Signals and Control Systems 1

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Outline

Signals and Systems
1. Introduction
2. Basic signals and operations
3. LTI systems
4. State variable models

Control
5. Feedback & Control

Discrete Time & Applications
6. Discrete Time
7. Filters
8. Applications and Exercises

Appendix
9. Appendix
10. MATLAB

Table of Contents

Signals and Systems

1. Introduction
2. Basic signals and operations
3. LTI systems
4. State variable models

Table of Contents

Introduction

1.1 General remarks
1.2 Signal Processing
1.3 Control Systems
1.4 Prerequisites
1.5 Outline
1.6 Assesment

Module handbook

Course Structure

- ▷ Basic signals, their properties and operations
- ▷ Linear time-invariant systems
- ▷ State variable models
- ▷ Linear feedback and control systems
- ▷ Sampling theorem and discrete Fourier-transform
- ▷ Structures for discrete time systems
- ▷ Continuous time and discrete time filters (FIR/IIR filter)
- ▷ Applications of the above
- ▷ Signal processing and control engineering with Matlab/Simulink and NUMPY

General remarks

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- Basic signals, their properties and operations
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General remarks

- The term *definition* is also used for important calculation rules and facts and not only in a strict mathematical sense.
- MATLAB and NUMPY are used for exercises. Nevertheless: Exam is pen & paper!

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Literature

- R. Bishop, R. Dorf: Modern Control Systems, Pearson Education, 2010
- Oppenheim, Willsky, Nawab, Signals and Systems, Pearson Education, 2013
- Oppenheim, Schafer, Discrete-Time Signal Processing, Pearson Education, 2014
- Ingle, V.K. and Proakis, J.G.; Digital Signal Processing Using MATLAB, Cengage Learning, 2016
- A. Asif, M. Mandal, Continuous and Discrete Time Signals and Systems, Cambridge University Press, 2007
- A. D. Poularikas, The Handbook of Formulas and Tables for Signal Processing

Table of Contents

Introduction

1.1 General remarks
1.2 Signal Processing
1.3 Control Systems
1.4 Prerequisites
1.5 Outline
1.6 Assesment

Signal Processing

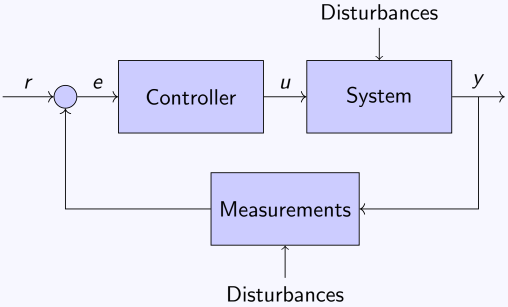
Typical architecture

Table of Contents

Introduction

1.1 General remarks
1.2 Signal Processing
1.3 Control Systems
1.4 Prerequisites
1.5 Outline
1.6 Assesment

Closed loop feedback control



Note: r is called Führungsgröße and y is called Regelgröße.

Table of Contents

Introduction

1.1 General remarks
1.2 Signal Processing
1.3 Control Systems
1.4 Prerequisites
1.5 Outline
1.6 Assesment

Prerequisites

Prerequisites

- Mathematics: Linear algebra, eigenvectors, eigenvalues, Taylor series, Fourier and Laplace transform,...
- NUMPY and/or MATLAB: Beginner level knowledge

Table of Contents

Introduction

1.1 General remarks
1.2 Signal Processing
1.3 Control Systems
1.4 Prerequisites
1.5 Outline
1.6 Assesment

Preliminary outline

Week	Unit	Topic
15	1	Introduction, Basics signals
16	2	LTI
17	3	State variable models
18	4	Feedback & Control
19	5	Exercises
20	–	Block week

Preliminary outline

Week	Unit	Topic
15	1	Introduction, Basics signals
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19	5	Exercises
20	–	Block week

Week	Unit	Topic
21	–	Pentecost
22	6	Exercises
23	7	Discrete Time
24	8	Filters
25	–	No Lecture (FB 10 Team Day)
26	9	Exercises
27	10	Applications
28	11	Exercises

Table of Contents

Introduction

1.1 General remarks
1.2 Signal Processing
1.3 Control Systems
1.4 Prerequisites
1.5 Outline
1.6 Assesment

Assessment of the course

Note: Different than in previous years.

