A_1, \dots, \frac{d^{n-1}}{dt^{n-1}}y(t=0) = A_{n-1}.$$

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DefinitionA **linear differential equation** is an equality involving one or more dependent variables, one independent variable, and one or more derivatives of the dependent variables, with respect to the independent variable.Williams et al., *Feedback and Control Systems***Definition**

The solution for an input $u(t) = 0$ is called the **free response**. The differential equation then has the form:

$$\sum_{i=0}^n a_i \frac{d^i}{dt^i} y(t) = 0.$$

Definition

The free response is given by a linear combination of the so-called **fundamental set**:

$$y(t) = \sum_{i=1}^n C_i y_i(t).$$

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The coefficients C_i are determined by the initial condition

$$y(0) = A_0, \quad \frac{d}{dt}y(t=0) = A_1, \dots, \frac{d^{n-1}}{dt^{n-1}}y(t=0) = A_{n-1}.$$

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Example

$$\dot{x}(t) = C_1 x(t)$$

has the solution

$$x = A_0 e^{C_1 t}$$

in case of $x(t=0) = A_0$.

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└ Mathematical foundations: Differential equations

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and is the forced response in case of $u(t)$ being the step function.**Definition**

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Mathematical foundations: Differential equations

3.2 Mathematical foundations: Differential equations

3.2.1 Definitions

3.2.2 Second-order differential equation

Definition

The system described by the second-order differential equation

$$\frac{d^2}{dt^2}y(t) + 2\sigma\omega_0 \frac{d}{dt}y(t) + \omega_0^2 y(t) = \omega_0^2 u(t)$$

has the unit step response

$$y(t) = 1 - \frac{e^{-\sigma\omega_0 t}}{\sqrt{1-\sigma^2}} \sin\left(\omega_0 \sqrt{1-\sigma^2} t + \phi_0\right),$$

with

$$\phi_0 = \tan^{-1}\left(\frac{\sqrt{1-\sigma^2}}{\sigma}\right)$$

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Exercise (#3.1)

Plot the unit step response ($x(t) = u(t)$, $y(t \leq 0) = 0$) of a system described by the second-order differential equations

$$\frac{d^2}{dt^2}y(t) + 2\sigma\omega_0 \frac{d}{dt}y(t) + \omega_0^2 y(t) = \omega_0^2 x(t)$$

for different values of ω_0 and σ by making of NUMPY or MATLAB. Use the equation shown on the former slide.

- ▷ What happens in case of negative σ ?
- ▷ What happens if you set $\phi_0 = 0$? Does the result fulfill the requirements of a unit step response (check initial conditions)?

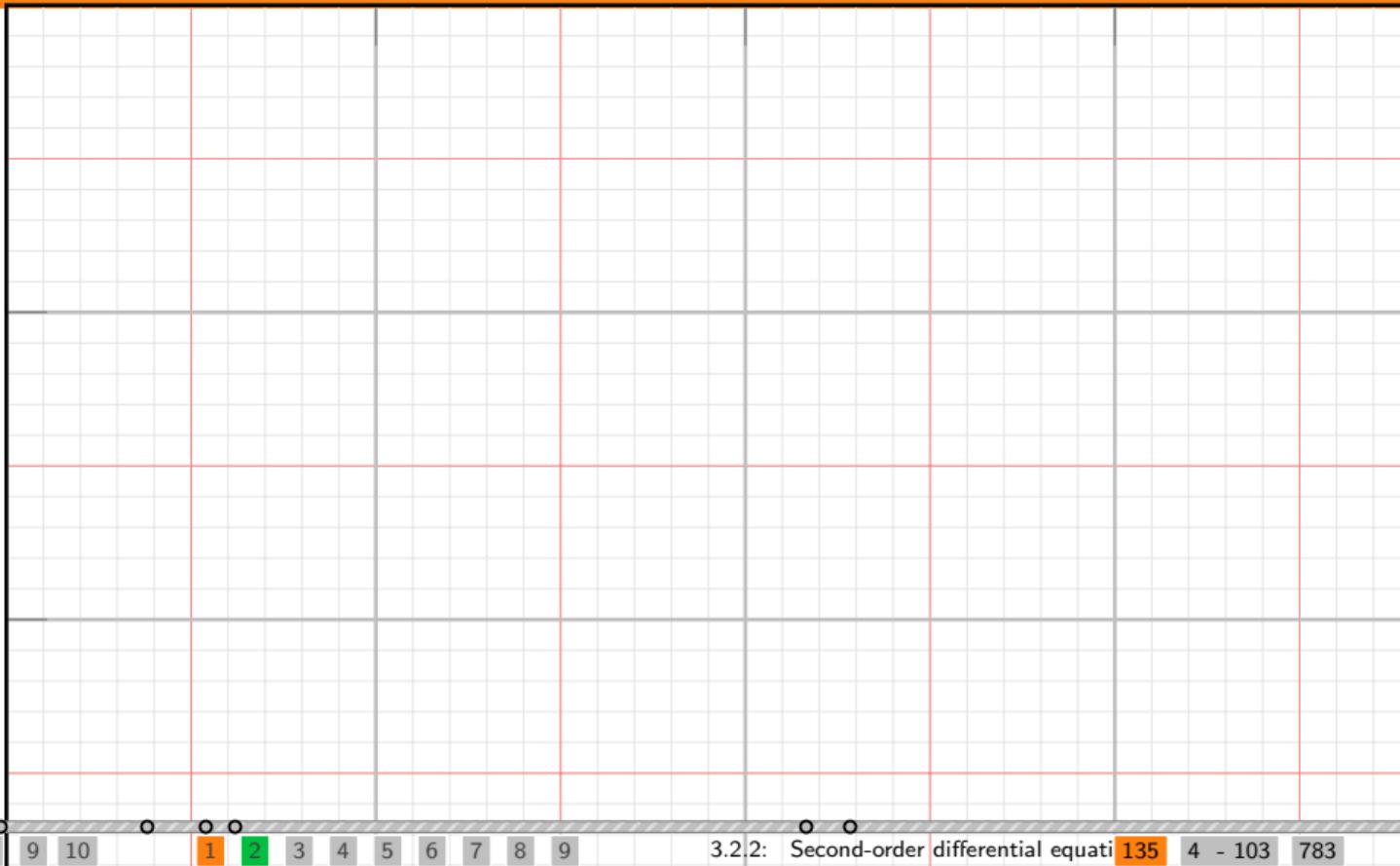
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Plot the unit step response ($x(t) = u(t)$, $y(t \leq 0) = 0$) of a system described by the second-order differential equation

$$\frac{d^2}{dt^2}y(t) + 2\zeta\omega_0 \frac{d}{dt}y(t) + \omega_0^2 y(t) = \omega_0^2 u(t)$$

for different values of ζ and ω_0 by making of Numpy or MATLAB. Use the equation shown on the former slide.

- > What happens in case of negative ζ ?
- > What happens if you set $\zeta_0 := 0$? Does the result fulfill the requirements of a unit step response (check initial conditions)?



└ LTI systems

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Reduction to second order

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for different values of ω_0 and σ by making of **Numerical** or MATLAB. Use the equation shown on the former slide.

- ▷ What happens in case of negative σ^2 ?
- ▷ What happens if you set $\omega_0 = 0$? Does the result fulfill the requirements of a unit step response (check initial conditions)?

Spoiler

We will later discuss how to reduce higher order differential equations to first/second order.

Definition

The system described by the second-order differential equation

$$\frac{d^2}{dt^2}y(t) + 2\sigma\omega_0 \frac{d}{dt}y(t) + \omega_0^2 y(t) = \omega_0^2 u(t)$$

has the unit step response

$$y(t) = 1 - \frac{\omega_0 e^{-\sigma t}}{\sqrt{1-\sigma^2}} \sin(\omega_0 \sqrt{1-\sigma^2} t + \phi_0),$$

with

$$\phi_0 = \tan^{-1}\left(\frac{\sqrt{1-\sigma^2}}{\sigma}\right)$$

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Definition

In the following we consider **LTI-systems** (Linear Time-Invariant)

- ▷ One can show that if we stimulate an LTI system with an input

$$x(t) = C_1 e^{j\omega_0 t}$$

the output will be of the form

$$y(t) = C_2 e^{j\omega_0 t} = H(\omega_0)x(t)$$

- ▷ Note: C_1 and C_2 are complex and ω_0 is a real number



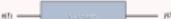
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Multiple (additive) inputs

We will now write the signals in slightly different form:

$$\begin{aligned} x(t) &= C_1 e^{j\omega t} = \frac{1}{2\pi} X(\omega) e^{j\omega t} \\ y(t) &= C_2 e^{j\omega t} = \frac{1}{2\pi} X(\omega) H(\omega) e^{j\omega t} \end{aligned}$$

The transfer function: Relationship between input and output in the frequency domain. For N-input signals:

$$\begin{aligned} x(t) &= \sum_i^N \frac{1}{2\pi} X_i(\omega_i) e^{j\omega_i t} \\ y(t) &= \sum_i^N \frac{1}{2\pi} X_i(\omega_i) H(\omega_i) e^{j\omega_i t} \end{aligned}$$

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Transfer function

In more general form:

$$x(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} X(\omega) e^{j\omega t} d\omega$$

$$y(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} X(\omega) H(\omega) e^{j\omega t} d\omega$$

It becomes obvious that the output signal is given by an inverse Fourier-Transform:

$$y(t) = \mathcal{F}^{-1} \{X(\omega)H(\omega)\}.$$

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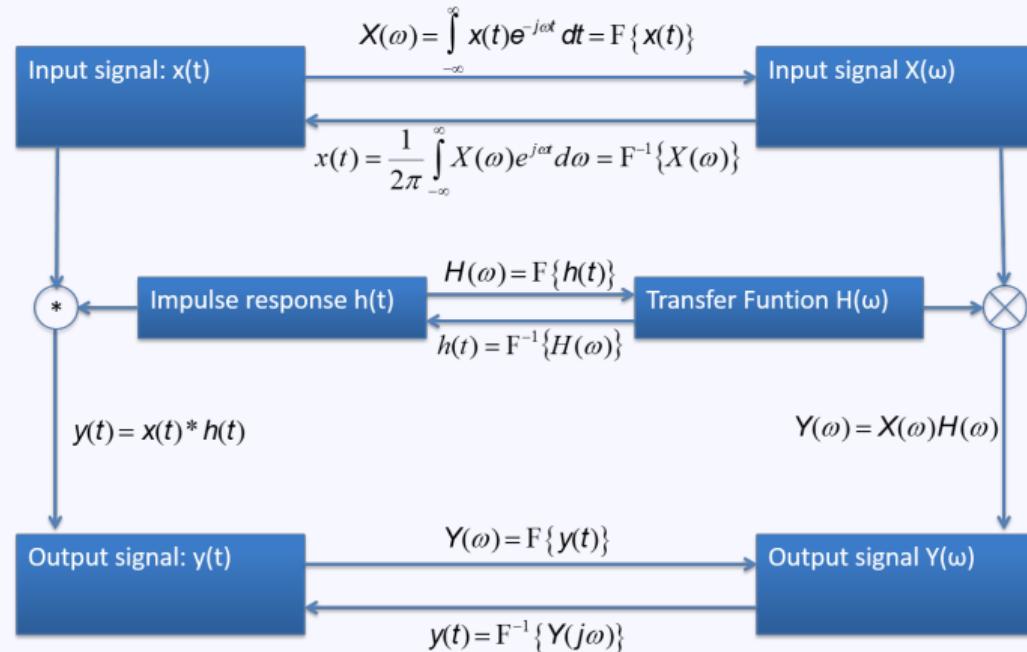
$$x(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} X(j\omega) e^{j\omega t} d\omega$$

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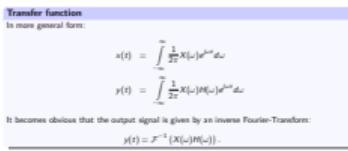
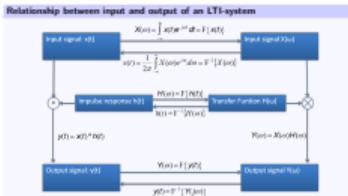
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Relationship between input and output of an LTI-system



Using the Laplace transform



Definition

Using the **Laplace Transform**, the output of an LTI system is given by

$$Y(s) = H(s)X(s) + Y_{init}(s),$$

with $Y_{init}(s)$ being terms due to all initial conditions. If one ignores all terms arising from initial conditions, then the **transfer function** is of the form

$$H(s) = \frac{Y(s)}{X(s)}.$$

- └ LTI systems
- └ Transfer function

Definition

Using the [Laplace Transform](#), the output of an LTI system is given by

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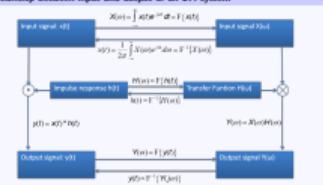
$$H(s) = \frac{Y(s)}{X(s)}$$

Exercise (#3.2)

Plot the step response for a system with the transfer function

$$H(s) = \frac{s - 1}{s^2 + 2s + 1}$$

using [NUMPY](#) or [MATLAB](#) with and without using the built-in function `step`.

Relationship between input and output of an LTI-system**Transfer function**

In more general form:

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} \frac{1}{2\pi} X(\omega)e^{j\omega t} d\omega \\ y(t) &= \int_{-\infty}^{\infty} \frac{1}{2\pi} X(\omega)H(\omega)e^{j\omega t} d\omega \end{aligned}$$

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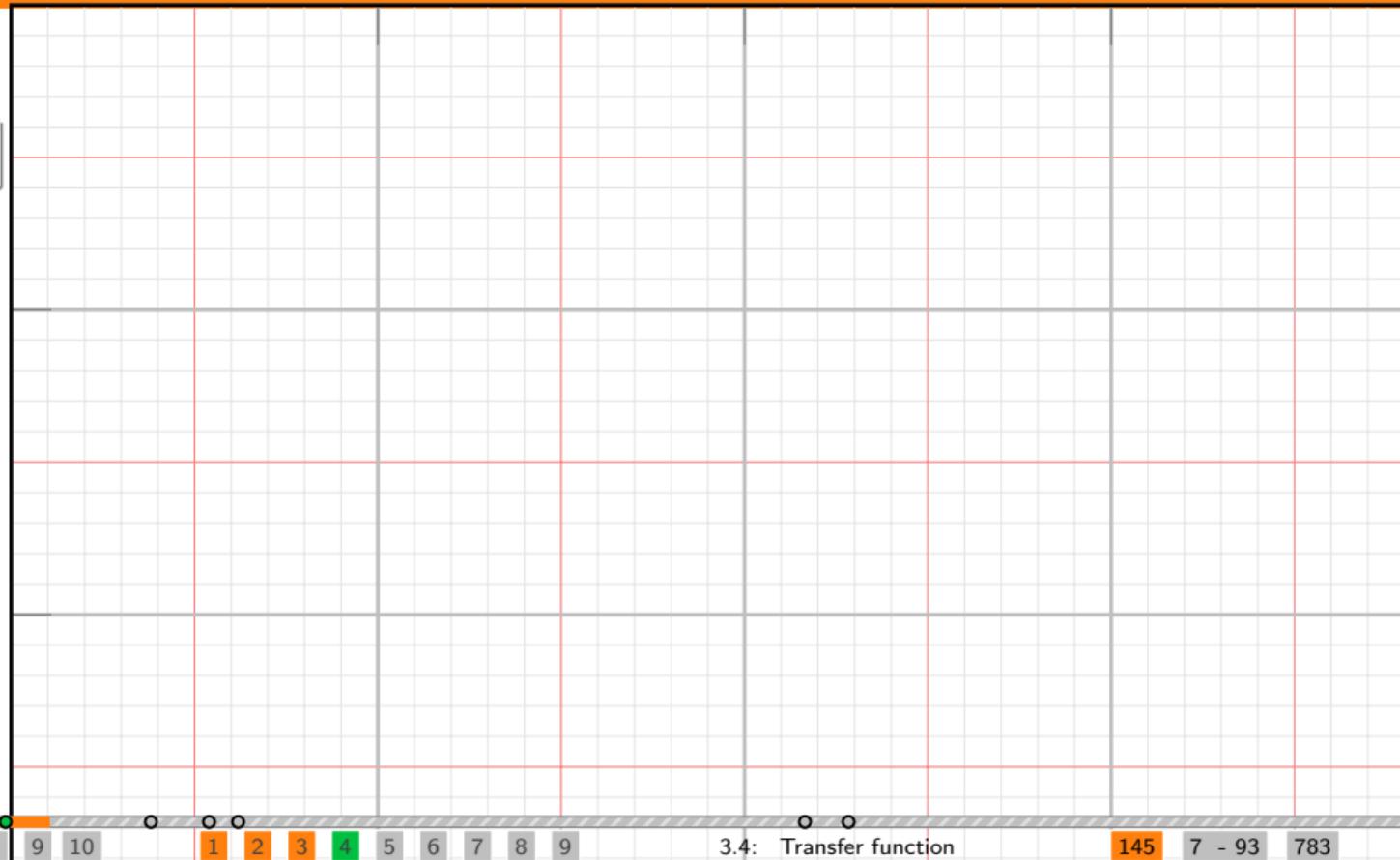
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Exercise (#3.2)

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using Numpy or MATLAB® with and without using the built-in function step.



