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State variable models

Example system

Mass spring damper system

- Second order Differential equation:

$$u = M\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + ky$$

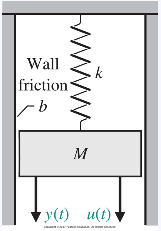
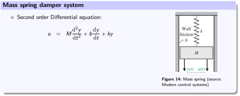


Figure 14: Mass spring (source: Modern control systems)

Reduction to first order



Introducing states

The equation

$$u \quad = \quad M\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + ky$$

can be reduced to a **system** of first order differential equations by using the states

$$\begin{aligned} x_1 &= y, \\ x_2 &= \frac{\mathrm{d}x_1}{\mathrm{d}t}. \end{aligned}$$

Reduction to first order

Introducing states

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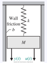

Figure 14: Mass spring (source: Modern control systems)

Matrix notation

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_2 \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -\frac{b}{M}x_2 - \frac{k}{M}x_1 + \frac{1}{M}u \end{aligned}$$

In matrix notation:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{M} \end{pmatrix} u}$$

General form

Matrix notation

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -\frac{b}{M}x_2 - \frac{k}{M}x_1 + \frac{1}{M}u$$

In matrix notation:

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Introducing states

The equation

$$u = M\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + ky$$

can be reduced to a **system** of first order differential equations by using the states

$$x_1 = y,$$
$$x_2 = \frac{\mathrm{d}x_1}{\mathrm{d}t}.$$

Mass spring damper system

- Second order Differential equation:

$$u = M\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + ky$$

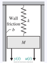

Figure 14: Mass spring (source: Modern control systems)

General form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ a_{2,1} & \dots & a_{2,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{bmatrix} b_{1,1} & \dots & b_{1,m} \\ b_{2,1} & \dots & b_{2,m} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \dots & b_{n,m} \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}$$

$$\boxed{\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}}$$

- Note: $\boldsymbol{A}$ and $\boldsymbol{B}$ are constants (no function of time)
- $\boldsymbol{x}$ is called state vector.
- LTI system

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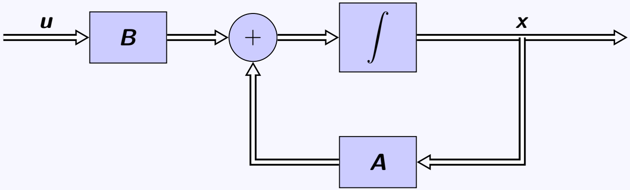
State variable models

Block diagram

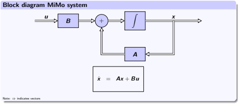
Block diagram MiMo system



$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

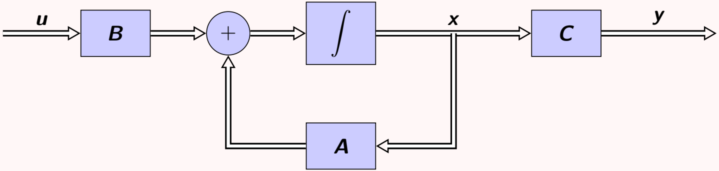
Note: $\Rightarrow$ indicates vectors

Measurement

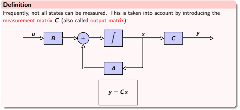


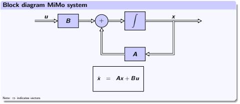
Definition

Frequently, not all states can be measured. This is taken into account by introducing the measurement matrix $\boldsymbol{C}$ (also called output matrix):



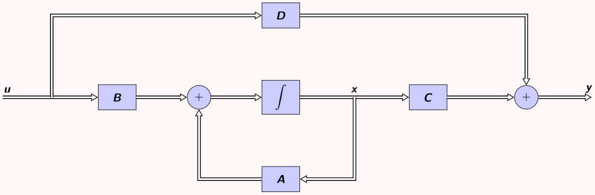
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$$



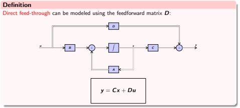


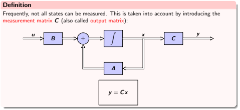
Definition

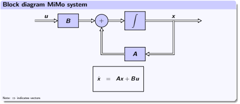
Direct feed-through can be modeled using the feedforward matrix $\boldsymbol{D}$:



$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}$$

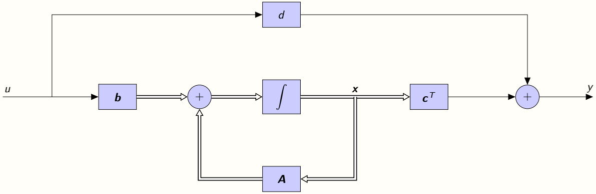






Properties

In case of single (scalar) input and single (scalar) output (SiSo) the following simplifications can be used:



$$
\begin{aligned}
y &= \boldsymbol{c}^T \boldsymbol{x} + du \\
\dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u
\end{aligned}
$$

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State variable models

Scalar first order DE

Definition

$$\dot{x} = ax + bu$$

Solution

$$\begin{aligned} x(t) &= e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)\mathrm{d}\tau \\ &= \underbrace{e^{at}k}_{\text{unforced response}} + \int_0^t e^{a(t-\tau)}bu(\tau)\mathrm{d}\tau \end{aligned}$$

Matrix first order DE

Definition

$$\dot{x} = ax + bu$$

Solution

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)\mathrm{d}\tau$$
$$= \underbrace{e^{at}k}_{\text{unforced response}} + \int_0^t e^{a(t-\tau)}bu(\tau)\mathrm{d}\tau$$

Definition

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

Unforced system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ a_{2,1} & \dots & a_{2,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$
$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$$

Matrix first order DE

Definition

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$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ a_{2,1} & \dots & a_{2,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$
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$$= \underbrace{e^{at}k}_{\text{unforced response}} + \int_0^t e^{a(t-\tau)}bu(\tau)\mathrm{d}\tau$$

Solution

The Ansatz

$$\boldsymbol{x}(t) = \boldsymbol{k}e^{\lambda t}$$

yields

$$\boldsymbol{k}\lambda e^{\lambda t} = \boldsymbol{A}\boldsymbol{k}e^{\lambda t}.$$

And thus the eigenvalue equations

$$\boldsymbol{k}\lambda = \boldsymbol{A}\boldsymbol{k}.$$

Matrix first order DE

Solution
The Ansatz
$$\boldsymbol{x}(t) = \boldsymbol{k}e^{\lambda t}$$
yields
$$\boldsymbol{k}\lambda e^{\lambda t} = \boldsymbol{A}\boldsymbol{k}e^{\lambda t}.$$
And thus the eigenvalue equations
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$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

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$$\dot{x} = ax + bu$$

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$$= \underbrace{e^{at}k}_{\text{unforced response}} + \int_0^t e^{a(t-\tau)}bu(\tau)\mathrm{d}\tau$$

Solution

$$(\boldsymbol{A} - \lambda \boldsymbol{I}) \cdot \boldsymbol{k} = 0.$$

The non trivial solution ($\boldsymbol{k} \neq \boldsymbol{0}$) exists only if

$$\boxed{\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0.}$$

Notes:

- $\boldsymbol{I}$ is the identity matrix.

Characteristic equation

Solution

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Notes:

- $\boldsymbol{I}$ is the identity matrix.