- Written exam at the end of the course (50 points).
- Practical part (50 points).
- Exam scheduled for: 2024-07-15
- Deadline practical part: 2024-08-31

Table of Contents

Signals and Systems

1. Introduction
2. Basic signals and operations
3. LTI systems
4. State variable models

Table of Contents

Basic signals and operations

2.1 Introduction
2.2 Operations
2.3 Basic signals
2.4 Frequency-domain representation
2.5 Signal Properties
2.6 Summary

Introduction

Content

- ▷ Basic signal operations
- ▷ Basic functions
- ▷ Frequency-domain representations
- ▷ Signal properties

Study goals

- ▷ Classify and name some basic functions and their properties
- ▷ Explain the basic concept of the Fourier transform
- ▷ Understand implications of basic signal operations (e.g. time scaling, time-shifting)

Table of Contents

Basic signals and operations

2.1 Introduction

2.2 Operations
- 2.2.1 Convolution
- 2.2.2 Correlation
- 2.2.3 Hilbert transform
- 2.2.4 Real & ideal signals

2.3 Basic signals

2.4 Frequency-domain representation

2.5 Signal Properties

2.6 Summary

Table of Contents

Operations

2.2 Operations
2.2.1 Convolution
2.2.2 Correlation
2.2.3 Hilbert transform
2.2.4 Real & ideal signals

Convolution

Definition

The convolution of two signals $f_1(t)$ and $f_2(t)$ is given by

$$f(t) = f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau.$$

Properties

- An intuitive interpretation is that $f(t)$ is the weighted average of f_1. In this case f_2 is the weighting function.
- Convolution can be used to implement a moving average (filter).
- Examples include smoothing e.g. in image processing

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Properties

$$\begin{aligned} f_1(t) * f_2(t) &= f_2(t) * f_1(t) \\ [f_1(t) * f_2(t)] * f_3(t) &= f_1(t) * [f_2(t) * f_3(t)] \\ [f_1(t) + f_2(t)] * f_3(t) &= f_1(t) * f_3(t) + f_2(t) * f_3(t) \end{aligned}$$

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Exercise (#2.1)

Proof $f_1(t) * f_2(t) = f_2(t) * f_1(t)$

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Operations

2.2 Operations
2.2.1 Convolution
2.2.2 Correlation
2.2.3 Hilbert transform
2.2.4 Real & ideal signals

Correlation

Definition

The cross-correlation between two signals:

$$R_{xy}(t) = (x \star y) \quad = \int_{-\infty}^{\infty} x^*(\tau) y(t+\tau) d\tau = \int_{-\infty}^{\infty} x^*(\tau - t) y(\tau) d\tau$$

with $x^*(t)$ being the conjugate complex of $x(t)$.

Properties

▷ The cross correlation is a measure of similarity of two signals.

Correlation

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Properties

- The cross correlation is a measure of similarity of two signals.

Properties

- $$R_{xy}(t) = R_{yx}(-t)$$
- $$|R_{xy}(t)| \leq \sqrt{R_{xx}(0)R_{yy}(0)} \leq 0.5R_{xx}(0) + 0.5R_{yy}(0)$$
- $$\lim_{t\to\pm\infty} R_{xy}(t) = 0$$

Exercise

Properties

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Exercise (#2.2)

Proof the relationship

$$R_{xy}(t) = x^*(-t) * y(t)$$

between correlation and convolution.

Exercise

Exercise (#2.2)

Proof the relationship

$$R_{xy}(t) \;=\; x^{*}(-t) * y(t)$$

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Table of Contents

Operations

2.2 Operations
2.2.1 Convolution
2.2.2 Correlation
2.2.3 Hilbert transform
2.2.4 Real & ideal signals

Hilbert transform

Definition

The Hilbert transform of a signal $s(t)$ is given by:

$$\mathcal{H}\left\{s(t)\right\} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{s(\tau)}{t-\tau}d\tau.$$

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Properties

$$\mathcal{H}\left\{s(t)\right\} = s(t) * \frac{1}{\pi t}$$

More properties of the Hilbert transform of a signal will be revisited later (see analytic signal).

Operations

2.2 Operations
2.2.1 Convolution
2.2.2 Correlation
2.2.3 Hilbert transform
2.2.4 Real & ideal signals

Real signals

- ▷ limited duration (start, end)
- ▷ continuous (no jumps)
- ▷ differentiable
- ▷ irregular course
- ▷ noisy

Ideal signals

- ▷ signals that can be described with relatively few parameters
- ▷ normally do not fulfill the requirements above
- ▷ example: sinus-function

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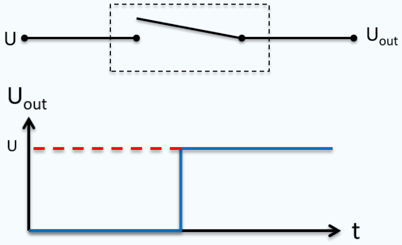
Example

Ideal switch:

As soon as the switch is switched to "on", the voltage jumps to its maximum.