Exercise (#3.2)

Plot the step response for a system with the transfer function

$$H(s) = \frac{s - 1}{s^2 + 2s + 1}$$

using **Neuro** or **MATLAB** with and without using the built-in function `step`.**Definition**One can use the **partial fraction expansion** to simplify **complex^a** functions of the form

$$A(x) = \frac{P(x)}{Q(x)}$$

as follows:

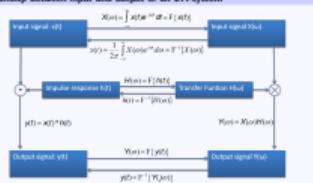
$$A(x) = \frac{P(x)}{Q(x)} = R(x) + \sum_{i=1}^k \sum_{j=1}^{n_i} \frac{a_{ij}}{(x - x_i)^j},$$

with n_i being the order of pole x_i . $R(x) = 0$ in case of $\deg(P) < \deg(Q)$.^aa complex function is a function $D \subset \mathbb{C} \rightarrow \mathbb{C}$.**Definition**Using the **Laplace Transform**, the output of an LTI system is given by

$$Y(s) = H(s)X(s) + Y_{\text{out}}(s),$$

with $Y_{\text{out}}(s)$ being terms due to all initial conditions. If one ignores all terms arising from initial conditions, then the **transfer function** is of the form

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Relationship between input and output of an LTI-system

- └ LTI systems
- └ Transfer function

Definition

One can use the **partial fraction expansion** to simplify **complex*** functions of the form

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with n_i being the order of pole s_i , $R(s) = 0$ in case of $\deg(P) < \deg(Q)$.

*a complex function is a function $D \subset \mathbb{C} \rightarrow \mathbb{C}$.

Properties

Rational transfer functions of the form

$$\begin{aligned} H(s) &= \frac{b_0 + b_1 s + \dots + b_M s^M}{a_0 + a_1 s + \dots + a_N s^N} \\ &= K \frac{(s - z_1)(s - z_2) \dots (s - z_M)}{(s - p_1)(s - p_2) \dots (s - z_N)} \end{aligned}$$

are completely characterized by the **gain factor** K and the poles (**natural frequencies**) and zeros.

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- └ LTI systems
- └ Transfer function

Properties

Rational transfer functions of the form

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$$A(s) = \frac{P(s)}{Q(s)}$$

as follows:

$$A(s) = \frac{P(s)}{Q(s)} = R(s) + \sum_{j=1}^n \sum_{i=1}^{n_j} \frac{a_{ij}}{(s - z_j)^i},$$

with n_j being the order of pole z_j . $R(s) \equiv 0$ in case of $\deg(P) < \deg(Q)$.*a complex function is a function $D \subset \mathbb{C} \rightarrow \mathbb{C}$.**Exercise (#3.3)**

A system is described by the differential equation (initial states of 0)

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x$$

- Calculate the transfer function $H(s)$
- Analytically calculate the impulse response
- Analytically calculate the (unit) step response

Exercise (#3.2)

Plot the step response for a system with the transfer function

$$H(s) = \frac{s-1}{s^2+2s+1}$$

using [NumPy](#) or [MATLAB](#) with and without using the built-in function `step`.

Exercise (#3.3)

A system is described by the differential equation (initial states of 0)

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = \frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x$$

- a) Calculate the transfer function $H(s)$
- b) Analytically calculate the impulse response
- c) Analytically calculate the (unit) step response

└ LTI systems

└ Impulse response

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- 3.2 Mathematical foundations: Differential equations
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- 3.5 Impulse response**
- 3.6 Causality
- 3.7 Stability
- 3.8 Network of systems
- 3.9 Graphical representation

Definition

The convolution is given by:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$$

It can be shown that

$$h(t) = \mathcal{F}^{-1}\{H(\omega)\}$$

is the reaction of the system to a $\delta(t)$ input – the so called **impulse response**.

- └ LTI systems
- └ Impulse response

Definition

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is the reaction of the system to a $\delta(t)$ input – the so called **impulse response**.

Exercise (#3.4)

Plot the impulse response for a system with the transfer function

$$H(s) = \frac{s - 1}{s^2 + 2s + 1}$$

using NUMPY or MATLAB^a.

^ause the *by hand* Laplace transform and built-in functions like `impulse`

- └ LTI systems
- └ Impulse response

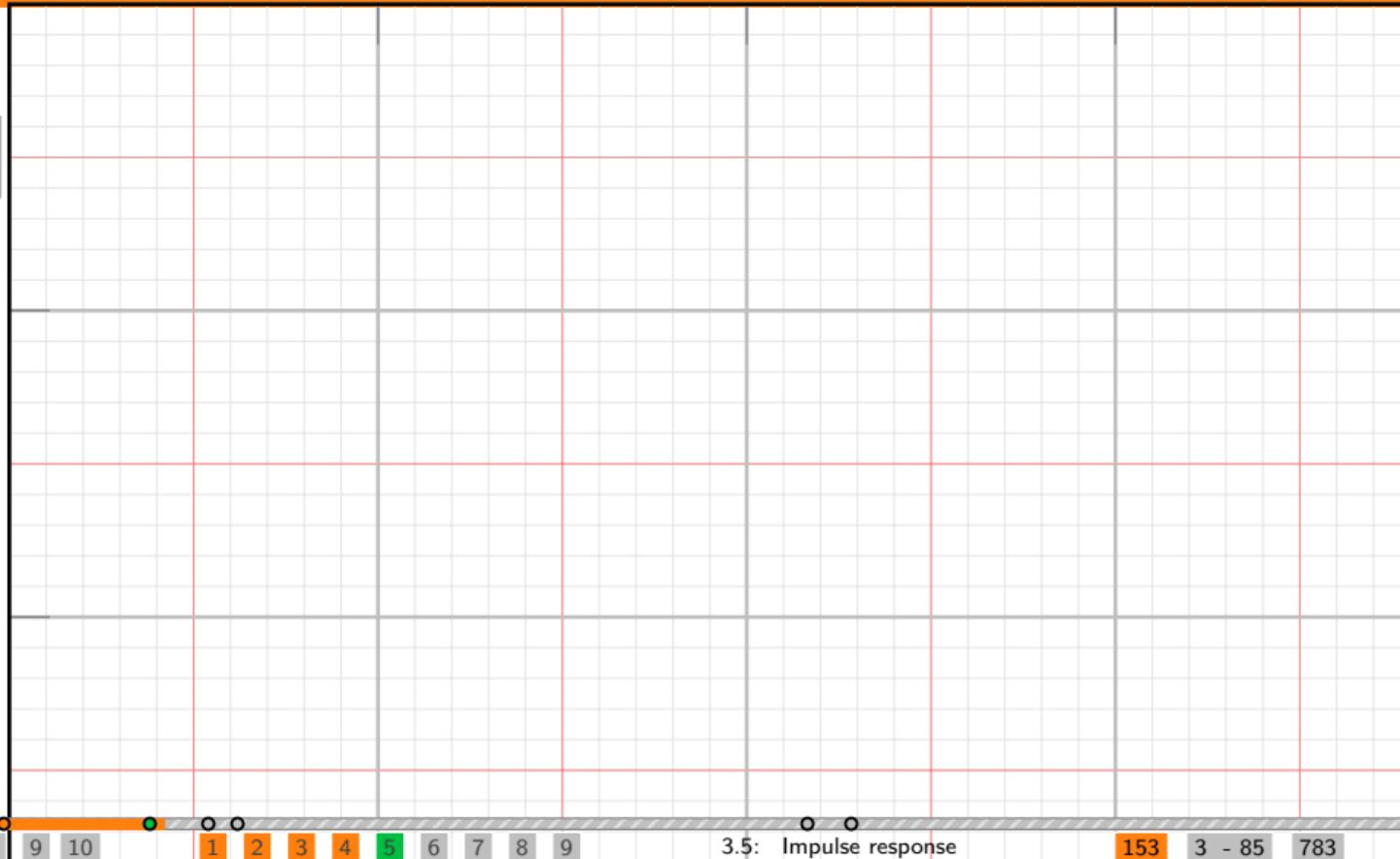
Exercise (#3.4)

Plot the impulse response for a system with the transfer function

$$H(s) = \frac{s-1}{s^2+2s+1}$$

using **Numerical** or **MATLAB®**.

*use the `b2` by `tf2ss` Laplace transform and built-in functions like `impulse`



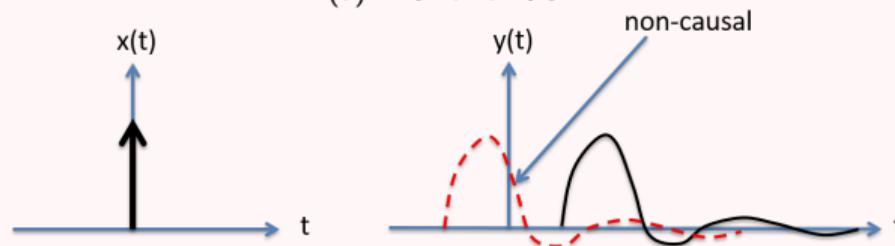
LTI systems

- 3.1 Introduction
- 3.2 Mathematical foundations: Differential equations
- 3.3 LTI systems
- 3.4 Transfer function
- 3.5 Impulse response
- 3.6 Causality**
- 3.7 Stability
- 3.8 Network of systems
- 3.9 Graphical representation

Definition

A system is called **causal** if

$$h(t) = 0 \text{ for } t < 0.$$



Non-Causal means that the reaction of the system comes before the excitation

LTI systems

- 3.1 Introduction
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- 3.7 Stability**
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 - 3.7.2 Stability criterion based upon poles of the transfer function
 - 3.7.3 Routh-Hurwitz stability criterion
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Stability

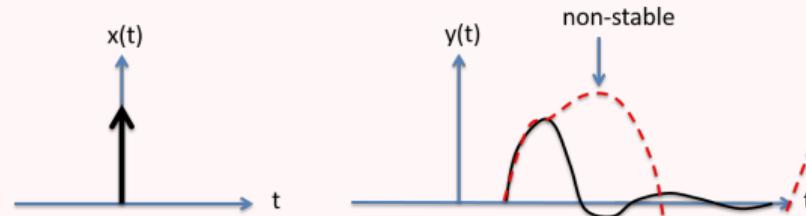
3.7 Stability

3.7.1 Definition

- 3.7.2 Stability criterion based upon poles of the transfer function
- 3.7.3 Routh-Hurwitz stability criterion

Definition

A system is called **stable** if for every time and value limited input signal^a the output approaches zero^b.



^a $x(t) = 0$ for $t < t_1$ and $t > t_2$ with $|x(t)| < M$

^b $y(t) \rightarrow 0$ for $t \rightarrow \infty$

Definition

A system is called **stable** if for every time and value limited input signal^a the output approaches zero^b.



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^b $v(t) \rightarrow 0$ for $t \rightarrow \infty$

Definition

A system is stable if

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

holds true.

└ LTI systems

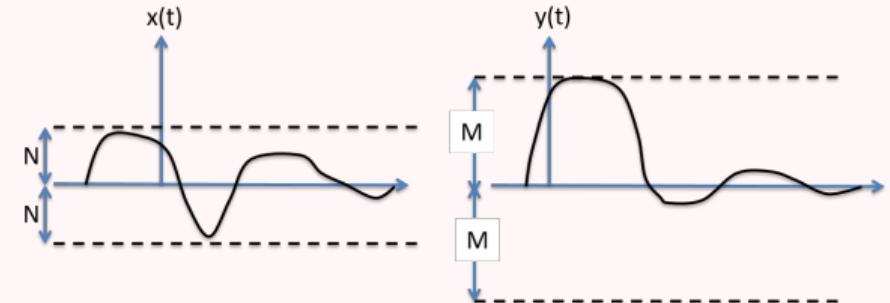
└ Stability

Definition

A system is stable if

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holds true.

DefinitionA system is called **stable** if for every time and value limited input signal^a the output approaches zero^b.^a $x(t) = 0$ for $t < t_0$ and $x > t_0$ with $|x(t)| < M$ ^b $y(t) \rightarrow 0$ for $t \rightarrow \infty$ **Definition**A causal and stable system has a bounded output in case of a bounded input (**BIBO**)

Stability

3.7 Stability

3.7.1 Definition

3.7.2 Stability criterion based upon poles of the transfer function

3.7.3 Routh-Hurwitz stability criterion

Example

Consider the transfer function

$$H(s) = \frac{1}{s+a} = \mathcal{L}\{e^{-at}\} = \mathcal{L}\left\{e^{-\Re\{a\}t}e^{-i\Im\{a\}t}\right\}:$$

Obviously,

$$\lim_{t \rightarrow \infty} h(t) = 0$$

for

$$\Re\{a\} > 0.$$

Note that $H(s)$ has a pole at $s = -a$.

- └ LTI systems
- └ Stability

Example

Consider the transfer function

$$H(s) = \frac{1}{s+a} = \mathcal{L}\{e^{-at}\} = \mathcal{L}\left\{e^{-\Re(s)t} e^{-j\Im(s)t}\right\};$$

Obviously,

$$\lim_{t \rightarrow \infty} h(t) = 0$$

for

$$\Re(s) > 0.$$

Note that $H(s)$ has a pole at $s = -a$.

Properties

Any causal system is absolutely stable if the poles of

$$H(s) = \frac{P(s)}{Q(s)}$$

lie in the left half space.

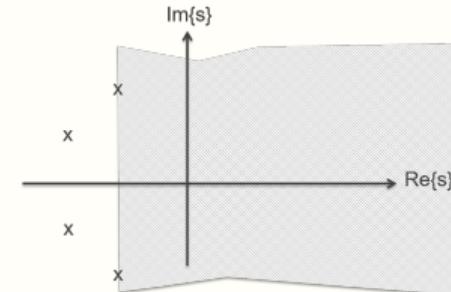


Figure 7: Poles and ROC

Note: This is only a yes/no classification.

└ LTI systems

└ Stability

Properties

Any causal system is absolutely stable if the poles of

$$H(s) = \frac{P(s)}{Q(s)}$$

lie in the left half space.



Figure 2: Poles and ROC

Note: This is only a pole classification

Example

Consider the transfer function

$$H(s) = \frac{1}{s+a} = \mathcal{L}\{e^{-at}\} = \mathcal{L}\{e^{-\Re(s)t}e^{-j\Im(s)t}\}:$$

Obviously,

$$\lim_{t \rightarrow \infty} h(t) = 0$$

for

$$\Re(s) > 0.$$

Note that $H(s)$ has a pole at $s = -a$.**Definition**

For a given causal system with

$$H(s) = \frac{P(s)}{Q(s)}$$

the equation

$$Q(s) = 0$$

is called the **characteristic equation** of the system.

└ LTI systems

└ Stability

Definition

For a given causal system with

$$H(s) = \frac{P(s)}{Q(s)}$$

the equation

$$Q(s) = 0$$

is called the **characteristic equation** of the system.**Example**

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

With poles at

$$s = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}.$$

For $p^2 - 4q < 0$ one gets poles at

$$s = -\frac{p}{2} \pm j\sqrt{q - \left(\frac{p}{2}\right)^2}.$$

Properties

Any causal system is absolutely stable if the poles of

$$H(s) = \frac{P(s)}{Q(s)}$$

lie in the left half space.



Figure 7: Pole and ROC.

Note: This is only a pole classification.

Example

Consider the transfer function

$$H(s) = \frac{1}{s+a} = \mathcal{L}\{e^{-at}\} = \mathcal{L}\{e^{-at}(1)e^{-j2\pi f_s t}\}.$$

Obviously,

$$\lim_{s \rightarrow \infty} H(s) = 0$$

for

$$\Re\{s\} > 0$$

Note that $H(s)$ has a pole at $s = -a$.

- └ LTI systems
- └ Stability

Example

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

With poles at

$$s = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

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Definition

For a given causal system with

$$H(s) = \frac{P(s)}{Q(s)}$$

the equation

$$Q(s) = 0$$

is called the **characteristic equation** of the system.**Properties**

Any causal system is absolutely stable if the poles of

$$H(s) = \frac{P(s)}{Q(s)}$$

lie in the left half space.



Now, this is only a sufficient condition.

