Table of Contents

LTI systems

Introduction

Content

- ▷ Differential equations
- ▷ LTI systems
- ▷ Frequency-domain representations

Study goals

- ▷ Describe LTI systems in time and frequency domain
- ▷ Use the concepts of transfer functions and impulse response
- ▷ Discuss stability of LTI systems
- ▷ Make use of Bode plots

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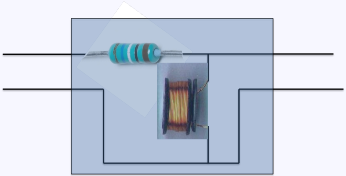
What is a system?



- ▷ A system is “something” with input and output
- ▷ The system “reacts” to its environment
- ▷ Usually, models are used for the description of systems. I.e. real-world behavior is simplified

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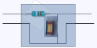

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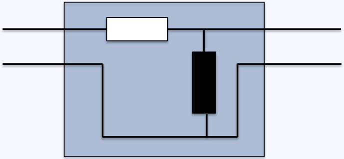

- Here we can define voltages as input and output
- A simple model for this circuit is shown above
- The output voltage can be calculated as follows (for open terminals):

$$U_{out} = U_{in} \frac{j\omega L}{R + j\omega L}.$$

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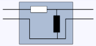

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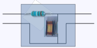

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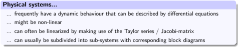
Physical systems...

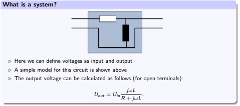
... frequently have a dynamic behaviour that can be described by differential equations

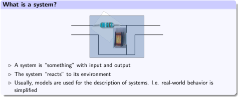
... might be non-linear

... can often be linearized by making use of the Taylor series / Jacobi-matrix

... can usually be subdivided into sub-systems with corresponding block diagrams







Network analysis

- In network analysis, we normally make the following assumptions:
  - Networks are linear
  - Networks are time invariant
- Analysis is usually done in the frequency domain:

$$U_{out} = U_{in}\frac{j\omega L}{R + j\omega L}.$$

- The fractional term can be interpreted as a transfer function $H(\omega)$: The frequency dependent reaction of the system to the input.

Table of Contents

LTI systems

Mathematical foundations: Differential equations

Ordinary differential equation

Definition

A ordinary differential equation is an equality involving one or more dependent variables, one independent variable, and one or more derivatives of the dependent variables, with respect to the independent variable.

Williams et al., Feedback and Control Systems

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Definition

An equation of the form

$$\sum_{i=0}^{n} a_i \frac{d^i}{dt^i} y(t) = \sum_{i=0}^{m} b_i \frac{d^i}{dt^i} u(t)$$

is called a linear constant-coefficient ordinary differential equation.

Definition

The initial condition is a set of equations of the form

$$y(0) = A_0,\ \frac{d}{dt} y(t=0) = A_1, \cdots \frac{d^{n-1}}{dt^{n-1}} y(t=0) = A_{n-1}.$$

Free response

Definition
An equation of the form

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Definition
A ordinary differential equation is an equality involving one or more dependent variables, one independent variable, and one or more derivatives of the dependent variables, with respect to the independent variable.
Williams et al., Feedback and Control Systems

Definition

The solution for an input $u(t) = 0$ is called the free response. The differential equation then has the form:

$$\sum_{i=0}^{n} a_i \frac{d^i}{dt^i} y(t) = 0.$$

Definition

The free response is given by a linear combination of the so-called fundamental set:

$$y(t) = \sum_{i=1}^{n} C_i y_i(t).$$

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Properties

The coefficients C_i are determined by the initial condition

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Example

$$\dot{x}(t) = C_1 x(t)$$

has the solution

$$x = A_0 e^{C_1 t}$$

in case of $x(t=0) = A_0$.

Forced response

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Definition

The forced response of a system described by an equation of the form

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is the solution for an initial condition of zero:

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Total response

Definition
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Unit step response

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Definition

The unit step response of a system described by an equation of the form

$$\sum_{i=0}^{n} a_i \frac{d^i}{dt^i} y(t) = \sum_{i=0}^{m} b_i \frac{d^i}{dt^i} u(t)$$

and is the forced response in case of $u(t)$ being the step function.

Table of Contents

Mathematical foundations: Differential equations

Second-order diff. eq.

Definition

The system described by the second-order differential equation

$$\frac{d^2}{dt^2}y(t) + 2\sigma\omega_0\frac{d}{dt}y(t) + \omega_0^2 y(t) = \omega_0^2 u(t)$$

has the unit step response

$$\begin{aligned} y(t) &= 1 - \frac{e^{-\sigma\omega_0 t}}{\sqrt{1-\sigma^2}}\sin\left(\omega_0\sqrt{1-\sigma^2}\,t + \phi_0\right), \\ \text{with} & \\ \phi_0 &= \tan^{-1}\left(\frac{\sqrt{1-\sigma^2}}{\sigma}\right) \end{aligned}$$

Exercise

Definition

The system described by the second-order differential equation

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Exercise (#3.1)

Plot the unit step response ($x(t) = u(t)$, $y(t \leq 0) = 0$) of a system described by the second-order differential equations

$$\frac{d^2}{dt^2}y(t) + 2\sigma\omega_0\frac{d}{dt}y(t) + \omega_0^2 y(t) = \omega_0^2 x(t)$$

for different values of ω_0 and σ by making of NUMPY or MATLAB. Use the equation shown on the former slide.

- What happens in case of negative σ?
- What happens if you set $\phi_0 = 0$? Does the result fulfill the requirements of a unit step response (check initial conditions)?

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Reduction to second order

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$$y(t) = 1 - \frac{e^{-\sigma\omega_0 t}}{\sqrt{1-\sigma^2}} \sin\left(\omega_0\sqrt{1-\sigma^2}t + \phi_0\right),$$

with

$$\phi_0 = \tan^{-1}\left(\frac{\sqrt{1-\sigma^2}}{\sigma}\right)$$

Spoiler

We will later discuss how to reduce higher order differential equations to first/second order.

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LTI Systems

Definition

In the following we consider LTI-systems (**L**inear **T**ime-**I**nvariant)

- One can show that if we stimulate an LTI system with an input

$$x(t) = C_1 e^{j\omega_0 t}$$

the output will be of the form

$$y(t) = C_2 e^{j\omega_0 t} = H(\omega_0) x(t)$$

- Note: C_1 and C_2 are complex and ω_0 is a real number

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Multiple (additive) inputs

We will now write the signals in slightly different form:

$$\begin{aligned} x(t) &= C_1 e^{j\omega t} = \frac{1}{2\pi} X(\omega) e^{j\omega t} \\ y(t) &= C_2 e^{j\omega t} = \frac{1}{2\pi} X(\omega) H(\omega) e^{j\omega t} \end{aligned}$$

The transfer function: Relationship between input and output in the frequency domain. For N-input signals:

$$\begin{aligned} x(t) &= \sum_i^N \frac{1}{2\pi} X_i(\omega_i) e^{j\omega_i t} \\ y(t) &= \sum_i^N \frac{1}{2\pi} X_i(\omega_i) H(\omega_i) e^{j\omega_i t} \end{aligned}$$

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LTI systems

Transfer function

In more general form:

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} \frac{1}{2\pi} X(\omega) e^{j\omega t} d\omega \\ y(t) &= \int_{-\infty}^{\infty} \frac{1}{2\pi} X(\omega) H(\omega) e^{j\omega t} d\omega \end{aligned}$$

It becomes obvious that the output signal is given by an inverse Fourier-Transform:

$$y(t) = \mathcal{F}^{-1}\left\{X(\omega)H(\omega)\right\}.$$

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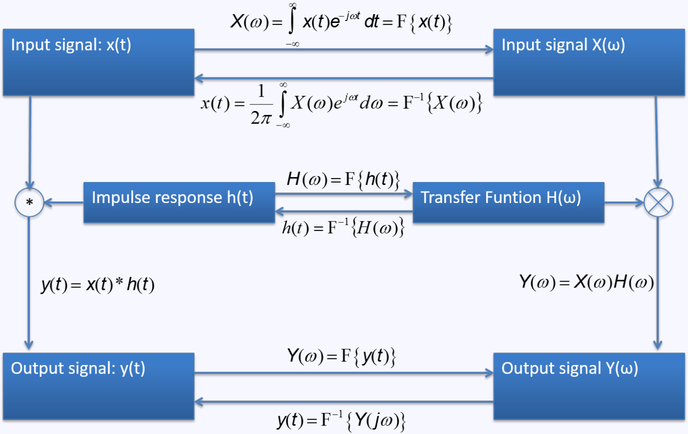
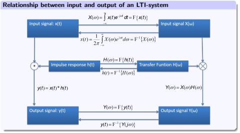
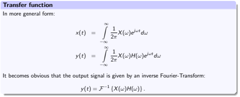
Relationship between input and output of an LTI-system



Using the Laplace tansform

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Definition

Using the Laplace Transform, the output of an LTI system is given by

$$Y(s) = H(s)X(s) + Y_{init}(s),$$

with $Y_{init}(s)$ being terms due to all initial conditions. If one ignores all terms arising from initial conditions, then the transfer function is of the form

$$H(s) = \frac{Y(s)}{X(s)}.$$

Exercise

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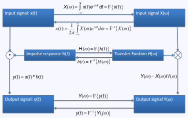

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Exercise (#3.2)

Plot the step response for a system with the transfer function

$$H(s) = \frac{s-1}{s^2 + 2s + 1}$$

using NUMPY or MATLAB with and without using the built-in function `step`.

Exercise

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Plot the step response for a system with the transfer function

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Partial fraction

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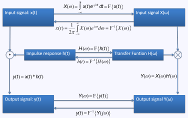
Relationship between input and output of an LTI-system



Definition

One can use the partial fraction expansion to simplify **complex**[a] functions of the form

$$A(x) = \frac{P(x)}{Q(x)}$$

as follows:

$$A(x) = \frac{P(x)}{Q(x)} = R(x) + \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{a_{ij}}{(x - x_i)^j},$$

with n_i being the order of pole x_i. $R(x) = 0$ in case of $\deg(P) < \deg(Q)$.

[a] a complex function is a function $D \subset \mathbb{C} \to \mathbb{C}$.

Zeros

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Properties

Rational transfer functions of the form

$$\begin{aligned} H(s) &= \frac{b_0 + b_1 s + \ldots + b_M s^M}{a_0 + a_1 s + \ldots a_N s^N} \\ &= K \frac{(s - z_1)(s - z_2)\ldots(s - z_M)}{(s - p_1)(s - p_2)\ldots(s - z_N)} \end{aligned}$$

are completely characterized by the gain factor K and the poles (natural frequencies) and zeros.

Properties

Rational transfer functions of the form

$$H(s) = \frac{b_0 + b_1 s + \ldots + b_M s^M}{a_0 + a_1 s + \ldots a_N s^N} = K \frac{(s-z_1)(s-z_2)\ldots(s-z_M)}{(s-p_1)(s-p_2)\ldots(s-z_N)}$$

are completely characterized by the gain factor K and the poles (natural frequencies) and zeros.

Definition

One can use the partial fraction expansion to simplify **complex**[a] functions of the form

$$A(x) = \frac{P(x)}{Q(x)}$$

as follows:

$$A(x) = \frac{P(x)}{Q(x)} = R(x) + \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{a_{ij}}{(x-x_i)^j},$$

with n_i being the order of pole x_i. $R(x) = 0$ in case of $\deg(P) < \deg(Q)$.

[a] a complex function is a function $D \subset \mathbb{C} \to \mathbb{C}$.

Exercise (#3.2)

Plot the step response for a system with the transfer function

$$H(s) = \frac{s-1}{s^2+2s+1}$$

using NumPy or MATLAB with and without using the built-in function `step`.

Exercise (#3.3)

A system is described by the differential equation (initial states of 0)

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 3\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} + 2x$$

a) Calculate the transfer function $H(s)$
b) Analytically calculate the impulse response
c) Analytically calculate the (unit) step response

Exercise

Exercise (#3.3)

A system is described by the differential equation (initial states of 0)

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 3\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} + 2x$$

a) Calculate the transfer function $H(s)$
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Table of Contents

LTI systems

Impulse response

Definition

The convolution is given by:

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

It can be shown that

$$h(t) = \mathcal{F}^{-1}\left\{H(\omega)\right\}$$

is the reaction of the system to a $\delta(t)$ input – the so called impulse response.

Exercise

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It can be shown that

$$h(t) = \mathcal{F}^{-1}\{H(\omega)\}$$

is the reaction of the system to a $\delta(t)$ input – the so called impulse response.

Exercise (#3.4)

Plot the impulse response for a system with the transfer function

$$H(s) = \frac{s-1}{s^2+2s+1}$$

using NUMPY or MATLAB[a].

[a] use the *by hand* Laplace transform and built-in functions like `impulse`

Exercise

Exercise (#3.4)

Plot the impulse response for a system with the transfer function

$$H(s) = \frac{s-1}{s^2+2s+1}$$

using NUMPY or MATLAB[a].

[a] use the *by hand* Laplace transform and built-in functions like `impulse`

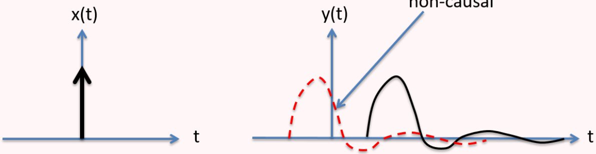
Table of Contents

LTI systems

Definition

A system is called causal if

$$h(t) = 0 \text{ for } t < 0.$$



Non-Causal means that the reaction of the system comes before the excitation

Table of Contents

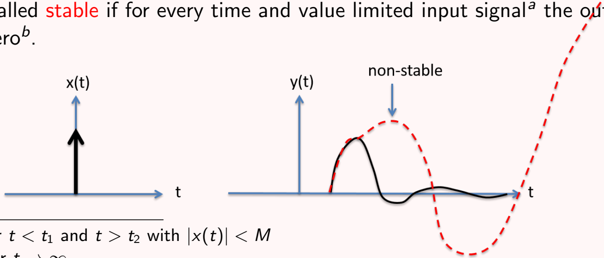
LTI systems

Table of Contents

Stability

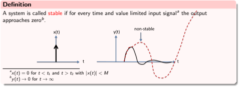
Definition

A system is called stable if for every time and value limited input signal[a] the output approaches zero[b].