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Ideal signals

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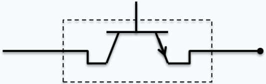
- ▷ normally do not fulfill the requirements above
- ▷ example: sinus-function

Example

Non-ideal switch (e.g. transistor used as a switch):

- ▷ delay and settling time are important parameters to describe a real switch
- ▷ Anyhow, using an ideal step function is feasible in most cases
- ▷ In the following you will learn how to describe the step function
- ▷ You will also learn how to describe the ideal switch as a system and derive measures to asses the impacts of non-ideal switching



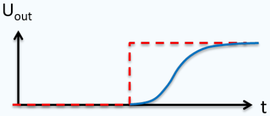


Table of Contents

Basic signals and operations

2.1 Introduction
2.2 Operations
2.3 Basic signals
2.4 Frequency-domain representation
2.5 Signal Properties
2.6 Summary

Definition

The ramp function is defined as follows:

$$f(t) = \frac{1}{2}(t + |t|)$$

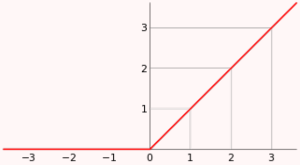


Figure: Wikipedia

Step function

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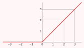

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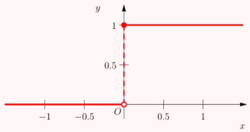
Definition

The Heaviside step function is defined as follows:

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 0.5 & \text{for } t = 0 \\ 1 & \text{else} \end{cases}$$



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Properties

The Heaviside step function is the derivative of the ramp function.

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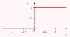

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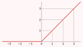
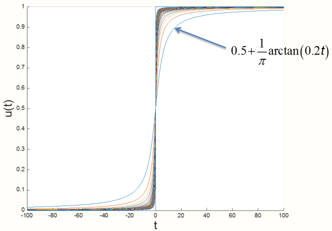


Figure: Wikipedia

Properties

The step function can be written as the limit of a sequence of functions (distributions):

$$u(t) = \lim_{n\to\infty} f_n(t) = 0.5 + \lim_{n\to\infty} \tan^{-1}(n \cdot a \cdot t)$$



Dirac delta function

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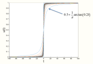

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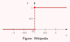

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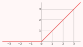


Figure: Wikipedia

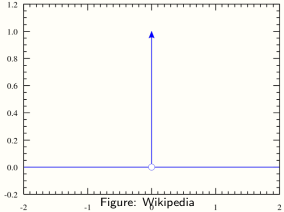
Properties

The derivative of the step function is the Dirac delta function:

$$\delta(t) = \begin{cases} \infty & \text{for } t = 0 \\ 0 & \text{else} \end{cases},$$

with

$$\int_{-\infty}^{\infty} \delta(at)dt = \frac{1}{|a|}$$



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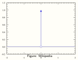

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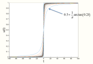

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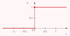

Figure: Wikipedia

Properties

The Heaviside step function is the derivative of the ramp function.

Properties

$$\begin{aligned} \int_{-\infty}^{\infty} f(t)\delta(t-T)dt &= f(T) \\ f(t)\delta(t-T) &= f(T)\delta(t-T) \\ f(t) * \delta(t-T) &= f(t-T) \end{aligned}$$

The Dirac delta function is a very useful tool to describe e.g. time-shifting and sampling (Dirac comb).

Rect function

Properties

$$\int_{-\infty}^{\infty} f(t)\delta(t-T)dt = f(T)$$
$$f(t)\delta(t-T) = f(T)\delta(t-T)$$
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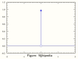

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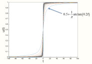


Properties

The rectangular $(\mathrm{rect}\,(t))$ function:

$$\mathrm{rect}\,(t) = \begin{cases} 1 & \text{for } |t| < 0.5 \\ 0.5 & \text{for } |t| = 0.5 \\ 0 & \text{else} \end{cases},$$

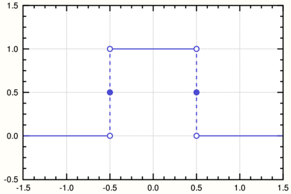
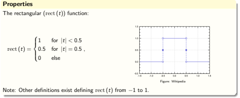
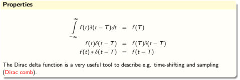


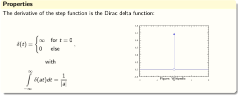
Figure: Wikipedia

Note: Other definitions exist defining $\mathrm{rect}\,(t)$ from -1 to 1.

Triangular function







Properties

The triangle function:

$$\mathrm{tria}\,(t) = \begin{cases} 1 - |t| & \text{for } |t| < 1 \\ 0 & \text{else} \end{cases},$$

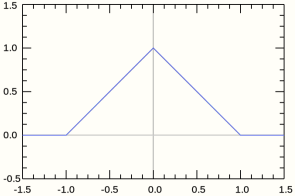


Figure: Wikipedia

Note: Other definitions exist defining $\mathrm{tria}\,(t)$ from -2 to 2.

