

State variable models

4.1 Example system

4.2 Block Diagram

4.3 First order differential equation

4.4 Laplace domain

4.5 Stability of State Variable Systems

4.6 Canonical forms

4.7 Discrete time

4.8 Exercises

Mass spring damper system

▷ Second order Differential equation:

$$u = M \frac{d^2y}{dt^2} + b \frac{dy}{dt} + ky$$

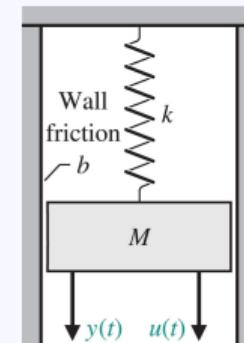


Figure 14: Mass spring (source:
Modern control systems)

Mass spring damper system

└ Second order Differential equation:
 $u = M \frac{d^2y}{dt^2} + b \frac{dy}{dt} + ky$

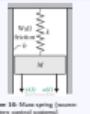


Figure 18: Mass spring [source: Modern control systems]

Introducing states

The equation

$$u = M \frac{d^2y}{dt^2} + b \frac{dy}{dt} + ky$$

can be reduced to a **system** of first order differential equations by using the states

$$\begin{aligned} x_1 &= y, \\ x_2 &= \frac{dx_1}{dt}. \end{aligned}$$

Introducing states

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$$u = M \frac{d^2y}{dt^2} + \frac{dy}{dt} + ky$$

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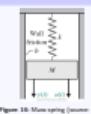
$$x_1 = y,$$

$$x_2 = \frac{dy}{dt}$$

Mass spring damper system

⇒ Second order Differential equation:

$$u = k \frac{d^2y}{dt^2} + \frac{dy}{dt} + ky$$

**Matrix notation**

$$\begin{aligned}\frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= -\frac{b}{M}x_2 - \frac{k}{M}x_1 + \frac{1}{M}u\end{aligned}$$

In matrix notation:

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{M}u \end{pmatrix}$$

Matrix notation

$$\begin{aligned}\frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= -\frac{b}{M}x_2 - \frac{b}{M}x_1 + \frac{1}{M}u\end{aligned}$$

In matrix notation:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{b}{M} & -\frac{b}{M} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{M} \end{pmatrix} u$$

General form

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ a_{2,1} & \dots & a_{2,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{bmatrix} b_{1,1} & \dots & b_{1,m} \\ b_{2,1} & \dots & b_{2,m} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \dots & b_{n,m} \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

▷ Note: \mathbf{A} and \mathbf{B} are constants (no function of time)

▷ \mathbf{x} is called **state vector**.

▷ LTI system

Mass spring damper system

▷ Second order Differential equation:

$$u = M \frac{d^2y}{dt^2} + b \frac{dy}{dt} + ky$$



Figure 10: Mass spring [source: Modern control systems]

State variable models

4.1 Example system

4.2 Block Diagram

4.3 First order differential equation

4.4 Laplace domain

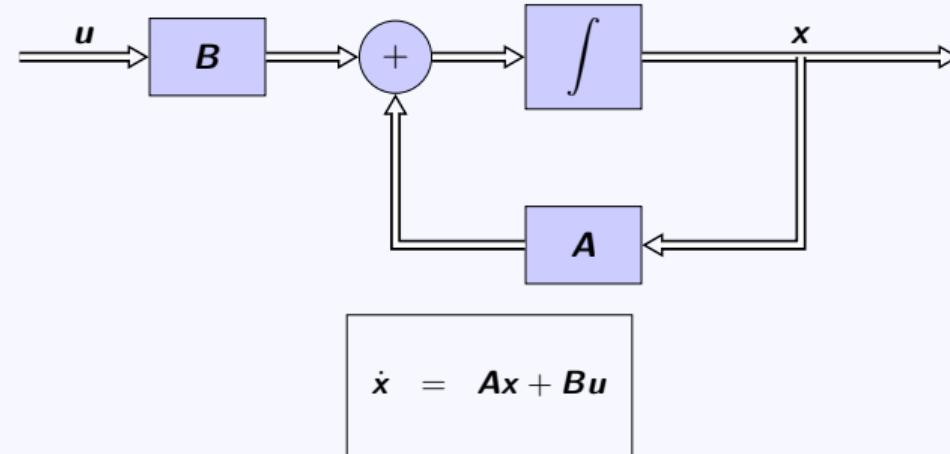
4.5 Stability of State Variable Systems

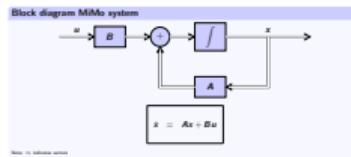
4.6 Canonical forms

4.7 Discrete time

4.8 Exercises

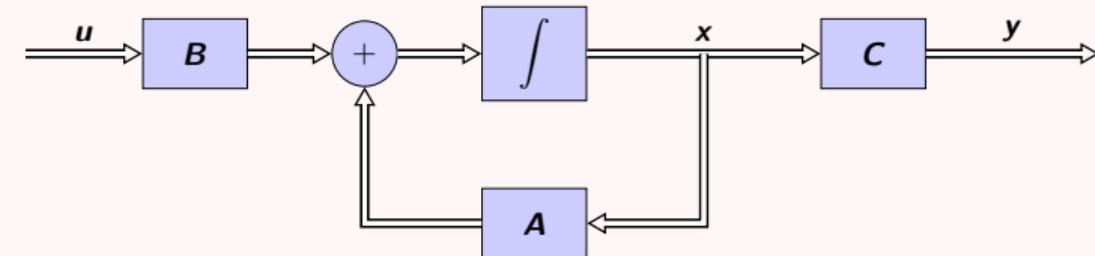
Block diagram MiMo system

Note: \Rightarrow indicates vectors



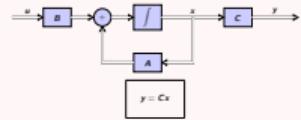
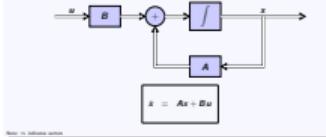
Definition

Frequently, not all states can be measured. This is taken into account by introducing the **measurement matrix C** (also called **output matrix**):

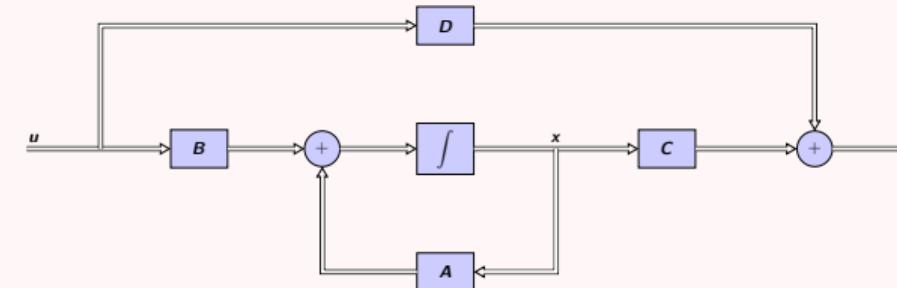


Definition

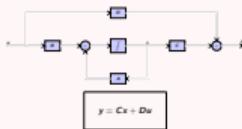
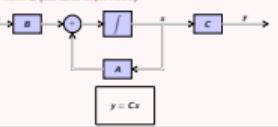
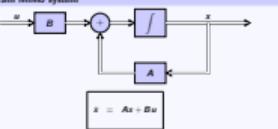
Frequently, not all states can be measured. This is taken into account by introducing the measurement matrix C (also called output matrix):

**Block diagram MiMo system****Definition**

Direct feed-through can be modeled using the feedforward matrix D :

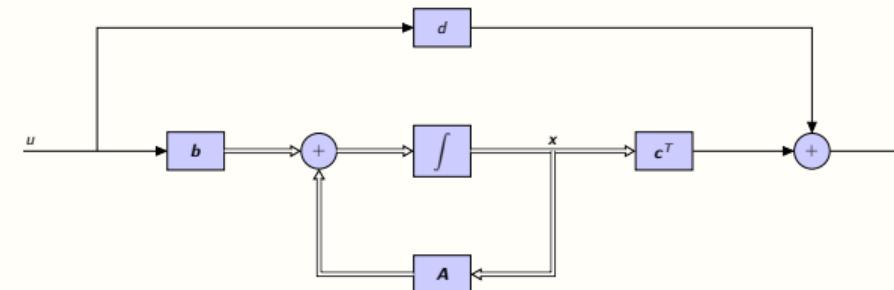


$$y = Cx + Du$$

DefinitionDirect feed-through can be modeled using the feedforward matrix D :**Definition**Frequently, not all states can be measured. This is taken into account by introducing the measurement matrix C (also called output matrix):**Block diagram MiMo system**

Properties

In case of single (scalar) input and single (scalar) output (**SiSo**) the following simplifications can be used:



$$y = c^T x + du$$

$$\dot{x} = Ax + bu$$

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Definition

$$\dot{x} = ax + bu$$

Solution

$$\begin{aligned}x(t) &= e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau \\&= \underbrace{e^{at}k}_{\text{unforced response}} + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau\end{aligned}$$

Definition

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Definition

$$\dot{x} = Ax + Bu$$

Unforced system

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ a_{2,1} & \dots & a_{2,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\dot{x} = Ax$$

Definition

$$\dot{x} = Ax + Bu$$

Unforced system

$$\begin{aligned}\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} &= \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ a_{2,1} & \cdots & a_{2,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ \dot{x} &= Ax\end{aligned}$$

Solution

The Ansatz

$$x(t) = ke^{\lambda t}$$

yields

$$k\lambda e^{\lambda t} = Ake^{\lambda t}.$$

And thus the eigenvalue equations

$$k\lambda = Ak.$$

└ State variable models

└ First order differential equation

Solution

The Ansatz

$$x(t) = k e^{\lambda t}$$

yields

$$k \lambda e^{\lambda t} = A k e^{\lambda t}$$

And thus the eigenvalue equation

$$k \lambda = A k$$

Solution

$$(A - \lambda I) \cdot k = 0.$$

The non trivial solution ($k \neq 0$) exists only if

$$\det(A - \lambda I) = 0.$$

Notes:

▷ I is the identity matrix.**Definition**

$$\dot{x} = Ax + Bu$$

Unforced system

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} &= \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ a_{2,1} & \dots & a_{2,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ x &= Ax \end{aligned}$$

Definition

$$\dot{x} = ax + bu$$

Solution

$$\begin{aligned} x(t) &= e^{at} x(0) + \int_0^t e^{a(t-\tau)} bu(\tau) d\tau \\ &= \underbrace{e^{at} x(0)}_{\text{unforced response}} + \int_0^t e^{a(t-\tau)} bu(\tau) d\tau \end{aligned}$$

└ State variable models

└ First order differential equation

Characteristic equation

Solution

$$(\mathbf{A} - \lambda \mathbf{I}) \cdot \mathbf{k} = 0.$$

The non trivial solution ($\mathbf{k} \neq 0$) exists only if

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0.$$

Note:

 \mathbf{I} is the identity matrix.