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$$\boldsymbol{x}(t) = \boldsymbol{k} e^{\lambda t}$$

yields

$$\boldsymbol{k} \lambda e^{\lambda t} = \boldsymbol{A} \boldsymbol{k} e^{\lambda t}.$$

And thus the eigenvalue equations

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Definition

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$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ a_{2,1} & \cdots & a_{2,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

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Definition

The equation

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$$

is called characteristic equation of the system defined by

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Matrix first order DE

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And thus the eigenvalue equations

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Definition

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

Solution

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\mathrm{d}\tau.,$$

with

$$e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}t + \frac{\boldsymbol{A}^2t^2}{2!} + \cdots \frac{\boldsymbol{A}^n t^n}{n!} + \cdots$$

Note: $\boldsymbol{I}$ is the identity matrix.

About $\exp(\boldsymbol{A}t)$

Definition

$$\dot{x} = \boldsymbol{A}x + \boldsymbol{B}u$$

Solution

$$\begin{aligned} x(t) &= e^{\boldsymbol{A}t}x(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}u(\tau)\mathrm{d}\tau, \\ &\text{with} \\ e^{\boldsymbol{A}t} &= \boldsymbol{I} + \boldsymbol{A}t + \frac{\boldsymbol{A}^2t^2}{2!} + \cdots \frac{\boldsymbol{A}^nt^n}{n!} + \cdots \end{aligned}$$

Note: $\boldsymbol{I}$ is the identity matrix.

Definition

The equation

$$\det(\lambda\boldsymbol{I} - \boldsymbol{A}) = 0$$

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$$\dot{x} = \boldsymbol{A}x.$$

Solution

$$(\boldsymbol{A} - \lambda\boldsymbol{I}) \cdot \boldsymbol{k} = 0.$$

The non trivial solution ($\boldsymbol{k} \neq \boldsymbol{0}$) exists only if

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About $\exp(\boldsymbol{A}t)$

It can be shown that for a square $(n \times n)$-matrix $\boldsymbol{A}$

$$\begin{aligned} e^{\boldsymbol{A}t} &= \boldsymbol{I} + \boldsymbol{A}t + \frac{\boldsymbol{A}^2t^2}{2!} + \cdots \frac{\boldsymbol{A}^nt^n}{n!} + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}\boldsymbol{A}^kt^k \\ &= \alpha_{n-1}(t)\boldsymbol{A}^{n-1}t^{n-1} + \alpha_{n-1}(t)\boldsymbol{A}^{n-2}t^{n-2} + \cdots \alpha_0(t)\boldsymbol{I} \\ &= \sum_{k=0}^{n-1} \alpha_k(t)\boldsymbol{A}^kt^k \end{aligned}$$

holds true. In this equation, $\alpha_0(t) \dots \alpha_{n-1}(t)$ are functions of t which need to be determined (see next slides).

About $\exp(\boldsymbol{A}t)$

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&= \alpha_{n-1}(t)\boldsymbol{A}^{n-1}t^{n-1} + \alpha_{n-1}(t)\boldsymbol{A}^{n-2}t^{n-2} + \cdots \alpha_0(t)\boldsymbol{I} \\
&= \sum_{k=0}^{n-1} \alpha_k(t)\boldsymbol{A}^k t^k
\end{aligned}
$$

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Definition

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

Solution

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\mathrm{d}\tau,$$

with

$$e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}t + \frac{\boldsymbol{A}^2 t^2}{2!} + \cdots \frac{\boldsymbol{A}^n t^n}{n!} + \cdots$$

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Definition

The equation

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$$

is called characteristic equation of the system defined by

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}.$$

About $\exp(\boldsymbol{A}t)$

Let $r(\lambda)$ be defined as follows:

$$r(\lambda) = \alpha_{n-1}(t)\lambda^{n-1} + \alpha_{n-2}(t)\lambda^{n-2} + \ldots + \alpha_1(t)\lambda + \alpha_0(t).$$

Then, $\alpha_k(t)$ can be determined by solving the equations shown on the next slides.

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holds true. In this equation, $\alpha_0(t) \ldots \alpha_{n-1}(t)$ are functions of t which need to be determined (see next slides).

Definition

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

Solution

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\mathrm{d}\tau,$$

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$$e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}t + \frac{\boldsymbol{A}^2t^2}{2!} + \cdots \frac{\boldsymbol{A}^n t^n}{n!} + \cdots$$

Note: $\boldsymbol{I}$ is the identity matrix.

About $\exp(\boldsymbol{A}t)$

For each eigenvalue λ_k of $\boldsymbol{A}t$ with multiplicity N, one gets N equations:

$$\begin{aligned} e^{\lambda_k} &= r(\lambda_k) \\ e^{\lambda_k} &= \left.\frac{\partial r(\lambda_k)}{\partial \lambda}\right|_{\lambda=\lambda_k} \\ \vdots \quad & \quad \vdots \\ e^{\lambda_k} &= \left.\frac{\partial^{N-1} r(\lambda_k)}{\partial \lambda^{N-1}}\right|_{\lambda=\lambda_k} \end{aligned}$$

For $N = 1$: $e^{\lambda_k} = r(\lambda_k)$.
For $N = 2$:

$$\begin{aligned} e^{\lambda_k} &= r(\lambda_k) \\ e^{\lambda_k} &= \left.\frac{\partial r(\lambda_k)}{\partial \lambda}\right|_{\lambda=\lambda_k} \end{aligned}$$

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About $\exp(\boldsymbol{A}t)$

Let $r(\lambda)$ be defined as follows:

$$r(\lambda) = \alpha_{n-1}(t)\lambda^{n-1} + \alpha_{n-2}(t)\lambda^{n-2} + \ldots + \alpha_1(t)\lambda + \alpha_0(t).$$

Then, $\alpha_k(t)$ can be determined by solving the equations shown on the next slides.

About $\exp(\boldsymbol{A}t)$

It can be shown that for a square $(n \times n)$-matrix $\boldsymbol{A}$

$$\begin{aligned} e^{\boldsymbol{A}t} &= \boldsymbol{I} + \boldsymbol{A}t + \frac{\boldsymbol{A}^2t^2}{2!} + \cdots \frac{\boldsymbol{A}^n t^n}{n!} + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}\boldsymbol{A}^k t^k \\ &= \alpha_{n-1}(t)\boldsymbol{A}^{n-1}t^{n-1} + \alpha_{n-1}(t)\boldsymbol{A}^{n-2}t^{n-2} + \cdots \alpha_0(t)\boldsymbol{I} \\ &= \sum_{k=0}^{n-1} \alpha_k(t)\boldsymbol{A}^k t^k \end{aligned}$$

holds true. In this equation, $\alpha_0(t) \ldots \alpha_{n-1}(t)$ are functions of t which need to be determined (see next slides).