Table of Contents

Basic signals and operations

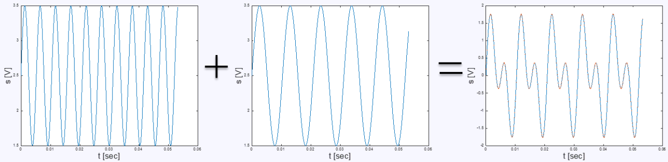
2.1 Introduction
2.2 Operations
2.3 Basic signals
2.4 Frequency-domain representation
2.4.1 Introduction
2.4.2 Fourier transform
2.4.3 Laplace transform
2.4.4 Relationship between Fourier and Laplace transform
2.4.5 Other transforms
2.5 Signal Properties
2.6 Summary

Table of Contents

Frequency-domain representation

2.4 Frequency-domain representation
2.4.1 Introduction
2.4.2 Fourier transform
2.4.3 Laplace transform
2.4.4 Relationship between Fourier and Laplace transform
2.4.5 Other transforms

Signal decomposition



In this block we will talk about:

- ▷ Is it possible to decompose arbitrary functions into sinus functions?
- ▷ What are the foundations of the "network analysis" approach?
- ▷ Is it possible to extend the "network analysis" approach to more general systems?

Table of Contents

Frequency-domain representation

2.4 Frequency-domain representation
2.4.1 Introduction
2.4.2 Fourier transform
2.4.3 Laplace transform
2.4.4 Relationship between Fourier and Laplace transform
2.4.5 Other transforms

Fourier transform

Definition

The inverse Fourier transform is given by

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega = \mathcal{F}^{-1}\{f(t)\}$$

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Properties

- The Fourier transform always exists for real-world signals. For idealized signals, the following condition is sufficient (absolute integrable function with l_1-norm existing):

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$$

- The Fourier transform vanishes for infinity (Riemann–Lebesgue lemma). I.e.

$$\lim_{|\omega|\to\infty} F(\omega) = 0$$

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Hint:

Slightly different definitions of the Fourier transform exist. Examples:

$$F(f) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi ft}dt = \mathcal{F}\{f(t)\}$$

and

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = \mathcal{F}\{f(t)\}$$

The one we use is called "Fourier transform non-unitary, angular frequency" e.g. in the English Wikipedia. Be sure to use tables for the correct definition if you apply the Fourier transform.

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The Fourier transform of $e^{j\omega_0 t}$:

$$\begin{aligned} \mathcal{F}\left\{e^{j\omega_0 t}\right\} &= \int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t} dt = \mathcal{F}\{f(t)\} \\ &= 2\pi\delta(\omega - \omega_0) \end{aligned}$$

Fourier transform

Time and frequency domain

Time domain signal description: f(t)

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \mathrm{F}\{f(t)\}$$

Frequency domain signal description: F(ω)

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This diagram will be be extended when we discuss systems!

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Compare the Fourier transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \mathcal{F}\left\{f(t)\right\}$$

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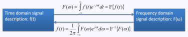

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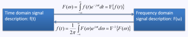

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\mathcal{F}\left\{f_1(t)f_2(t)\right\} &= \frac{1}{2\pi}F_1(\omega)*F_2(\omega) \\
\mathcal{F}\left\{f_1(t)*f_2(t)\right\} &= F_1(\omega)F_2(\omega) \\
\mathcal{F}^{-1}\left\{F(\omega-\omega_0)\right\} &= e^{j\omega_0 t}\mathcal{F}^{-1}\left\{F(\omega)\right\}=e^{j\omega_0 t}f(t) \\
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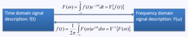

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Fourier transform and Correlation

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Notes:

- ▷ The tri-function is the triangular function defined from -1 to 1.
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Fourier transform

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Calculate the relationship between $S(\omega)$ and $S(-\omega)$ for
- a real signal (no imaginary part)
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Exercise (#2.4)

Calculate the Fourier transform of the following functions (no need to solve the integral):

- $$x(t)=\mathrm{rect}\left(\frac{t-t_0}{T}\right)$$
- $$x(t)=\mathrm{rect}\left(\frac{t-t_0}{T}\right)\sin(\omega_0 t)$$
- $$x(t)=\sin(\omega_0 t)u(t)e^{-\frac{t}{\tau}}$$

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- an imaginary signal (no real part)

Properties: Dirac comb

- $$\mathcal{F}\left\{\sum_{n=-\infty}^{\infty}\delta(t-nT)\right\} = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi k}{T}\right)$$
- $\sum_{n=-\infty}^{\infty}\delta(t-nT)$ is the so-called Dirac comb and will be used later when we discuss sampling.