Definition

The equation

$$\det(\lambda \mathbf{I} - \mathbf{A}) = 0$$

is called **characteristic equation** of the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}.$$

Solution

The Ansatz

$$\mathbf{x}(t) = \mathbf{k} e^{\lambda t}$$

yields

$$\mathbf{k} \lambda e^{\lambda t} = \mathbf{A} \mathbf{k} e^{\lambda t}.$$

And thus the eigenvalue equation

$$\mathbf{k} \lambda = \mathbf{A} \mathbf{k}$$

Definition

$$\mathbf{x} = \mathbf{Ax} + \mathbf{Bu}$$

Unforced system

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} &= \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ a_{2,1} & \dots & a_{2,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ \dot{\mathbf{x}} &= \mathbf{Ax} \end{aligned}$$

Definition

The equation

$$\det(\lambda I - \mathbf{A}) = 0$$

is called **characteristic equation** of the system defined by

$$\dot{\mathbf{x}} = \mathbf{Ax}$$

Definition

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

Solution

$$(\mathbf{A} - \lambda I) \cdot \mathbf{k} = 0.$$

The non trivial solution ($\mathbf{k} \neq 0$) exists only if

$$\det(\mathbf{A} - \lambda I) = 0.$$

Note: \mathbf{k} is the identity matrix.**Solution**

$$\mathbf{x}(t) = e^{\mathbf{At}} \mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)} \mathbf{B} \mathbf{u}(\tau) d\tau,$$

with

$$e^{\mathbf{At}} = \mathbf{I} + \mathbf{At} + \frac{\mathbf{A}^2 t^2}{2!} + \cdots + \frac{\mathbf{A}^n t^n}{n!} + \cdots$$

Note: \mathbf{I} is the identity matrix.**Solution**

The Ansatz

$$\mathbf{x}(t) = \mathbf{k} e^{\lambda t}$$

yields

$$\mathbf{k} \lambda e^{\lambda t} = \mathbf{A} \mathbf{k} e^{\lambda t}.$$

And thus the eigenvalue equations

$$\mathbf{k} \lambda = \mathbf{Ak}$$

Definition

$$\dot{x} = Ax + Bu$$

Solution

$$x(t) = e^{\mathbf{At}}x(0) + \int_0^t e^{\mathbf{A}(t-\tau)}Bu(\tau)d\tau,$$

$$e^{\mathbf{At}} = I + \mathbf{At} + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \dots$$

Note: I is the identity matrix.**Definition**

The equation

$$\det(\lambda I - A) = 0$$

is called **characteristic equation** of the system defined by

$$\dot{x} = Ax.$$

About $\exp(\mathbf{At})$ It can be shown that for a square ($n \times n$)-matrix A

$$\begin{aligned} e^{\mathbf{At}} &= I + \mathbf{At} + \frac{\mathbf{A}^2 t^2}{2!} + \cdots + \frac{\mathbf{A}^n t^n}{n!} + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} \mathbf{A}^k t^k \\ &= \alpha_{n-1}(t)\mathbf{A}^{n-1}t^{n-1} + \alpha_{n-2}(t)\mathbf{A}^{n-2}t^{n-2} + \cdots + \alpha_0(t)\mathbf{I} \\ &= \sum_{k=0}^{n-1} \alpha_k(t)\mathbf{A}^k t^k \end{aligned}$$

holds true. In this equation, $\alpha_0(t) \dots \alpha_{n-1}(t)$ are functions of t which need to be determined (see next slides).**Solution**

$$(A - \lambda I) \cdot k = 0.$$

The non trivial solution ($k \neq 0$) exists only if

$$\det(A - \lambda I) = 0.$$

Note: I is the identity matrix.

└ State variable models

└ First order differential equation

About $\exp(At)$

About $\exp(At)$ It can be shown that for a square ($n \times n$)-matrix A

$$\begin{aligned} e^{At} &= I + At + \frac{A^2 t^2}{2!} + \cdots + \frac{A^n t^n}{n!} + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k \\ &= \alpha_{n-1}(t)A^{n-1}t^{n-1} + \alpha_{n-2}(t)A^{n-2}t^{n-2} + \cdots + \alpha_0(t)I \\ &= \sum_{k=0}^{n-1} \alpha_k(t)A^k t^k \end{aligned}$$

holds true. In this equation, $\alpha_0(t), \dots, \alpha_{n-1}(t)$ are functions of t which need to be determined (see next slides).**Definition**

$$\dot{x} = Ax + Bu$$

Solution

$$\begin{aligned} x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau, \\ \text{with } e^{At} &= I + At + \frac{A^2 t^2}{2!} + \cdots + \frac{A^n t^n}{n!} + \cdots \end{aligned}$$

Note: If x is the identity matrix**Definition**

The equation

$$\det(\lambda I - A) = 0$$

is called **characteristic equation** of the system defined by

$$\dot{x} = Ax.$$

About $\exp(At)$

Let $r(\lambda)$ be defined as follows:

$$r(\lambda) = \alpha_{n-1}(t)\lambda^{n-1} + \alpha_{n-2}(t)\lambda^{n-2} + \cdots + \alpha_1(t)\lambda + \alpha_0(t).$$

Then, $\alpha_k(t)$ can be determined by solving the equations shown on the next slides.

└ State variable models

└ First order differential equation

About $\exp(At)$

About $\exp(At)$ Let $r(\lambda)$ be defined as follows:

$$r(\lambda) := \alpha_{n-1}(t)\lambda^{n-1} + \alpha_{n-2}(t)\lambda^{n-2} + \dots + \alpha_1(t)\lambda + \alpha_0(t).$$

Then, $\alpha_k(t)$ can be determined by solving the equations shown on the next slides.

About $\exp(At)$

For each eigenvalue λ_k of At with multiplicity N , one gets N equations:

$$\begin{aligned} e^{\lambda_k} &= r(\lambda_k) \\ e^{\lambda_k} &= \left. \frac{\partial r(\lambda_k)}{\partial \lambda} \right|_{\lambda=\lambda_k} \\ &\vdots && \vdots \\ e^{\lambda_k} &= \left. \frac{\partial^{N-1} r(\lambda_k)}{\partial \lambda^{N-1}} \right|_{\lambda=\lambda_k} \end{aligned}$$

About $\exp(At)$ It can be shown that for a square ($n \times n$)-matrix A

$$\begin{aligned} e^{At} &= I + At + \frac{A^2 t^2}{2!} + \dots + \frac{A^n t^n}{n!} + \dots = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k \\ &= \alpha_{n-1}(t)A^{n-1}t^{n-1} + \alpha_{n-2}(t)A^{n-2}t^{n-2} + \dots + \alpha_0(t)I \\ &= \sum_{k=0}^{n-1} \alpha_k(t)A^k t^k \end{aligned}$$

holds true. In this equation, $\alpha_0(t), \dots, \alpha_{n-1}(t)$ are functions of t which need to be determined (use next slides).**Definition**

$$x = Ax + Bu$$

Solution

$$\begin{aligned} x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau, \\ \text{with } e^{At} &= I + At + \frac{A^2 t^2}{2!} + \dots + \frac{A^n t^n}{n!} + \dots \end{aligned}$$

Note: I is the identity matrix.For $N = 1$: $e^{\lambda_k} = r(\lambda_k)$.For $N = 2$:

$$\begin{aligned} e^{\lambda_k} &= r(\lambda_k) \\ e^{\lambda_k} &= \left. \frac{\partial r(\lambda_k)}{\partial \lambda} \right|_{\lambda=\lambda_k} \end{aligned}$$

└ State variable models

└ First order differential equation

About $\exp(\mathbf{At})$

About $\exp(\mathbf{At})$ For each eigenvalue λ_0 of $\mathbf{A}t$ with multiplicity N , one gets N equations:

$$\begin{aligned} e^{\lambda_0 t} &= r(\lambda_0) \\ \dot{e}^{\lambda_0 t} &= \frac{d(r(\lambda_0))}{d\lambda} \Big|_{\lambda=\lambda_0} \\ &\vdots \\ &\vdots \\ \ddot{e}^{\lambda_0 t} &= \frac{d^{N-1}(r(\lambda_0))}{d\lambda^{N-1}} \Big|_{\lambda=\lambda_0} \end{aligned}$$

For $N = 1$: $e^{\lambda_0 t} = r(\lambda_0)$.For $N = 2$:

$$\begin{aligned} e^{\lambda_0 t} &= r(\lambda_0) \\ \dot{e}^{\lambda_0 t} &= \frac{d(r(\lambda_0))}{d\lambda} \Big|_{\lambda=\lambda_0} \end{aligned}$$

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 9 & 1 \end{bmatrix}.$$

About $\exp(\mathbf{At})$ Let $r(\lambda)$ be defined as follows:

$$r(\lambda) := \alpha_{m-1}(t)\lambda^{m-1} + \alpha_{m-2}(t)\lambda^{m-2} + \dots + \alpha_1(t)\lambda + \alpha_0(t).$$

Then, $\alpha_n(t)$ can be determined by solving the equations shown on the next slides.

$$e^{\mathbf{At}} = \alpha_1 \mathbf{At} + \alpha_0 \mathbf{I} = \begin{bmatrix} \alpha_1 t + \alpha_0 & \alpha_1 t \\ 9\alpha_1 t & \alpha_1 t + \alpha_0 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 4t$, $\lambda_2 = -2t$ and thus

$$\begin{aligned} e^{4t} &= 4t\alpha_1 + \alpha_0 \\ e^{-2t} &= -2t\alpha_1 + \alpha_0 \end{aligned}$$

About $\exp(\mathbf{At})$ It can be shown that for a square ($n \times n$)-matrix \mathbf{A}

$$\begin{aligned} e^{\mathbf{At}} &= I + \mathbf{At} + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{1}{k!} \mathbf{A}^k t^k \\ &= \alpha_{m-1}(t)\mathbf{A}^{m-1}t^{m-1} + \alpha_{m-2}(t)\mathbf{A}^{m-2}t^{m-2} + \dots + \alpha_0(t)\mathbf{I} \\ &= \sum_{k=0}^{m-1} \alpha_k(t)\mathbf{A}^k t^k \end{aligned}$$

holds true. In this equation, $\alpha_0(t), \dots, \alpha_{m-1}(t)$ are functions of t which need to be determined (see next slides).