[a] $x(t) = 0$ for $t < t_1$ and $t > t_2$ with $|x(t)| < M$
[b] $y(t) \to 0$ for $t \to \infty$

Criterion for Stability



Definition

A system is stable if

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

holds true.

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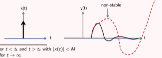

[a] $x(t)=0$ for $t<t_1$ and $t>t_2$ with $|x(t)|<M$
[b] $y(t)\to 0$ for $t\to\infty$

Definition

A causal and stable system has a bounded output in case of a bounded input (BIBO)

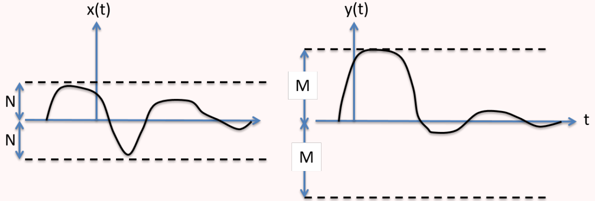

Table of Contents

Stability

Example

Consider the transfer function

$$H(s) = \frac{1}{s+a} = \mathcal{L}\left\{e^{-at}\right\} = \mathcal{L}\left\{e^{-\Re\{a\}t} e^{-i\Im\{a\}t}\right\}:$$

Obviously,

$$\lim_{t\to\infty} h(t) = 0$$

for

$$\Re\left\{a\right\} > 0.$$

Note that $H(s)$ has a pole at $s = -a$.

Stability

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Properties

Any causal system is absolutely stable if the poles of

$$H(s) = \frac{P(s)}{Q(s)}$$

lie in the left half space.

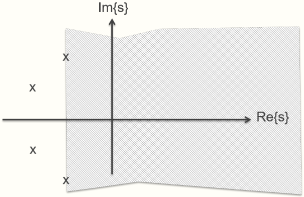

Figure 7: Poles and ROC

Note: This is only a yes/no classification.

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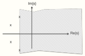

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For a given causal system with

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the equation

$$Q(s) = 0$$

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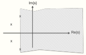

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Example

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

With poles at

$$s = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}.$$

For $p^2 - 4q < 0$ one gets poles at

$$s = -\frac{p}{2} \pm j\sqrt{q - \left(\frac{p}{2}\right)^2}.$$

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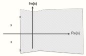

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Figure 7: Poles and ROC

Example

Consider the transfer function

$$H(s) = \frac{1}{s^2+ps+q}$$

with poles at $-1 \pm j$:

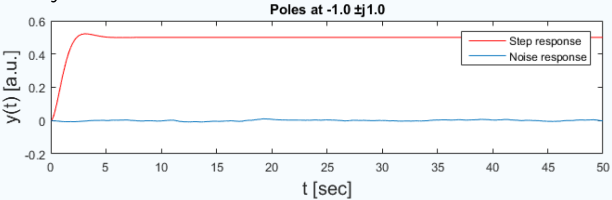

Figure 8: Step response and response to Gaussian noise (std. dev.: 0.01)

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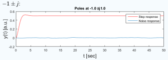

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For a given causal system with

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Example

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

with poles at $-0.1 \pm j$:

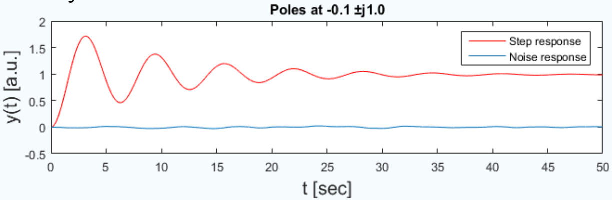

Figure 9: Step response and response to Gaussian noise (std. dev.: 0.01)

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Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

with poles at $-0.1 \pm j$:

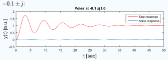

Figure 9: Step response and response to Gaussian noise (std. dev.: 0.01)

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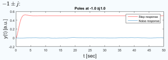

Figure 8: Step response and response to Gaussian noise (std. dev.: 0.01)

Example

Consider the transfer function

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Example

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

with poles at $-1 \pm 10j$:

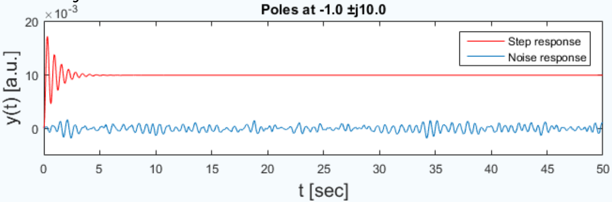
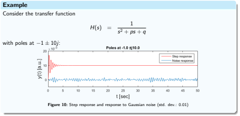
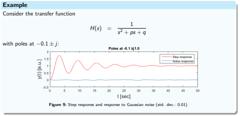
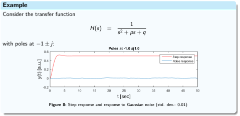


Figure 10: Step response and response to Gaussian noise (std. dev.: 0.01)







Example

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

with poles at $-0.1 \pm 10j$:

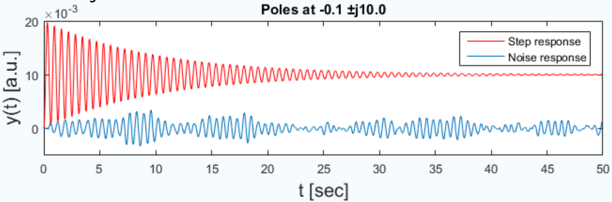
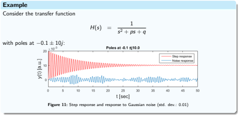
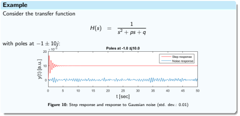
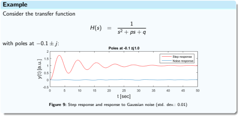


Figure 11: Step response and response to Gaussian noise (std. dev.: 0.01)







Example

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

with poles at $\pm 5j$:

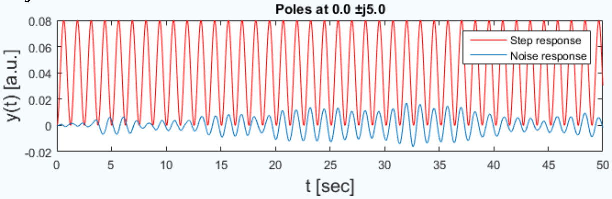
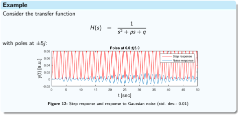
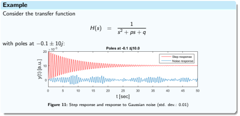
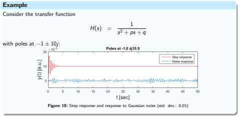


Figure 12: Step response and response to Gaussian noise (std. dev.: 0.01)







Exercise (#3.5)

Consider the transfer function

$$H(s) \;=\; \frac{1}{s^2 + ps + q}$$

with poles at $\alpha \pm j\beta$. Plot the step response and the response to Gaussian noise (std. dev.: 0.1) for the values of α and β given below by making use of NumPy or MATLAB. Plot the result up to $t = 40\,\mathrm{s}$.

case	α	β
1	-10	1
2	-1	1
3	-1	10
4	1	1
5	10	1

Exercise

Exercise (#3.5)