Example

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 9 & 1 \end{bmatrix}.$$

$$e^{\boldsymbol{A}t} = \alpha_1 \boldsymbol{A}t + \alpha_0 \boldsymbol{I} = \begin{bmatrix} \alpha_1 t + \alpha_0 & \alpha_1 t \\ 9\alpha_1 t & \alpha_1 t + \alpha_0 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 4t$, $\lambda_2 = -2t$ and thus

$$\begin{aligned} e^{4t} &= 4t\alpha_1 + \alpha_0 \\ e^{-2t} &= -2t\alpha_1 + \alpha_0 \end{aligned}$$

About $\exp(At)$

Example

$$A = \begin{bmatrix} 1 & 1 \\ 9 & 1 \end{bmatrix}.$$

$$e^{At} = \alpha_1 At + \alpha_0 I = \begin{bmatrix} \alpha_1 t + \alpha_0 & \alpha_1 t \\ 9\alpha_1 t & \alpha_1 t + \alpha_0 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 4t$, $\lambda_2 = -2t$ and thus

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About $\exp(At)$

For each eigenvalue λ_k of At with multiplicity N, one gets N equations:

$$\begin{aligned} e^{\lambda_k} &= r(\lambda_k) \\ e^{\lambda_k} &= \left.\frac{\partial r(\lambda_k)}{\partial \lambda}\right|_{\lambda=\lambda_k} \\ &\vdots \\ e^{\lambda_k} &= \left.\frac{\partial^{N-1} r(\lambda_k)}{\partial \lambda^{N-1}}\right|_{\lambda=\lambda_k} \end{aligned}$$

For $N=1$: $e^{\lambda_k} = r(\lambda_k)$.
For $N=2$:

$$\begin{aligned} e^{\lambda_k} &= r(\lambda_k) \\ e^{\lambda_k} &= \left.\frac{\partial r(\lambda_k)}{\partial \lambda}\right|_{\lambda=\lambda_k} \end{aligned}$$

About $\exp(At)$

Let $r(\lambda)$ be defined as follows:

$$r(\lambda) = \alpha_{n-1}(t)\lambda^{n-1} + \alpha_{n-2}(t)\lambda^{n-2} + \ldots + \alpha_1(t)\lambda + \alpha_0(t).$$

Then, $\alpha_k(t)$ can be determined by solving the equations shown on the next slides.

Example

Solving

$$\begin{aligned} e^{4t} &= 4t\alpha_1 + \alpha_0 \\ e^{-2t} &= -2t\alpha_1 + \alpha_0 \end{aligned}$$

leads to

$$\begin{aligned} \alpha_1 &= \frac{1}{6t}\left(e^{4t} - e^{-2t}\right) \\ \alpha_0 &= \frac{1}{3}\left(e^{4t} + 2e^{-2t}\right) \end{aligned} \quad \Rightarrow \quad e^{At} = \frac{1}{6}\begin{bmatrix} 3e^{4t} + 3e^{-2t} & e^{4t} - e^{-2t} \\ 9e^{4t} - 9e^{-2t} & 3e^{4t} + 3e^{-2t} \end{bmatrix}.$$

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State variable models

Scalar first order DE

Definition and Laplace transform

$$
\begin{aligned}
\dot{x} &= ax + bu \\
sX(s) - x(0) &= aX(s) + bU(s) \\
X(S) &= \frac{x(0)}{s-a} + \frac{b}{s-a}U(s)
\end{aligned}
$$

Scalar first order DE

Definition and Laplace transform

$$\begin{aligned}\dot{x} &= ax + bu\\ sX(s) - x(0) &= aX(s) + bU(s)\\ X(S) &= \frac{x(0)}{s-a} + \frac{b}{s-a}U(s)\end{aligned}$$

Solution by inverse Laplace transform

$$\begin{aligned}x(t) &= e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)\mathrm{d}\tau\\ &= \underbrace{e^{at}k}_{\text{unforced response}} + \int_0^t e^{a(t-\tau)}bu(\tau)\mathrm{d}\tau\end{aligned}$$

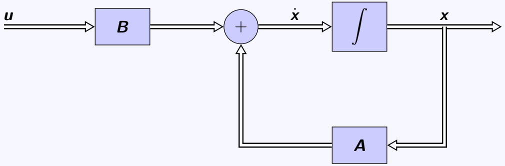
Matrix first order DE

Solution by inverse Laplace transform

$$\begin{aligned} x(t) &= e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)\mathrm{d}\tau \\ &= \underbrace{e^{at}k}_{\text{unforced response}} + \int_0^t e^{a(t-\tau)}bu(\tau)\mathrm{d}\tau \end{aligned}$$

Definition and Laplace transform

$$\begin{aligned} \dot{x} &= ax + bu \\ sX(s) - x(0) &= aX(s) + bU(s) \\ X(S) &= \frac{x(0)}{s-a} + \frac{b}{s-a}U(s) \end{aligned}$$

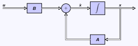


With the assumption $\boldsymbol{u}(0) = \boldsymbol{x}(0) = \mathbf{0}$:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\ s\boldsymbol{X}(s) &= \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{B}\boldsymbol{U}(s) \\ (s\boldsymbol{I} - \boldsymbol{A})\,\boldsymbol{X}(s) &= \boldsymbol{B}\boldsymbol{U}(s) \\ \boldsymbol{X}(s) &= [s\boldsymbol{I} - \boldsymbol{A}]^{-1}\,\boldsymbol{B}\boldsymbol{U}(s) = \boldsymbol{\Phi}(s)\boldsymbol{B}\boldsymbol{U}(s) \end{aligned}$$

Note: $\boldsymbol{X}(s)$ and $\boldsymbol{U}(s)$ are vectors (capital letters indicate frequency-domain).

Laplace domain



With the assumption $\boldsymbol{u}(0) = \boldsymbol{x}(0) = \boldsymbol{0}$:

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\
s\boldsymbol{X}(s) &= \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{B}\boldsymbol{U}(s) \\
(s\boldsymbol{I} - \boldsymbol{A})\,\boldsymbol{X}(s) &= \boldsymbol{B}\boldsymbol{U}(s) \\
\boldsymbol{X}(s) &= [s\boldsymbol{I} - \boldsymbol{A}]^{-1}\,\boldsymbol{B}\boldsymbol{U}(s) = \boldsymbol{\Phi}(s)\boldsymbol{B}\boldsymbol{U}(s)
\end{aligned}
$$

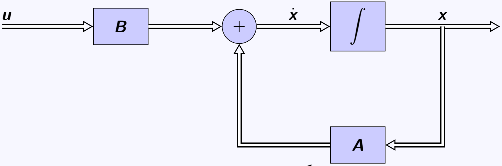
Note: $\boldsymbol{X}(s)$ and $\boldsymbol{U}(s)$ are vectors (capital letters indicate frequency-domain).