└ State variable models

└ First order differential equation

About $\exp(At)$

Example

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix},$$

$$e^{At} = \alpha_1 \mathbf{A}t + \alpha_0 \mathbf{I} = \begin{bmatrix} 4t^2 + \alpha_0 & \alpha_1 t \\ 0 & 1 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 4t$, $\lambda_2 = -2t$ and thus

$$\begin{aligned} e^{\lambda_1 t} &= 4t\alpha_1 + \alpha_0 \\ e^{\lambda_2 t} &= -2t\alpha_1 + \alpha_0 \end{aligned}$$

Example

Solving

$$\begin{aligned} e^{4t} &= 4t\alpha_1 + \alpha_0 \\ e^{-2t} &= -2t\alpha_1 + \alpha_0 \end{aligned}$$

leads to

$$\begin{aligned} \alpha_1 &= \frac{1}{6t} (e^{4t} - e^{-2t}) \quad \Rightarrow \quad e^{At} = \frac{1}{6} \begin{bmatrix} 3e^{4t} + 3e^{-2t} & e^{4t} - e^{-2t} \\ 9e^{4t} - 9e^{-2t} & 3e^{4t} + 3e^{-2t} \end{bmatrix}. \\ \alpha_0 &= \frac{1}{3} (e^{4t} + 2e^{-2t}) \end{aligned}$$

About $\exp(At)$ For each eigenvalue λ_k of A with multiplicity N , one gets N equations:

$$\begin{aligned} e^{\lambda_k t} &= r(\lambda_k) \\ e^{\lambda_k t} &= \left. \frac{\partial r(\lambda_k)}{\partial \lambda} \right|_{\lambda=\lambda_k} \\ &\vdots \\ e^{\lambda_k t} &= \left. \frac{\partial^{N-1} r(\lambda_k)}{\partial \lambda^{N-1}} \right|_{\lambda=\lambda_k} \end{aligned}$$

For $N=1$: $e^{\lambda_k t} = r(\lambda_k)$.For $N=2$:

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About $\exp(At)$ Let $r(\lambda)$ be defined as follows:

$$r(\lambda) = \alpha_{n-2}(t)\lambda^{n-1} + \alpha_{n-3}(t)\lambda^{n-2} + \dots + \alpha_1(t)\lambda + \alpha_0(t).$$

Then, $\alpha_k(t)$ can be determined by solving the equations shown on the next slides.

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Definition and Laplace transform

$$\begin{aligned}\dot{x} &= ax + bu \\ sX(s) - x(0) &= aX(s) + bU(s) \\ X(s) &= \frac{x(0)}{s-a} + \frac{b}{s-a} U(s)\end{aligned}$$

Definition and Laplace transform

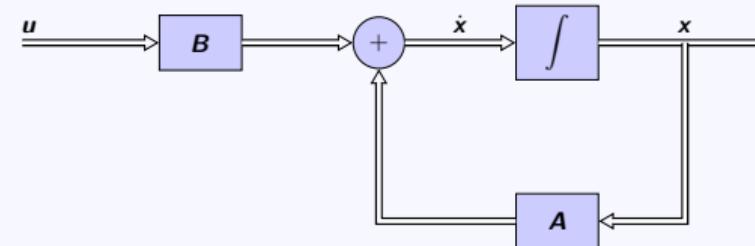
$$\begin{aligned}\dot{x} &= ax + bu \\ x(t) - x(0) &= \int_0^t aX(\tau) + bu(\tau) d\tau \\ X(s) &= \frac{x(0)}{s-a} + \frac{b}{s-a} U(s)\end{aligned}$$

Solution by inverse Laplace transform

$$\begin{aligned}x(t) &= e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau \\ &= \underbrace{e^{at}k}_{\text{unforced response}} + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau\end{aligned}$$

Solution by inverse Laplace transform

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau \\ = \underbrace{e^{at}x(0)}_{\text{unforced response}} + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau$$



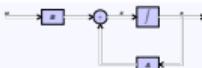
Definition and Laplace transform

$$\begin{aligned}\dot{x} &= ax + bu \\ sX(s) - x(0) &= aX(s) + bU(s) \\ X(s) &= \frac{x(0)}{s-a} + \frac{b}{s-a}U(s)\end{aligned}$$

With the assumption $\mathbf{u}(0) = \mathbf{x}(0) = \mathbf{0}$:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} \\ s\mathbf{X}(s) &= \mathbf{AX}(s) + \mathbf{BU}(s) \\ (sI - \mathbf{A})\mathbf{X}(s) &= \mathbf{BU}(s) \\ \mathbf{X}(s) &= [sI - \mathbf{A}]^{-1} \mathbf{BU}(s) = \Phi(s)\mathbf{BU}(s)\end{aligned}$$

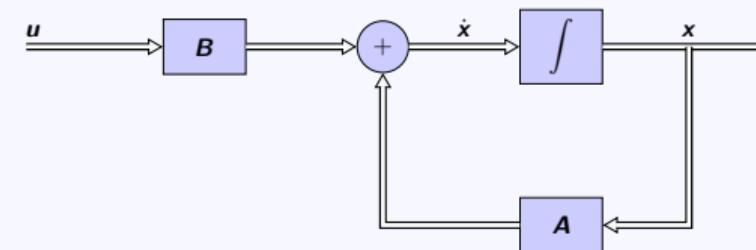
Note: $\mathbf{X}(s)$ and $\mathbf{U}(s)$ are vectors (capital letters indicate frequency-domain).



With the assumption $u(0) = x(0) = 0$

$$\begin{aligned} x &= Ax + Bu \\ x(s) &= AX(s) + BU(s) \\ (sI - A)x(s) &= BU(s) \\ X(s) &= [sI - A]^{-1}BU(s) = \Phi(s)BU(s) \end{aligned}$$

Note: $\Phi(s)$ and $\Phi(t)$ are defined in the inverse Laplace transform section.



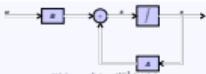
$$\begin{aligned} X(s) &= [sI - A]^{-1}BU(s) \\ X(s) &= \Phi(s)BU(s) \\ \Phi(t) &= e^{At} \end{aligned}$$

Note: Compare with

Definition and Laplace transform

$$\begin{aligned} \dot{x} &= ax + bu \\ x(s) - x(0) &= sX(s) + bu(s) \\ X(s) &= \frac{x(0)}{s-a} + \frac{b}{s-a}U(s) \end{aligned}$$

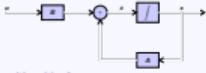
$$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}.$$



$$\begin{aligned} X(s) &= [sI - A]^{-1} B U(s) \\ X(s) &= \Phi(s) B U(s) \\ \phi(t) &= e^{At} \end{aligned}$$

Note: Compare with

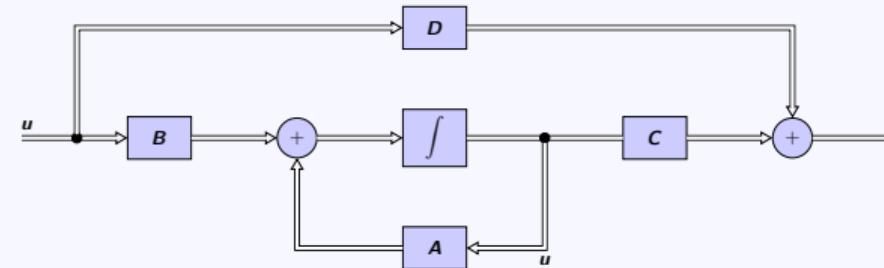
$$\mathcal{L}\{e^{-st}\} = \frac{1}{s - z}$$

With the assumption $x(0) = x'(0) = 0$

$$\begin{aligned} x &= Ax + Bu \\ xK(s) &= AX(s) + BU(s) \\ (sI - A)X(s) &= BU(s) \\ X(s) &= [sI - A]^{-1}BU(s) = \Phi(s)BU(s) \end{aligned}$$

Note: $X(s)$ and $X'(s)$ are vectors (capital letters indicate frequency domain)

MiMo

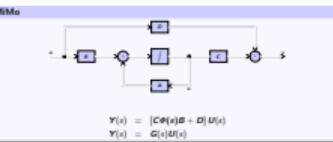
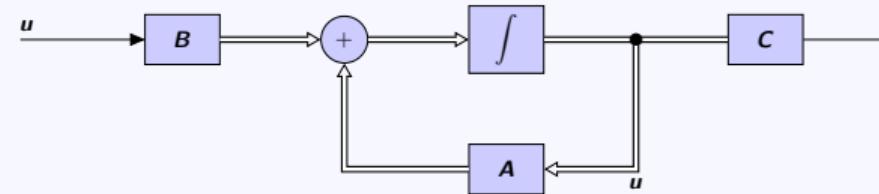


$$Y(s) = [C\Phi(s)B + D] U(s)$$

$$Y(s) = G(s)U(s)$$

Solution by inverse Laplace transform

$$\begin{aligned} x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\ &= \underbrace{\sum_{k=0}^n b_k}_{\text{unforced response}} + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \end{aligned}$$

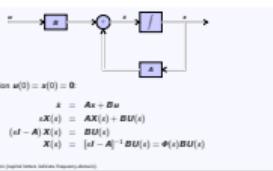
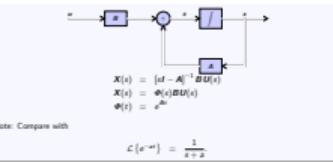
**SiSo without direct feedthrough**For **SiSo** systems without direct feedthrough:

$$Y(s) = [C\Phi(s)B] U(s)$$

$$Y(s) = G(s)U(s),$$

with

$$\Phi(s) = [sI - A]^{-1}$$



State variable models

4.1 Example system

4.2 Block Diagram

4.3 First order differential equation

4.4 Laplace domain

4.5 Stability of State Variable Systems

4.6 Canonical forms

4.7 Discrete time

4.8 Exercises

Definition

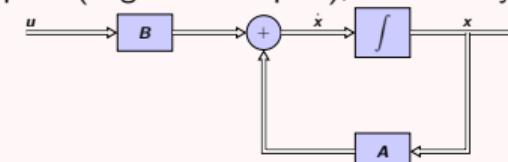
The equation

$$\mathbf{X}(s) = [sI - \mathbf{A}]^{-1} \mathbf{B}U(s)$$

can be used to determine the stability of the system. If all of the roots of the **characteristic equation**

$$\det(sI - \mathbf{A}) = 0$$

are located in the left half space (negative real part), then the system is **stable**.