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

with poles at $\alpha \pm j\beta$. Plot the step response and the response to Gaussian noise (std. dev.: 0.1) for the values of α and β given below by making use of NUMPY or MATLAB. Plot the result up to $t = 40\,\mathrm{s}$.

case	α	β
1	-10	1
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Poles

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case	α	β
1	-10	1
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4	1	1
5	10	1

Example

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

with poles at $\pm 5j$:

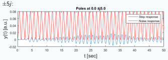

Figure 12: Step response and response to Gaussian noise (std. dev.: 0.01)

Example

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

with poles at $-0.1 \pm 10j$:

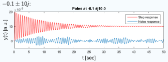

Figure 11: Step response and response to Gaussian noise (std. dev.: 0.01)

Properties

- ▷ A real pole in the left-half plane: Exponentially decaying component
- ▷ A real pole in the right-half plane: Exponentially increasing component
- ▷ A complex pole in the left-half plane: Exponentially decaying oscillatory component
- ▷ A complex pole in the right-half plane: Exponentially increasing oscillatory component
- ▷ A pole on the imaginary axis: Marginally stable
- ▷ Differential equations with real-valued coefficients yield poles that are pairwise complex conjugates

Poles

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- A pole on the imaginary axis: Marginally stable
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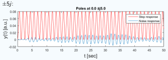

Figure 12: Step response and response to Gaussian noise (std. dev.: 0.01)

Marginally stable

Figure 13: Source: Schlecky Silberstein

Table of Contents

Stability

Definition

Given is a transfer function of the form $H(s) = \frac{P(s)}{Q(s)}$, with

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots a_1 s + a_0.$$

The Routh array is then of the form

$$\begin{array}{l|llll}
s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \cdots \\
s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \\
s^0 & h_{n-1} & & &
\end{array}$$

See next slides for definition of $b_{n-1}, b_{n-3}, \ldots$.

Routh-Hurwitz

Definition

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$$\begin{array}{l|llll} s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \cdots \\ s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ s^0 & h_{n-1} & & & \end{array}$$

See next slides for definition of $b_{n-1}, b_{n-3}, \dots$.

Properties

$$\begin{aligned}
b_{n-1} &= -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} \\
b_{n-3} &= -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix} \\
\vdots &= \vdots \\
c_{n-1} &= -\frac{1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix} \\
d_{n-1} &= -\frac{1}{c_{n-1}} \begin{vmatrix} b_{n-1} & b_{n-3} \\ c_{n-1} & c_{n-3} \end{vmatrix} \\
\vdots & \vdots \quad \vdots
\end{aligned}$$

Routh-Hurwitz

Properties

$$
\begin{aligned}
b_{n-1} &= -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} \\
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\vdots &= \vdots \\
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\vdots & \vdots \vdots
\end{aligned}
$$

Definition

Given is a transfer function of the form $H(s) = \frac{P(s)}{Q(s)}$, with

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \dots a_1 s + a_0.$$

The Routh array is then of the form

$$
\begin{array}{l|llll}
s^n & a_n & a_{n-2} & a_{n-4} & \dots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \dots \\
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s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \dots \\
\vdots & \vdots & \vdots & \vdots & \\
s^0 & h_{n-1} & & &
\end{array}
$$

See next slides for definition of $b_{n-1}, b_{n-3}, \dots$.

Definition

The Routh-Hurwitz stability criterion states that the number of roots of $Q(s)$ with positive real parts is equal to the number of changes in the sign of the first column

$$\begin{bmatrix} a_n & a_{n-1} & b_{n-1} & c_{n-1} & \dots & h_{n-1} \end{bmatrix}^T$$

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The following cases need to be distinguished:

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2. There is a zero in the first column, but at least one other element in the corresponding row is unequal to zero.
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4. Repeated roots of $Q(s)$ on the $j\omega$ axis

Routh-Hurwitz stab. criterion

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Example: $Q(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0$

Routh array:

$$\begin{array}{l|ll}
s^3 & a_3 & a_1 \\
s^2 & a_2 & a_0 \\
s^1 & b_1 = \frac{a_2 a_1 - a_0 a_3}{a_2} & 0 \\
s^0 & c_1 = \frac{b_1 a_0}{b_1} = a_0 & 0
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It follows that a necessary and sufficient condition for stability is that the coefficients are positive and that $a_2 a_1 > a_0 a_3$.

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Example: $Q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$

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$$\begin{array}{l|lll} s^5 & 1 & 2 & 11 \\ s^3 & 2 & 4 & 10 \\ s^3 & \boxed{0} & 6 & 0 \\ s^2 & c & 10 & 0 \\ s^1 & d & 0 & 0 \\ s^0 & 10 & 0 & 0 \end{array}$$

The 0 in the third row can be replaced by ϵ with $0 < \epsilon \ll 1$ (see next slide)

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s^3	1	4
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Stable for $0 < K < 8$. For $K = 8$: See next slide.

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This case occurs if singularities are symmetrically located about the origin of the s-plane: $(s+\sigma)(s-\sigma)$ or $(s+j\omega)(s-j\omega)$.

Routh-Hurwitz

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Utilizing the row preceding the row of zeros to form an auxiliary equation:

$$U(s) = 2s^2 + 8s^0 = 0,$$

With $s = -j2$ and $s = j2$ the roots of $U(s)$.

Routh-Hurwitz

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Example: $Q(s) = s^3 + 2s^2 + 4s + 8$

By taking the derivative of the the auxiliary equation, one can form a new array

s^3	1	4
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Polynomial long division of $Q(s)$ by $U(s)$ leads to:

$$Q(s) = U(s)\left[1 + \frac{1}{2}s\right] = (s + j2)(s - j2)\left[s + 2\right]$$

Routh-Hurwitz

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Properties

For the following forms of $Q(s)$ one can directly give a stability criterion:

$Q(s)$	Criterion
$s^2 + bs + 1$	$b > 0$
$as^2 + bs + c$	all coefficients positive
$s^3 + bs^2 + cs + 1$	$bc - 1 > 0$
$s^3 + a_2 s^2 + a_1 s + a_0$	$a_2 a_1 > a_0$ for positive coefficients
$s^4 + bs^3 + cs^2 + ds + 1$	$bcd - d^2 - b^2 > 0$
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Properties

For systems of the form

$$G(s)=e^{-sT}\frac{P(s)}{Q(s)}$$

with a time delay one needs to approximate e^{-sT} e.g. by making use of the Pade approximation

$$e^{-sT}\approx\frac{1-\frac{sT}{2}}{1+\frac{sT}{2}}.$$

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Exercise (#3.6)

Is the system with the transfer function

$$G(s) = \frac{3}{3s^3+12s^2+24s+36}$$

stable? Test by making use of NUMPY or MATLAB (e.g using `roots`).

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Exercise (#3.7)

Is the system with the transfer function

$$G(s) = \frac{1}{s^4 + s^3 - s \pm 1}$$

stable?

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Exercise (#3.7)

Is the system with the transfer function

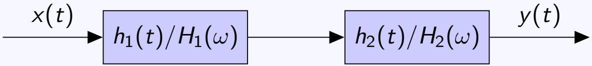
$$G(s) = \frac{1}{s^4 + s^3 - s \pm 1}$$

stable?