Exercise (#2.5)

Calculate the following convolutions directly in the time domain and by making use of the frequency domain:

- $$f(t) = (\mathrm{rect} * \mathrm{rect})(t) = \mathrm{rect}\,(t) * \mathrm{rect}\,(t)$$
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Exercise (#2.4)
Calculate the Fourier transform of the following functions (no need to solve the integral):

- $$x(t) = \mathrm{rect}\left(\frac{t - t_0}{T}\right)$$
- $$x(t) = \mathrm{rect}\left(\frac{t - t_0}{T}\right) \sin(\omega_0 t)$$
- $$x(t) = \sin(\omega_0 t) u(t) e^{-\frac{t}{\tau}}$$

Exercise (#2.3)
Calculate the relationship between $S(\omega)$ and $S(-\omega)$ for
- a real signal (no imaginary part)
- an imaginary signal (no real part)

Exercise (#2.6)

Calculate the cross correlation

$$R_{xy}(\tau) = (x(t) \star y(t))(\tau)$$

for the following signals:

- $$x(t) = \mathrm{rect}\,(t)\,, y(t) = \mathrm{rect}\,(t)$$
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Analytic signal

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Definition

An analytic signal $s_a(t)$ is a complex-valued function that has no negative frequency components.

Properties

Applying the Fourier transform to an Hilbert transformed signal:

$$\mathcal{F}\left\{\mathcal{H}\left\{s(t)\right\}\right\} = -j \cdot \mathrm{sgn}\left(\omega\right)\mathcal{F}\left\{s(t)\right\}(\omega)$$

Thus one can calculate the analytic signal by making use of the Hilbert transform:

$$\begin{aligned} s_a(t) &= s(t) + j\mathcal{H}\left\{s(t)\right\} \\ S_a(\omega) &= S(\omega) + \mathrm{sgn}\left(\omega\right)S(\omega) \end{aligned}$$

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Example

Let $s(t) = \sin(\omega_0 t)$, then

$$\begin{aligned} S(\omega) &= -j\pi\left(\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right) \\ S_a(\omega) &= -j\pi\left(\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right) \\ &\quad + \left[-j\pi\left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right)\right] \\ &= -j\pi\left(\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right) + \delta(\omega-\omega_0) - \delta(\omega+\omega_0) \\ &= -j\pi\delta(2\omega-\omega_0) \end{aligned}$$

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An analytic signal can be expressed in terms of time-variant magnitude and phase angle:

$$s_a(t) = s_m(t)e^{j\Phi(t)},$$

with

$$\begin{aligned}
s_m(t) &= |s_a(t)| \\
&\text{and} \\
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Exercise (#2.7)

Show that for an amplitude modulated signal[a]

$$s_{\mathrm{am}}(t) = \sin(\omega_0 t)a(t)$$

the Hilbert transform can be used to find the envelope.

[a] More specifically: Double-sideband suppressed-carrier modulation

Exercise

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Table of Contents

Frequency-domain representation

2.4 Frequency-domain representation
- 2.4.1 Introduction
- 2.4.2 Fourier transform
- 2.4.3 Laplace transform
- 2.4.4 Relationship between Fourier and Laplace transform
- 2.4.5 Other transforms

Introduction

Step function

Example

Consider this simple example of switching a signal at $t = 0$.

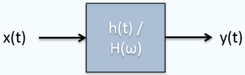


Figure 1: A system

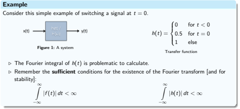
$$h(t) = \begin{cases} 0 & \text{for } t < 0 \\ 0.5 & \text{for } t = 0 \\ 1 & \text{else} \end{cases}$$

Transfer function

- The Fourier integral of $h(t)$ is problematic to calculate.
- Remember the **sufficient** conditions for the existence of the Fourier transform [and for stability]:

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty \qquad \int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

Step function



Introduction

Example

The way we solved such issues has been the use of sequences of functions (distributions):

$$h(t) = \lim_{n \to 0} \frac{1}{1 + e^{-t/n}}$$

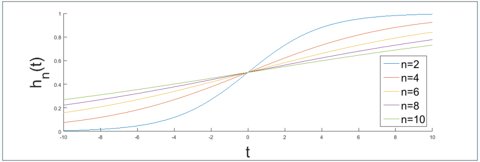
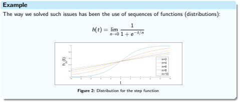
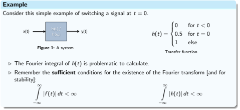


Figure 2: Distribution for the step function

Step function





Introduction

Example

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$$h(t) = \lim_{n\to 0} u(t)e^{-t/n}$$

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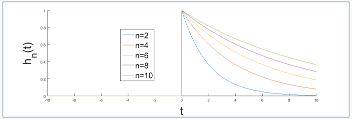

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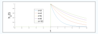

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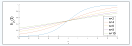

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Laplace transform

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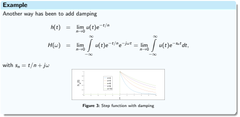
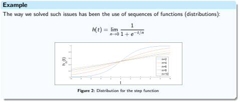