State variable models

- 4.1 Example system
- 4.2 Block Diagram
- 4.3 First order differential equation
- 4.4 Laplace domain
- 4.5 Stability of State Variable Systems

4.6 Canonical forms

- 4.6.1 SiSo transfer function
- 4.6.2 Phase variable canonical form
- 4.6.3 Input feedforward canonical form

4.7 Discrete time

4.8 Exercises

Canonical forms

4.6 Canonical forms

4.6.1 SiSo transfer function

4.6.2 Phase variable canonical form

4.6.3 Input feedforward canonical form

Exemplary SiSo transfer function

$$\begin{aligned} G(s) = \frac{Y(s)}{U(s)} &= \frac{[b_0 + b_1 s^1 + b_2 s^2]}{[a_0 + a_1 s^1 + a_2 s^2 + s^3]} \\ &= \frac{[b_0 s^{-3} + b_1 s^{-2} + b_2 s^{-1}]}{[a_0 s^{-3} + a_1 s^{-2} + a_2 s^{-1} + 1]} \end{aligned}$$

Expanding with $Z(s)$:

$$G(s) = \frac{[b_0 + b_1 s^1 + b_2 s^3]}{[a_0 + a_1 s^1 + a_2 s^2 + s^3]} \cdot \frac{Z(s)}{Z(s)}$$

Also used:

$$Y(s) = \frac{b_2 s^0 + b_1 s^1 + b_0 s^2}{a_2 s^0 + a_1 s^1 + a_0 s^2 + s^3} U(s)$$

Exemplary SiSo transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{[b_0 + b_1 s^1 + b_2 s^2]}{[a_0 + a_1 s^1 + a_2 s^2 + s^3]}$$

$$= \frac{[b_0 s^{-3} + b_1 s^{-2} + b_2 s^{-1}]}{[a_0 s^{-3} + a_1 s^{-2} + a_2 s^{-1} + 1]}$$

Expanding with $Z(s)$:

$$G(s) = \frac{[b_0 + b_1 s^1 + b_2 s^2]}{[a_0 + a_1 s^1 + a_2 s^2 + s^3]} \cdot \frac{Z(s)}{Z(s)}$$

and

$$Y(s) = \frac{s^3 + a_1 s^2 + a_2 s + a_0}{s^3 + a_1 s^2 + a_2 s^1 + 1} Z(s)$$

SiSo transfer function

$$Y(s) = [b_0 + b_1 s^1 + b_2 s^2] Z(s)$$

$$U(s) = [a_0 + a_1 s^1 + a_2 s^2 + s^3] Z(s)$$

In the time domain:

$$y = b_0 z(t) + b_1 \frac{d}{dt} z(t) + b_2 \frac{d^2}{dt^2} z(t)$$

$$u = a_0 z(t) + a_1 \frac{d}{dt} z(t) + a_2 \frac{d^2}{dt^2} z(t) + \frac{d^3}{dt^3} z(t)$$

Canonical forms

4.6 Canonical forms

4.6.1 SiSo transfer function

4.6.2 Phase variable canonical form

4.6.3 Input feedforward canonical form

SiSo transfer function

$$\begin{aligned}y &= b_0 z(t) + b_1 \frac{d}{dt} z(t) + b_2 \frac{d^2}{dt^2} z(t) \\u &= a_0 z(t) + a_1 \frac{d}{dt} z(t) + a_2 \frac{d^2}{dt^2} z(t) + \frac{d^3}{dt^3} z(t)\end{aligned}$$

Introducing state variables (also called phase variables):

$$x_1 = z(t), x_2 = \frac{d}{dt} z(t) \text{ and } x_3 = \frac{d^2}{dt^2} z(t)$$

Note that usually the states do not represent physical values.

SiSo transfer function

$$\begin{aligned}y &= b_0z(t) + b_1 \frac{d}{dt}z(t) + b_2 \frac{d^2}{dt^2}z(t) \\u &= a_0z(t) + a_1 \frac{d}{dt}z(t) + a_2 \frac{d^2}{dt^2}z(t) + \frac{d^3}{dt^3}z(t)\end{aligned}$$

Introducing state variables (also called phase variables):

$$x_1 = z(t), x_2 = \frac{d}{dt}z(t) \text{ and } x_3 = \frac{d^2}{dt^2}z(t)$$

Note that usually the states do not represent physical values.

SiSo transfer function

$$y = b_0x_1 + b_1x_2 + b_2x_3$$

$$u = a_0x_1 + a_1x_2 + a_2x_3 + \frac{d}{dt}x_3$$

The resulting system of first order differential equations:

$$\frac{d}{dt}x_1 = x_2$$

$$\frac{d}{dt}x_2 = x_3$$

$$\frac{d}{dt}x_3 = -a_0x_1 - a_1x_2 - a_2x_3 + u$$

SISO transfer function

$$\begin{aligned} y &= b_0x_0 + b_1x_1 + b_2x_2 \\ u &= a_0x_0 + a_1x_1 + a_2x_2 + \frac{d}{dt}x_3 \end{aligned}$$

The resulting system of first order differential equations:

$$\begin{aligned} \frac{dx_0}{dt} &= x_2 \\ \frac{dx_1}{dt} &= x_3 \\ \frac{dx_2}{dt} &= -a_0x_0 - a_1x_1 - a_2x_2 - \frac{d}{dt}x_3 \end{aligned}$$

SISO transfer function

$$\begin{aligned} y &= b_0x(t) + b_1\frac{dx}{dt}(t) + b_2\frac{d^2x}{dt^2}(t) \\ u &= a_0x(t) + a_1\frac{dx}{dt}(t) + a_2\frac{d^2x}{dt^2}(t) + \frac{d^3x}{dt^3}(t) \end{aligned}$$

Introducing state variables (also called phase variables):

$$x_1 = x(t), x_2 = \frac{dx}{dt}(t) \text{ and } x_3 = \frac{d^2x}{dt^2}(t)$$

Note that usually the states do not represent physical values.

As matrix equations

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \mathbf{C}^T \mathbf{x} = [b_0 \quad b_1 \quad b_2] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Phase variable canonical form

As matrix equations

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = C^T \mathbf{x} = [b_0 \ b_1 \ b_2] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

SISO transfer function

$$y = b_0 x_0 + b_1 x_1 + b_2 x_2$$

$$u = -a_0 x_0 - a_1 x_1 - a_2 x_2 + \frac{d}{dt} x_0$$

The resulting system of first order differential equations

$$\frac{d}{dt} x_0 = x_2$$

$$\frac{d}{dt} x_1 = x_3$$

$$\frac{d}{dt} x_2 = -a_0 x_0 - a_1 x_1 - a_2 x_2 + u$$

SISO transfer function

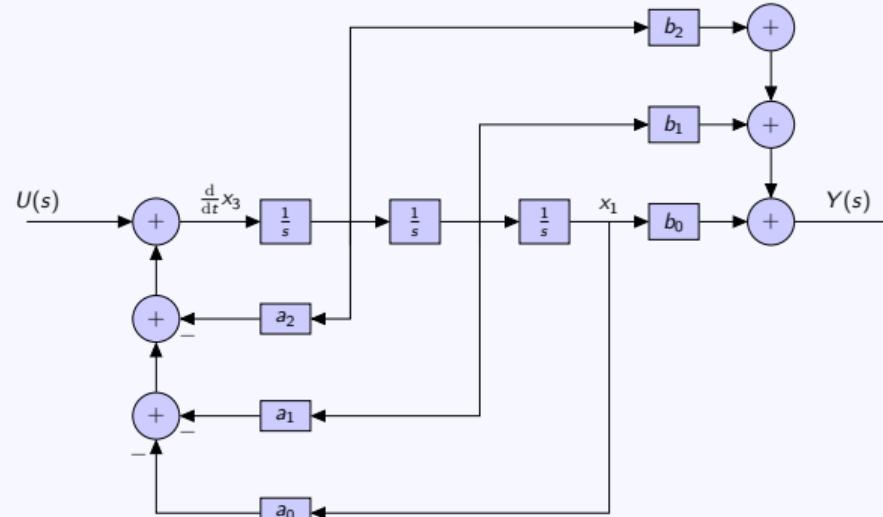
$$y = b_0 x(t) + b_1 \frac{d}{dt} x(t) + b_2 \frac{d^2}{dt^2} x(t)$$

$$u = a_0 x(t) + a_1 \frac{d}{dt} x(t) + a_2 \frac{d^2}{dt^2} x(t) + \frac{d^3}{dt^3} x(t)$$

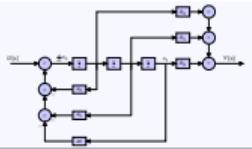
Introducing state variables (also called phase variables):

$$x_0 = x(t), x_1 = \frac{d}{dt} x(t) \text{ and } x_2 = \frac{d^2}{dt^2} x(t)$$

Note that usually the states do not represent physical values.



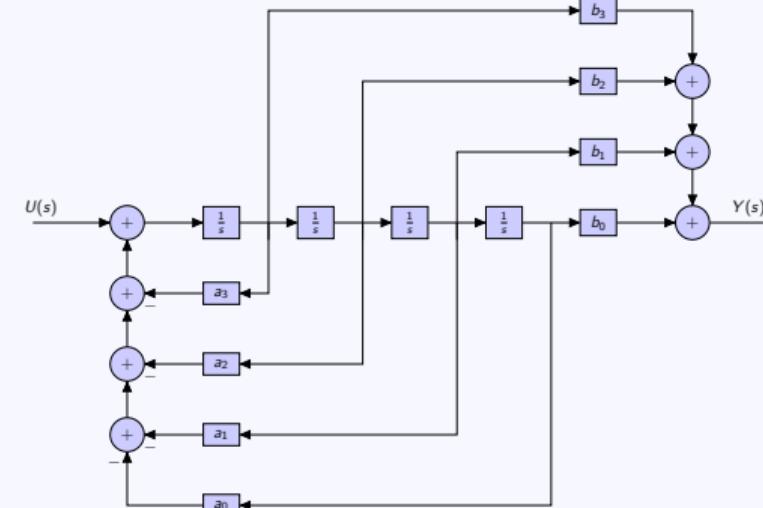
Phase variable canonical form



Increasing order

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = C^T \mathbf{x} = [b_0 \quad b_1 \quad b_2] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



SISO transfer function

$$y = b_0 x_0 + b_1 x_1 + b_2 x_2$$

$$u = a_0 x_0 + a_1 x_1 + a_2 x_2 + \frac{d}{dt} x_2$$

The resulting system of first order differential equations:

$$\begin{aligned} \frac{dx_0}{dt} &= x_2 \\ \frac{dx_1}{dt} &= x_3 \\ \frac{dx_2}{dt} &= -a_0 x_0 - a_1 x_1 - a_2 x_2 + u \end{aligned}$$

Canonical forms

4.6 Canonical forms

4.6.1 SiSo transfer function

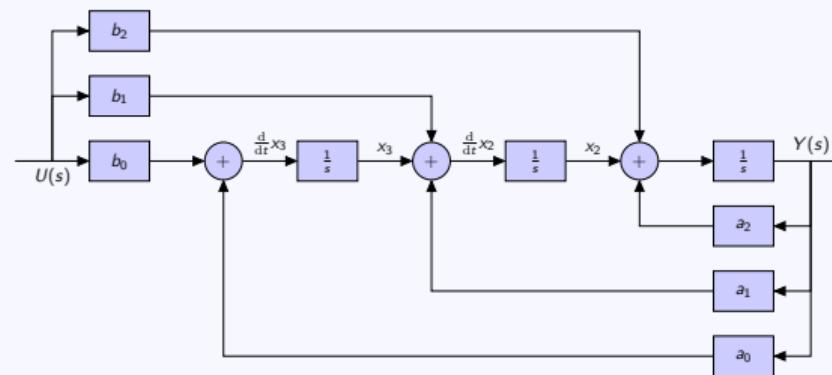
4.6.2 Phase variable canonical form

4.6.3 Input feedforward canonical form

Alternative flow graph

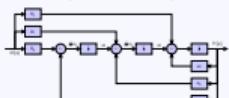
It can be shown (e.g. by using **Mason's gain formular**) that the following flow diagram realizes the transfer function

$$G(s) = \frac{[b_0 s^{-3} + b_1 s^{-2} + b_2 s^{-1}]}{[a_0 s^{-3} + a_1 s^{-2} + a_2 s^{-1} + 1]}.$$



It can be shown (e.g. by using Mason's gain formula) that the following flow diagram realizes the transfer function

$$G(s) = \frac{[b_0 s^2 + b_1 s^2 + b_2 s^1]}{[a_0 s^3 + a_1 s^2 + a_2 s^1 + 1]}$$



SiSo transfer function

$$\begin{aligned} y &= b_0 z + b_1 \frac{d}{dt} z + b_2 \frac{d^2}{dt^2} z \\ u &= a_0 z + a_1 \frac{d}{dt} z + a_2 \frac{d^2}{dt^2} z + \frac{d^3}{dt^3} z \end{aligned}$$

Feeding forward the input signal $u(t)$ allows us to use the output as state x_1 .