Example

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

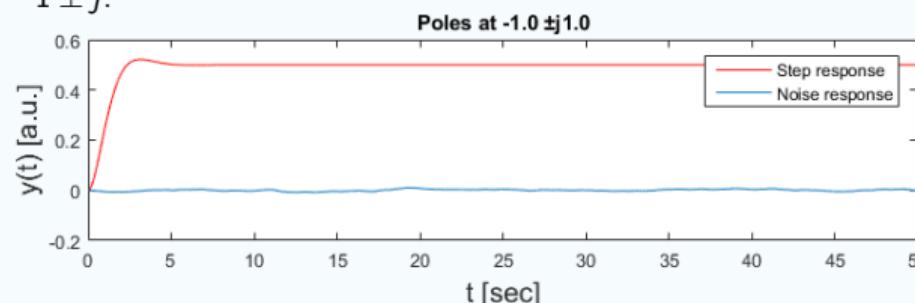
with poles at $-1 \pm j$:

Figure 8: Step response and response to Gaussian noise (std. dev.: 0.01)

- └ LTI systems
- └ Stability

Example

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

with poles at $-1 \pm j$:

Figure 8: Step response and response to Gaussian noise (std. dev.: 0.01)

Example

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

With poles at

$$s = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

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Definition

For a given causal system with

$$H(s) = \frac{P(s)}{Q(s)}$$

the equation

$$Q(s) = 0$$

is called the **characteristic equation** of the system.**Example**

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

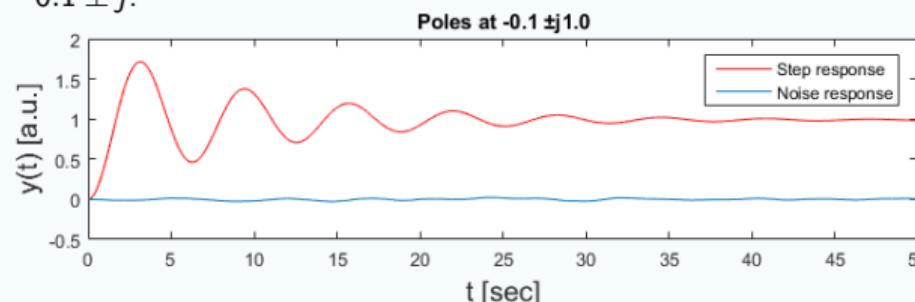
with poles at $-0.1 \pm j$:

Figure 9: Step response and response to Gaussian noise (std. dev.: 0.01)

- └ LTI systems
- └ Stability

Example

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

with poles at $-0.1 \pm j$:

Figure 9: Step response and response to Gaussian noise (std. dev.: 0.01)

Example

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

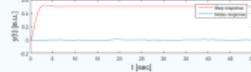
with poles at $-1 \pm j$:

Figure 8: Step response and response to Gaussian noise (std. dev.: 0.01)

Example

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

With poles at

$$\lambda = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - 4q}$$

For $p^2 - 4q < 0$ one gets poles at

$$\lambda = -\frac{p}{2} \pm j\sqrt{q - \left(\frac{p}{2}\right)^2}$$

Example

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

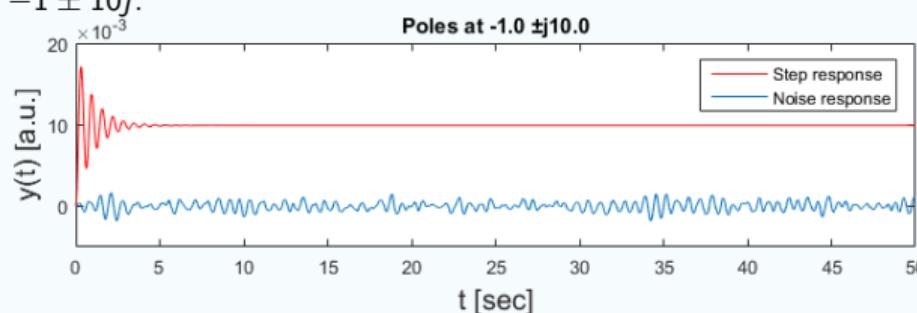
with poles at $-1 \pm 10j$:

Figure 10: Step response and response to Gaussian noise (std. dev.: 0.01)

- └ LTI systems
- └ Stability

Example

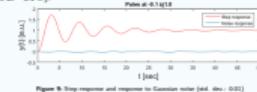
Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

with poles at $-1 \pm j10$:**Example**

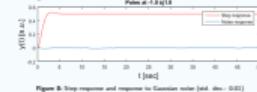
Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

with poles at $-0.1 \pm j10$:**Example**

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

with poles at $-1 \pm j1$:**Example**

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

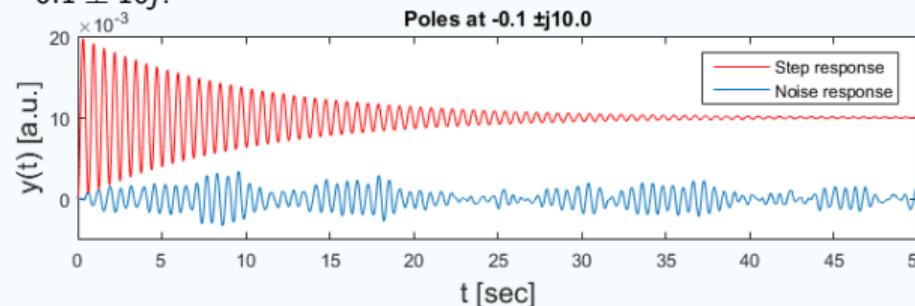
with poles at $-0.1 \pm 10j$:

Figure 11: Step response and response to Gaussian noise (std. dev.: 0.01)

- └ LTI systems
- └ Stability

Example

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

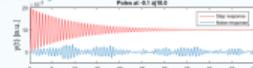
with poles at $-0.1 \pm j0$:

Figure 11: Step response and response to Gaussian noise (std. dev.: 0.01)

Example

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

with poles at $-1 \pm j0$:

Figure 12: Step response and response to Gaussian noise (std. dev.: 0.01)

Example

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

with poles at $-0.1 \pm j1$:

Figure 13: Step response and response to Gaussian noise (std. dev.: 0.01)

Example

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

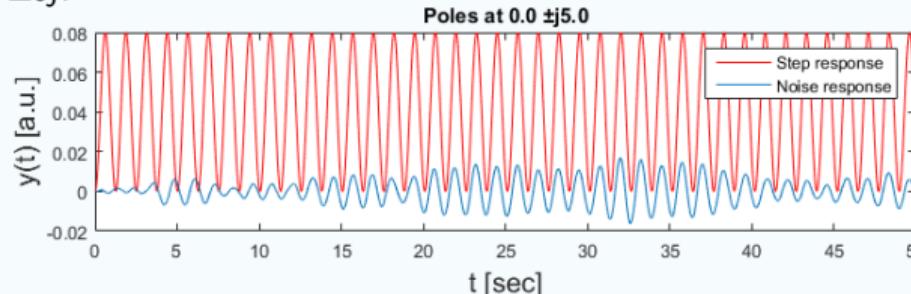
with poles at $\pm 5j$:

Figure 14: Step response and response to Gaussian noise (std. dev.: 0.01)

└ LTI systems

└ Stability

Example

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

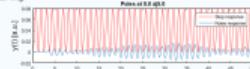
with poles at $\pm j\beta$ 

Figure 12: Step response and response to Gaussian noise (std. dev.: 0.01)

Example

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

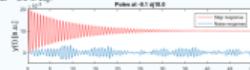
with poles at $-0.1 \pm j\beta$ 

Figure 13: Step response and response to Gaussian noise (std. dev.: 0.1)

Example

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

with poles at $-1 \pm j\beta$ 

Figure 14: Step response and response to Gaussian noise (std. dev.: 0.01)

Exercise (#3.5)

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

with poles at $\alpha \pm j\beta$. Plot the step response and the response to Gaussian noise (std. dev.: 0.1) for the values of α and β given below by making use of NUMPY or MATLAB. Plot the result up to $t = 40$ s.

case	α	β
1	-10	1
2	-1	1
3	-1	10
4	1	1
5	10	1

└ LTI systems

└ Stability

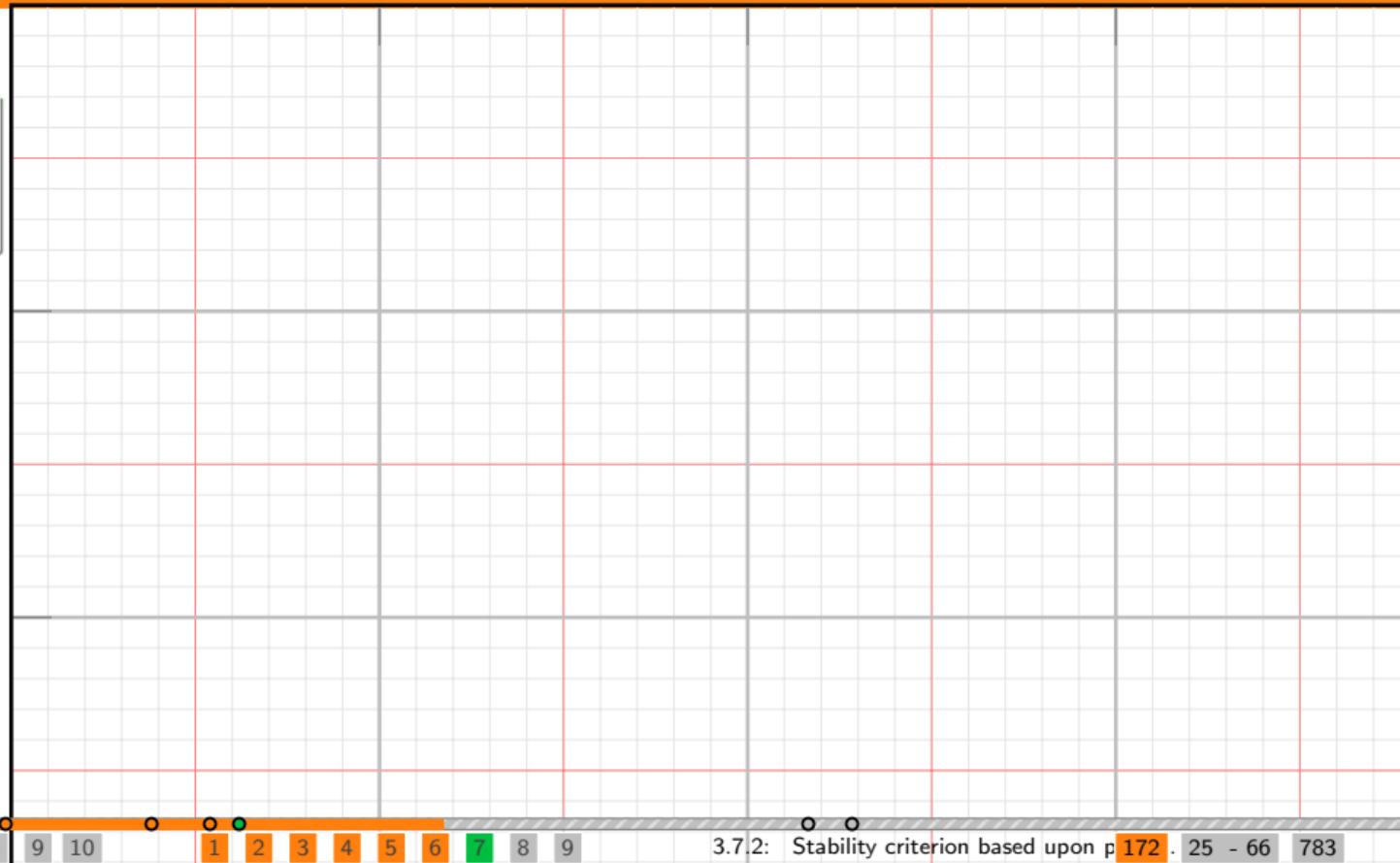
Exercise (#3.5)

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

with poles at $s = -\bar{\alpha} \pm j\bar{\beta}$. Plot the step response and the response to Gaussian noise (std. dev.: 0.1) for the values of α and β given below by making use of Nturyv or MATLAB. Plot the results up to $t = 40$.

Case	$\bar{\alpha}$	$\bar{\beta}$
1	100	1
2	-1	10
3	-1	100
4	-1	1
5	10	1



Exercise (#3.5)

Consider the transfer function

$$H(s) = \frac{1}{s^2 + \beta s + \alpha}$$

with poles at $\alpha \pm j\beta$. Plot the step response and the response to Gaussian noise [std. dev.: 0.1] for the values of α and β given below by making use of Numerical or MATLAB. Plot the result up to $t = 40$.

case	m	n
1	-10	1
2	-1	1
3	-2	15
4	0	1
5	10	1

Example

Consider the transfer function

$$H(s) = \frac{1}{s^2 + 0.0 s + 0}$$

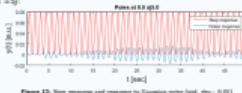
with poles at $\pm j5$:

Figure 12: Step response and response to Gaussian noise [std. dev.: 0.1]

Example

Consider the transfer function

$$H(s) = \frac{1}{s^2 + 0.1 s + 0}$$

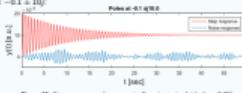
with poles at $-0.1 \pm j0.1$:

Figure 13: Step response and response to Gaussian noise [std. dev.: 0.01]

Properties

- ▷ A real pole in the left-half plane: Exponentially decaying component
- ▷ A real pole in the right-half plane: Exponentially increasing component
- ▷ A complex pole in the left-half plane: Exponentially decaying oscillatory component
- ▷ A complex pole in the right-half plane: Exponentially increasing oscillatory component
- ▷ A pole on the imaginary axis: Marginally stable
- ▷ Differential equations with real-valued coefficients yield poles that are pairwise complex conjugates

Properties

- ▷ A real pole in the left-half plane: Exponentially decaying component
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- ▷ A complex pole in the left-half plane: Exponentially decaying oscillatory component
- ▷ A complex pole in the right-half plane: Exponentially increasing oscillatory component
- ▷ A pole on the imaginary axis: Marginally stable
- ▷ Differential equations with real-valued coefficients yield poles that are pairwise complex conjugates

Marginally stable

**Exercise (#3.5)**

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

with poles at $\alpha \pm j\beta$. Plot the step response and the response to Gaussian noise [std. dev.: 0.1] for the values of α and β given below by making use of Nusmv or MATLAB. Plot the result up to $t = 40$.

case	α	β
1	-10	3
2	-10	5
3	-1	10
4	1	1
5	10	1

Example

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

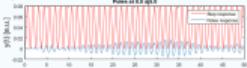
with poles at $\pm 5j$:

Figure 13: Step response and response to Gaussian noise [std. dev.: 0.01]

Figure 13: Source: Schlecke Silberstein

Stability

3.7 Stability

3.7.1 Definition

3.7.2 Stability criterion based upon poles of the transfer function

3.7.3 Routh-Hurwitz stability criterion

Definition

Given is a transfer function of the form $H(s) = \frac{P(s)}{Q(s)}$, with

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0.$$

The Routh array is then of the form

s^n	a_n	a_{n-2}	a_{n-4}	...
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	...
s^{n-2}	b_{n-1}	b_{n-3}	b_{n-5}	...
s^{n-3}	c_{n-1}	c_{n-3}	c_{n-5}	...
:	:	:	:	
s^0	h_{n-1}			

See next slides for definition of b_{n-1}, b_{n-3}, \dots

Definition

Given is a transfer function of the form $H(s) = \frac{P(s)}{Q(s)}$, with:

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_2 s + a_1$$

The Routh array is then of the form

s^{n-1}	a_{n-2}	a_{n-4}	\dots
s^{n-2}	a_{n-1}	a_{n-3}	\dots
s^{n-3}	a_{n-2}	a_{n-4}	\dots
s^{n-4}	a_{n-1}	a_{n-3}	\dots
\vdots	\vdots	\vdots	\vdots
s^0	a_{n-1}	a_{n-3}	\dots

See next slides for definition of a_{n-1}, a_{n-3}, \dots

Properties

$$b_{n-1} = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_{n-3} = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}$$

$$\vdots = \vdots$$

$$c_{n-1} = -\frac{1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}$$

$$d_{n-1} = -\frac{1}{c_{n-1}} \begin{vmatrix} b_{n-1} & b_{n-3} \\ c_{n-1} & c_{n-3} \end{vmatrix}$$

$$\vdots \quad \vdots \quad \vdots$$

Properties

$$\begin{aligned} d_{n-1} &= -\frac{1}{d_{n-1}} \begin{vmatrix} d_n & d_{n-2} \\ d_{n-1} & d_{n-3} \end{vmatrix} = \frac{d_n d_{n-2} - d_{n-1}^2}{d_{n-1}} \\ d_{n-2} &= \frac{1}{d_{n-1}} \begin{vmatrix} d_n & d_{n-3} \\ d_{n-1} & d_{n-4} \end{vmatrix} \\ \vdots &= \vdots \\ c_{n-1} &= -\frac{1}{d_{n-1}} \begin{vmatrix} d_{n-1} & d_{n-2} \\ d_{n-2} & d_{n-3} \end{vmatrix} \\ d_{n-3} &= -\frac{1}{c_{n-1}} \begin{vmatrix} d_{n-1} & d_{n-2} \\ c_{n-1} & d_{n-3} \end{vmatrix} \\ \vdots &= \vdots \end{aligned}$$

Definition

The **Routh-Hurwitz stability criterion** states that the number of roots of $Q(s)$ with positive real parts is equal to the number of changes in the sign of the first column

$$[a_n \quad a_{n-1} \quad b_{n-1} \quad c_{n-1} \quad \dots \quad h_{n-1}]^T$$

in the Routh array.

Definition

Given is a transfer function of the form $H(s) = \frac{P(s)}{Q(s)}$, with

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_2 s + a_0$$

The **Routh array** is then of the form

s^n	a_n	a_{n-2}	a_{n-4}	\dots
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	\dots
s^{n-2}	a_{n-2}	a_{n-4}	a_{n-6}	\dots
s^{n-3}	a_{n-3}	a_{n-5}	a_{n-7}	\dots
s^0	a_{n-4}	\vdots	\vdots	\vdots

See next slides for definition of d_{n-1}, d_{n-2}, \dots

Definition

The Routh-Hurwitz stability criterion states that the number of roots of $Q(s)$ with positive real parts is equal to the number of changes in the sign of the first column.

$$[a_0 \ a_{0-1} \ a_{0-2} \ a_{0-3} \ \dots \ a_{0-n-1}]^T$$

in the Routh array.