Solution by inverse Laplace transform

$$
\begin{aligned}
x(t) &= e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)\mathrm{d}\tau \\
&= \underbrace{e^{at}k}_{\text{unforced response}} + \int_0^t e^{a(t-\tau)}bu(\tau)\mathrm{d}\tau
\end{aligned}
$$

Definition and Laplace transform

$$
\begin{aligned}
\dot{x} &= ax + bu \\
sX(s) - x(0) &= aX(s) + bU(s) \\
X(S) &= \frac{x(0)}{s-a} + \frac{b}{s-a}U(s)
\end{aligned}
$$

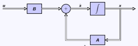


$$
\begin{aligned}
\boldsymbol{X}(s) &= [s\boldsymbol{I} - \boldsymbol{A}]^{-1}\,\boldsymbol{B}\boldsymbol{U}(s) \\
\boldsymbol{X}(s) &= \boldsymbol{\Phi}(s)\boldsymbol{B}\boldsymbol{U}(s) \\
\boldsymbol{\Phi}(t) &= e^{\boldsymbol{A}t}
\end{aligned}
$$

Note: Compare with

$$
\mathcal{L}\left\{e^{-at}\right\} = \frac{1}{s+a}.
$$

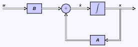
Laplace domain



$$\boldsymbol{X}(s) = [s\boldsymbol{I} - \boldsymbol{A}]^{-1}\boldsymbol{B}\boldsymbol{U}(s)$$
$$\boldsymbol{X}(s) = \boldsymbol{\Phi}(s)\boldsymbol{B}\boldsymbol{U}(s)$$
$$\boldsymbol{\Phi}(t) = e^{\boldsymbol{A}t}$$

Note: Compare with

$$\mathcal{L}\left\{e^{-at}\right\} = \frac{1}{s+a}.$$



With the assumption $\boldsymbol{u}(0) = \boldsymbol{x}(0) = \boldsymbol{0}$:

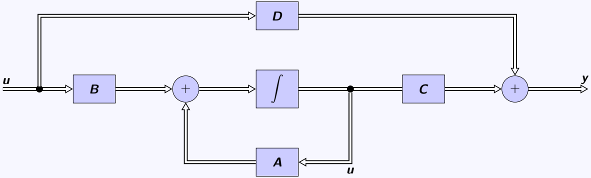
$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\ s\boldsymbol{X}(s) &= \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{B}\boldsymbol{U}(s) \\ (s\boldsymbol{I} - \boldsymbol{A})\,\boldsymbol{X}(s) &= \boldsymbol{B}\boldsymbol{U}(s) \\ \boldsymbol{X}(s) &= [s\boldsymbol{I} - \boldsymbol{A}]^{-1}\boldsymbol{B}\boldsymbol{U}(s) = \boldsymbol{\Phi}(s)\boldsymbol{B}\boldsymbol{U}(s) \end{aligned}$$

Note: $\boldsymbol{X}(s)$ and $\boldsymbol{U}(s)$ are vectors (capital letters indicate frequency-domain).

Solution by inverse Laplace transform

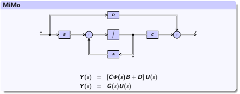
$$\begin{aligned} x(t) &= e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)\mathrm{d}\tau \\ &= \underbrace{e^{at}k}_{\text{unforced response}} + \int_0^t e^{a(t-\tau)}bu(\tau)\mathrm{d}\tau \end{aligned}$$

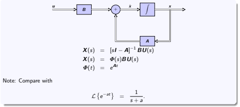
MiMo

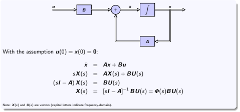


$$\begin{aligned} \boldsymbol{Y}(s) &= [\boldsymbol{C}\boldsymbol{\Phi}(s)\boldsymbol{B} + \boldsymbol{D}]\,\boldsymbol{U}(s) \\ \boldsymbol{Y}(s) &= \boldsymbol{G}(s)\boldsymbol{U}(s) \end{aligned}$$

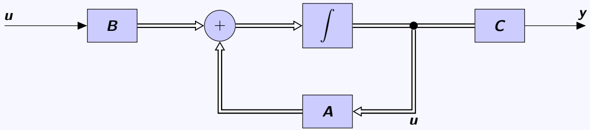
Laplace domain







SiSo without direct feedthrough



For SiSo systems without direct feedthrough:

$$
\begin{aligned}
Y(s) &= [\boldsymbol{C}\boldsymbol{\Phi}(\boldsymbol{s})\boldsymbol{B}]\, U(s) \\
Y(s) &= G(s)U(s), \\
&\text{with} \\
\boldsymbol{\Phi}(\boldsymbol{s}) &= [s\boldsymbol{I} - \boldsymbol{A}]^{-1}
\end{aligned}
$$

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State variable models

Laplace domain

Definition

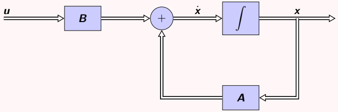
The equation

$$\boldsymbol{X}(s) \;=\; [s\boldsymbol{I} - \boldsymbol{A}]^{-1}\,\boldsymbol{B}\boldsymbol{U}(s)$$

can be used to determine the stability of the system. If all of the roots of the characteristic equation

$$\det\left(s\boldsymbol{I} - \boldsymbol{A}\right) = 0$$

are located in the left half space (negative real part), then the system is stable.



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Canonical forms

Transfer function

Exemplary SiSo transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\left[b_0 + b_1 s^1 + b_2 s^2\right]}{\left[a_0 + a_1 s^1 + a_2 s^2 + s^3\right]} = \frac{\left[b_0 s^{-3} + b_1 s^{-2} + b_2 s^{-1}\right]}{\left[a_0 s^{-3} + a_1 s^{-2} + a_2 s^{-1} + 1\right]}$$

Expanding with $Z(s)$:

$$G(s) = \frac{\left[b_0 + b_1 s^1 + b_2 s^3\right]}{\left[a_0 + a_1 s^1 + a_2 s^2 + s^3\right]} \cdot \frac{Z(s)}{Z(s)}$$

Also used:

$$Y(s) = \frac{b_2 s^0 + b_1 s^1 + b_0 s^2}{a_2 s^0 + a_1 s^1 + a_0 s^2 + s^3} U(s)$$

Transfer function

Exemplary SiSo transfer function

$$G(s)=\frac{Y(s)}{U(s)} = \frac{\left[b_0+b_1s^1+b_2s^2\right]}{\left[a_0+a_1s^1+a_2s^2+s^3\right]} = \frac{\left[b_0s^{-3}+b_1s^{-2}+b_2s^{-1}\right]}{\left[a_0s^{-3}+a_1s^{-2}+a_2s^{-1}+1\right]}$$

Expanding with $Z(s)$:

$$G(s) = \frac{\left[b_0+b_1s^1+b_2s^3\right]}{\left[a_0+a_1s^1+a_2s^2+s^3\right]}\cdot\frac{Z(s)}{Z(s)}$$

Also used:

$$Y(s) = \frac{b_2s^0+b_1s^1+b_0s^2}{a_2s^0+a_1s^1+a_0s^2+s^3}U(s)$$

SiSo transfer function

$$\begin{aligned} Y(s) &= \left[b_0+b_1s^1+b_2s^2\right]Z(s) \\ U(s) &= \left[a_0+a_1s^1+a_2s^2+s^3\right]Z(s) \end{aligned}$$

In the time domain:

$$\begin{aligned} y &= b_0z(t)+b_1\frac{\mathrm{d}}{\mathrm{d}t}z(t)+b_2\frac{\mathrm{d}^2}{\mathrm{d}t^2}z(t) \\ u &= a_0z(t)+a_1\frac{\mathrm{d}}{\mathrm{d}t}z(t)+a_2\frac{\mathrm{d}^2}{\mathrm{d}t^2}z(t)+\frac{\mathrm{d}^3}{\mathrm{d}t^3}z(t) \end{aligned}$$

Canonical forms

Transfer function

SiSo transfer function

$$
\begin{aligned}
y &= b_0 z(t) + b_1 \frac{\mathrm{d}}{\mathrm{d}t} z(t) + b_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2} z(t) \\
u &= a_0 z(t) + a_1 \frac{\mathrm{d}}{\mathrm{d}t} z(t) + a_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2} z(t) + \frac{\mathrm{d}^3}{\mathrm{d}t^3} z(t)
\end{aligned}
$$

Introducing state variables (also called phase variables):

$$
x_1 = z(t), x_2 = \frac{\mathrm{d}}{\mathrm{d}t} z(t) \text{ and } x_3 = \frac{\mathrm{d}^2}{\mathrm{d}t^2} z(t)
$$

Note that usually the states do not represent physical values.