SiSo transfer function

$$\begin{aligned}y &= b_0x + b_1 \frac{d}{dt}x + b_2 \frac{d^2}{dt^2}x \\u &= a_0x + a_1 \frac{d}{dt}x + a_2 \frac{d^2}{dt^2}x + a_3 \frac{d^3}{dt^3}x\end{aligned}$$

Feeding forward the input signal $u(t)$ allows us to use the output as state x_0 .

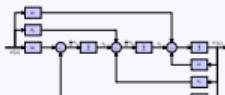
SiSo transfer function

Introducing state variables:

$$\begin{aligned}x_1 &= y \\ \frac{d}{dt}x_1 &= -a_2x_1 + x_3 + b_2u \\ \frac{d}{dt}x_2 &= -a_1x_1 + x_2 + b_1u \\ \frac{d}{dt}x_3 &= -a_0 + b_0u\end{aligned}$$

It can be shown (e.g. by using Mason's gain formula) that the following flow diagram realizes the transfer function

$$G(s) = \frac{b_0s^3 + b_1s^2 + b_2s + b_3}{a_0s^3 + a_1s^2 + a_2s + a_3 + 1}$$



SISO transfer function

Introducing state variables:

$$\begin{aligned}x_1 &= y \\ \frac{dx_1}{dt} &= -a_2 x_1 + a_1 x_2 + b_2 u \\ \frac{dx_2}{dt} &= -a_1 x_1 + a_0 x_2 + b_1 u \\ \frac{dx_3}{dt} &= -a_0 x_1 + b_0 u\end{aligned}$$

As matrix equation

$$\begin{aligned}y(t) &= [1 \ 0 \ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} u(t)\end{aligned}$$

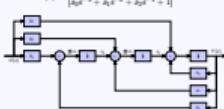
SISO transfer function

$$\begin{aligned}y &= b_0 x + b_1 \frac{dx}{dt} + b_2 \frac{d^2 x}{dt^2} \\ u &= a_0 x + a_1 \frac{dx}{dt} + a_2 \frac{d^2 x}{dt^2} + \frac{d^3 x}{dt^3}\end{aligned}$$

Feeding forward the input signal $u(t)$ allows us to use the output as state x_1 .

It can be shown (e.g. by using Mason's gain formula) that the following flow diagram realizes the transfer function

$$G(s) = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{a_0 s^3 + a_1 s^2 + a_2 s + a_3 + 1}$$



State variable models

- 4.1 Example system
- 4.2 Block Diagram
- 4.3 First order differential equation
- 4.4 Laplace domain
- 4.5 Stability of State Variable Systems
- 4.6 Canonical forms
- 4.7 Discrete time**
- 4.8 Exercises

Continuous time model

$$\dot{\mathbf{x}}(t) = \mathbf{A}_c \mathbf{x}(t) + \mathbf{B}_c \mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}_c \mathbf{x}(t) + \mathbf{D}_c \mathbf{u}(t)$$

$$\mathbf{x}(t) = \exp(\mathbf{A}_c(t - t_0)) \mathbf{x}(t_0) + \int_{t_0}^t \exp(\mathbf{A}_c(t - \tau)) \mathbf{B}_c \mathbf{u}(\tau) d\tau$$

Thus

$$\begin{aligned}\mathbf{x}((k+1)T) &= \exp(\mathbf{A}_c((k+1)T - kT)) \mathbf{x}(kT) \\ &\quad + \int_{kT}^{(k+1)T} \exp(\mathbf{A}_c(kT - \tau)) \mathbf{B}_c \mathbf{u}(\tau) d\tau \\ &= \exp(\mathbf{A}_c(T)) \mathbf{x}(kT) \\ &\quad + \int_{kT}^{(k+1)T} \exp(\mathbf{A}_c(kT - \tau)) \mathbf{B}_c \mathbf{u}(\tau) d\tau\end{aligned}$$

Continuous time model

$$\begin{aligned} \dot{x}(t) &= \mathbf{A}_c x(t) + \mathbf{B}_c u(t) \\ y(t) &= \mathbf{C}_c x(t) + \mathbf{D}_c u(t) \\ x(t) &= \exp(\mathbf{A}_c(t - t_0)) x(t_0) + \int_{t_0}^t \exp(\mathbf{A}_c(t - \tau)) \mathbf{B}_c u(\tau) d\tau \end{aligned}$$

Thus

$$\begin{aligned} x((k+1)T) &= \exp(\mathbf{A}_c((k+1)T - kT)) x(kT) \\ &\quad + \int_{kT}^{(k+1)T} \exp(\mathbf{A}_c(kT - \tau)) \mathbf{B}_c u(\tau) d\tau \\ &= \exp(\mathbf{A}_c(T)) x(kT) \\ &\quad + \int_{kT}^{(k+1)T} \exp(\mathbf{A}_c(kT - \tau)) \mathbf{B}_c u(\tau) d\tau \end{aligned}$$

Zero-order hold assumption

$$x((k+1)T) = \exp(\mathbf{A}_c(T)) x(kT) + \int_{kT}^{(k+1)T} \exp(\mathbf{A}_c(kT - \tau)) \mathbf{B}_c u(\tau) d\tau$$

Based on the assumption that the *analog \Leftrightarrow digital converter* gives constant values in between time steps (*zero-order hold*) one gets

$$x_{k+1} = \mathbf{A}_d x_k + \mathbf{B}_d u_k$$

$$\mathbf{A}_d = \exp(\mathbf{A}_c T)$$

$$\mathbf{B}_d = \int_0^T \exp(\mathbf{A}_c(T - \tau)) \mathbf{B}_c d\tau$$

Zero-order hold assumption

$$x([(k+1)T]) = \exp(\mathbf{A}_c(T))x(kT) + \int_{kT}^{(k+1)T} \exp(\mathbf{A}_c(kT-\tau))\mathbf{B}_cu(\tau)d\tau$$

Based on the assumption that the analog or digital converter gives constant values in between time steps (zero-order hold) one gets

$$\begin{aligned} x_{k+1} &= \mathbf{A}_d x_k + \mathbf{B}_d u_k \\ \mathbf{A}_d &= \exp(\mathbf{A}_c T) \\ \mathbf{B}_d &= \int_0^T \exp(\mathbf{A}_c(T-\tau))\mathbf{B}_cu(\tau)d\tau \end{aligned}$$

Complete description

$$x_{k+1} = \mathbf{A}_d x_k + \mathbf{B}_d u_k$$

$$y_k = \mathbf{C}_d x_k + \mathbf{D}_d u_k$$

$$\mathbf{A}_d = \exp(\mathbf{A}_c T)$$

$$\mathbf{B}_d = \int_0^T \exp(\mathbf{A}_c(T-\tau))\mathbf{B}_c d\tau$$

$$\mathbf{C}_d = \mathbf{C}_c$$

$$\mathbf{D}_d = \mathbf{D}_c.$$

Continuous time model

$$\begin{aligned} \dot{x}(t) &= \mathbf{A}_c x(t) + \mathbf{B}_c u(t) \\ y(t) &= \mathbf{C}_c x(t) + \mathbf{D}_c u(t) \\ x(t) &= \exp(\mathbf{A}_c(t-n))x(n) + \int_n^t \exp(\mathbf{A}_c(t-\tau))\mathbf{B}_c u(\tau)d\tau \end{aligned}$$

Thus

$$\begin{aligned} x([(k+1)T]) &= \exp(\mathbf{A}_c([(k+1)T-kT]))x(kT) \\ &\quad + \int_{kT}^{(k+1)T} \exp(\mathbf{A}_c(kT-\tau))\mathbf{B}_c u(\tau)d\tau \\ &= \exp(\mathbf{A}_c(T))x(kT) \\ &\quad + \int_{kT}^{(k+1)T} \exp(\mathbf{A}_c(kT-\tau))\mathbf{B}_c u(\tau)d\tau \end{aligned}$$

Complete description

$$\begin{aligned}x_{k+1} &= \mathbf{A}_d x_k + \mathbf{B}_d u_k \\y_k &= \mathbf{C}_d x_k + \mathbf{D}_d u_k \\ \mathbf{A}_d &= \exp(\mathbf{A}_e T) \\ \mathbf{B}_d &= \int_0^T \exp(\mathbf{A}_e (T-\tau)) \mathbf{B}_e d\tau \\ \mathbf{C}_d &= \mathbf{C}_e \\ \mathbf{D}_d &= \mathbf{D}_e\end{aligned}$$

Example

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1} + 8s^{-2} + 6s^{-3}}{1 + 8s^{-1} + 16s^{-2} + 6s^{-3}}$$

Using phase variable state model^a one gets:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -16 & -8 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [6 \ 8 \ 2] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Zero-order hold assumption

$$x((k+1)T) = \exp(\mathbf{A}_e T) x(kT) + \int_{kT}^{(k+1)T} \exp(\mathbf{A}_e (kT-\tau)) \mathbf{B}_e u(\tau) d\tau$$

Based on the assumption that the analog to digital converter gives constant values in between time steps (zero-order hold) one gets

$$\begin{aligned}x_{k+1} &= \mathbf{A}_d x_k + \mathbf{B}_d u_k \\ \mathbf{A}_d &= \exp(\mathbf{A}_e T) \\ \mathbf{B}_d &= \int_0^T \exp(\mathbf{A}_e (T-\tau)) \mathbf{B}_e d\tau\end{aligned}$$

Continuous time model

$$\begin{aligned}x(t) &= \mathbf{A}_e x(t) + \mathbf{B}_e u(t) \\y(t) &= \mathbf{C}_e x(t) + \mathbf{D}_e u(t) \\x(t) &= \exp(\mathbf{A}_e (t-t_0)) x(t_0) + \int_{t_0}^t \exp(\mathbf{A}_e (t-\tau)) \mathbf{B}_e u(\tau) d\tau\end{aligned}$$

Thus

$$\begin{aligned}x((k+1)T) &= \exp(\mathbf{A}_e ((k+1)T - kT)) x(kT) \\&\quad + \int_{kT}^{(k+1)T} \exp(\mathbf{A}_e (kT-\tau)) \mathbf{B}_e u(\tau) d\tau \\&= \exp(\mathbf{A}_e (T)) x(kT) \\&\quad + \int_{kT}^{(k+1)T} \exp(\mathbf{A}_e (kT-\tau)) \mathbf{B}_e u(\tau) d\tau\end{aligned}$$

^aSee e.g. Dorf, Bishop for relationship between transfer function and state model

Example

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1} + 8s^{-2} + 6s^{-3}}{1 + 8s^{-1} + 16s^{-2} + 6s^{-3}}$$

Using phase variable state model* one gets:

$$\begin{aligned} x &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -16 & -8 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= [B \quad B \quad 2] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

*See e.g. Diaf, Bindapur for relationship between transfer function and state model

Example (cont.)