Table of Contents

LTI systems

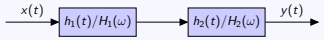
Cascaded systems



$$
\begin{aligned}
H(\omega) &= H_1(\omega)H_2(\omega) \\
h(t) &= h_1(t) * h_2(t)
\end{aligned}
$$

Note: This is only valid, if the output of the first system is not changed by adding the second system (second system does not "load" the first system).

Cascaded systems



$$H(\omega) = H_1(\omega)H_2(\omega)$$
$$h(t) = h_1(t) * h_2(t)$$

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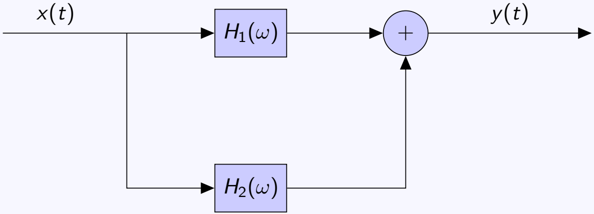
Parallel systems



$$\begin{aligned} H(\omega) &= H_1(\omega) + H_2(\omega) \\ h(t) &= h_1(t) + h_2(t) \end{aligned}$$

Table of Contents

LTI systems

Table of Contents

Graphical representation

Transfer function

Definition

The transfer function can also be written as follows

$$H(\omega) = e^{-A(\omega)} e^{-jB(\omega)}.$$

With

$$A(\omega) = \ln \frac{1}{|H(\omega)|}$$

the damping of the system and

$$B(\omega) = -\mathrm{arc}(H(\omega))$$

the phase of the system.

Bode diagrams

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A Bode plot shows the frequency response of a system as a function of the logarithm of the frequency.

Bode diagrams

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the phase of the system.

Properties

$$H(\omega) = K \frac{(j\omega - z_1)(j\omega - z_2)\cdots(j\omega - z_m)}{(j\omega - p_1)(j\omega - p_2)\cdots(j\omega - p_n)}$$

Notes:

- In MATLAB, use *zp2tf* to convert zero-pole-gain parameters to the transfer function form
- Note $\log(a \cdot b) = \log(a) + \log(b)$

Bode diagrams

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the damping of the system and

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the phase of the system.

Example: Constant

For $k \in \mathbb{R}$:

Term	Magnitude	Phase
$K > 0$	$20\,\log_{10}(K)$	0^o
$K < 0$	$20\log_{10}(\lvert K\rvert)$	$\pm 180^o$

Bode diagrams

Example: Constant

For $k \in \mathbb{R}$:

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$K > 0$	$20\log_{10}(K)$	0^o
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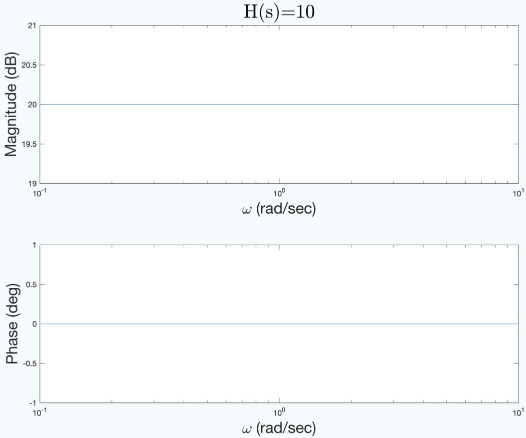
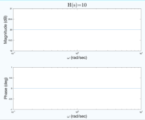


Bode diagrams

Example



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$K > 0$	$20\log_{10}(K)$	0^o
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Notes:

- In MATLAB, use *zp2tf* to convert zero-pole-gain parameters to the transfer function form
- Note $\log(a \cdot b) = \log(a) + \log(b)$

Example: Pole at origin

$$\frac{1}{s}$$

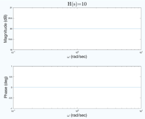
Magnitude	Phase
▷ -20 dB/dec ▷ 0 dB at $\omega = 1$	▷ -90^o for all ω

Example: Pole at origin

$$\frac{1}{s}$$

Magnitude	Phase
▷ -20 dB/dec ▷ 0 dB at $\omega = 1$	▷ $-90°$ for all ω

Example



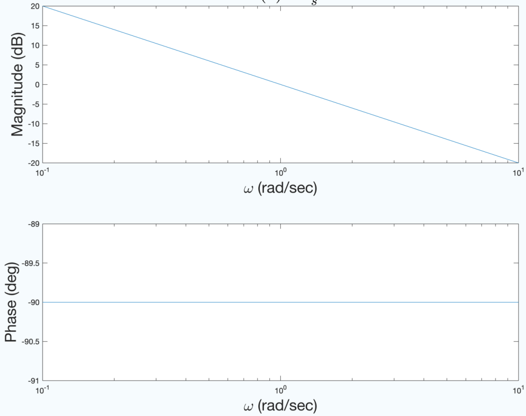
Example: Constant

For $k \in \mathbb{R}$:

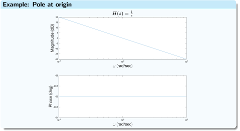
Term	Magnitude	Phase
$K > 0$	$20\log_{10}(K)$	$0°$
$K < 0$	$20\log_{10}(\lvert K \rvert)$	$\pm 180°$

Example: Pole at origin

$$H(s) = \frac{1}{s}$$



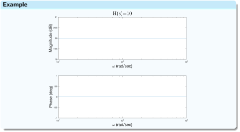
Bode diagrams



Example: Pole at origin

$\frac{1}{s}$

Magnitude	Phase
▷ -20 dB/dec ▷ 0 dB at $\omega = 1$	▷ -90^o for all ω



Example: Pole at zero

s

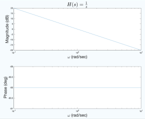
Magnitude	Phase
▷ $+20$ dB/dec ▷ 0 dB at $\omega = 1$	▷ $+90^o$ for all ω

Example: Pole at zero

s

Magnitude	Phase
▷ +20 dB/dec ▷ 0 dB at $\omega = 1$	▷ $+90°$ for all ω

Example: Pole at origin

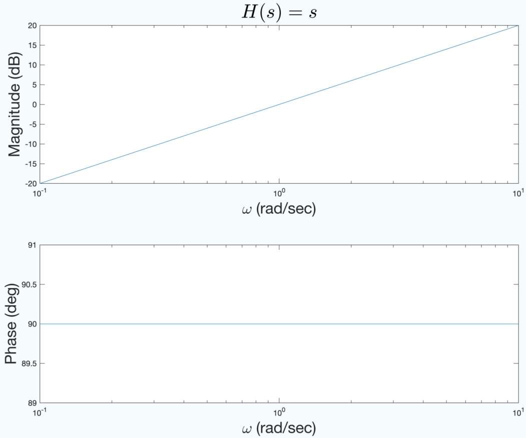


Example: Pole at origin

$\frac{1}{s}$

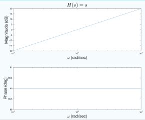
Magnitude	Phase
▷ -20 dB/dec ▷ 0 dB at $\omega = 1$	▷ $-90°$ for all ω