Transfer function

SiSo transfer function

$$
\begin{aligned}
y &= b_0 z(t) + b_1 \frac{\mathrm{d}}{\mathrm{d}t} z(t) + b_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2} z(t) \\
u &= a_0 z(t) + a_1 \frac{\mathrm{d}}{\mathrm{d}t} z(t) + a_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2} z(t) + \frac{\mathrm{d}^3}{\mathrm{d}t^3} z(t)
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$$x_1 = z(t), x_2 = \frac{\mathrm{d}}{\mathrm{d}t} z(t) \text{ and } x_3 = \frac{\mathrm{d}^2}{\mathrm{d}t^2} z(t)$$

Note that usually the states do not represent physical values.

SiSo transfer function

$$
\begin{aligned}
y &= b_0 x_1 + b_1 x_2 + b_2 x_3 \\
u &= a_0 x_1 + a_1 x_2 + a_2 x_3 + \frac{\mathrm{d}}{\mathrm{d}t} x_3
\end{aligned}
$$

The resulting system of first order differential equations:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} x_1 &= x_2 \\
\frac{\mathrm{d}}{\mathrm{d}t} x_2 &= x_3 \\
\frac{\mathrm{d}}{\mathrm{d}t} x_3 &= -a_0 x_1 - a_1 x_2 - a_2 x_3 + u
\end{aligned}
$$

Transfer function

SiSo transfer function

$$\begin{aligned} y &= b_0x_1 + b_1x_2 + b_2x_3 \\ u &= a_0x_1 + a_1x_2 + a_2x_3 + \frac{\mathrm{d}}{\mathrm{d}t}x_3 \end{aligned}$$

The resulting system of first order differential equations:

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SiSo transfer function

$$\begin{aligned} y &= b_0z(t) + b_1\frac{\mathrm{d}}{\mathrm{d}t}z(t) + b_2\frac{\mathrm{d}^2}{\mathrm{d}t^2}z(t) \\ u &= a_0z(t) + a_1\frac{\mathrm{d}}{\mathrm{d}t}z(t) + a_2\frac{\mathrm{d}^2}{\mathrm{d}t^2}z(t) + \frac{\mathrm{d}^3}{\mathrm{d}t^3}z(t) \end{aligned}$$

Introducing state variables (also called phase variables):

$$x_1 = z(t), x_2 = \frac{\mathrm{d}}{\mathrm{d}t}z(t) \text{ and } x_3 = \frac{\mathrm{d}^2}{\mathrm{d}t^2}z(t)$$

Note that usually the states do not represent physical values.

As matrix equations

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \boldsymbol{C}^T\boldsymbol{x} = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

Phase variable canonical form

As matrix equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\-a_0 & -a_1 & -a_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}u(t)$$

$$y(t) = \boldsymbol{C}^T\boldsymbol{x} = \begin{bmatrix}b_0 & b_1 & b_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

SiSo transfer function

$$y = b_0x_1 + b_1x_2 + b_2x_3$$

$$u = a_0x_1 + a_1x_2 + a_2x_3 + \frac{\mathrm{d}}{\mathrm{d}t}x_3$$

The resulting system of first order differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}x_1 = x_2$$

$$\frac{\mathrm{d}}{\mathrm{d}t}x_2 = x_3$$

$$\frac{\mathrm{d}}{\mathrm{d}t}x_3 = -a_0x_1 - a_1x_2 - a_2x_3 + u$$

SiSo transfer function

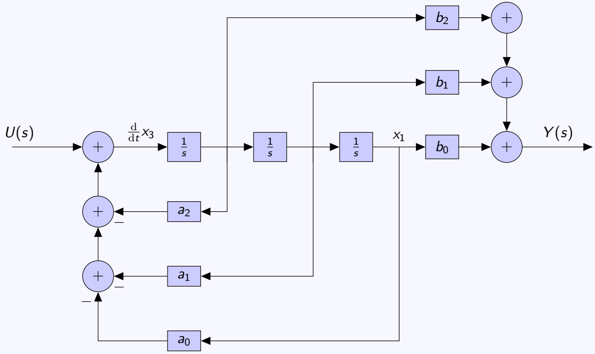
$$y = b_0z(t) + b_1\frac{\mathrm{d}}{\mathrm{d}t}z(t) + b_2\frac{\mathrm{d}^2}{\mathrm{d}t^2}z(t)$$

$$u = a_0z(t) + a_1\frac{\mathrm{d}}{\mathrm{d}t}z(t) + a_2\frac{\mathrm{d}^2}{\mathrm{d}t^2}z(t) + \frac{\mathrm{d}^3}{\mathrm{d}t^3}z(t)$$

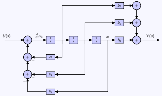
Introducing state variables (also called phase variables):

$$x_1 = z(t), x_2 = \frac{\mathrm{d}}{\mathrm{d}t}z(t) \text{ and } x_3 = \frac{\mathrm{d}^2}{\mathrm{d}t^2}z(t)$$

Note that usually the states do not represent physical values.



Phase variable canonical form



As matrix equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \boldsymbol{C}^T\boldsymbol{x} = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

SiSo transfer function

$$y = b_0x_1 + b_1x_2 + b_2x_3$$

$$u = a_0x_1 + a_1x_2 + a_2x_3 + \frac{\mathrm{d}}{\mathrm{d}t}x_3$$

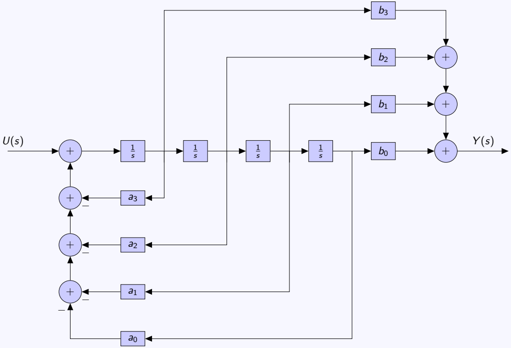
The resulting system of first order differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}x_1 = x_2$$

$$\frac{\mathrm{d}}{\mathrm{d}t}x_2 = x_3$$

$$\frac{\mathrm{d}}{\mathrm{d}t}x_3 = -a_0x_1 - a_1x_2 - a_2x_3 + u$$

Increasing order

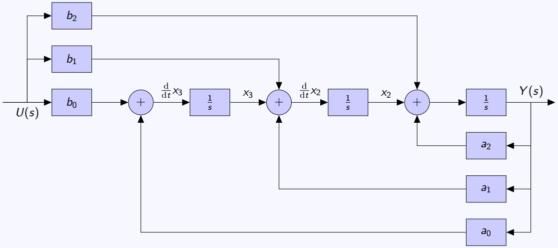


Canonical forms

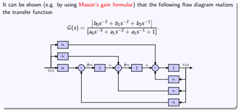
Alternative flow graph

It can be shown (e.g. by using Mason's gain formular) that the following flow diagram realizes the transfer function

$$G(s) = \frac{\left[b_0 s^{-3} + b_1 s^{-2} + b_2 s^{-1}\right]}{\left[a_0 s^{-3} + a_1 s^{-2} + a_2 s^{-1} + 1\right]}.$$



Transfer function



SiSo transfer function

$$
\begin{aligned}
y &= b_0 z + b_1 \frac{\mathrm{d}}{\mathrm{d}t} z + b_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2} z \\
u &= a_0 z + a_1 \frac{\mathrm{d}}{\mathrm{d}t} z + a_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2} z + \frac{\mathrm{d}^3}{\mathrm{d}t^3} z
\end{aligned}
$$

Feeding forward the input signal $u(t)$ allows us to use the output as state x_1.