Definition

- The unilateral (or one-sided) Laplace transform is defined by

$$\mathcal{L}\left\{s(t)\right\} = \int_{0}^{\infty} s(t)e^{-st}dt = F(s),$$

$$s = \sigma + j\omega$$

- The set of values for which $\mathcal{L}\left\{s(t)\right\}$ converges is called the region of convergence (ROC) (half space in case of one-sided transform)

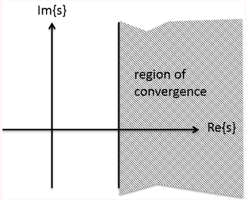
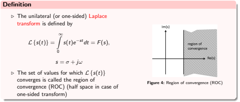


Figure 4: Region of convergence (ROC)

Inverse Laplace transform



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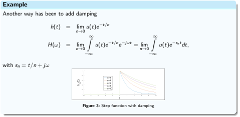
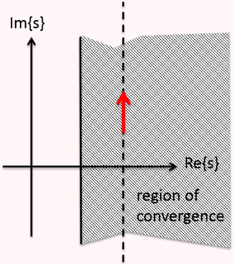


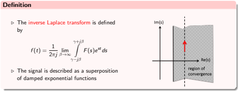
Definition

- The inverse Laplace transform is defined by

$$f(t) = \frac{1}{2\pi j} \lim_{\beta\to\infty} \int_{\gamma-j\beta}^{\gamma+j\beta} F(s)e^{st}\,ds$$

- The signal is described as a superposition of damped exponential functions

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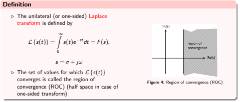
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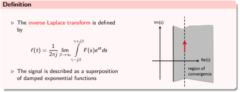

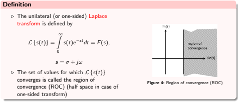

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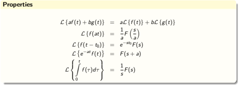
Laplace transform



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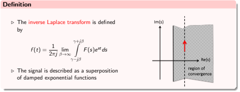

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The integration is done along the vertical line $\Re(\sigma) = c$ that lies entirely within the region of convergence of F.

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\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} &= \frac{1}{s}F(s)
\end{aligned}$$

Example

Consider

$$\begin{aligned}
\mathcal{L}\left\{f(t) = e^{-at}\right\} &= \frac{1}{s+a} \\
\lim_{s\to 0} sF(s) &= \lim_{s\to 0}\frac{s}{s+a} = 0
\end{aligned}$$

If $\lim_{t\to\infty} f(t)$ exists, then

$$\begin{aligned}
\lim_{t\to\infty} f(t) &= \lim_{s\to 0} sF(s) \\
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This holds true only for $\Re\{a\} > 0$ (Pole in left half space).

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Properties

$$\begin{aligned}\mathcal{L}\left\{\frac{\partial}{\partial t}f(t)\right\} &= sF(s)-f(0_+)\\ \mathcal{L}\left\{-tx(t)\right\} &= \frac{\partial}{\partial s}X(s)\\ f(0_+) &= \lim_{d\to 0} f(|d|)\\ f(0) &= \frac{1}{2}f(0_+)\\ \lim_{t\to\infty} f(t) &= \lim_{s\to 0} sF(s)\ (\text{if}\ \lim_{t\to\infty} f(t)\ \text{exists})\end{aligned}$$

Properties

$$\mathcal{L}\left\{f(t)g(t)\right\} = \frac{1}{2\pi j}\lim_{T\to\infty}\int_{c-jT}^{c+jT} F(\sigma)G(s-\sigma)d\sigma$$

The integration is done along the vertical line $\Re(\sigma)=c$ that lies entirely within the region of convergence of F.

Properties

$$\begin{aligned}\mathcal{L}\left\{\delta(t)\right\} &= 1\\ \mathcal{L}\left\{u(t)\right\} &= \frac{1}{s},\ \text{for}\ \Re\{s\}>0\\ \mathcal{L}\left\{t\right\} &= \frac{1}{s^2},\ \text{for}\ \Re\{s\}>0\\ \mathcal{L}\left\{e^{-at}\right\} &= \frac{1}{s+a}\end{aligned}$$

Note: Unilateral Laplace transform

Properties

$$\mathcal{L}\{\delta(t)\} = 1$$
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Properties

$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$$
$$\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}$$
$$\mathcal{L}\{\sin(\omega t)e^{-at}\} = \frac{\omega}{(s+a)^2+\omega^2}$$
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Exercise

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Exercise (#2.8)

Calculate the unilateral Laplace transforms of the following signals:

- ▷ $\delta(t)$
- ▷ $tu(t)$
- ▷ $u(t)e^{-at}$
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Note:

$$\int te^{-st}dt = -\frac{e^{-st}(st+1)}{s^2} + C$$

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$$\begin{aligned}
\mathcal{L}\{\delta(t)\} &= 1 \\
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Exercise (#2.9)

Calculate the inverse Laplace transforms of the following signals:

- ▷ $X(s) = \frac{2s+1}{s+2}$, for $\Re(s) > -2$
- ▷ $X(s) = \frac{s^3+2s^2+6}{s^2+3s}$, for $\Re(s) > 0$

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Table of Contents

Frequency-domain representation

2.4 Frequency-domain representation
2.4.1 Introduction
2.4.2 Fourier transform
2.4.3 Laplace transform
2.4.4 Relationship between Fourier and Laplace transform
2.4.5 Other transforms

Laplace and Fourier

Properties

If

$f(t < 0) = 0$

and if

the imaginary axis belongs to the ROC,

then

$$\mathcal{L}\left\{f(t)\right\}(s = j\omega) = \mathcal{F}\left\{f(t)\right\}$$

Note: Frequently, the Fourier transform is written as follows:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}$$

Frequency-domain representation

2.4 Frequency-domain representation
2.4.1 Introduction
2.4.2 Fourier transform
2.4.3 Laplace transform
2.4.4 Relationship between Fourier and Laplace transform
2.4.5 Other transforms

Other transforms

Continuous transforms (discrete counterparts exist)

- Fourier series (esp. periodic signals)
- Sine and cosine transforms (in case of symmetric signals)
- Short-time Fourier transform (spectrogram)
- Gabor-Transform (e.g. computer vision)
- Wavelet transform (e.g. JPEG)

Discrete transforms discussed in later chapters

- Discrete Fourier transform
- Z-transform

Table of Contents

Basic signals and operations

2.1 Introduction
2.2 Operations
2.3 Basic signals
2.4 Frequency-domain representation
2.5 Signal Properties
  2.5.1 Energy
  2.5.2 Bandwidth
  2.5.3 Mean square error
2.6 Summary

Table of Contents

Signal Properties

2.5 Signal Properties
2.5.1 Energy
2.5.2 Bandwidth
2.5.3 Mean square error

Energy

Definition

In signal processing, the energy E_s of a continuous-time signal $s(t)$ is defined as follows:

$$E_s = \int_{-\infty}^{\infty} |s(t)|^2 \, dt = \int_{-\infty}^{\infty} f^*(\tau) f(\tau) d\tau.$$

Parseval Equation

Definition

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$$E_s = \int_{-\infty}^{\infty} |s(t)|^2 \, dt = \int_{-\infty}^{\infty} f^*(\tau) f(\tau) d\tau.$$

Energy in time and frequency domain

The Parseval equation:

$$E_s = \int_{-\infty}^{\infty} f^*(\tau) f(\tau) d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{F^*(\omega) F(\omega)}_{\text{spectral energy density}} \, d\omega$$

The energy of a signal can be calculated either in the time or in the frequency domain. Note that using different definitions of the Fourier Transform makes the factor $\frac{1}{2\pi}$ obsolete.

Signal Properties

2.5 Signal Properties
2.5.1 Energy
2.5.2 Bandwidth
2.5.3 Mean square error

Bandwidth

Definition

The bandwidth of a signal can be defined as follows: The bandwidth is the range of positive frequencies in which *most* of the energy of the signal lies. A common definition is the 3 dB bandwidth $W_{3\mathrm{dB}}$ based on the power spectrum of the signal.

Definition

A signal is called band-limited if

$$|X(\omega)| = 0 \text{ for } |\omega| > \omega_M.$$

For band-limited signals, it is natural to define ω_M as the bandwidth.

Rect function

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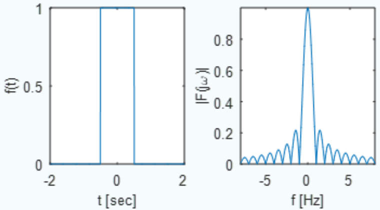
Example

$$\mathcal{F}\{\mathrm{rect}(at)\} = \frac{1}{|a|}\mathrm{sinc}\left(\frac{\omega}{2\pi a}\right)$$



Note: Different definitions for the sinc−function do exist.
Here: $\mathrm{sinc} = \frac{\sin(\pi x)}{\pi x}$

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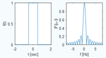

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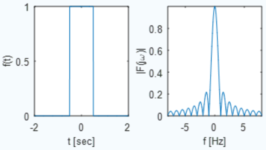


Figure 5: $\mathrm{rect}\,(t)$-function

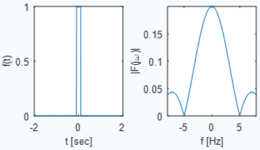
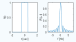


Figure 6: $\mathrm{rect}\,(5t)$-function

Corollary

- ▷ A band-limited signal cannot be time-limited
- ▷ A time-limited signal cannot be band-limited

Example



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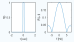

Figure 6: rect $(5t)$-function

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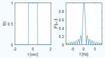

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For band-limited signals, it is natural to define ω_M as the bandwidth.