Properties

$$\begin{aligned} a_{0-1} &= -\frac{1}{a_{0-1}} \begin{vmatrix} a_0 & a_{0-2} \\ a_{0-1} & a_{0-3} \end{vmatrix} = \frac{a_{0-1}a_{0-2} - a_0a_{0-3}}{a_{0-1}} \\ a_{0-3} &= -\frac{1}{a_{0-1}} \begin{vmatrix} a_0 & a_{0-4} \\ a_{0-1} & a_{0-5} \end{vmatrix} \\ \vdots &= \vdots \\ a_{0-n-1} &= -\frac{1}{a_{0-1}} \begin{vmatrix} a_{0-1} & a_{0-n} \\ a_{0-2} & a_{0-n-1} \end{vmatrix} \\ a_{0-n} &= -\frac{1}{a_{0-1}} \begin{vmatrix} a_{0-2} & a_{0-n} \\ a_{0-3} & a_{0-n-1} \end{vmatrix} \\ \vdots &= \vdots \end{aligned}$$

Definition

Given is a transfer function of the form $H(s) = \frac{N(s)}{D(s)}$, with

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$$

The Routh array is then of the form

$$\begin{array}{ccccccc} s^n & a_0 & a_{0-2} & a_{0-4} & \dots \\ s^{n-1} & a_1 & a_{0-1} & a_{0-3} & \dots \\ s^{n-2} & a_{n-2} & a_{0-2} & a_{0-4} & \dots \\ s^{n-3} & a_{n-1} & a_{0-1} & a_{0-3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s^0 & a_{n-1} & & & \dots \end{array}$$

See next slides for definition of a_{0-1}, a_{0-2}, \dots

Properties

The following cases need to be distinguished:

- There is no zero in the first column**
- There is a zero in the first column, but at least one other element in the corresponding row is unequal to zero.
- There is a zero in the first column and all other elements in the corresponding row are zero as well.
- Repeated roots of $Q(s)$ on the $j\omega$ axis

Properties

- The following cases need to be distinguished:
1. There is no zero in the first column.
 2. There is one zero in the first column, but at least one other element in the corresponding row is nonzero to zero.
 3. There is a zero in the first column and all other elements in the corresponding row are zero as well.
 4. Repeated roots of $Q(s)$ on the $j\omega$ -axis.

Definition

The Routh-Hurwitz stability criterion states that the number of roots of $Q(s)$ with positive real parts is equal to the number of changes in the sign of the first column.

$$\begin{bmatrix} a_0 & a_{n-1} & a_{n-2} & \dots & a_{n-1} \end{bmatrix}^T$$

in the Routh array.

Example: $Q(s) = a_3s^3 + a_2s^2 + a_1s + a_0$

Routh array:

s^3	a_3	a_1
s^2	a_2	a_0
s^1	$b_1 = \frac{a_2 a_1 - a_0 a_3}{a_2}$	0
s^0	$c_1 = \frac{b_1 a_0}{b_1} = a_0$	0

It follows that a necessary and sufficient condition for stability is that the coefficients are positive and that $a_2 a_1 > a_0 a_3$.

Properties

$$\begin{aligned} b_{n-2} &= -\frac{1}{a_{n-1}} \begin{vmatrix} a_0 & a_{n-2} \\ a_{n-1} & a_{n-2} \end{vmatrix} = \frac{a_{n-1}a_{n-2} - a_0a_{n-1}}{a_{n-1}} \\ b_{n-3} &= -\frac{1}{a_{n-2}} \begin{vmatrix} a_0 & a_{n-3} \\ a_{n-1} & a_{n-3} \end{vmatrix} \\ \vdots & \\ b_{n-1} &= -\frac{1}{a_{n-1}} \begin{vmatrix} a_0 & a_{n-1} \\ a_{n-1} & a_{n-1} \end{vmatrix} \\ d_{n-2} &= \frac{1}{c_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-2} \\ c_{n-1} & a_{n-2} \end{vmatrix} \\ d_{n-3} &= \frac{1}{c_{n-2}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ c_{n-1} & a_{n-3} \end{vmatrix} \\ \vdots & \end{aligned}$$

Example: $Q(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0$

Routh array:

s^3	a_3	a_1
s^2	a_2	a_0
s^1	$b_1 = \frac{a_1 - a_3}{a_2}$	0
s^0	$c_1 = \frac{a_0 - b_1 a_2}{a_1}$	0

It follows that a necessary and sufficient condition for stability is that the coefficients are positive and that $a_2 a_3 > a_0 a_1$.

Properties

The following cases need to be distinguished:

1. There is no zero in the first column
2. **There is a zero in the first column, but at least one other element in the corresponding row is unequal to zero.**
3. There is a zero in the first column and all other elements in the corresponding row are zero as well.
4. Repeated roots of $Q(s)$ on the $j\omega$ axis

Properties

The following cases need to be distinguished:

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Definition

The Routh-Hurwitz stability criterion states that the number of roots of $Q(s)$ with positive real parts is equal to the number of changes in the sign of the first column

$$[a_n \ a_{n-1} \ a_{n-2} \ \dots \ a_{n-1}]^T$$

in the Routh array.



Properties

The following cases need to be distinguished:

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4. Repeated roots of $Q(s)$ on the $j\omega$ -axis

Example: $Q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$

Routh array:

s^5	1	2	11
s^3	2	4	10
s^3	0	6	0
s^2	c	10	0
s^1	d	0	0
s^0	10	0	0

Example: $Q(s) = a_5s^5 + a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0$

Routh array:

s^5	a_5	a_1
s^4	a_4	a_0
s^3	$b_3 = \frac{a_4a_0 - a_1a_2}{a_5}$	0
s^2	$c_1 = \frac{a_3}{b_3} = a_0$	0

It follows that a necessary and sufficient condition for stability is that the coefficients are positive and that $a_2a_4 > a_1a_3$.

The 0 in the third row can be replaced by ϵ with $0 < \epsilon \ll 1$ (see next slide)

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Example: $Q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$

Routh array:

s^5	1	2	11
s^4	2	4	10
s^3	c	6	0
s^2	d	10	0
s^1		0	0
s^0	10	0	0

The 0 in the third row can be replaced by ϵ with $0 < \epsilon \ll 1$ (see next slide)**Example:** $Q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$

Routh array:

s^5	1	2	11
s^4	2	4	10
s^3	ϵ	6	0
s^2	$\frac{4\epsilon - 12}{\epsilon} < 0$	10	0
s^1	$\frac{6\epsilon - 10\epsilon}{\epsilon}$	0	0
s^0	10	0	0

Properties

- The following cases need to be distinguished:
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 - There is a zero in the first column, but at least one other element in the corresponding row is unequal to zero.
 - There is a zero in the first column and all other elements in the corresponding row are zero as well.
 - Repeated roots of $Q(s)$ on the $j\omega$ -axis

Example: $Q(s) = a_0s^3 + a_1s^2 + a_2s + a_3$

Routh array:

s^3	a_0	a_1
s^2	a_2	a_3
s^1	$b_1 = \frac{a_1a_2 - a_0a_3}{a_2}$	0
s^0	$c_1 = \frac{a_3b_1}{a_2} = a_0$	0

It follows that a necessary and sufficient condition for stability is that the coefficients are positive and that $a_0a_2 > a_1a_3$.

Two sign changes and thus unstable.

Example: $Q(s) = s^6 + 2s^5 + 2s^4 + 4s^3 + 11s + 10$

Routh array:

s^6	1	2	11
s^5	2	4	10
s^4	0	6	0
s^3	10	10	0
s^2	0	0	0
s^1	10	0	0
s^0	0	0	0

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s^2	d	0	0
s^1	10	0	0
s^0	0	0	0

The 0 in the third row can be replaced by ϵ with $0 < \epsilon \ll 1$ (see next slide)

Properties

The following cases need to be distinguished:

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4. Repeated roots of $Q(s)$ on the $j\omega$ axis

Properties

The following cases need to be distinguished:

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Properties:

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 4. Repeated roots of $Q(s)$ on the $j\omega$ axis

Example: $Q(s) = s^3 + 2s^2 + 4s + K$

Routh array:

s^3	1	2	11
s^2	2	4	10
s^1	0	6	0
s^0	10	0	0
	0	0	0

Two sign changes and thus unstable.

Example: $Q(s) = s^3 + 2s^2 + 4s^2 + 11s + 10$

Routh array:

s^3	1	4
s^2	2	K
s^1	$\frac{8-K}{2}$	0
s^0	K	0

Example: $Q(s) = s^3 + 2s^2 + 2s^3 + 4s^2 + 11s + 10$

Routh array:

s^3	1	2	11
s^2	2	4	10
s^1	0	0	0
s^0	10	0	0
	0	0	0

The 0 in the third row can be replaced by ϵ with $0 < \epsilon \ll 1$ (see next slide)

Stable for $0 < K < 8$. For $K = 8$: See next slide.

Example: $Q(s) = s^3 + 2s^2 + 4s + K$

Routh array:

s^3	1	4
s^2	2	K
s^1	0	
s^0	0	

Stable for $0 < K < 8$. For $K \geq 8$: See next slide.

Properties

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 4. Repeated roots of $Q(s)$ on the $j\omega$ -axis

Example: $Q(s) = s^3 + 2s^2 + 4s + 8$

Routh array:

s^3	1	4
s^2	2	8
s^1	0	0
s^0	8	0

This case occurs if singularities are symmetrically located about the origin of the s -plane:
 $(s + \sigma)(s - \sigma)$ or $(s + j\omega)(s - j\omega)$.

Example: $Q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$

Routh array:

s^5	1	22
s^4	2	10
s^3	0	0
s^2	$\frac{22}{2} < 0$	10
s^1	0	0
s^0	10	0

Two sign changes and thus unstable.

Example: $Q(s) = s^3 + 2s^2 + 4s + 8$

Routh array:

s^3	1	4
s^2	2	8
s^1	0	0
s^0		

This case occurs if singularities are symmetrically located about the origin of the s -plane:
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Example: $Q(s) = s^3 + 2s^2 + 4s + K$

Routh array:

s^3	1	4
s^2	2	K
s^1	0	0
s^0		

Stable for $0 < K < 8$. For $K = 8$: See next slide.

Example: $Q(s) = s^3 + 2s^2 + 4s + 8$

s^3	1	4
s^2	2	8
s^1	0	0

Utilizing the row preceding the row of zeros to form an auxiliary equation:

$$U(s) = 2s^2 + 8s^0 = 0,$$

With $s = -j2$ and $s = j2$ the roots of $U(s)$.

Properties

- The following cases need to be distinguished:
1. There is no zero in the first column
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Example: $Q(s) = s^3 + 2s^2 + 4s + 8$

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s^2	2	
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s^0	0	

This case occurs if singularities are symmetrically located about the origin of the s -plane:

$$(s + \sigma)(s - \sigma) \text{ or } (s + j\omega)(s - j\omega).$$

Example: $Q(s) = s^3 + 2s^2 + 4s + 8$

By taking the derivative of the the auxiliary equation, one can form a new array

s^3	1	4
s^2	2	8
s^1	2	0
s^0	8	0

Polynomial long division of $Q(s)$ by $U(s)$ leads to:

$$Q(s) = U(s) \left[1 + \frac{1}{2}s \right] = (s + j2)(s - j2)[s + 2]$$

Example: $Q(s) = s^3 + 2s^2 + 4s + K$

Routh array:		
s^3	1	4
s^2	2	K
s^1	0	0
s^0	0	0

Stable for $0 < K < 8$. For $K \geq 8$: See next slide.

Example: $Q(s) = s^3 + 2s^2 + 4s + 8$

By taking the derivative of the auxiliary equation, one can form a new array:

s^3	1	4
s^2	0	0
s^1	0	0
s^0	8	0

Polynomial long division of $Q(s)$ by $U(s)$ leads to:

$$Q(s) = U(s) \left[1 + \frac{1}{2}s \right] = (s + j2)(s - j2)(s + 2)$$

Example: $Q(s) = s^3 + 2s^2 + 4s + 8$

Utilizing the row preceding the row of zeros to form an auxiliary equation:

$$U(s) = 2s^2 + 8s^0 = 0,$$

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Routh array:

s^3	1	4
s^2	2	0
s^1	0	0
s^0	8	0

This case occurs if singularities are symmetrically located about the origin of the s -plane:
 $(s + \sigma)(s - \sigma)$ or $(s + j\omega)(s - j\omega)$

Properties

For the following forms of $Q(s)$ one can directly give a stability criterion:

$Q(s)$	Criterion
$s^2 + bs + 1$	$b > 0$
$as^2 + bs + c$	all coefficients positive
$s^3 + bs^2 + cs + 1$	$bc - 1 > 0$
$s^3 + a_2s^2 + a_1s + a_0$	$a_2a_1 > a_0$ for positive coefficients
$s^4 + bs^3 + cs^2 + ds + 1$	$bcd - d^2 - b^2 > 0$
$s^4 + bs^3 + cs^2 + ds + e$	all coefficients positive and $bcd > d^2 + b^2e$

PropertiesFor the following forms of $Q(s)$ one can directly give a stability criterion:

$Q(s)$	Criterion
$s^2 + 2s + c$	$\Delta > 0$
$s^2 + bs + c$	all coefficients positive
$s^2 + bs^2 + cs + 1$	$bc - 1 > 0$
$s^2 + 2s^2 + 2s + \lambda_2$	$\lambda_2 \lambda_1 > \lambda_2$ for positive coefficients
$s^2 + 2s^2 + cs^2 + bs + 1$	$b\lambda_2 - c\lambda^2 - b^2 > 0$
$s^2 + bs^2 + cs^2 + bs + c$	all coefficients positive and $b\lambda_2 > c^2 + b^2\lambda_1$

Example: $Q(s) = s^3 + 2s^2 + 4s + 8$

By taking the derivative of the auxiliary equation, one can form a new array

s^3	1	0
s^2	2	0
s^1	4	0
s^0	8	0

Polynomial long division of $Q(s)$ by $U(s)$ leads to:

$$Q(s) = U(s) \left[1 + \frac{1}{2}s \right] = (s + j2)(s - j2)(s + 2)$$

Example: $Q(s) = s^3 + 2s^2 + 4s + 8$

s^3	1	0
s^2	2	0
s^1	4	0
s^0	8	0

Utilizing the row preceding the row of zeros to form an auxiliary equation:

$$U(s) = 2s^2 + 8s^0 = 0,$$

With $s = -j2$ and $s = j2$ the roots of $U(s)$.

Properties

For systems of the form

$$G(s) = e^{-sT} \frac{P(s)}{Q(s)}$$

with a time delay one needs to approximate e^{-sT} e.g. by making use of the **Pade approximation**

$$e^{-sT} \approx \frac{1 - \frac{sT}{2}}{1 + \frac{sT}{2}}.$$

Properties

For systems of the form

$$G(s) = e^{-sT} \frac{P(s)}{Q(s)}$$

with a time delay one needs to approximate e^{-sT} e.g. by making use of the [Padé approximation](#)

$$e^{-sT} \approx \frac{1 - \frac{sT}{2}}{1 + \frac{sT}{2}}$$

Exercise (#3.6)

Is the system with the transfer function

$$G(s) = \frac{3}{3s^3 + 12s^2 + 24s + 36}$$

stable? Test by making use of NUMPY or MATLAB (e.g using roots).