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1} + 8s^{-2} + 6s^{-3}}{1 + 8s^{-1} + 16s^{-2} + 6s^{-3}}$$

Using $T = 3$ one gets:

$$\mathbf{A}_d = \begin{bmatrix} 0.3214 & 0.1950 & 0.0259 \\ -0.1555 & -0.03933 & -0.0123 \\ 0.0740 & 0.0419 & 0.0054 \end{bmatrix}$$

and

$$\mathbf{B}_d = [0.1131 \quad 0.0259 \quad -0.0108]^T$$

Zero-order hold assumption

$$x((k+1)T) = \exp(\mathbf{A}_d(T))x(kT) + \int_{kT}^{(k+1)T} \exp(\mathbf{A}_d(kT-\tau))\mathbf{B}_d u(\tau)d\tau$$

Based on the assumption that the analog to digital converter gives constant values in between time steps (zero-order hold) one gets

$$\begin{aligned} x_{k+1} &= \mathbf{A}_d x_k + \mathbf{B}_d u_k \\ \mathbf{A}_d &= \exp(\mathbf{A}, T) \\ \mathbf{B}_d &= \int_0^T \exp(\mathbf{A}_d(T-\tau)) \mathbf{B}_d d\tau \end{aligned}$$

└ State variable models

└ Discrete time

Example (cont.)

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1} + 8s^{-2} + 6s^{-3}}{1 + 8s^{-1} + 16s^{-2} + 6s^{-3}}$$

Using $T = 3$ one gets:

$$\mathbf{A}_d = \begin{bmatrix} 0.3214 & 0.1950 & 0.0299 \\ -0.1955 & -0.0303 & -0.0123 \\ 0.0740 & 0.0419 & 0.0024 \end{bmatrix}$$

and

$$\mathbf{B}_d = [0.1131 \quad 0.0299 \quad -0.0024]^T$$

Example (cont.)

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1} + 8s^{-2} + 6s^{-3}}{1 + 8s^{-1} + 16s^{-2} + 6s^{-3}}$$

Using $T = 5$ one gets:

$$\mathbf{A}_d = \begin{bmatrix} 0.1218 & 0.0741 & 0.0099 \\ -0.0592 & -0.0360 & -0.0048 \\ 0.0287 & 0.0175 & 0.0023 \end{bmatrix}$$

and

$$\mathbf{B}_d = [0.1464 \quad 0.0098 \quad -0.0023]^T$$

Complete description

$$\begin{aligned} x_{1,1} &= \mathbf{A}_x x_1 + \mathbf{B}_x u_1 \\ y_1 &= \mathbf{C}_x x_1 + \mathbf{D}_x u_1 \\ \mathbf{A}_x &= \exp(\mathbf{A} \cdot T) \\ \mathbf{B}_x &= \int_0^T \exp(\mathbf{A}_x(T-\tau)) \mathbf{B}_d d\tau \\ \mathbf{C}_x &= \mathbf{C}_x \\ \mathbf{D}_x &= \mathbf{D}_x \end{aligned}$$

└ State variable models

└ Discrete time

Example (cont.)

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1} + 8s^{-2} + 6s^{-3}}{1 + 8s^{-1} + 16s^{-2} + 8s^{-3}}$$

Using $T = 5$ one gets:

$$A_d = \begin{bmatrix} 0.1218 & 0.0741 & 0.0099 \\ -0.0592 & -0.0360 & -0.0048 \\ 0.0287 & 0.0175 & 0.0023 \end{bmatrix}$$

and

$$B_d = [0.1464 \quad 0.0398 \quad -0.0023]^T$$

Example

Now, consider

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1} + 8s^{-2} + 6s^{-3}}{1 + 8s^{-1} + s^{-2} + s^{-3}}$$

Example (cont.)

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1} + 8s^{-2} + 6s^{-3}}{1 + 8s^{-1} + 16s^{-2} + 8s^{-3}}$$

Using $T = 3$ one gets:

$$A_d = \begin{bmatrix} 0.3214 & 0.1890 & 0.0299 \\ -0.1505 & -0.0383 & -0.0123 \\ 0.0740 & 0.0419 & 0.0054 \end{bmatrix}$$

and

$$B_d = [0.1131 \quad 0.0259 \quad -0.0008]^T$$

Using phase variable state model one gets:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -8 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [6 \quad 8 \quad 2] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1} + 8s^{-2} + 6s^{-3}}{1 + 8s^{-1} + 16s^{-2} + 8s^{-3}}$$

Using phase variable state model* one gets:

$$x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -16 & -8 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [6 \quad 8 \quad 2] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

*See e.g. Ljung, Blockup for relationship between transfer function and state model

└ State variable models

└ Discrete time

Example

Now, consider

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1} + 8s^{-2} + 6s^{-3}}{1 + 8s^{-1} + s^{-2} + 6s^{-3}}$$

Using phase variable state model one gets:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [6 \ 8 \ 2] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example (cont.)

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1} + 8s^{-2} + 6s^{-3}}{1 + 8s^{-1} + s^{-2} + s^{-3}}$$

Using phase variable state model and $T = 3$ one gets:

$$\mathbf{A}_d = \begin{bmatrix} 0.5666 & 2.1239 & 0.2601 \\ -0.2601 & 0.3065 & 0.0430 \\ -0.0430 & -0.3031 & -0.037 \end{bmatrix}$$

and

$$\mathbf{B}_d = [0.4338 \ 0.2602 \ 0.0445]^T$$

Example (cont.)

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1} + 8s^{-2} + 6s^{-3}}{1 + 8s^{-1} + 16s^{-2} + 6s^{-3}}$$

Using $T = 3$ one gets:

$$\mathbf{A}_d = \begin{bmatrix} 0.3214 & 0.1950 & 0.0259 \\ -0.1595 & -0.0383 & -0.0123 \\ 0.0740 & 0.0419 & 0.0054 \end{bmatrix}$$

and

$$\mathbf{B}_d = [0.1131 \ 0.0259 \ -0.0309]^T$$

Example (cont.)

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1} + 8s^{-2} + 6s^{-3}}{1 + 8s^{-1} + s^{-2} + s^{-3}}$$

Using phase variable state model and $T = 3$ one gets:

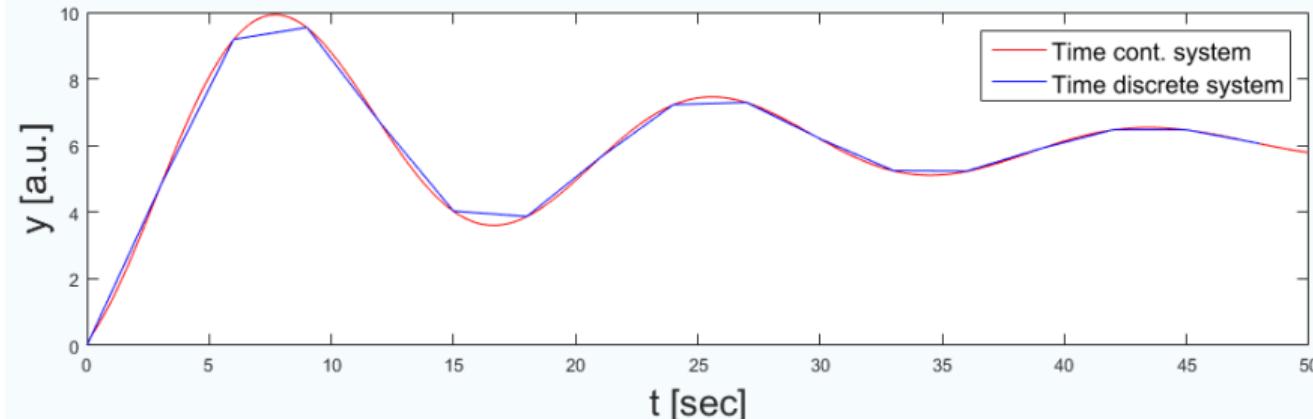
$$A_d = \begin{bmatrix} 0.5966 & 2.1239 & 0.2801 \\ -0.2601 & 0.3095 & 0.0430 \\ -0.0430 & -0.3531 & -0.037 \end{bmatrix}$$

and

$$B_d = [0.4130 \ 0.2862 \ 0.0446]^T$$

Example (cont.)

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1} + 8s^{-2} + 6s^{-3}}{1 + 8s^{-1} + s^{-2} + s^{-3}}$$



Example

Now consider

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1} + 8s^{-2} + 6s^{-3}}{1 + 8s^{-1} + s^{-2} + s^{-3}}$$

Using phase variable state model one gets:

$$\begin{aligned} x &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= [0 \ 8 \ 2] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

Example (cont.)

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1} + 8s^{-2} + 6s^{-3}}{1 + 8s^{-1} + 2s^{-2} + 6s^{-3}}$$

Using $T = 5$ one gets:

$$A_d = \begin{bmatrix} 0.1218 & 0.0741 & 0.0099 \\ 0.0592 & -0.0360 & -0.0048 \\ 0.0287 & 0.0175 & 0.0023 \end{bmatrix}$$

and

$$B_d = [0.1464 \ 0.0398 \ -0.0023]^T$$

Exercise (#4.1)

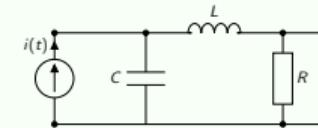
Check example file E04_07.m.

State variable models

- 4.1 Example system
- 4.2 Block Diagram
- 4.3 First order differential equation
- 4.4 Laplace domain
- 4.5 Stability of State Variable Systems
- 4.6 Canonical forms
- 4.7 Discrete time
- 4.8 Exercises**

Exercise (#4.2)

Given is the schematic shown below.



One can derive the following set of equations:

$$C \frac{d}{dt} u_c(t) = i(t) - i_L(t)$$

$$L \frac{d}{dt} i_L(t) = -R i_L(t) + u_c(t)$$

$$u_{out}(t) = R i_L(t)$$

Write this in state variable form and plot the step response making use of NUMPY or MATLAB. Assume $R = 50 \Omega$, $C = 1 \text{ nF}$ and $L = 1 \text{ mH}$ and zero initial state. Compare to a direct solution making use of the inverse Laplace transform.