Transfer function

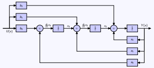
SiSo transfer function

$$\begin{aligned} y &= b_0 z + b_1 \frac{\mathrm{d}}{\mathrm{d}t} z + b_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2} z \\ u &= a_0 z + a_1 \frac{\mathrm{d}}{\mathrm{d}t} z + a_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2} z + \frac{\mathrm{d}^3}{\mathrm{d}t^3} z \end{aligned}$$

Feeding forward the input signal $u(t)$ allows us to use the output as state x_1.

It can be shown (e.g. by using Mason's gain formular) that the following flow diagram realizes the transfer function

$$G(s) = \frac{[b_0 s^{-3} + b_1 s^{-2} + b_2 s^{-1}]}{[a_0 s^{-3} + a_1 s^{-2} + a_2 s^{-1} + 1]}.$$



SiSo transfer function

Introducing state variables:

$$\begin{aligned} x_1 &= y \\ \frac{\mathrm{d}}{\mathrm{d}t} x_1 &= -a_2 x_1 + x_3 + b_2 u \\ \frac{\mathrm{d}}{\mathrm{d}t} x_2 &= -a_1 x_1 + x_2 + b_1 u \\ \frac{\mathrm{d}}{\mathrm{d}t} x_3 &= -a_0 + b_0 u \end{aligned}$$

Transfer function

SiSo transfer function

Introducing state variables:

$$
\begin{aligned}
x_1 &= y \\
\frac{\mathrm{d}}{\mathrm{d}t}x_1 &= -a_2 x_1 + x_3 + b_2 u \\
\frac{\mathrm{d}}{\mathrm{d}t}x_2 &= -a_1 x_1 + x_2 + b_1 u \\
\frac{\mathrm{d}}{\mathrm{d}t}x_3 &= -a_0 + b_0 u
\end{aligned}
$$

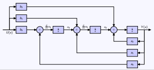
SiSo transfer function

$$
\begin{aligned}
y &= b_0 z + b_1 \frac{\mathrm{d}}{\mathrm{d}t} z + b_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2} z \\
u &= a_0 z + a_1 \frac{\mathrm{d}}{\mathrm{d}t} z + a_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2} z + \frac{\mathrm{d}^3}{\mathrm{d}t^3} z
\end{aligned}
$$

Feeding forward the input signal $u(t)$ allows us to use the output as state x_1.

It can be shown (e.g. by using Mason's gain formular) that the following flow diagram realizes the transfer function

$$G(s) = \frac{\left[b_0 s^{-3} + b_1 s^{-2} + b_2 s^{-1}\right]}{\left[a_0 s^{-3} + a_1 s^{-2} + a_2 s^{-1} + 1\right]}.$$



As matrix equation

$$
y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} u(t)
$$

Table of Contents

State variable models

Continuous time model

Continuous time model

$$\begin{aligned}
\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_c\boldsymbol{x}(t) + \boldsymbol{B}_c\boldsymbol{u}(t) \\
\boldsymbol{y}(t) &= \boldsymbol{C}_c\boldsymbol{x}(t) + \boldsymbol{D}_c\boldsymbol{u}(t) \\
\boldsymbol{x}(t) &= \exp\left(\boldsymbol{A}_c\left(t - t_0\right)\right)\boldsymbol{x}(t_0) + \int_{t_0}^{t} \exp\left(\boldsymbol{A}_c\left(t - \tau\right)\right)\boldsymbol{B}_c\boldsymbol{u}(\tau)d\tau
\end{aligned}$$

Thus

$$\begin{aligned}
\boldsymbol{x}((k+1)T) &= \exp\left(\boldsymbol{A}_c\left((k+1)T - kT\right)\right)\boldsymbol{x}(kT) \\
&\quad + \int_{kT}^{(k+1)T} \exp\left(\boldsymbol{A}_c\left(kT - \tau\right)\right)\boldsymbol{B}_c\boldsymbol{u}(\tau)d\tau \\
&= \exp\left(\boldsymbol{A}_c\left(T\right)\right)\boldsymbol{x}(kT) \\
&\quad + \int_{kT}^{(k+1)T} \exp\left(\boldsymbol{A}_c\left(kT - \tau\right)\right)\boldsymbol{B}_c\boldsymbol{u}(\tau)d\tau
\end{aligned}$$

Discrete time

Continuous time model

$$
\begin{aligned}
\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_c\boldsymbol{x}(t) + \boldsymbol{B}_c\boldsymbol{u}(t) \\
\boldsymbol{y}(t) &= \boldsymbol{C}_c\boldsymbol{x}(t) + \boldsymbol{D}_c\boldsymbol{u}(t) \\
\boldsymbol{x}(t) &= \exp\left(\boldsymbol{A}_c\left(t-t_0\right)\right)\boldsymbol{x}(t_0) + \int_{t_0}^{t}\exp\left(\boldsymbol{A}_c\left(t-\tau\right)\right)\boldsymbol{B}_c\boldsymbol{u}(\tau)d\tau
\end{aligned}
$$

Thus

$$
\begin{aligned}
\boldsymbol{x}((k+1)T) &= \exp\left(\boldsymbol{A}_c\left((k+1)T-kT\right)\right)\boldsymbol{x}(kT) \\
&\quad + \int_{kT}^{(k+1)T}\exp\left(\boldsymbol{A}_c\left(kT-\tau\right)\right)\boldsymbol{B}_c\boldsymbol{u}(\tau)d\tau \\
&= \exp\left(\boldsymbol{A}_c\left(T\right)\right)\boldsymbol{x}(kT) \\
&\quad + \int_{kT}^{(k+1)T}\exp\left(\boldsymbol{A}_c\left(kT-\tau\right)\right)\boldsymbol{B}_c\boldsymbol{u}(\tau)d\tau
\end{aligned}
$$

Zero-order hold assumption

$$
\boldsymbol{x}((k+1)T) = \exp\left(\boldsymbol{A}_c\left(T\right)\right)\boldsymbol{x}(kT) + \int_{kT}^{(k+1)T}\exp\left(\boldsymbol{A}_c\left(kT-\tau\right)\right)\boldsymbol{B}_c\boldsymbol{u}(\tau)d\tau
$$