PropertiesFor the following forms of $Q(s)$ one can directly give a stability criterion:

$Q(s)$	Criterion
$s^2 + bs + 1$	$b > 0$
$as^2 + bs + c$	all coefficients positive
$s^3 + bs^2 + cs + 1$	$bc - 1 > 0$
$s^3 + as_2 s^2 + as_1 + a_0$	$a_2 a_0 > a_1$ for positive coefficients
$s^4 + bs^3 + cs^2 + ds + 1$	$bcd - d^2 - b^2 > 0$
$s^4 + bs^3 + cs^2 + ds + a$	all coefficients positive and $bcd - d^2 - b^2 > 0$

Example: $Q(s) = s^3 + 2s^2 + 4s + 8$

By taking the derivative of the auxiliary equation, one can form a new array

s^3	1	4
s^2	2	0
s^1	0	0
s^0	8	0

Polynomial long division of $Q(s)$ by $U(s)$ leads to:

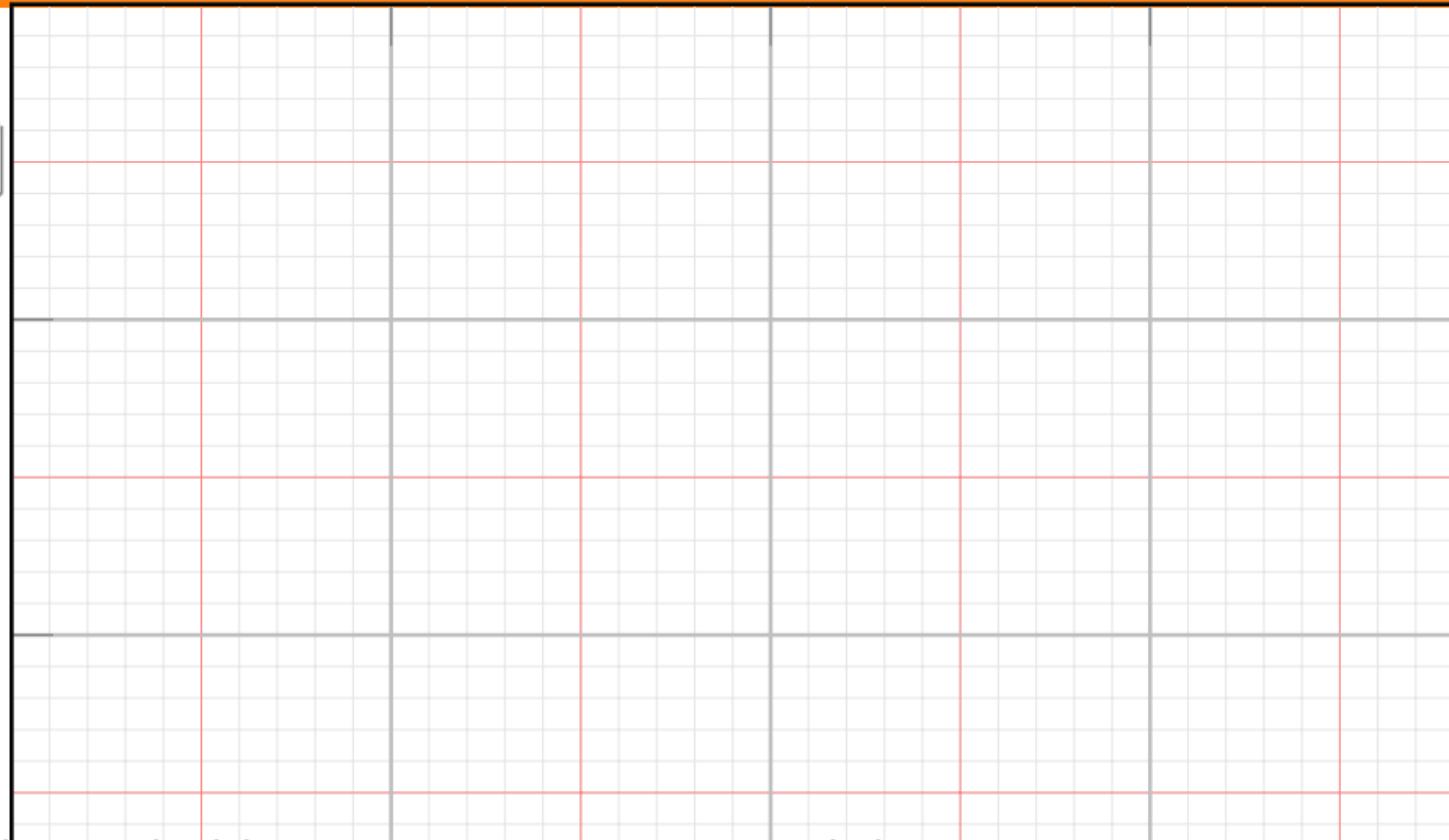
$$Q(s) = U(s) \left[1 + \frac{1}{2}s \right] = (s + j2)(s - j2)(s + 2)$$

Exercise (#3.6)

Is the system with the transfer function

$$G(s) = \frac{3}{3s^3 + 12s^2 + 24s + 36}$$

stable? Test by making use of Newton or MATLAB (e.g. using roots).



Exercise (#3.6)

Is the system with the transfer function

$$G(s) = \frac{3}{3s^2 + 12s + 24s + 36}$$

stable? Test by making use of **Numerov** or **MATLAB** (e.g. using `roots`).**Exercise (#3.7)**

Is the system with the transfer function

$$G(s) = \frac{1}{s^4 + s^3 - s \pm 1}$$

stable?

Properties

For systems of the form

$$G(s) = e^{-sT} \frac{P(s)}{Q(s)}$$

with a time delay one needs to approximate e^{-sT} e.g. by making use of the **Pade approximation**

$$e^{-sT} \approx \frac{1 - \frac{sT}{2}}{1 + \frac{sT}{2}}$$

PropertiesFor the following forms of $Q(s)$ one can directly give a stability criterion:

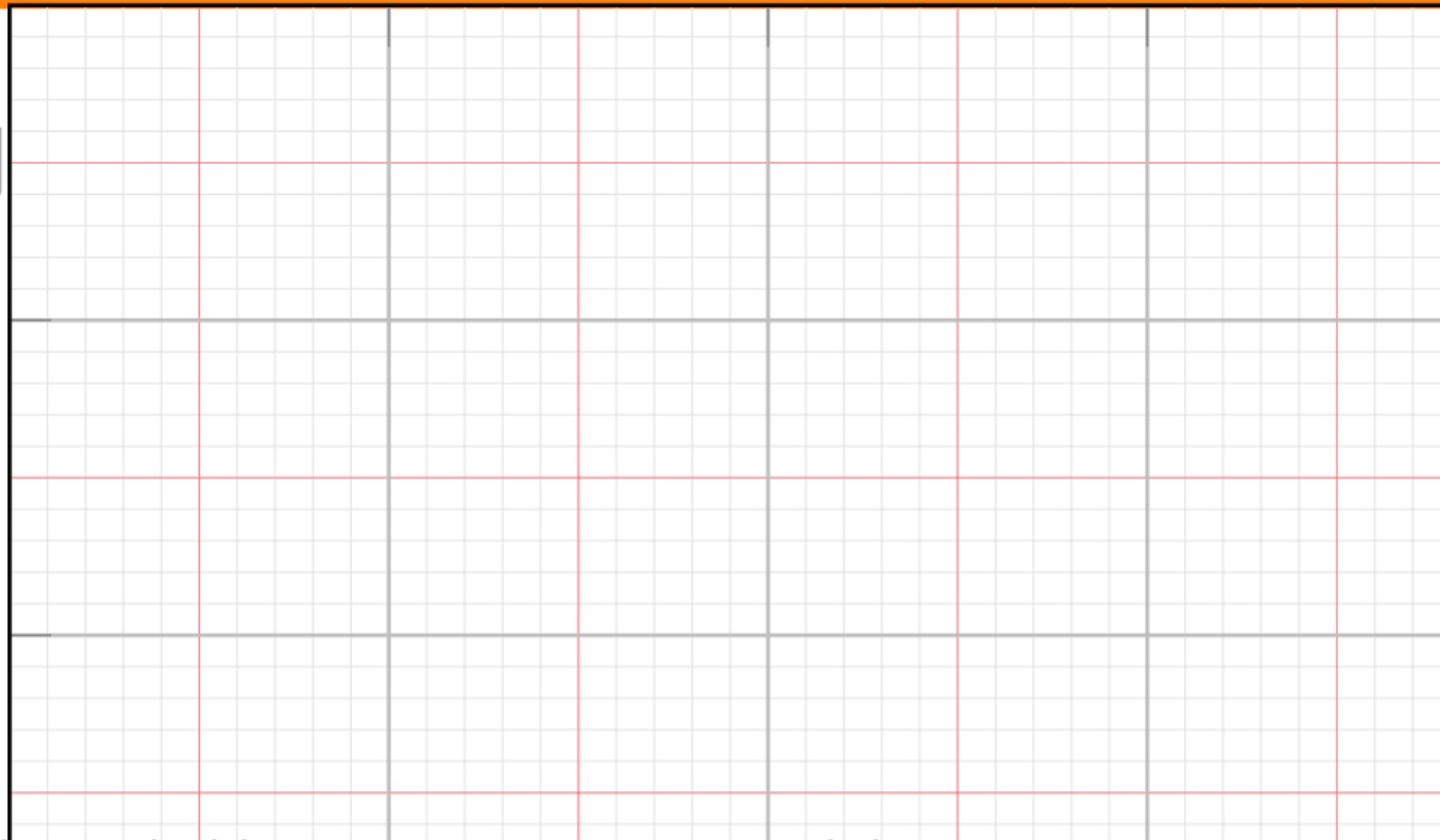
$Q(s)$	Criterion
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$s^2 + as_2s^2 + as_1 + a_0$	$a_2a_1 > a_0$ for positive coefficients
$s^4 + bs^3 + cs^2 + ds + 1$	$bcd - d^2 - b^2 > 0$
$s^4 + bs^3 + cs^2 + ds + a$	all coefficients positive and $bcd > d^2 + b^2a$

Exercise (#3.7)

Is the system with the transfer function

$$G(s) = \frac{1}{s^2 + s^2 - 4 \pm 1}$$

stable?



LTI systems

- 3.1 Introduction
- 3.2 Mathematical foundations: Differential equations
- 3.3 LTI systems
- 3.4 Transfer function
- 3.5 Impulse response
- 3.6 Causality
- 3.7 Stability
- 3.8 Network of systems**
- 3.9 Graphical representation

Cascaded systems



$$H(\omega) = H_1(\omega)H_2(\omega)$$

$$h(t) = h_1(t) * h_2(t)$$

Note: This is only valid, if the output of the first system is not changed by adding the second system (second system does not "load" the first system).

Cascaded systems

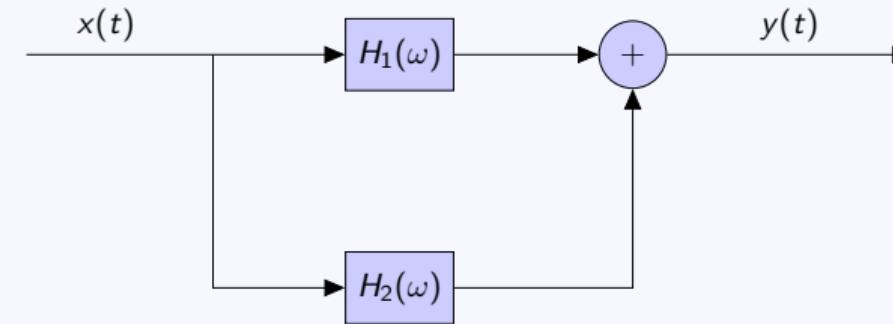


$$H(\omega) = H_1(\omega)H_2(\omega)$$

$$h(t) = h_1(t) * h_2(t)$$

Note: This is only valid, if the output of the first system is not changed by adding the second system (second system does not "load" the first system).

Parallel systems



$$H(\omega) = H_1(\omega) + H_2(\omega)$$

$$h(t) = h_1(t) + h_2(t)$$

LTI systems

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- 3.9 Graphical representation**
 - 3.9.1 Bode plots
 - 3.9.2 Root locus analysis
 - 3.9.3 Nyquist diagrams

Graphical representation

3.9 Graphical representation

3.9.1 Bode plots

3.9.2 Root locus analysis

3.9.3 Nyquist diagrams

Definition

The transfer function can also be written as follows

$$H(\omega) = e^{-A(\omega)} e^{-jB(\omega)}.$$

With

$$A(\omega) = \ln \frac{1}{|H(\omega)|}$$

the **damping** of the system and

$$B(\omega) = -\text{arc}(H(\omega))$$

the **phase** of the system.

Definition

The transfer function can also be written as follows

$$H(\omega) = e^{-j\theta(\omega)} e^{j\phi(\omega)},$$

With

$$\theta(\omega) = \ln \frac{1}{|H(\omega)|}$$

the **damping** of the system and

$$\phi(\omega) = -\arg(H(\omega))$$

the **phase** of the system.

Definition

A **Bode plot** shows the frequency response of a system as a function of the logarithm of the frequency.

Definition

A **Bode plot** shows the frequency response of a system as a function of the logarithm of the frequency.

Properties

$$H(\omega) = K \frac{(j\omega - z_1)(j\omega - z_2) \cdots (j\omega - z_m)}{(j\omega - p_1)(j\omega - p_2) \cdots (j\omega - p_n)}$$

Definition

The transfer function can also be written as follows

$$H(\omega) = e^{-A[\omega]} e^{-B[\omega]}$$

With

$$A[\omega] = \ln \frac{1}{|H(\omega)|}$$

the **damping** of the system and

$$B[\omega] = -\arg(H(\omega))$$

the **phase** of the system.

Notes:

- ▷ In MATLAB, use `zp2tf` to convert zero-pole-gain parameters to the transfer function form
- ▷ Note $\log(a \cdot b) = \log(a) + \log(b)$

Properties

$$H(\omega) = K \frac{(\omega - z_1)(\omega - z_2) \dots (\omega - z_n)}{(\omega - p_1)(\omega - p_2) \dots (\omega - p_n)}$$

- Notes:
- In MATLAB, use `zp2tf` to convert zero-pole-gain parameters to the transfer function form
 - Note $\log(a \cdot b) = \log(a) + \log(b)$

Example: Constant

For $k \in \mathbb{R}$:

Term	Magnitude	Phase
$K > 0$	$20 \log_{10}(K)$	0°
$K < 0$	$20 \log_{10}(K)$	$\pm 180^\circ$

Definition
A **Bode plot** shows the frequency response of a system as a function of the logarithm of the frequency.

Definition
The transfer function can also be written as follows

$$H(\omega) = e^{-A(\omega)} e^{-j\theta(\omega)}$$

With

$$A(\omega) = \ln \left| \frac{1}{|H(\omega)|} \right|$$

the **damping** of the system and

$$\theta(\omega) = -\arctan(H(\omega))$$

the **phase** of the system

Example

Example: Constant

For $k \in \mathbb{R}$:

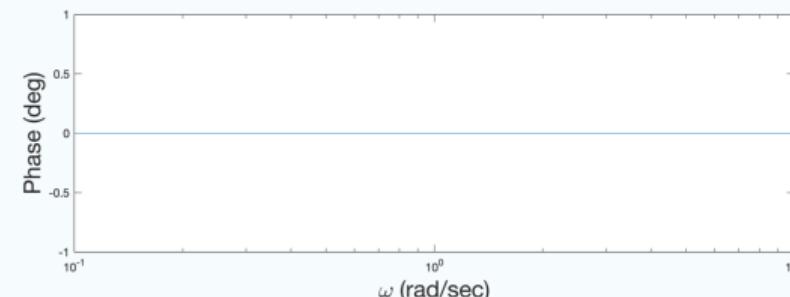
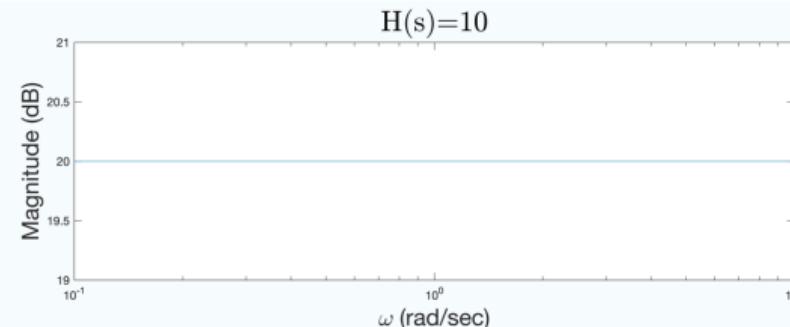
Term	Magnitude	Phase
$K > 0$	$20 \log_{10}(K)$	0°
$K < 0$	$20 \log_{10}(K)$	$\pm 180^\circ$

Properties

$$P(\omega) = K \frac{(\omega - \omega_1)(\omega - \omega_2) \cdots (\omega - \omega_n)}{(\omega - \mu_1)(\omega - \mu_2) \cdots (\omega - \mu_n)}$$

Notes:

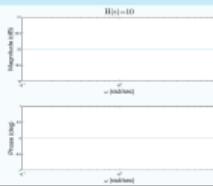
- ▷ In MATLAB, use `zp2tf` to convert zero-pole-gain parameters to the transfer function form
- ▷ Note $\log(a \cdot b) = \log(a) + \log(b)$



Definition

A **Bode plot** shows the frequency response of a system as a function of the logarithm of the frequency.