Exercise (#4.2)

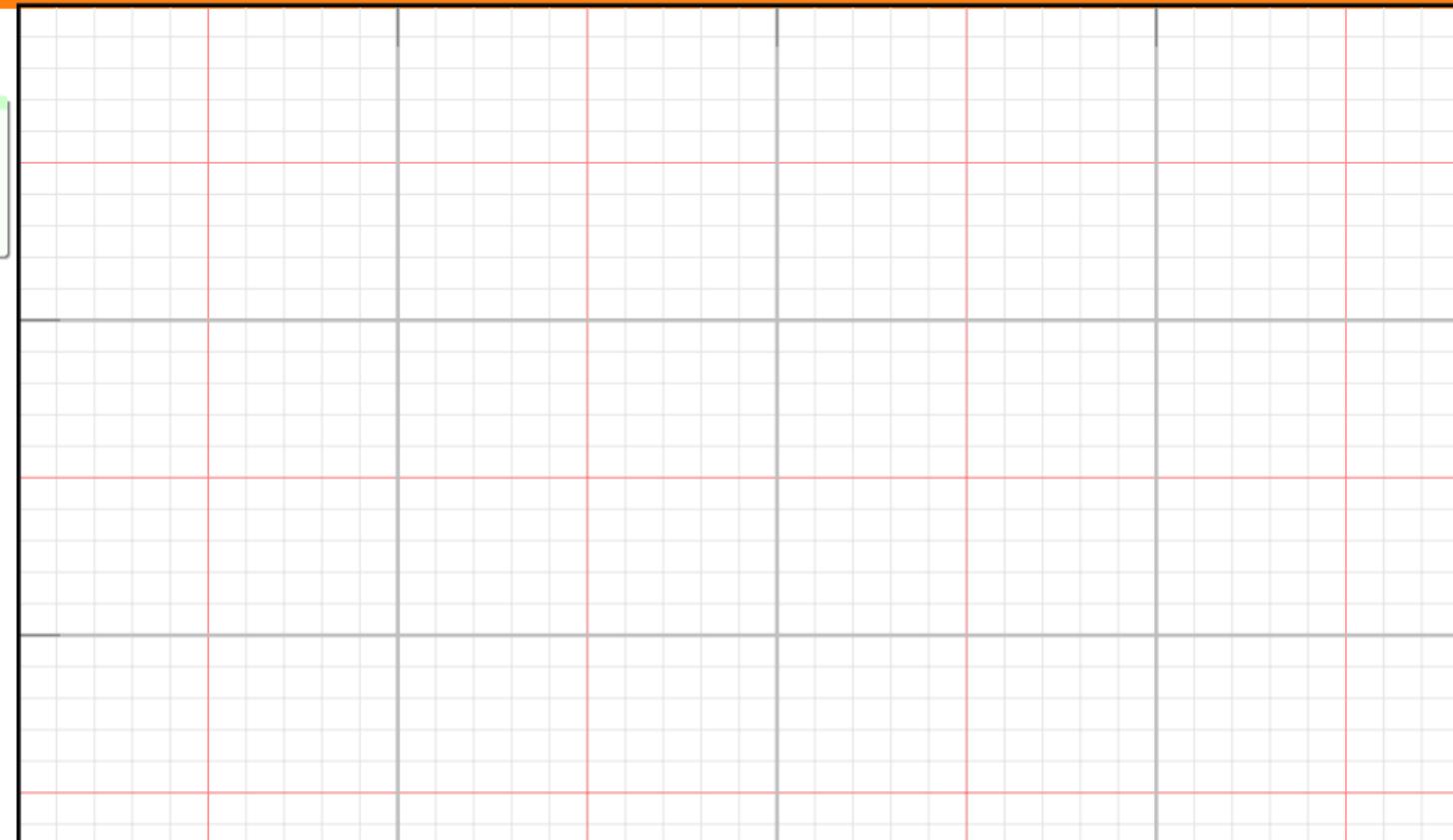
Given is the schematic shown below.



One can derive the following set of equations:

$$\begin{aligned} C \frac{d\phi}{dt}(t) &= i(t) - k_1(t) \\ i \frac{d\phi}{dt}(t) &= -R_0(t) + u_s(t) \\ u_{out}(t) &= R_0(t) \end{aligned}$$

Write this in state variable form and plot the step response making use of MATLAB or MATLAB. Assume $R = 50\Omega$, $C = 1\text{nF}$ and $L = 1\text{mH}$ and zero initial state. Compare to a direct solution making use of the inverse Laplace transform.



Exercise (#4.2)

Given is the schematic shown below.



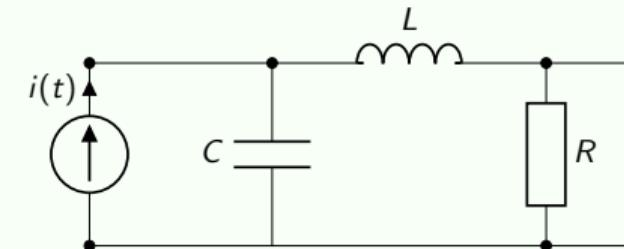
One can derive the following set of equations:

$$\begin{aligned} C\ddot{\phi}(t) &= i(t) - \dot{\phi}(t) \\ L\ddot{\phi}(t) &= -R\dot{\phi}(t) + \phi(t) \\ \phi_0(t) &= \theta_0(t) \end{aligned}$$

Write this in state variable form and plot the step response making use of `Nexterrv` or MATLAB. Assume $R = 5\Omega$, $C = 1\text{nF}$ and $L = 1\text{mH}$ and zero initial state. Compare to a direct solution making use of the inverse Laplace transform.

Exercise (#4.3)

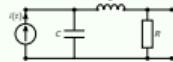
Given is the schematic shown below.



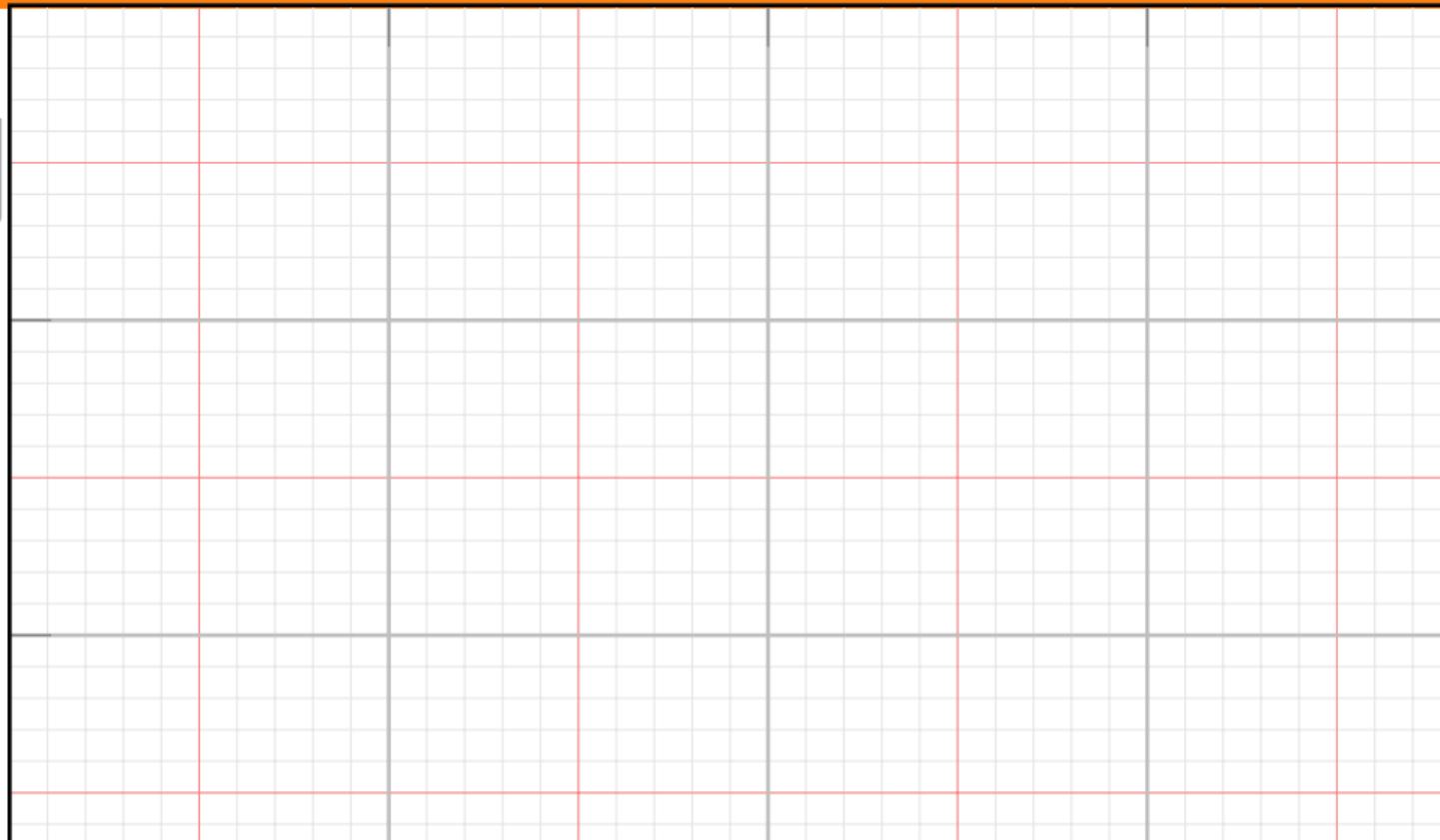
Solve the characteristic equation.

Exercise (#4.3)

Given is the schematic shown below.

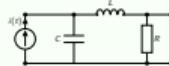


Solve the characteristic equation.



Exercise (#4.3)

Given is the schematic shown below.



Solve the characteristic equation.

Exercise (#4.2)

Given is the schematic shown below.

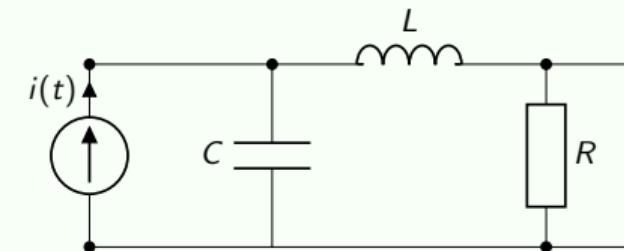


One can derive the following set of equations:

$$\begin{aligned} C\frac{du_0(t)}{dt} &= i(t) - i_L(t) \\ i_L(t) &= -Ri(t) + u_0(t) \\ u_0(t) &= R_i(t) \end{aligned}$$

Write this in state variable form and plot the step response making use of *MatLab*. Assume $R = 50\Omega$, $C = 1\text{nF}$ and $i = 1\text{mH}$ and zero initial state. Compare to a direct solution making use of the inverse Laplace transform.**Exercise (#4.4)**

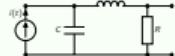
Given is the schematic shown below.