Based on the assumption that the *analog* $\Leftrightarrow$ *digital converter* gives constant values in between time steps (*zero-order hold*) one gets

$$
\begin{aligned}
\boldsymbol{x}_{k+1} &= \boldsymbol{A}_d\boldsymbol{x}_k + \boldsymbol{B}_d\boldsymbol{u}_k \\
\boldsymbol{A}_d &= \exp\left(\boldsymbol{A}_c T\right) \\
\boldsymbol{B}_d &= \int_0^T \exp\left(\boldsymbol{A}_c\left(T-\tau\right)\right)\boldsymbol{B}_c d\tau
\end{aligned}
$$

Discrete time

Zero-order hold assumption

$$\boldsymbol{x}((k+1)T) = \exp(\boldsymbol{A}_c(T))\,\boldsymbol{x}(kT) + \int_{kT}^{(k+1)T} \exp(\boldsymbol{A}_c(kT-\tau))\,\boldsymbol{B}_c\boldsymbol{u}(\tau)d\tau$$

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$$\begin{aligned}\boldsymbol{x}_{k+1} &= \boldsymbol{A}_d\boldsymbol{x}_k + \boldsymbol{B}_d\boldsymbol{u}_k\\ \boldsymbol{A}_d &= \exp(\boldsymbol{A}_c T)\\ \boldsymbol{B}_d &= \int_0^T \exp(\boldsymbol{A}_c(T-\tau))\,\boldsymbol{B}_c d\tau\end{aligned}$$

Continuous time model

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_c\boldsymbol{x}(t) + \boldsymbol{B}_c\boldsymbol{u}(t)\\ \boldsymbol{y}(t) &= \boldsymbol{C}_c\boldsymbol{x}(t) + \boldsymbol{D}_c\boldsymbol{u}(t)\\ \boldsymbol{x}(t) &= \exp(\boldsymbol{A}_c(t-t_0))\,\boldsymbol{x}(t_0) + \int_{t_0}^{t} \exp(\boldsymbol{A}_c(t-\tau))\,\boldsymbol{B}_c\boldsymbol{u}(\tau)d\tau\end{aligned}$$

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$$\begin{aligned}\boldsymbol{x}((k+1)T) &= \exp(\boldsymbol{A}_c((k+1)T-kT))\,\boldsymbol{x}(kT)\\ &\quad + \int_{kT}^{(k+1)T} \exp(\boldsymbol{A}_c(kT-\tau))\,\boldsymbol{B}_c\boldsymbol{u}(\tau)d\tau\\ &= \exp(\boldsymbol{A}_c(T))\,\boldsymbol{x}(kT)\\ &\quad + \int_{kT}^{(k+1)T} \exp(\boldsymbol{A}_c(kT-\tau))\,\boldsymbol{B}_c\boldsymbol{u}(\tau)d\tau\end{aligned}$$

Complete description

$$\begin{aligned}\boldsymbol{x}_{k+1} &= \boldsymbol{A}_d\boldsymbol{x}_k + \boldsymbol{B}_d\boldsymbol{u}_k\\ \boldsymbol{y}_k &= \boldsymbol{C}_d\boldsymbol{x}_k + \boldsymbol{D}_d\boldsymbol{u}_k\\ \boldsymbol{A}_d &= \exp(\boldsymbol{A}_c T)\\ \boldsymbol{B}_d &= \int_0^T \exp(\boldsymbol{A}_c(T-\tau))\,\boldsymbol{B}_c d\tau\\ \boldsymbol{C}_d &= \boldsymbol{C}_c\\ \boldsymbol{D}_d &= \boldsymbol{D}_c.\end{aligned}$$

Complete description

$$\begin{aligned} \boldsymbol{x}_{k+1} &= \boldsymbol{A}_d \boldsymbol{x}_k + \boldsymbol{B}_d \boldsymbol{u}_k \\ \boldsymbol{y}_k &= \boldsymbol{C}_d \boldsymbol{x}_k + \boldsymbol{D}_d \boldsymbol{u}_k \\ \boldsymbol{A}_d &= \exp(\boldsymbol{A}_c T) \\ \boldsymbol{B}_d &= \int_0^T \exp(\boldsymbol{A}_c (T-\tau)) \boldsymbol{B}_c d\tau \\ \boldsymbol{C}_d &= \boldsymbol{C}_c \\ \boldsymbol{D}_d &= \boldsymbol{D}_c. \end{aligned}$$

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$$\begin{aligned} \boldsymbol{x}((k+1)T) &= \exp(\boldsymbol{A}_c((k+1)T - kT))\boldsymbol{x}(kT) \\ &\quad + \int_{kT}^{(k+1)T} \exp(\boldsymbol{A}_c(kT-\tau)) \boldsymbol{B}_c \boldsymbol{u}(\tau) d\tau \\ &= \exp(\boldsymbol{A}_c(T))\boldsymbol{x}(kT) \\ &\quad + \int_{kT}^{(k+1)T} \exp(\boldsymbol{A}_c(kT-\tau)) \boldsymbol{B}_c \boldsymbol{u}(\tau) d\tau \end{aligned}$$

Example

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1} + 8s^{-2} + 6s^{-3}}{1 + 8s^{-1} + 16s^{-2} + 6s^{-3}}$$

Using phase variable state model[a] one gets:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -16 & -8 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 6 & 8 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

[a] See e.g. Dorf, Bishop for relationship between transfer function and state model

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Example (cont.)

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1}+8s^{-2}+6s^{-3}}{1+8s^{-1}+16s^{-2}+6s^{-3}}$$

Using $T = 3$ one gets:

$$\boldsymbol{A}_d = \begin{bmatrix} 0.3214 & 0.1950 & 0.0259 \\ -0.1555 & -0.03933 & -0.0123 \\ 0.0740 & 0.0419 & 0.0054 \end{bmatrix}$$

and

$$\boldsymbol{B}_d = \begin{bmatrix} 0.1131 & 0.0259 & -0.0108 \end{bmatrix}^T$$

Example

Example (cont.)

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$$\begin{aligned} \boldsymbol{x}_{k+1} &= \boldsymbol{A}_d\boldsymbol{x}_k + \boldsymbol{B}_d\boldsymbol{u}_k \\ \boldsymbol{y}_k &= \boldsymbol{C}_d\boldsymbol{x}_k + \boldsymbol{D}_d\boldsymbol{u}_k \\ \boldsymbol{A}_d &= \exp(\boldsymbol{A}_c T) \\ \boldsymbol{B}_d &= \int_0^T \exp(\boldsymbol{A}_c(T-\tau))\,\boldsymbol{B}_c d\tau \\ \boldsymbol{C}_d &= \boldsymbol{C}_c \\ \boldsymbol{D}_d &= \boldsymbol{D}_c. \end{aligned}$$

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Example

Now, consider

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1}+8s^{-2}+6s^{-3}}{1+8s^{-1}+s^{-2}+s^{-3}}$$

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Example (cont.)

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Example (cont.)

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Using phase variable state model and $T = 3$ one gets:

$$\boldsymbol{A}_d = \begin{bmatrix} 0.5666 & 2.1239 & 0.2601 \\ -0.2601 & 0.3065 & 0.0430 \\ -0.0430 & -0.3031 & -0.037 \end{bmatrix}$$

and

$$\boldsymbol{B}_d = \begin{bmatrix} 0.4338 & 0.2602 & 0.0445 \end{bmatrix}^T$$

Example

Example (cont.)