Example: Pole at zero



Bode diagrams

Example: Pole at zero

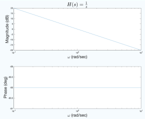


Example: Pole at zero

s

Magnitude	Phase
▷ +20 dB/dec ▷ 0 dB at $\omega = 1$	▷ $+90°$ for all ω

Example: Pole at origin



Example: Real Pole

$$\frac{1}{\frac{s}{\omega_0} + 1}$$

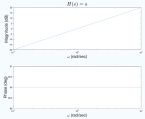
▷ Low frequency asymptote at 0 dB ▷ high frequency asymptote at -20 dB/dec ▷ connect asymptotic lines at ω_0	▷ Low frequency asymptote at 0^o ▷ High frequency asymptote at -90^o ▷ connect with straight line from $0.1\,\omega_0$ to $10\,\omega_0$

Example: Real Pole

$$\frac{1}{\frac{s}{\omega_0}+1}$$

▷ Low frequency asymptote at 0 dB ▷ high frequency asymptote at -20 dB/dec ▷ connect asymptotic lines at ω_0	▷ Low frequency asymptote at 0° ▷ High frequency asymptote at −90° ▷ connect with straight line from 0.1 ω_0 to 10 ω_0

Example: Pole at zero

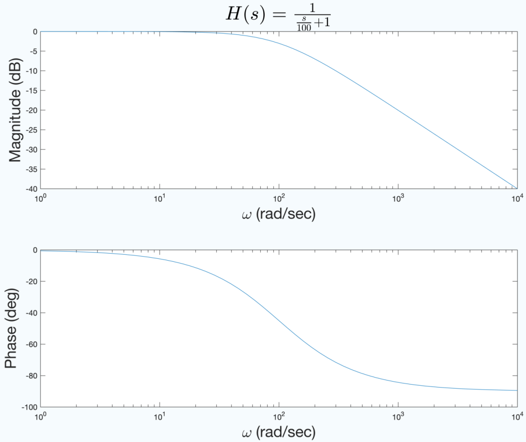


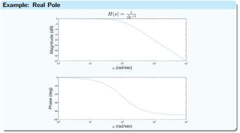
Example: Pole at zero

$$s$$

Magnitude	Phase
▷ +20 dB/dec ▷ 0 dB at $\omega = 1$	▷ +90° for all ω

Example: Real Pole



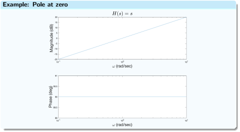


Example: Real Pole

$$\frac{1}{\frac{s}{\omega_0}+1}$$

- ▷ Low frequency asymptote at 0 dB
- ▷ high frequency asymptote at -20 dB/dec
- ▷ connect asymptotic lines at ω_0

- ▷ Low frequency asymptote at 0^o
- ▷ High frequency asymptote at -90^o
- ▷ connect with straight line from $0.1\,\omega_0$ to $10\,\omega_0$



Example: Real Zero

$$\frac{s}{\omega_0}+1$$

Magnitude	Phase
▷ Low frequency asymptote at 0 dB ▷ high frequency asymptote at $+20$ dB/dec ▷ connect asymptotic lines at ω_0	▷ Low frequency asymptote at 0^o ▷ High frequency asymptote at $+90^o$ ▷ connect with straight line from $0.1\,\omega_0$ to $10\,\omega_0$

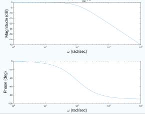
Example: Real Zero

$$\frac{s}{\omega_0}+1$$

Magnitude	Phase
▷ Low frequency asymptote at 0 dB ▷ high frequency asymptote at +20 dB/dec ▷ connect asymptotic lines at ω_0	▷ Low frequency asymptote at 0° ▷ High frequency asymptote at +90° ▷ connect with straight line from $0.1\,\omega_0$ to $10\,\omega_0$

Example: Real Pole

$H(s) = \frac{1}{\frac{s}{100}+1}$



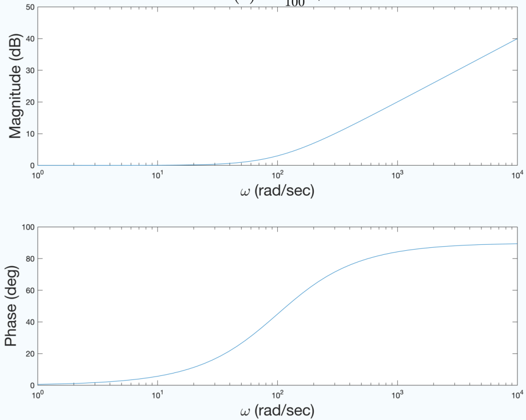
Example: Real Pole

$$\frac{1}{\frac{s}{\omega_0}+1}$$

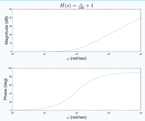
▷ Low frequency asymptote at 0 dB ▷ high frequency asymptote at -20 dB/dec ▷ connect asymptotic lines at ω_0	▷ Low frequency asymptote at 0° ▷ High frequency asymptote at $-90°$ ▷ connect with straight line from $0.1\,\omega_0$ to $10\,\omega_0$

Example: Real Zero

$$H(s) = \frac{s}{100}+1$$



Example: Real Zero

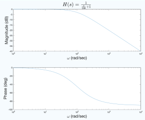


Example: Real Zero

$$\frac{s}{\omega_0} + 1$$

Magnitude	Phase
▷ Low frequency asymptote at 0 dB ▷ high frequency asymptote at +20 dB/dec ▷ connect asymptotic lines at ω_0	▷ Low frequency asymptote at 0° ▷ High frequency asymptote at +90° ▷ connect with straight line from $0.1\,\omega_0$ to $10\,\omega_0$

Example: Real Pole



Example: Underdamped Pole

$$\frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2\sigma\left(\frac{s}{\omega_0}\right) + 1}$$

- ▷ Low freq. asymptote at 0 dB
- ▷ high freq. asymptote at -40 dB/dec
- ▷ connect asymptotic lines at ω_0
- ▷ draw peak at $\omega_p = \omega_0\sqrt{1 - 2\sigma^2}$ with amplitude $-20\log_{10} 2\sigma\sqrt{1-\sigma^2}$
- ▷ $\sigma < \frac{1}{\sqrt{2}}$.

- ▷ Low freq. asymptote at 0^o
- ▷ High freq. asymptote at -180^o
- ▷ Connect with straight line from $\frac{\omega_0}{2}\log_{10}\left(\frac{2}{\sigma}\right)$ to $\frac{2\omega_0}{\log_{10}\left(\frac{2}{\sigma}\right)}$

Note: underdamped means that an oscillation takes place.

Example: Underdamped Pole

$$\frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2\sigma\left(\frac{s}{\omega_0}\right) + 1}$$

- Low freq. asymptote at 0 dB
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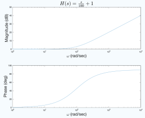
- $\sigma < \frac{1}{\sqrt{2}}$.

- Low freq. asymptote at 0°
- High freq. asymptote at −180°
- Connect with straight line from $\frac{\omega_0}{2}\log_{10}\left(\frac{2}{\sigma}\right)$ to $\frac{2\omega_0}{\log_{10}\left(\frac{2}{\sigma}\right)}$

Note: underdamped means that an oscillation takes place.