Example



Example: Constant

For $k \in \mathbb{R}$:

Term	Magnitude	Phase
$K > 0$	$20 \log_{10}(K)$	0°
$K < 0$	$20 \log_{10}(K)$	-180°

Properties

$$H(\omega) = K \frac{(\omega - z_1)(\omega - z_2) \cdots (\omega - z_n)}{(\omega - p_1)(\omega - p_2) \cdots (\omega - p_n)}$$

- Notes:
- ▷ In MATLAB, use `zp2tf` to convert zero-pole-gain parameters to the transfer function form
 - ▷ Note $\log(a \cdot b) = \log(a) + \log(b)$

$$\frac{1}{s}$$

Example: Pole at origin

Magnitude

- ▷ -20 dB/dec
- ▷ $0 \text{ dB at } \omega = 1$

Phase

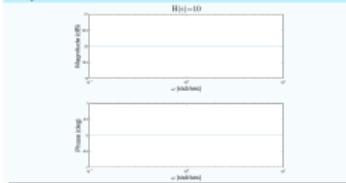
- ▷ -90° for all ω

- └ LTI systems
- └ Graphical representation

Example: Pole at origin

$s = \frac{1}{s}$	
Magnitude	Phase
$\geq -20 \text{ dB/sec}$	$\geq -90^\circ \text{ for all } \omega$
$\geq 0 \text{ dB at } \omega = 1$	

Example



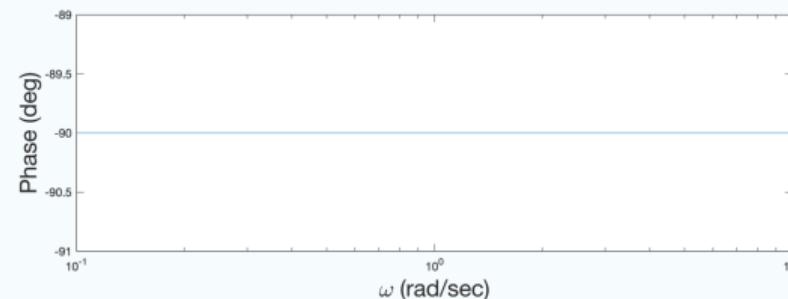
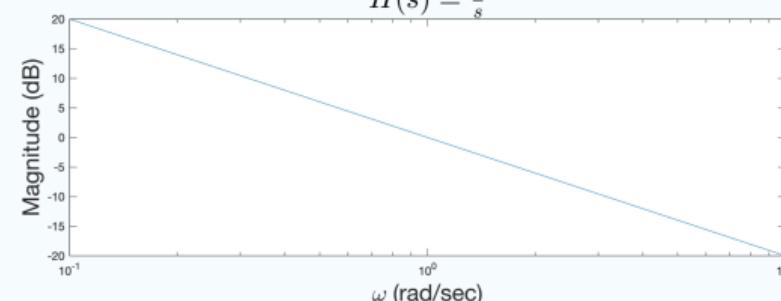
Example: Constant

For $K \in \mathbb{R}$:

Term	Magnitude	Phase
$K > 0$	$20 \log_{10}(K)$	0°
$K < 0$	$20 \log_{10}(K)$	$\pm 180^\circ$

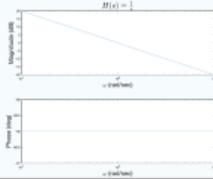
Example: Pole at origin

$$H(s) = \frac{1}{s}$$



- └ LTI systems
- └ Graphical representation

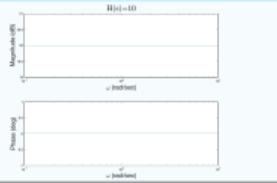
Example: Pole at origin



Example: Pole at origin

Magnitude	Phase
▷ -20 dB/dec	▷ -90° for all ω
▷ 0 dB at $\omega = 1$	

Example



Example: Pole at zero

 s

Magnitude

- ▷ $+20 \text{ dB/dec}$
- ▷ 0 dB at $\omega = 1$

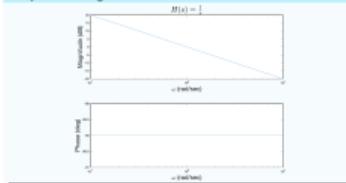
Phase

- ▷ $+90^\circ$ for all ω

Example: Pole at zero



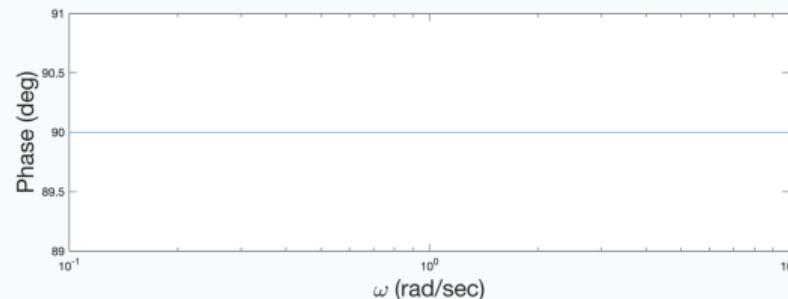
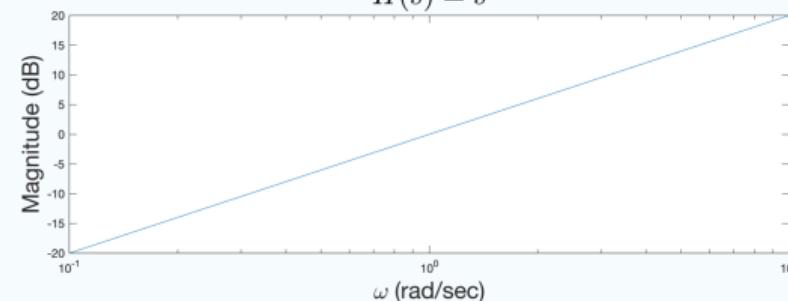
Example: Pole at origin



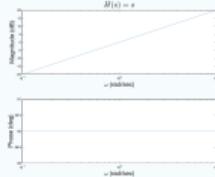
Example: Pole at origin

**Example: Pole at zero**

$$H(s) = s$$



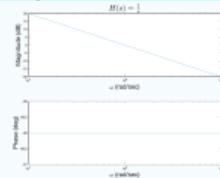
Example: Pole at zero



Example: Pole at zero

s	
Magnitude	Phase
$\approx +20 \text{ dB/dec}$	$\approx -90^\circ \text{ for all } \omega$
$\approx 0 \text{ dB at } \omega = 1$	

Example: Pole at origin



Example: Real Pole

$$\frac{1}{\frac{s}{\omega_0} + 1}$$

- ▷ Low frequency asymptote at 0 dB
- ▷ high frequency asymptote at -20 dB/dec
- ▷ connect asymptotic lines at ω_0

- ▷ Low frequency asymptote at 0°
- ▷ High frequency asymptote at -90°
- ▷ connect with straight line from $0.1\omega_0$ to $10\omega_0$

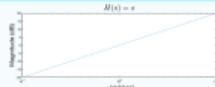
└ LTI systems

└ Graphical representation

Example: Real Pole

$$\frac{1}{\frac{s}{100} + 1}$$

- ▷ Low frequency asymptote at 0 dB
- ▷ high frequency asymptote at -20 dB/dec
- ▷ connect asymptotic lines at ω_0
- ▷ Low frequency asymptote at 0°
- ▷ High frequency asymptote at -90°
- ▷ connect with straight line from 0.1 rad/sec to 10 rad/sec

Example: Pole at zero**Example: Pole at zero**

$$s$$

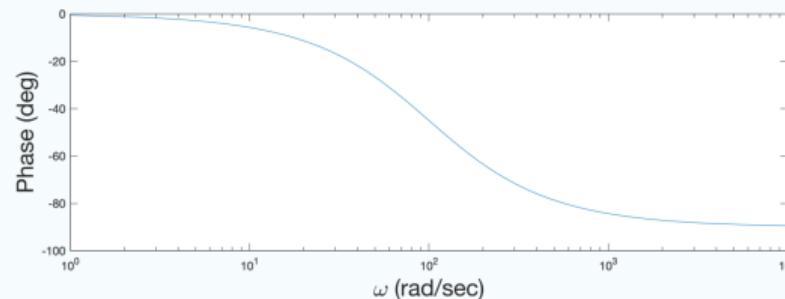
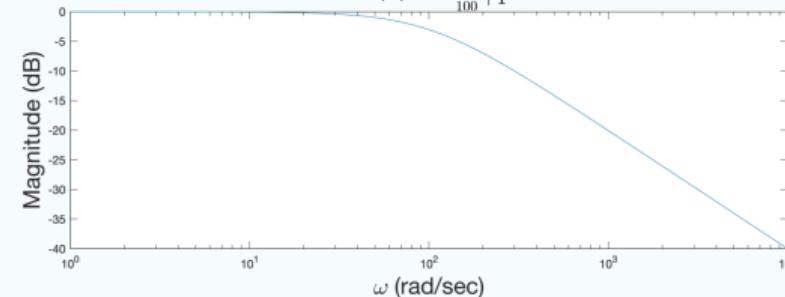
Magnitude

Phase

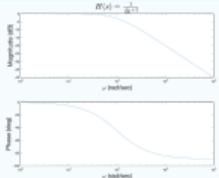
- ▷ +20 dB/dec
- ▷ 0 dB at $\omega = 1$

▷ -90° for all ω **Example: Real Pole**

$$H(s) = \frac{1}{\frac{s}{100} + 1}$$



Example: Real Pole

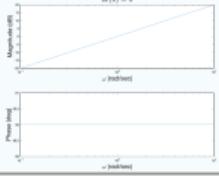


Example: Real Pole

$$\frac{1}{s+1}$$

- ▷ Low frequency asymptote at 0 dB
- ▷ high frequency asymptote at -20 dB/dec
- ▷ connect asymptotic lines at ω_0
- ▷ Low frequency asymptote at 0°
- ▷ High frequency asymptote at -90°
- ▷ connect with straight line from $0.1\omega_0$ to $10\omega_0$

Example: Pole at zero



Example: Real Zero

$$\frac{s}{\omega_0} + 1$$

Magnitude

- ▷ Low frequency asymptote at 0 dB
- ▷ high frequency asymptote at $+20$ dB/dec
- ▷ connect asymptotic lines at ω_0

Phase

- ▷ Low frequency asymptote at 0°
- ▷ High frequency asymptote at $+90^\circ$
- ▷ connect with straight line from $0.1\omega_0$ to $10\omega_0$

Example: Real Zero

$$\frac{s}{\omega_0} + 1$$

Magnitude

- ▷ Low frequency asymptote at 0 dB
- ▷ High frequency asymptote at +20 dB/dec
- ▷ connect asymptotic lines at ω_0

- ▷ Low frequency asymptote at 0°
- ▷ High frequency asymptote at +90°
- ▷ connect with straight line from 0.1. ω_0 to 10. ω_0

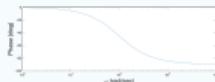
Example: Real Pole

$$H(s) = \frac{1}{s^2 + 1}$$

Magnitude (dB)



Phase (deg)



Example: Real Pole

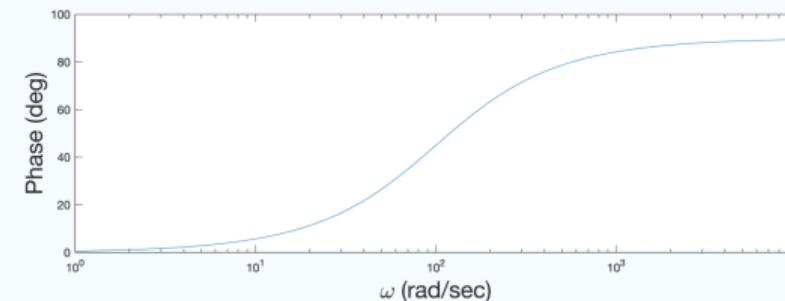
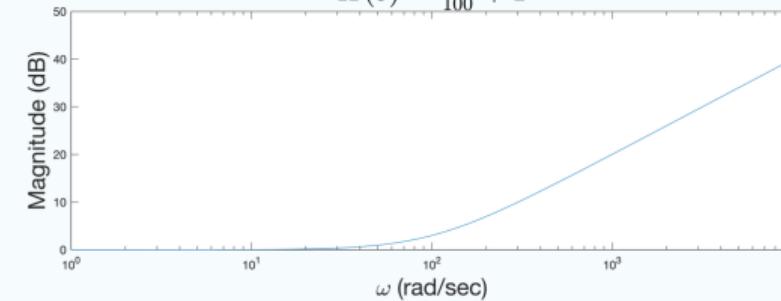
$$\frac{1}{\omega_0^2 s + 1}$$

- ▷ Low frequency asymptote at 0 dB
- ▷ High frequency asymptote at -20 dB/dec
- ▷ connect asymptotic lines at ω_0

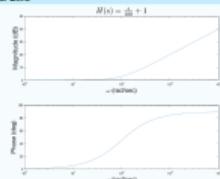
- ▷ Low frequency asymptote at 0°
- ▷ High frequency asymptote at -90°
- ▷ connect with straight line from 0.1. ω_0 to 10. ω_0

Example: Real Zero

$$H(s) = \frac{s}{100} + 1$$



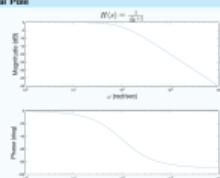
Example: Real Zero



Example: Real Zero

Magnitude	Phase
Low frequency asymptote at 0 dB	Low frequency asymptote at 0°
high frequency asymptote at +20 dB/dec	High frequency asymptote at $+90^\circ$
connect with straight line from 0.1 ω_0 to 10 ω_0	connect with straight line from 0.1 ω_0 to 10 ω_0

Example: Real Pole



Example: Underdamped Pole

$$\frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2\sigma\left(\frac{s}{\omega_0}\right) + 1}$$

- ▷ Low freq. asymptote at 0 dB
- ▷ high freq. asymptote at -40 dB/dec
- ▷ connect asymptotic lines at ω_0
- ▷ draw peak at $\omega_p = \omega_0\sqrt{1 - 2\sigma^2}$ with amplitude $-20\log_{10} 2\sigma\sqrt{1 - \sigma^2}$
- ▷ $\sigma < \frac{1}{\sqrt{2}}$.

- ▷ Low freq. asymptote at 0°
- ▷ High freq. asymptote at -180°
- ▷ Connect with straight line from $\frac{\omega_0}{2}\log_{10}\left(\frac{2}{\sigma}\right)$ to $\frac{2\omega_0}{\log_{10}\left(\frac{2}{\sigma}\right)}$

Note: underdamped means that an oscillation takes place.

- └ LTI systems
- └ Graphical representation

Example: Undamped Pole

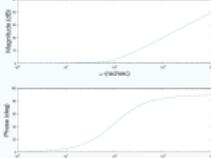
$$\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1$$

- ▷ Low freq. asymptote at 0 dB
- ▷ high freq. asymptote at -40 dB/dec
- ▷ connect asymptotic lines at ω_0
- ▷ draw peak at $\omega_p = \omega_0\sqrt{1-2\zeta^2}$ with amplitude $-20\log_{10}(2\zeta\sqrt{1-\zeta^2})$
- ▷ $\zeta < \frac{1}{2}$

Note: undamped means that an oscillation takes place.

Example: Real Zero

$$H(s) = \frac{1}{s} + 1$$



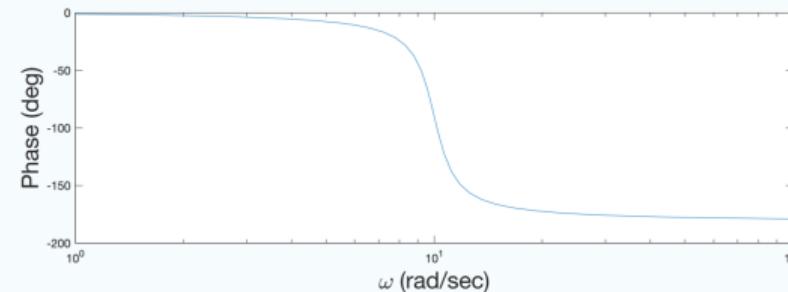
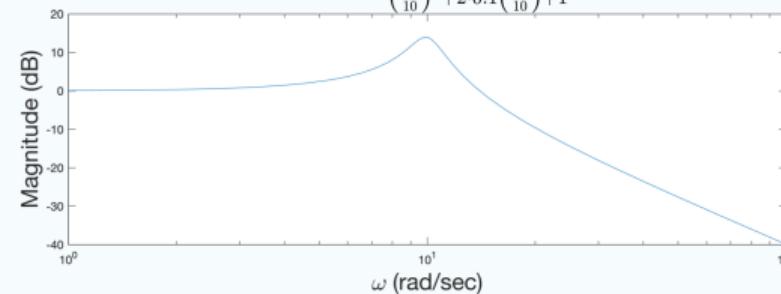
Example: Real Zero

$$\frac{s}{\omega_0} + 1$$

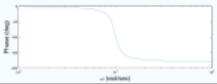
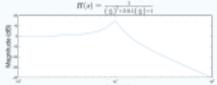
- | Magnitude | Phase |
|--|---|
| ▷ Low frequency asymptote at 0 dB | ▷ Low frequency asymptote at 0° |
| ▷ high frequency asymptote at +20 dB/dec | ▷ High frequency asymptote at +90° |
| ▷ connect asymptotic lines at ω_0 | ▷ connect with straight line from 0.1 ω_0 to 10 ω_0 |

Example

$$H(s) = \frac{1}{\left(\frac{s}{10}\right)^2 + 2 \cdot 0.1 \left(\frac{s}{10}\right) + 1}$$



Example



Example: Underdamped Pole

$$\left(\frac{s}{\omega_0}\right)^2 + 2\sigma \left(\frac{s}{\omega_0}\right) + 1$$

- ▷ Low freq. asymptote at 0 dB
 - ▷ high freq. asymptote at +40 dB/dec
 - ▷ connect asymptotic lines at ω_0
 - ▷ draw peak at $\omega_p = \omega_0 \sqrt{1-2\sigma^2}$ with amplitude $-20 \log_{10} 2\sigma \sqrt{1-\sigma^2}$
 - ▷ $\sigma < \frac{1}{\sqrt{2}}$
- Note: underdamped means that an oscillation takes place.

Example: Real Zero



Example: Underdamped Zero

$$\left(\frac{s}{\omega_0}\right)^2 + 2\sigma \left(\frac{s}{\omega_0}\right) + 1$$

- ▷ Low freq. asymptote at 0 dB
- ▷ high freq. asymptote at +40 dB/dec
- ▷ connect asymptotic lines at ω_0
- ▷ draw dip at $\omega_p = \frac{\omega_0}{\sqrt{1-2\sigma^2}}$ with amplitude $20 \log_{10} 2\sigma \sqrt{1-\sigma^2}$
- ▷ $\sigma < \frac{1}{\sqrt{2}}$.