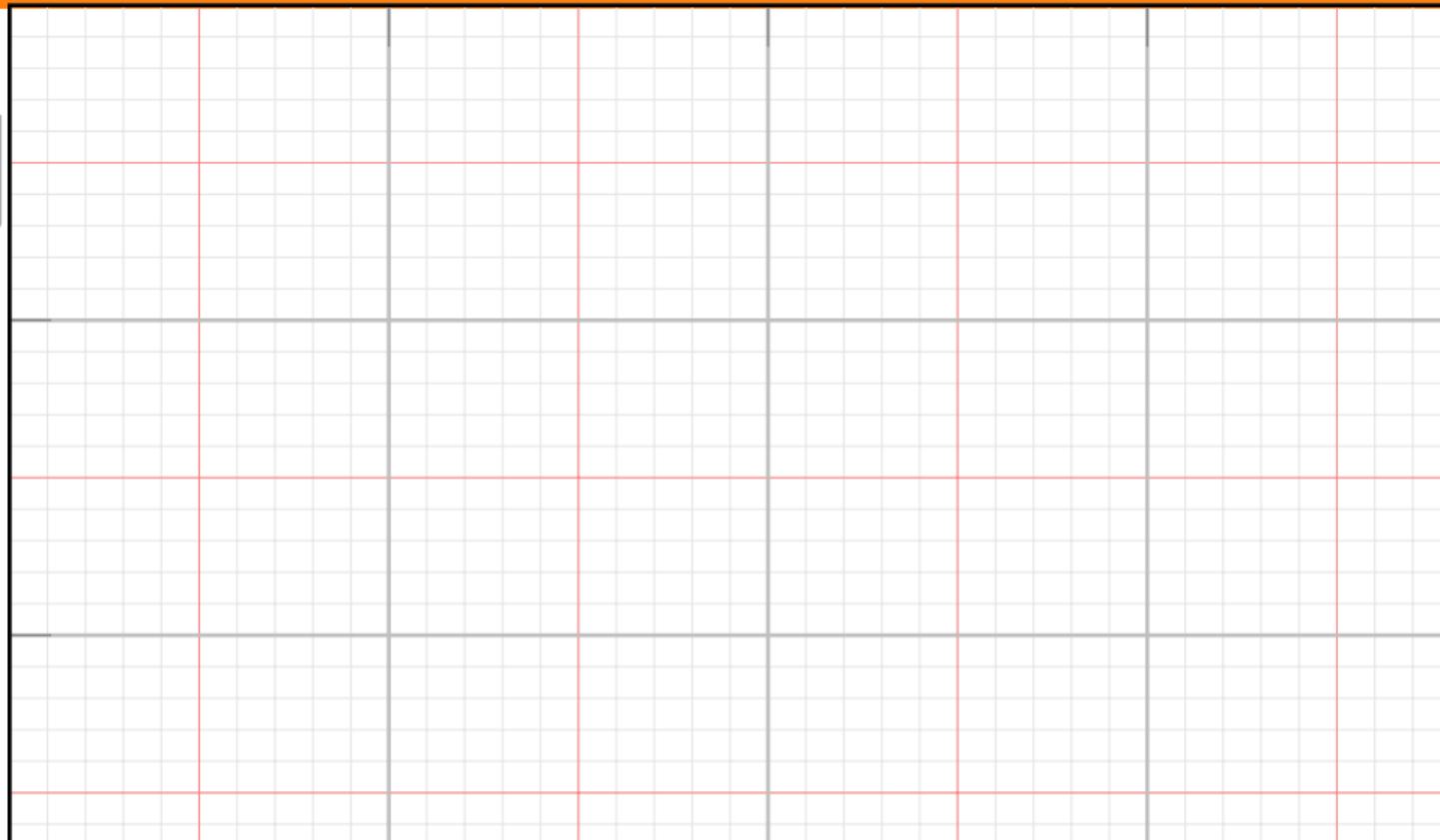
Sketch the phase variable canonical form and determine the coefficients by making use of the transfer function.

Exercise (#4.4)

Given is the schematic shown below.

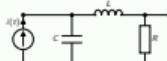


Sketch the phase variable canonical form and determine the coefficients by making use of the transfer function.



Exercise (#4.4)

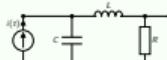
Given is the schematic shown below.



Sketch the phase variable canonical form and determine the coefficients by making use of the transfer function.

Exercise (#4.3)

Given is the schematic shown below.



Solve the characteristic equation.

Exercise (#4.2)

Given is the schematic shown below.



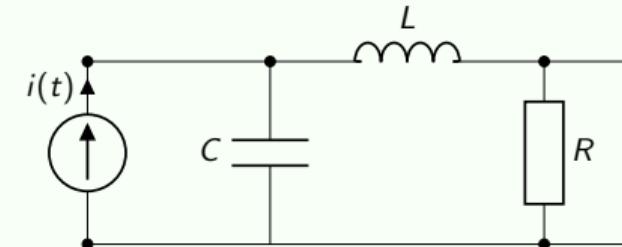
One can derive the following set of equations:

$$\begin{aligned} C \frac{d}{dt} u_0(t) &= i(t) - i_L(t) \\ i_L(t) &= -R_L u_0(t) + i_R(t) \\ u_{00}(t) &= R_L i_R(t) \end{aligned}$$

Write this in state variable form and plot the step response making use of Nisimur or MATLAB. Assume $R = 50 \Omega$, $C = 1 \text{ mF}$ and $L = 1 \text{ mH}$ and zero initial state. Compare to a direct solution making use of the inverse Laplace transform.

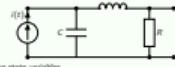
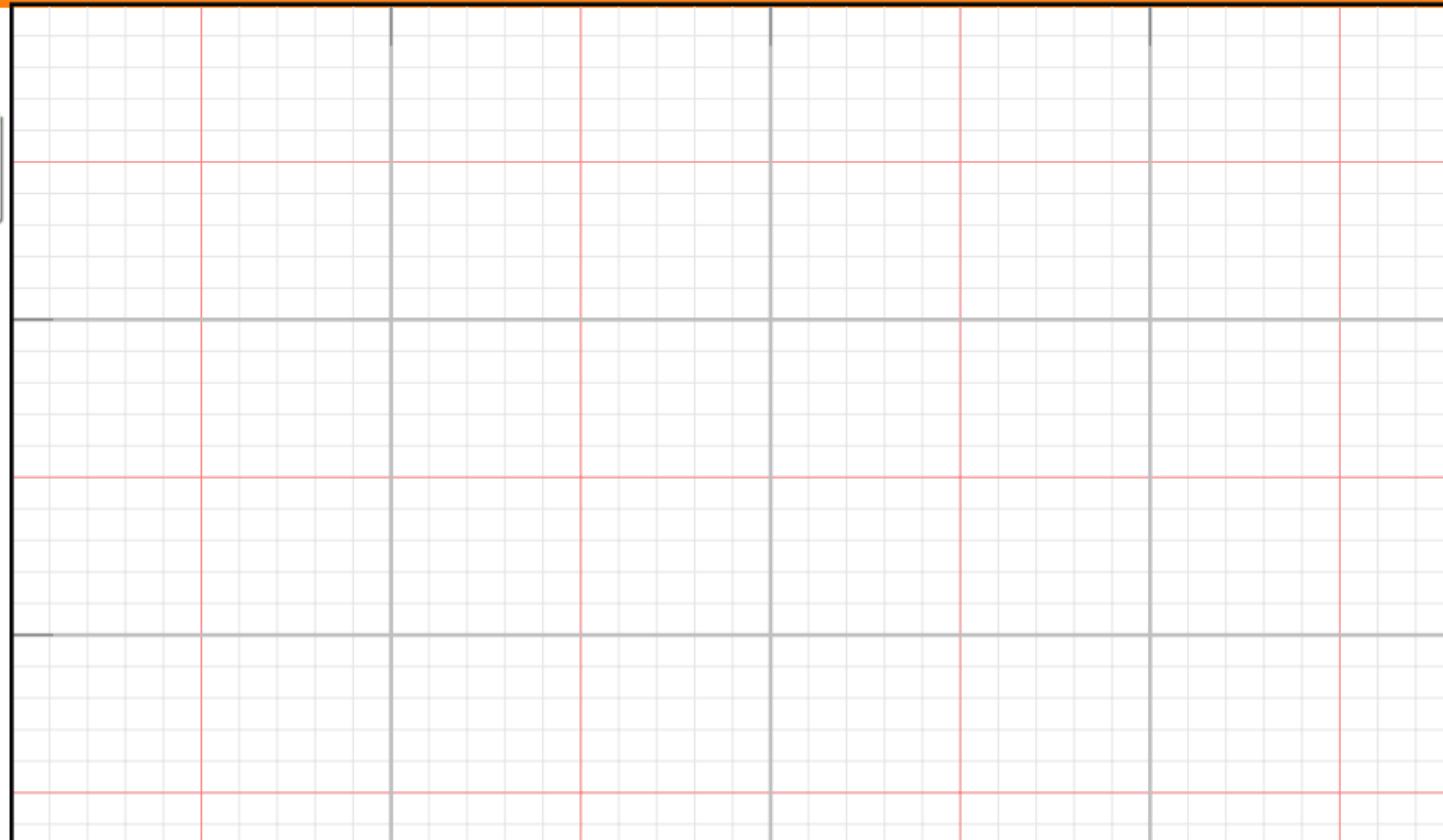
Exercise (#4.5)

Given is the schematic shown below.

Find $H(s) = \frac{U_r(s)}{I(s)}$ using state variables.

Exercise (#4.5)

Given is the schematic shown below:

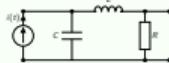
Find $H(s) = \frac{V_o}{V_i}$ using state variables.

└ State variable models

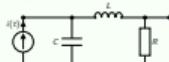
└ Exercises

Exercise (#4.5)

Given is the schematic shown below.

Find $H(s) = \frac{U(s)}{u(t)}$ using state variables.**Exercise (#4.4)**

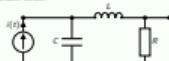
Given is the schematic shown below.



Sketch the phase variable canonical form and determine the coefficients by making use of the transfer function.

Exercise (#4.3)

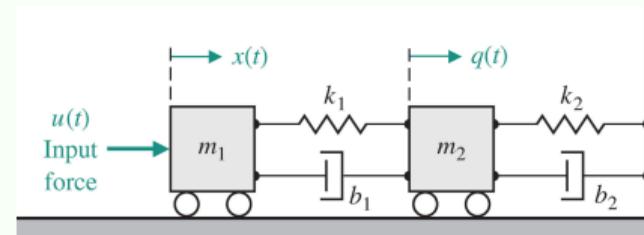
Given is the schematic shown below.



Solve the characteristic equation.

Exercise (#4.6)

Given is the system shown below.



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Determine a state space representation of the system taking $u(t)$ as input and $q(t)$ as output and under the assumption that friction can be neglected. Plot the step and frequency response of the system using NUMPY^a or MATLAB. Simulate the response to a sine wave with an amplitude of 1 N and a frequency of 0.5 Hz and 0.2 Hz, respectively as well.

^aUse $m_1 = 300 \text{ g}$, $m_2 = 100 \text{ g}$, $k_1 = 1 \text{ N m}^{-1}$, $k_2 = 1 \text{ N m}^{-1}$, $b_1 = 0.1 \text{ kg s}^{-1}$, $b_2 = 0.1 \text{ kg s}^{-1}$

Exercise (#4.6)

Given is the system shown below:



Determine a state space representation of the system taking $u(t)$ as input and $q(t)$ as output and under the assumption that friction can be neglected. Plot the step and frequency response of the system using "Numerov" or MATLAB. Simulate the response to a sine wave with an amplitude of 1 N and a frequency of 0.5 Hz and 0.2 Hz, respectively as well.

*Use $m_1 = 100 \text{ g}$, $m_2 = 100 \text{ g}$, $k_1 = 1 \text{ N m}^{-1}$, $k_2 = 1 \text{ N m}^{-1}$, $b_1 = 0.1 \text{ kg s}^{-1}$, $b_2 = 0.1 \text{ kg s}^{-1}$