Example: Real Zero

$H(s) = \frac{s}{100} + 1$



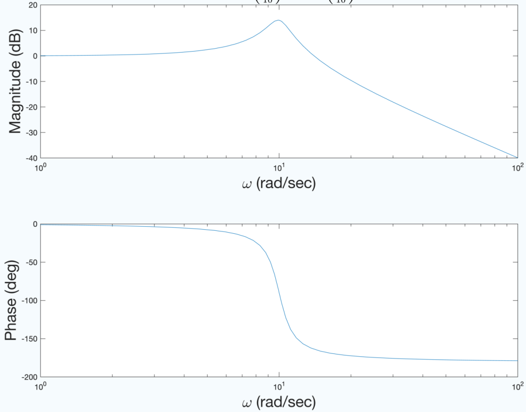
Example: Real Zero

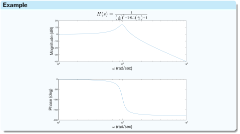
$$\frac{s}{\omega_0} + 1$$

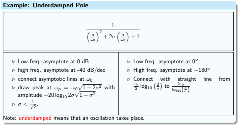
Magnitude	Phase
▷ Low frequency asymptote at 0 dB ▷ high frequency asymptote at +20 dB/dec ▷ connect asymptotic lines at ω_0	▷ Low frequency asymptote at 0° ▷ High frequency asymptote at +90° ▷ connect with straight line from $0.1\,\omega_0$ to $10\,\omega_0$

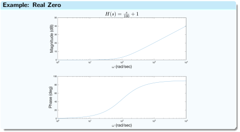
Example

$$H(s) = \frac{1}{\left(\frac{s}{10}\right)^2 + 2{\cdot}0.1\left(\frac{s}{10}\right) + 1}$$









Example: Underdamped Zero

$$\left(\frac{s}{\omega_0}\right)^2 + 2\sigma\left(\frac{s}{\omega_0}\right) + 1$$

- ▷ Low freq. asymptote at 0 dB
- ▷ high freq. asymptote at +40 dB/dec
- ▷ connect asymptotic lines at ω_0
- ▷ draw dip at $\omega_p = \frac{\omega_0}{\sqrt{1-2\sigma^2}}$ with amplitude $20\log_{10} 2\sigma\sqrt{1-\sigma^2}$
- ▷ $\sigma < \frac{1}{\sqrt{2}}$.

- ▷ Low freq. asymptote at 0^o
- ▷ High freq. asymptote at $+180^o$
- ▷ Connect with straight line from $\frac{\omega_0}{2}\log_{10}\left(\frac{2}{\sigma}\right)$ to $\frac{2\omega_0}{\log_{10}\left(\frac{2}{\sigma}\right)}$

Example: Underdamped Zero

$$\left(\frac{s}{\omega_0}\right)^2 + 2\sigma\left(\frac{s}{\omega_0}\right) + 1$$

- Low freq. asymptote at 0 dB
- high freq. asymptote at +40 dB/dec
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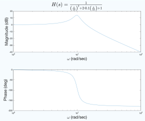
- $\sigma < \frac{1}{\sqrt{2}}$.

- Low freq. asymptote at 0°
- High freq. asymptote at +180°
- Connect with straight line from $\frac{\omega_0}{2}\log_{10}\left(\frac{2}{\sigma}\right)$ to $\frac{2\omega_0}{\log_{10}\left(\frac{2}{\sigma}\right)}$

Example



Example: Underdamped Pole

$$\frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2\sigma\left(\frac{s}{\omega_0}\right) + 1}$$

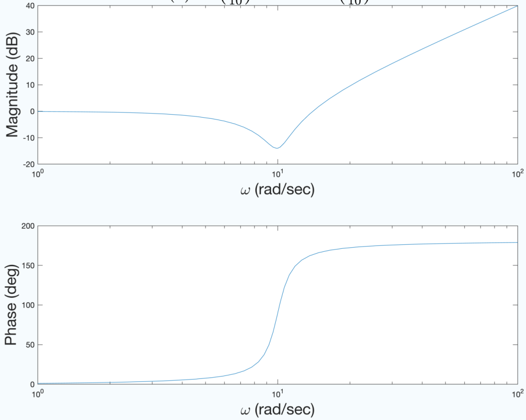
- Low freq. asymptote at 0 dB
- high freq. asymptote at -40 dB/dec
- connect asymptotic lines at ω_0
- draw peak at $\omega_p = \omega_0\sqrt{1-2\sigma^2}$ with amplitude $-20\log_{10} 2\sigma\sqrt{1-\sigma^2}$
- $\sigma < \frac{1}{\sqrt{2}}$.

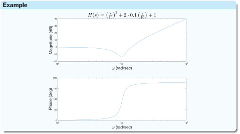
- Low freq. asymptote at 0°
- High freq. asymptote at −180°
- Connect with straight line from $\frac{\omega_0}{2}\log_{10}\left(\frac{2}{\sigma}\right)$ to $\frac{2\omega_0}{\log_{10}\left(\frac{2}{\sigma}\right)}$

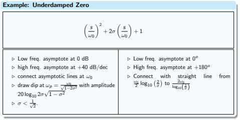
Note: underdamped means that an oscillation takes place.

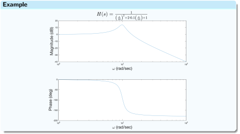
Example

$$H(s) = \left(\frac{s}{10}\right)^2 + 2\cdot 0.1\left(\frac{s}{10}\right) + 1$$









Exercise (#3.8)

Plot the Bode plot for the transfer function

$$H(s) = \frac{s-1}{s^2+2s+2}$$

using NUMPY or MATLAB with and without using the built-in function bode.

Exercise

Exercise (#3.8)

Plot the Bode plot for the transfer function

$$H(s) = \frac{s-1}{s^2+2s+2}$$

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Exercise (#3.8)

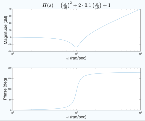
Plot the Bode plot for the transfer function

$$H(s) = \frac{s-1}{s^2+2s+2}$$

using NUMPY or MATLAB with and without using the built-in function bode.

Example

$H(s) = \left(\frac{s}{10}\right)^2 + 2 \cdot 0.1 \left(\frac{s}{10}\right) + 1$



Example: Underdamped Zero

$$\left(\frac{s}{\omega_0}\right)^2 + 2\sigma\left(\frac{s}{\omega_0}\right) + 1$$

- Low freq. asymptote at 0 dB
- high freq. asymptote at +40 dB/dec
- connect asymptotic lines at ω_0
- draw dip at $\omega_p = \frac{\omega_0}{\sqrt{1-2\sigma^2}}$ with amplitude $20\log_{10} 2\sigma\sqrt{1-\sigma^2}$
- $\sigma < \frac{1}{\sqrt{2}}$.