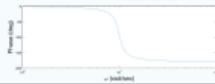
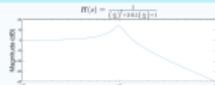
- ▷ Low freq. asymptote at 0°
- ▷ High freq. asymptote at $+180^\circ$
- ▷ Connect with straight line from $\log_{10}(\frac{2}{\sigma})$ to $\frac{2\omega_0}{\log_{10}(\frac{2}{\sigma})}$

Example: Undamped Zero

$$\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right) + 1$$

- ▷ Low freq. asymptote at 0 dB
- ▷ high freq. asymptote at +40 dB/dec
- ▷ connect asymptotic lines at ω_0
- ▷ draw dip at $\omega_p = \sqrt{\frac{1-2\zeta^2}{\zeta^2}}$ with an amplitude $-20\log_{10}2\zeta\sqrt{1-\zeta^2}$
- ▷ $\zeta < \frac{1}{\sqrt{2}}$
- ▷ Low freq. asymptote at 0°
- ▷ High freq. asymptote at +180°
- ▷ Connect with straight line from $\log_{10}\left(\frac{s}{\omega_0}\right)$ to $\frac{-20\log_{10}2\zeta}{\sqrt{1-\zeta^2}}$

Example



Example: Undamped Pole

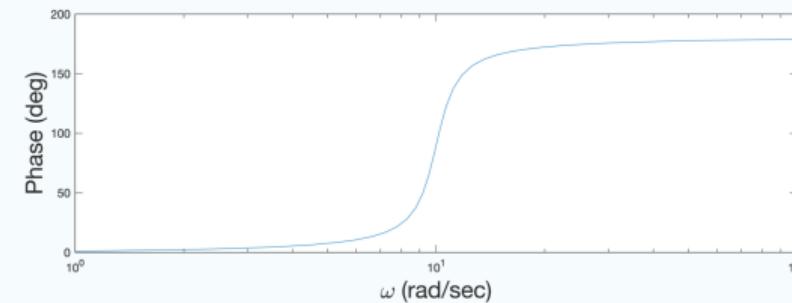
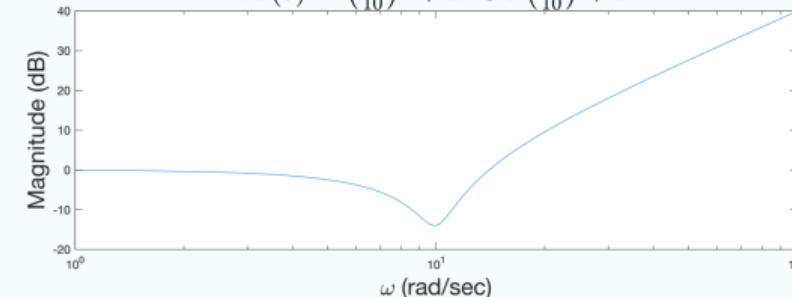
$$\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right) + 1$$

- ▷ Low freq. asymptote at 0 dB
- ▷ high freq. asymptote at -40 dB/dec
- ▷ connect asymptotic lines at ω_0
- ▷ draw peak at $\omega_p = \omega_0\sqrt{1-2\zeta^2}$ with amplitude $-20\log_{10}2\zeta\sqrt{1-\zeta^2}$
- ▷ $\zeta < \frac{1}{\sqrt{2}}$
- ▷ Low freq. asymptote at 0°
- ▷ High freq. asymptote at -180°
- ▷ Connect with straight line from $\log_{10}\left(\frac{s}{\omega_0}\right)$ to $\frac{-20\log_{10}2\zeta}{\sqrt{1-\zeta^2}}$

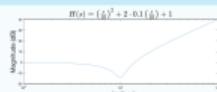
Note: undamped means that an oscillation takes place.

Example

$$H(s) = \left(\frac{s}{10}\right)^2 + 2 \cdot 0.1 \left(\frac{s}{10}\right) + 1$$



- └ LTI systems
- └ Graphical representation

Example**Example: Underdamped Zero**

$$\left(\frac{s}{\omega_n}\right)^2 + 2\zeta \left(\frac{s}{\omega_n}\right) + 1$$

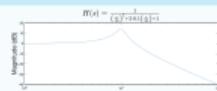
- ▷ Low freq. asymptote at 0 dB
- ▷ high freq. asymptote at -40 dB/dec
- ▷ connect asymptotic lines at ω_0
- ▷ draw dip at $\omega_d = \frac{\omega_0}{\sqrt{1-\zeta^2}}$ with amplitude $20\log_{10}(2\zeta\sqrt{1-\zeta^2})$
- ▷ $\omega < \frac{\omega_0}{\sqrt{1-\zeta^2}}$
- ▷ Low freq. asymptote at 0°
- ▷ High freq. asymptote at +180°
- ▷ Connect with straight line from $\log_{10}(\frac{s}{\omega_0})$ to $\frac{\omega_0}{\sqrt{1-\zeta^2}}(j)$

$$H(s) = \frac{s - 1}{s^2 + 2s + 2}$$

Exercise (#3.8)

Plot the Bode plot for the transfer function

using NUMPY or MATLAB with and without using the built-in function bode.

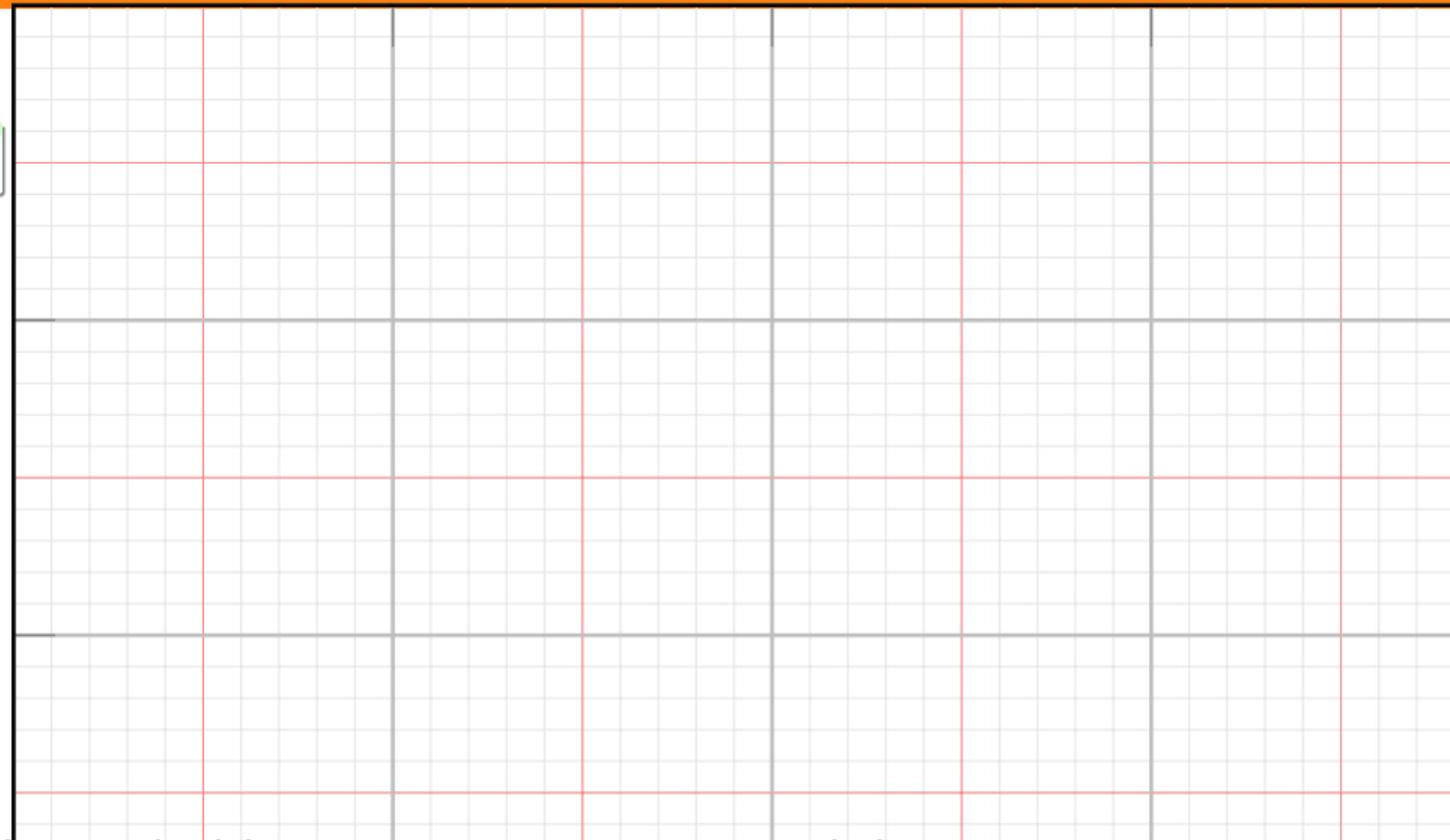
Example

Exercise (#3.8)

Plot the Bode plot for the transfer function

$$H(s) = \frac{s - 1}{s^2 + 2s + 2}$$

using Numpy or MATLAB with and without using the built-in function bode



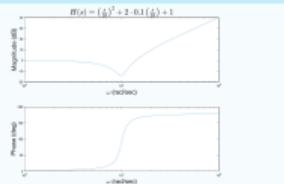
Exercise (#3.8)

Plot the Bode plot for the transfer function

$$H(s) = \frac{s - 1}{s^2 + 2s + 2}$$

using Numpy or MATLAB with and without using the built-in function bode.

Example



Example: Underdamped Zero

$$\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1$$

- | | |
|---|--|
| <ul style="list-style-type: none"> ▷ Low freq. asymptote at 0 dB ▷ high freq. asymptote at +40 dB/dec ▷ connect asymptotic lines at ω_0 ▷ draw dip at $\omega_{pe} = \sqrt{1 - \zeta^2}$ with an amplitude $20\log_{10}(2\pi\sqrt{1 - \zeta^2})$ ▷ $\omega = \omega_n \sqrt{1 - \zeta^2}$ | <ul style="list-style-type: none"> ▷ Low freq. asymptote at 0° ▷ High freq. asymptote at +180° ▷ Connect with straight line from $\text{Phase}(\omega_n)$ to $\text{Phase}(\omega)$ |
|---|--|

Exercise (#3.9)

Sketch the Bode plot for the following transfer functions:

a)

$$H(s) = 1 + \frac{s}{10}$$

b)

$$H(s) = \frac{1}{1 + s/100}$$

c)

$$H(s) = \frac{10^4(1 + s)}{(10 + s)(100 + s)}$$

Exercise (#3.9)

Sketch the Bode plot for the following transfer functions:

a)

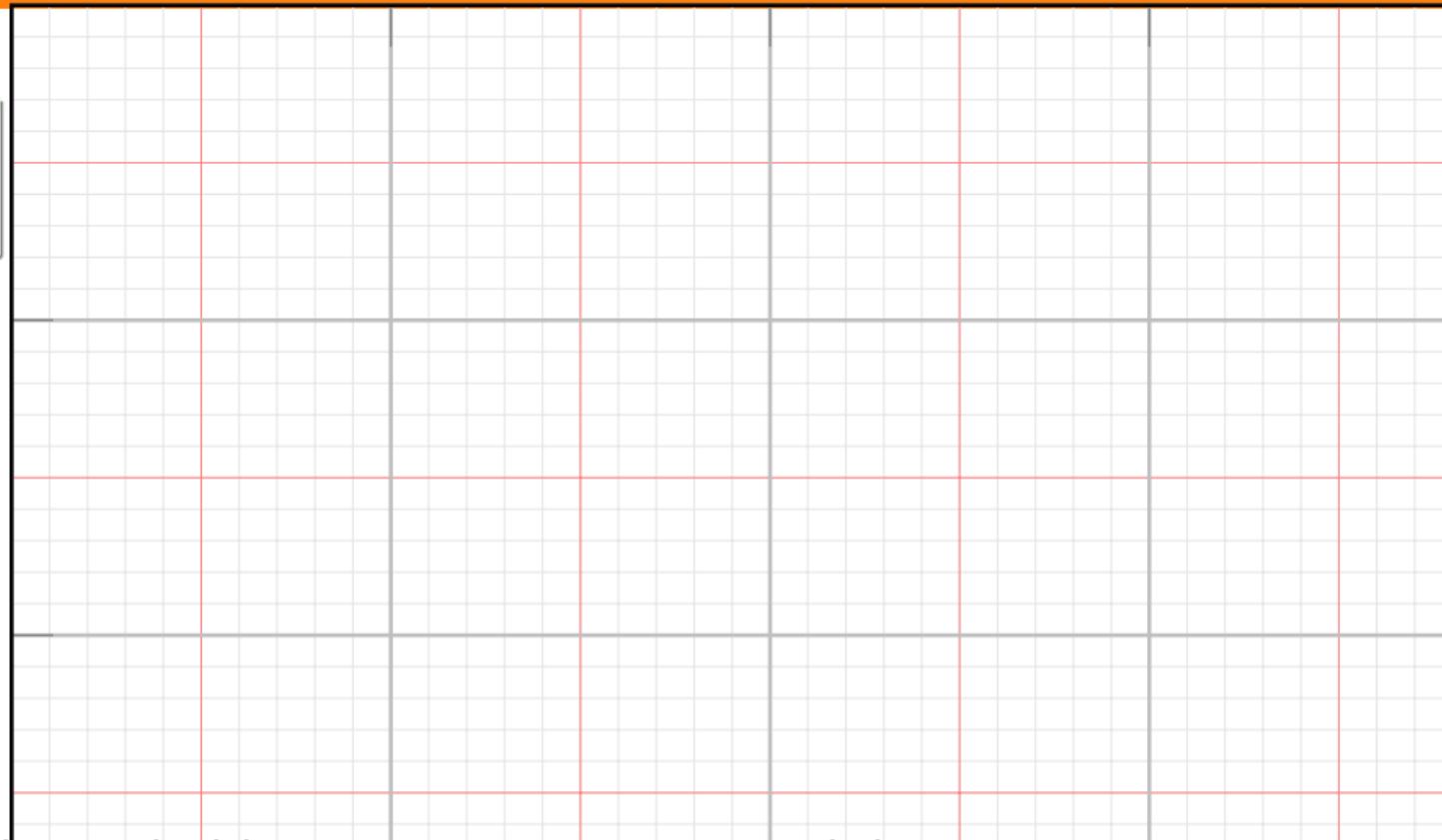
$$H(s) = 1 + \frac{s}{2s}$$

b)

$$H(s) = \frac{1}{1 + s/100}$$

c)

$$H(s) = \frac{10^3(1 + s)}{(10 + s)(100 + s)}$$



Graphical representation

3.9 Graphical representation

3.9.1 Bode plots

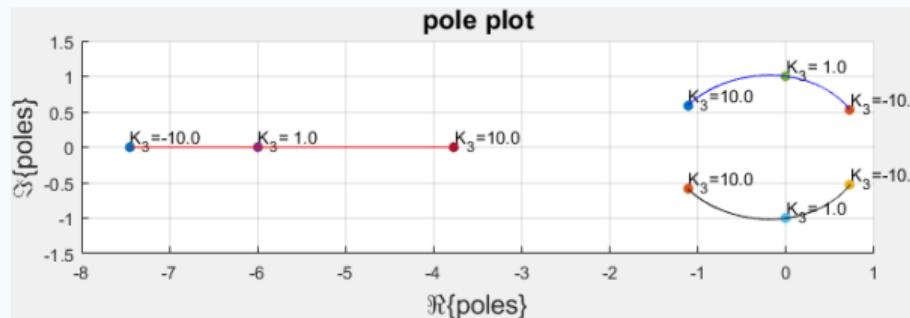
3.9.2 Root locus analysis

3.9.3 Nyquist diagrams

Example

Given is a system (plant) with following transfer function:

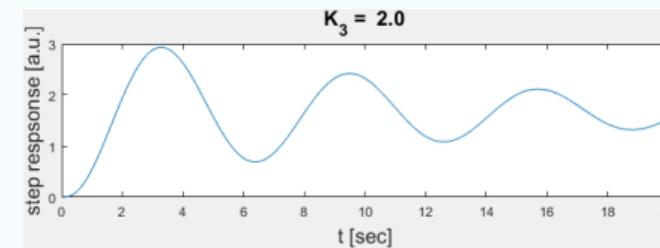
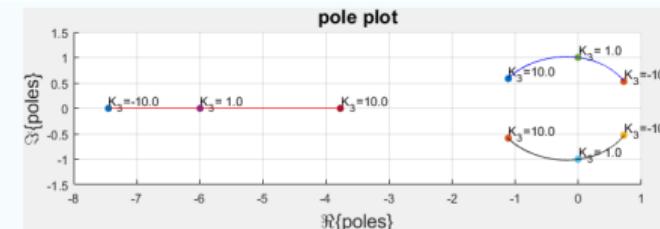
$$H(s) = \frac{10}{s^3 + 6s^2 + K_3 s + 6}.$$