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1} + 8s^{-2} + 6s^{-3}}{1 + 8s^{-1} + s^{-2} + s^{-3}}$$

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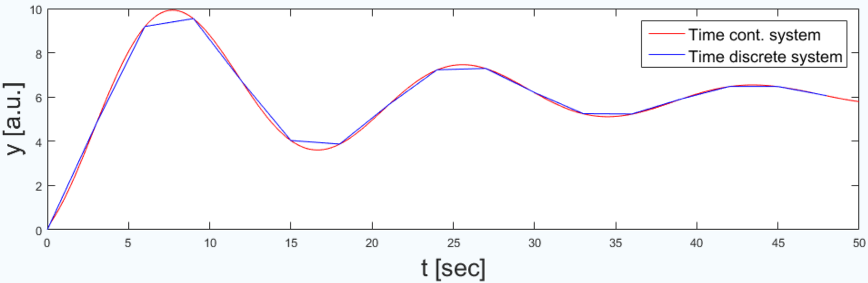


Exercise (#4.1)

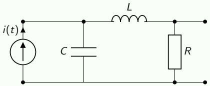
Check example file `E04_07.m`.

Table of Contents

State variable models

Exercise (#4.2)

Given is the schematic shown below.



One can derive the following set of equations:

$$
\begin{aligned}
C\frac{\mathrm{d}}{\mathrm{d}t}u_c(t) &= i(t) - i_L(t)\\
L\frac{\mathrm{d}}{\mathrm{d}t}i_L(t) &= -Ri_L(t) + u_c(t)\\
u_{out}(t) &= Ri_L(t)
\end{aligned}
$$

Write this in state variable form and plot the step response making use of NUMPY or MATLAB. Assume $R = 50\,\Omega$, $C = 1\,\mathrm{nF}$ and $L = 1\,\mathrm{mH}$ and zero initial state. Compare to a direct solution making use of the inverse Laplace transform.

Exercise

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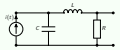
Given is the schematic shown below.



One can derive the following set of equations:

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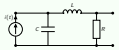

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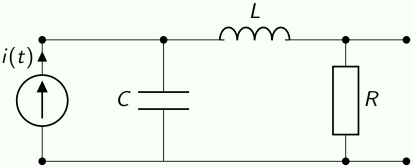
Given is the schematic shown below.

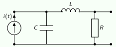


Solve the characteristic equation.

Exercise

Exercise (#4.3)

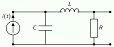
Given is the schematic shown below.



Solve the characteristic equation.

Exercise (#4.3)

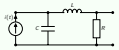
Given is the schematic shown below.



Solve the characteristic equation.

Exercise (#4.2)

Given is the schematic shown below.



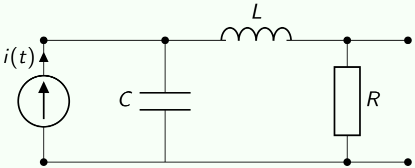
One can derive the following set of equations:

$$
\begin{aligned}
C\frac{d}{dt}u_c(t) &= i(t) - i_L(t)\\
L\frac{d}{dt}i_L(t) &= -Ri_L(t) + u_c(t)\\
u_{out}(t) &= Ri_L(t)
\end{aligned}
$$

Write this in state variable form and plot the step response making use of NUMPY or MATLAB. Assume $R = 50\,\Omega$, $C = 1\,\text{nF}$ and $L = 1\,\text{mH}$ and zero initial state. Compare to a direct solution making use of the inverse Laplace transform.

Exercise (#4.4)

Given is the schematic shown below.

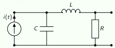


Sketch the phase variable canonical form and determine the coefficients by making use of the transfer function.

Exercise

Exercise (#4.4)

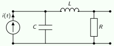
Given is the schematic shown below.



Sketch the phase variable canonical form and determine the coefficients by making use of the transfer function.

Exercise (#4.4)

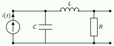
Given is the schematic shown below.



Sketch the phase variable canonical form and determine the coefficients by making use of the transfer function.

Exercise (#4.3)

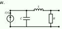
Given is the schematic shown below.



Solve the characteristic equation.

Exercise (#4.2)

Given is the schematic shown below.



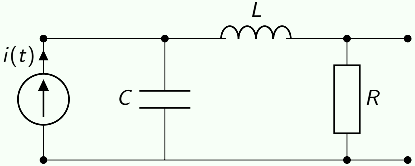
One can derive the following set of equations:

$$
\begin{aligned}
C\frac{d}{dt}u_c(t) &= i(t) - i_L(t)\\
L\frac{d}{dt}i_L(t) &= -Ri_L(t) + u_c(t)\\
u_{out}(t) &= Ri_L(t)
\end{aligned}
$$

Write this in state variable form and plot the step response making use of NUMPY or MATLAB. Assume $R = 50\,\Omega$, $C = 1\,\mathrm{nF}$ and $L = 1\,\mathrm{mH}$ and zero initial state. Compare to a direct solution making use of the inverse Laplace transform.

Exercise (#4.5)

Given is the schematic shown below.

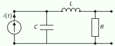


Find $H(s) = \frac{U_r(s)}{I(s)}$ using state variables.

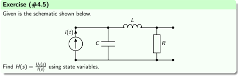
Exercise

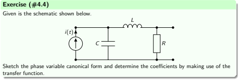
Exercise (#4.5)

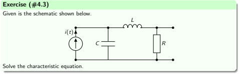
Given is the schematic shown below.



Find $H(s) = \frac{U_r(s)}{I(s)}$ using state variables.

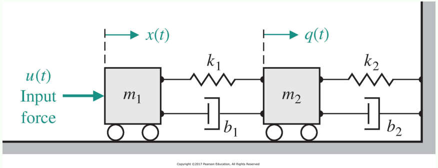






Exercise (#4.6)

Given is the system shown below.



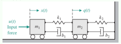
Determine a state space representation of the system taking $u(t)$ as input and $q(t)$ as output and under the assumption that friction can be neglected. Plot the step and frequency response of the system using NumPy[a] or MATLAB. Simulate the response to a sine wave with an amplitude of 1 N and a frequency of 0.5 Hz and 0.2 Hz, respectively as well.

[a] Use $m_1 = 300\,\mathrm{g}$, $m_2 = 100\,\mathrm{g}$, $k_1 = 1\,\mathrm{N\,m^{-1}}$, $k_2 = 1\,\mathrm{N\,m^{-1}}$, $b_1 = 0.1\,\mathrm{kg\,s^{-1}}$, $b_2 = 0.1\,\mathrm{kg\,s^{-1}}$

Exercise

Exercise (#4.6)

Given is the system shown below.



Determine a state space representation of the system taking $u(t)$ as input and $q(t)$ as output and under the assumption that friction can be neglected. Plot the step and frequency response of the system using NUMPY[a] or MATLAB. Simulate the response to a sine wave with an amplitude of 1 N and a frequency of 0.5 Hz and 0.2 Hz, respectively as well.

[a] Use $m_1 = 300\,\mathrm{g}$, $m_2 = 100\,\mathrm{g}$, $k_1 = 1\,\mathrm{N\,m^{-1}}$, $k_2 = 1\,\mathrm{N\,m^{-1}}$, $b_1 = 0.1\,\mathrm{kg\,s^{-1}}$, $b_2 = 0.1\,\mathrm{kg\,s^{-1}}$