- Low freq. asymptote at 0°
- High freq. asymptote at +180°
- Connect with straight line from $\frac{\omega_0}{2}\log_{10}\left(\frac{2}{\sigma}\right)$ to $\frac{2\omega_0}{\log_{10}\left(\frac{2}{\sigma}\right)}$

Exercise (#3.9)

Sketch the Bode plot for the following transfer functions:

a)

$$H(s) = 1 + \frac{s}{10}$$

b)

$$H(s) = \frac{1}{1+s/100}$$

c)

$$H(s) = \frac{10^4(1+s)}{(10+s)(100+s)}$$

Exercise

Exercise (#3.9)

Sketch the Bode plot for the following transfer functions:

a)

$$H(s) = 1 + \frac{s}{10}$$

b)

$$H(s) = \frac{1}{1 + s/100}$$

c)

$$H(s) = \frac{10^4(1+s)}{(10+s)(100+s)}$$

Table of Contents

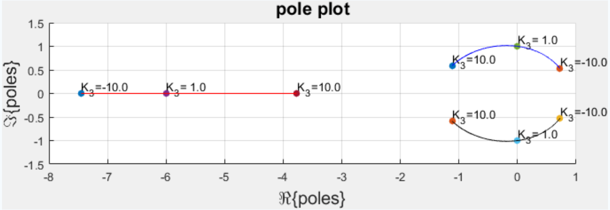
Graphical representation

Root locus analysis

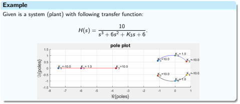
Example

Given is a system (plant) with following transfer function:

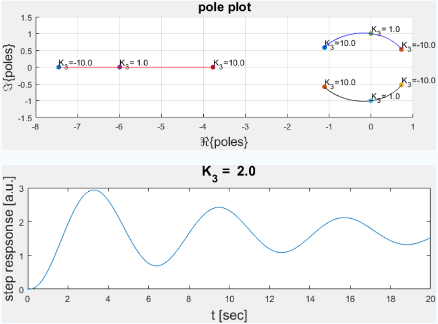
$$H(s) = \frac{10}{s^3 + 6s^2 + K_3 s + 6}.$$



Root locus analysis



Example



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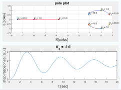

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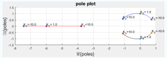
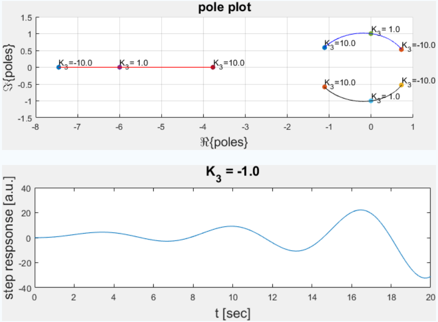
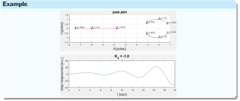
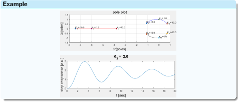
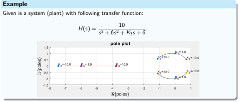


Example




Example


Example


Example
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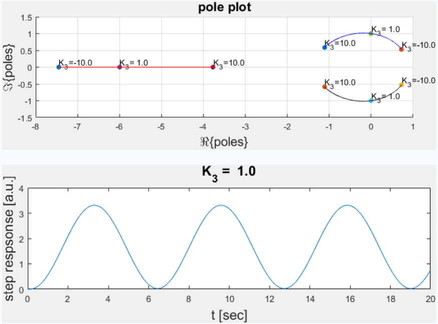
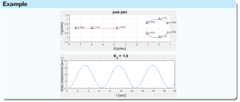
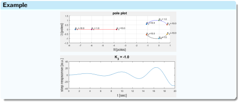
$H(s) = \frac{10}{s^3 + 6s^2 + K_3 s + 6}$
pole plot

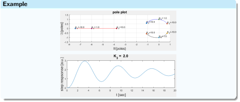
Example


pole plot
$\Im$\{poles\}
$\Re$\{poles\}
K_3=-10.0
K_3= 1.0
K_3=10.0
K_3=-10.0
K_3= 1.0
K_3=10.0
K_3=-10.0
K_3= 1.0
K_3=10.0
K_3 = 1.0
step respsonse [a.u.]
t [sec]

Root locus analysis



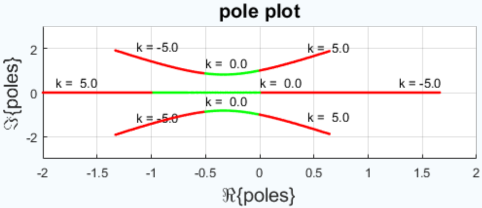




Example

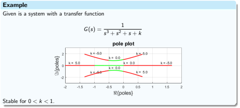
Given is a system with a transfer function

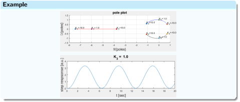
$$G(s) = \frac{1}{s^3 + s^2 + s + k}$$

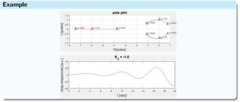


Stable for $0 < k < 1$.

Dominant pole







Definition

The slowest pole of a system is the one with the closest distance to the imaginary axis.

Definition

A pole is called dominant pole if it is much slower than the other poles and dominates the transient response.

Exercise

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Given is a system with a transfer function

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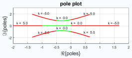
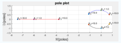
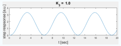


Stable for $0 < k < 1$.

Example





Exercise (#3.10)

Plot the step repsonses for a system with poles at $-10 \pm j2$, $-1 \pm j1$ and for a system with poles at $-1 \pm j1$. Normalize the step responses so that the final values of both become 1. Make use of NUMPY or MATLAB.

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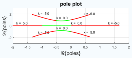
Given is a system with a transfer function

$$G(s) = \frac{1}{s^3 + s^2 + s + k}$$



Stable for $0 < k < 1$.

Exercise (#3.11)

A system has the transfer function

$$H(s) = \frac{1}{(s+2-j)(s+2+j)(s^2+20s+104)}$$

- ▷ Determine the dominant pole(s).
- ▷ Determine the final value of the step response.
- ▷ Approximate the system with a second order system.
- ▷ Plot the step response of the original and approximated system using NUMPY or MATLAB.

Exercise

Exercise (#3.11)

A system has the transfer function

$$H(s) = \frac{1}{(s+2-j)(s+2+j)(s^2+20s+104)}$$

- Determine the dominant pole(s).
- Determine the final value of the step response.
- Approximate the system with a second order system.
- Plot the step response of the original and approximated system using NUMPY or MATLAB.

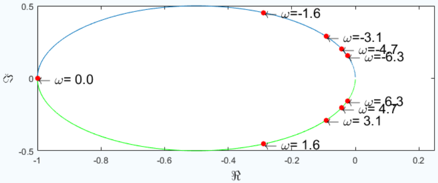
Graphical representation

Nyquist diagrams

Definition

The Nyquist diagram is a plot of the frequency response. Usually, the real part of the transfer function is plotted on the $x-$axis and the imaginary part is plotted on the $y-$axis.

Example: $G(s) = \frac{1}{s-1}$



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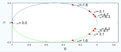
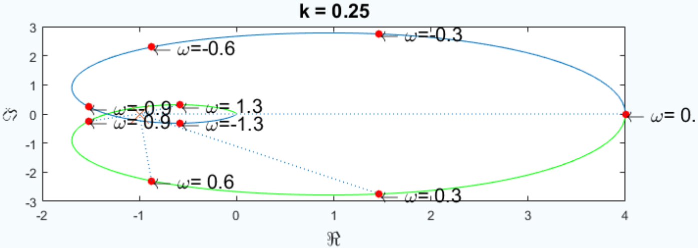


Example

Given is a system with a transfer function

$$G(s) = \frac{10}{s^3 + s^2 + s + 0.5}$$



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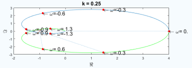

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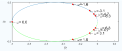

Example

Given is a system with a transfer function

$$G(s) = \frac{10}{s^3 + s^2 + s - 0.5}$$