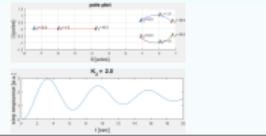
Example

Given is a system (plant) with following transfer function:

$$H(s) = \frac{10}{s^2 + 6s + K_3 s + 6}$$

**Example**

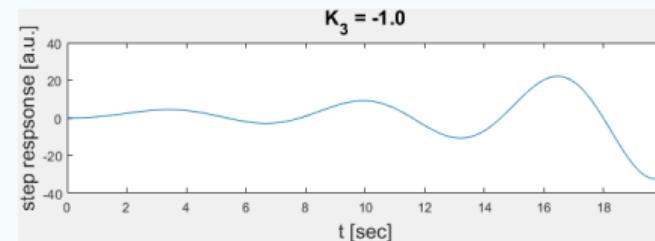
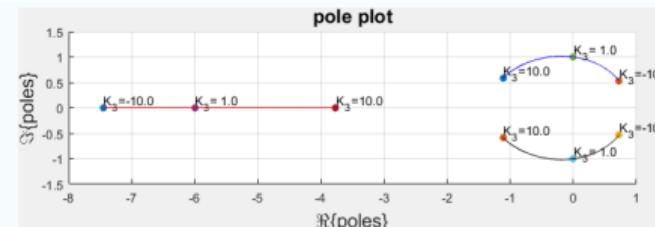
Example



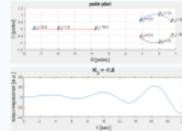
Exam

Given is a system (plant) with following transfer function

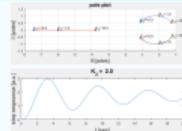
$$H(s) = \frac{10}{s^3 + 8s^2 + K_{38}s + 0}$$



Example



Example



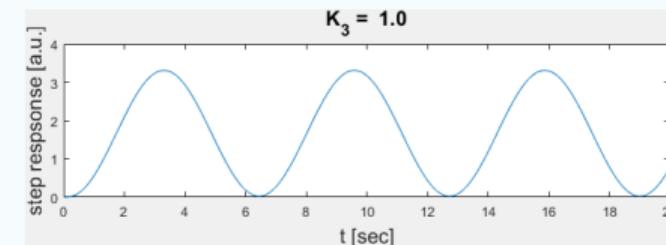
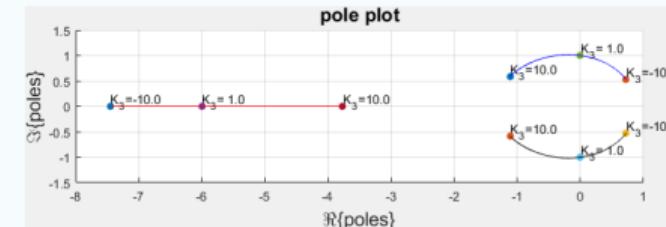
Example

Given is a system (plant) with following transfer function:

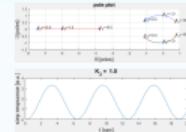
$$H(s) = \frac{10}{s^2 + 6s + 10s + 6}$$



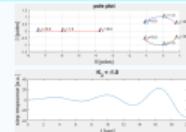
Example



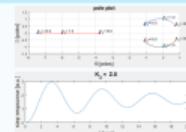
Example



Example



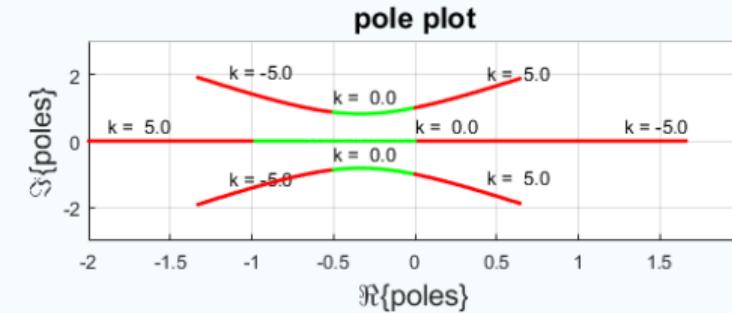
Example



Example

Given is a system with a transfer function

$$G(s) = \frac{1}{s^3 + s^2 + s + k}$$

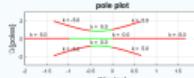


Stable for $0 < k < 1$.

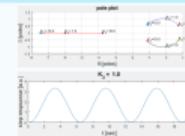
Example

Given is a system with a transfer function

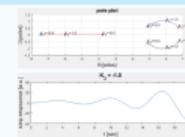
$$G(s) = \frac{1}{s^2 + s^2 + s + k}$$

Stable for $0 < k < 1$.**Definition**

The **slowest pole** of a system is the one with the closest distance to the imaginary axis.

Example**Definition**

A pole is called **dominant pole** if it is much slower than the other poles and dominates the transient response.

Example

Definition

The **closest pole** of a system is the one with the closest distance to the imaginary axis.

Definition

A pole is called **dominant pole** if it is much slower than the other poles and dominates the transient response.

Example

Given is a system with a transfer function

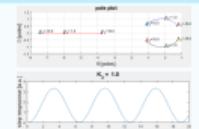
$$G(s) = \frac{1}{s^2 + s^2 + s + K}$$



Stable for $0 < K < 1$.

Exercise (#3.10)

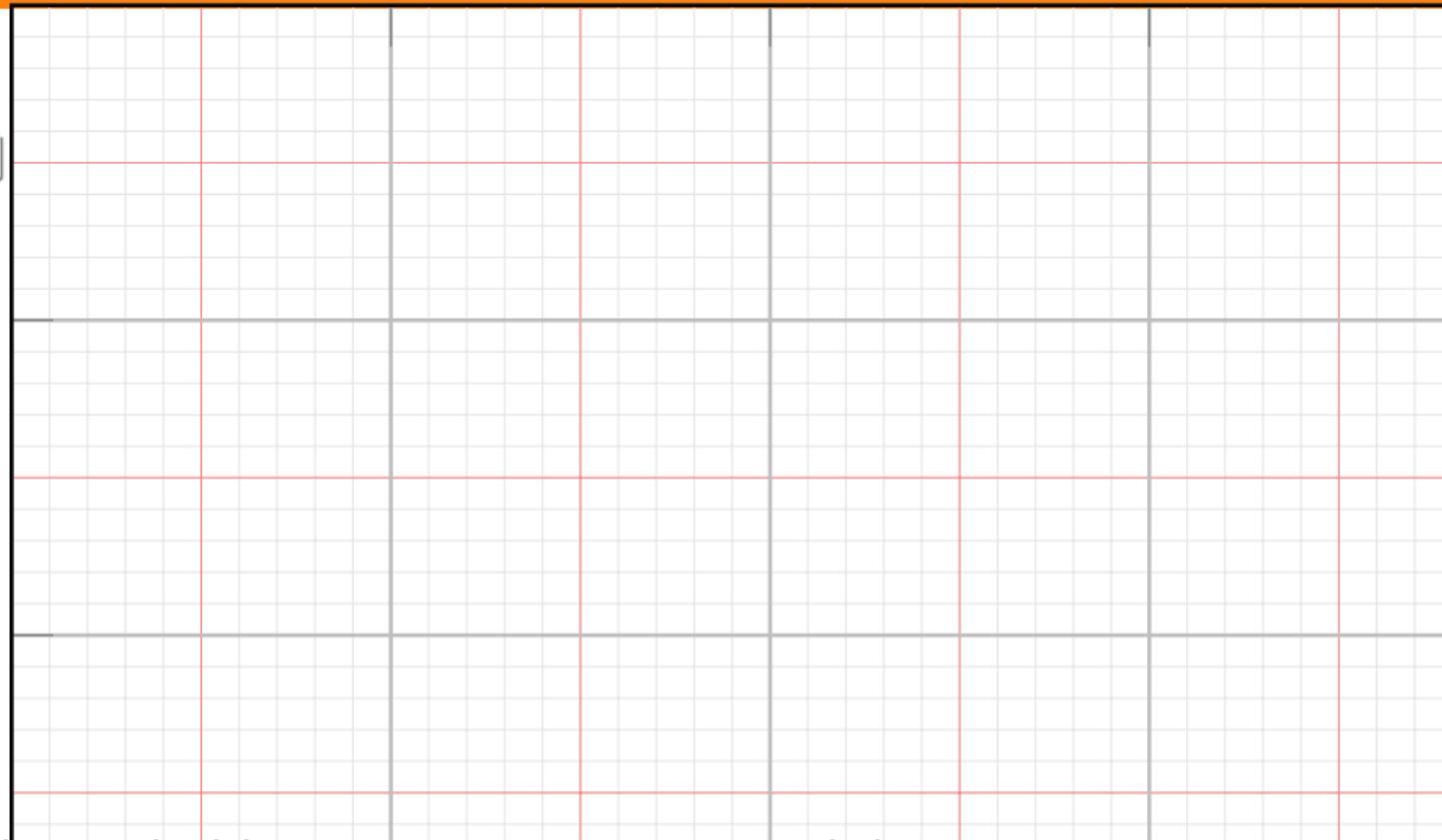
Plot the step responses for a system with poles at $-10 \pm j2$, $-1 \pm j1$ and for a system with poles at $-1 \pm j1$. Normalize the step responses so that the final values of both become 1. Make use of NUMPY or MATLAB.

Example

Exercise (#3.10)

Plot the step responses for a system with poles at $-10 \pm j2$, $-1 \pm j1$ and for a system with poles at $-1 \pm j1$. Normalize the step responses so that the final values of both become 1.

Make use of Nyström or MATLAB.



Exercise (#3.10)

Plot the step responses for a system with poles at $-10 \pm j2$, $-1 \pm j1$ and for a system with poles at $-1 \pm j1$. Normalize the step responses so that the final value of both become 1.
Make use of Numpy or MATLAB.

Definition

The **closest pole** of a system is the one with the closest distance to the imaginary axis.

Definition

A pole is called **dominant pole** if it is much slower than the other poles and dominates the transient response.

Example

Given is a system with a transfer function

$$G(s) = \frac{1}{s^2 + 2s + s + k}$$



Stable for $0 < k < 1$.

Exercise (#3.11)

A system has the transfer function

$$H(s) = \frac{1}{(s + 2 - j)(s + 2 + j)(s^2 + 20s + 104)}$$

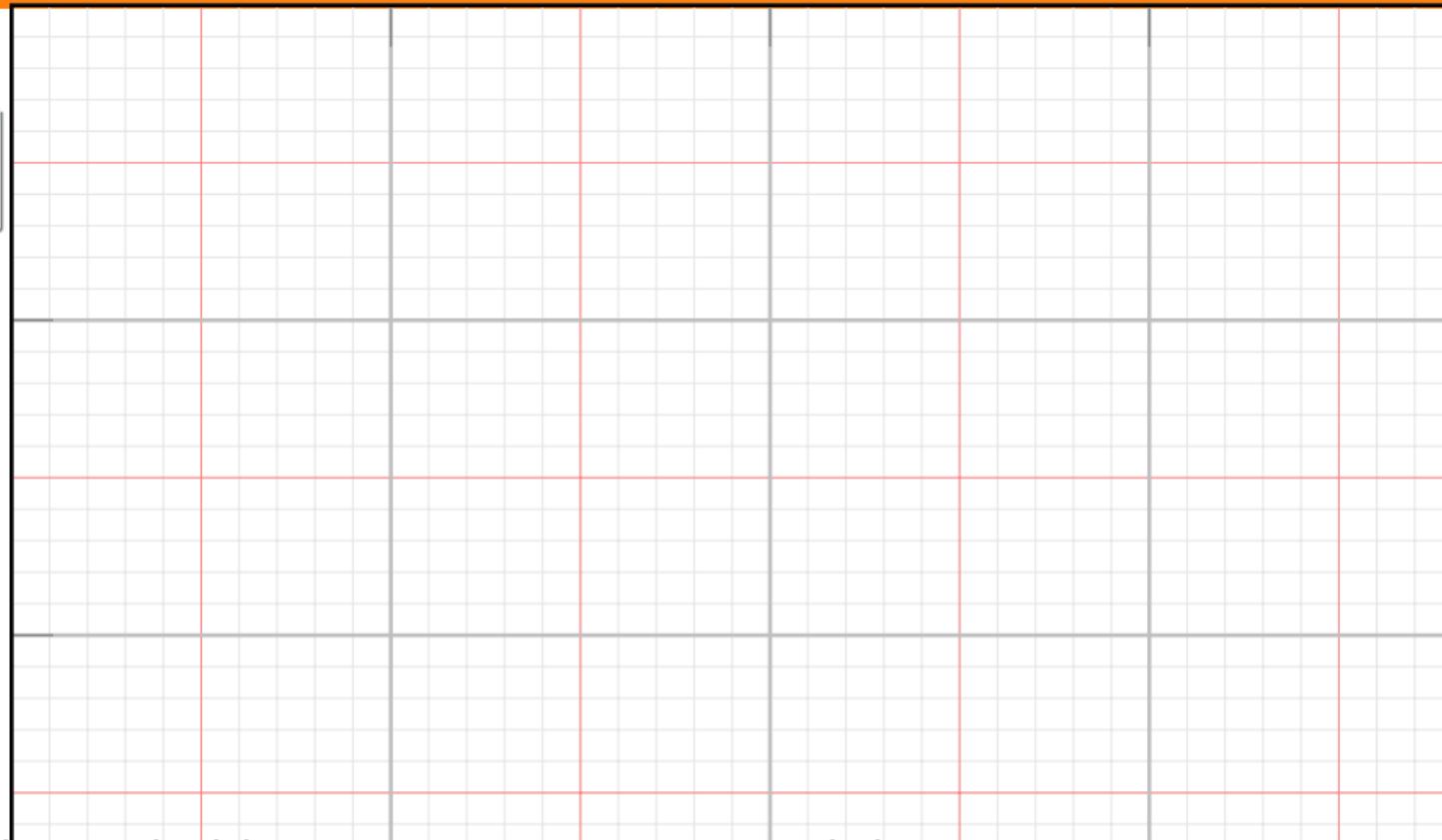
- ▷ Determine the **dominant pole(s)**.
- ▷ Determine the final value of the step response.
- ▷ Approximate the system with a second order system.
- ▷ Plot the step response of the original and approximated system using NUMPY or MATLAB.

Exercise (#3.11)

A system has the transfer function

$$H(s) = \frac{1}{(s+2-j)(s+2+j)(s^2 + 20s + 104)}$$

- o Determine the **dominant pole(s)**.
- o Determine the final value of the step response.
- o Approximate the system with a second order system.
- o Plot the step response of the original and approximated system using **Numerical or MATLAB**.



Graphical representation

3.9 Graphical representation

3.9.1 Bode plots

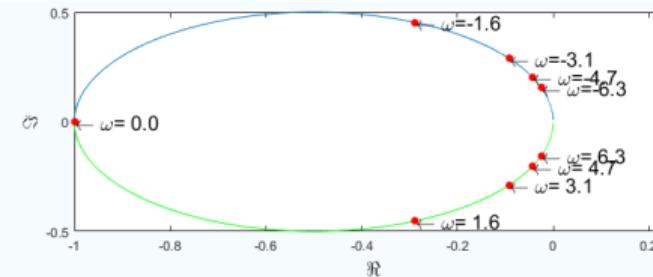
3.9.2 Root locus analysis

3.9.3 Nyquist diagrams

Definition

The **Nyquist diagram** is a plot of the frequency response. Usually, the real part of the transfer function is plotted on the x -axis and the imaginary part is plotted on the y -axis.

Example: $G(s) = \frac{1}{s-1}$



Definition

The Nyquist diagram is a plot of the frequency response. Usually, the real part of the transfer function is plotted on the x -axis and the imaginary part is plotted on the y -axis.

Example

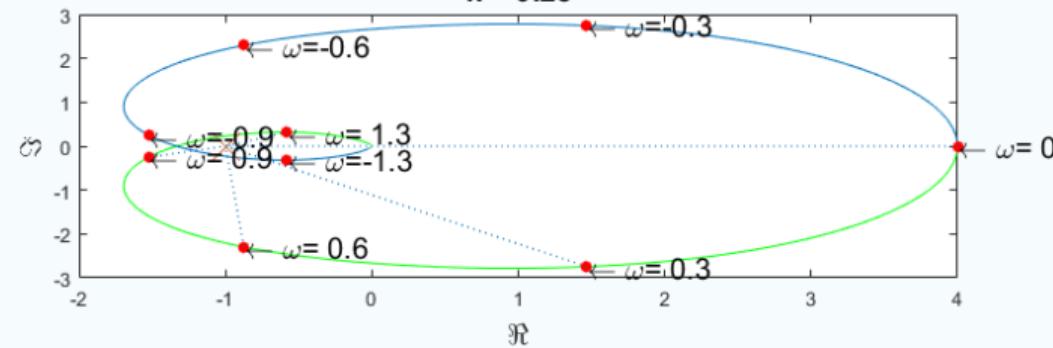
Example: $G(s) = \frac{1}{s^3 + s^2 + s + 0.5}$

**Example**

Given is a system with a transfer function

$$G(s) = \frac{10}{s^3 + s^2 + s + 0.5}$$

$k = 0.25$



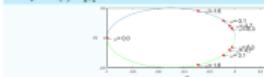
Example

Given is a system with a transfer function

$$G(s) = \frac{10}{s^3 + s^2 + s - 0.5}$$

**Definition**

The Nyquist diagram is a plot of the frequency response. Usually, the real part of the transfer function is plotted on the x-axis and the imaginary part on the y-axis.

Example:**Example**

Given is a system with a transfer function

$$G(s) = \frac{10}{s^3 + s^2 + s - 0.5}$$