Exercise (#2.10)

Given is the modulated signal[a] $s(t) = c(t)m(t)$ with a carrier

$$c(t) = A\cos(\omega_c t)$$

and a signal message

$$m(t) = \text{sinc}(t) + \text{sinc}^2(t).$$

- Find the frequency representation of $s(t)$
- Find the bandwidth of the modulated signal.

[a] so called double sideband amplitude modulation

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Table of Contents

Signal Properties

2.5 Signal Properties
2.5.1 Energy
2.5.2 Bandwidth
2.5.3 Mean square error

Mean square error

Let

$$e(t) = f_1(t) - f_2(t)$$

be an error function. One can then calculate the mean square error either in the time or in the frequency domain:

$$\int_{-\infty}^{\infty} |f_1(\tau) - f_2(\tau)|^2 \, d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F_1(\omega) - F_2(\omega)|^2 \, d\omega$$

Table of Contents

Basic signals and operations

2.1 Introduction
2.2 Operations
2.3 Basic signals
2.4 Frequency-domain representation
2.5 Signal Properties
2.6 Summary

Summary

After this block...

you should be able to understand the implications of the following basic operations in the time and in the frequency domain:

- ▷ Amplitude scaling of signals
- ▷ Addition and multiplication of signals
- ▷ Differentiation of signals
- ▷ Integration of signals
- ▷ Time scaling of signals
- ▷ Reflection of signals
- ▷ Time-shifting of signals
- ▷ Convolution
- ▷ Energy of signals

Table of Contents

Signals and Systems

1. Introduction
2. Basic signals and operations
3. LTI systems
4. State variable models

Table of Contents

LTI systems

3.1 Introduction
3.2 Mathematical foundations: Differential equations
3.3 LTI systems
3.4 Transfer function
3.5 Impulse response
3.6 Causality
3.7 Stability
3.8 Network of systems
3.9 Graphical representation

Introduction

Content

- ▷ Differential equations
- ▷ LTI systems
- ▷ Frequency-domain representations

Study goals

- ▷ Describe LTI systems in time and frequency domain
- ▷ Use the concepts of transfer functions and impulse response
- ▷ Discuss stability of LTI systems
- ▷ Make use of Bode plots

Content
- Differential equations
- LTI systems
- Frequency-domain representations

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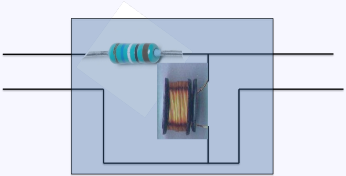
What is a system?



- A system is “something” with input and output
- The system “reacts” to its environment
- Usually, models are used for the description of systems. I.e. real-world behavior is simplified

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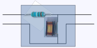

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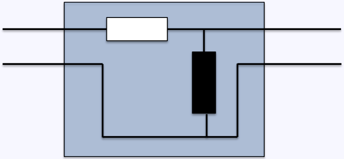

- Here we can define voltages as input and output
- A simple model for this circuit is shown above
- The output voltage can be calculated as follows (for open terminals):

$$U_{out} = U_{in} \frac{j\omega L}{R + j\omega L}.$$

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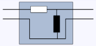

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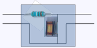

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Physical systems...

... frequently have a dynamic behaviour that can be described by differential equations

... might be non-linear

... can often be linearized by making use of the Taylor series / Jacobi-matrix

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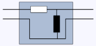

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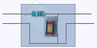

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Network analysis

- In network analysis, we normally make the following assumptions:
  - Networks are linear
  - Networks are time invariant
- Analysis is usually done in the frequency domain:

$$U_{out} = U_{in}\frac{j\omega L}{R + j\omega L}.$$

- The fractional term can be interpreted as a transfer function $H(\omega)$: The frequency dependent reaction of the system to the input.

Table of Contents

LTI systems

3.1 Introduction
3.2 Mathematical foundations: Differential equations
3.2.1 Definitions
3.2.2 Second-order differential equation
3.3 LTI systems
3.4 Transfer function
3.5 Impulse response
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Mathematical foundations: Differential equations

3.2 Mathematical foundations: Differential equations
3.2.1 Definitions
3.2.2 Second-order differential equation

Ordinary differential equation

Definition

A ordinary differential equation is an equality involving one or more dependent variables, one independent variable, and one or more derivatives of the dependent variables, with respect to the independent variable.

Williams et al., Feedback and Control Systems

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Definition

An equation of the form

$$\sum_{i=0}^{n} a_i \frac{d^i}{dt^i} y(t) = \sum_{i=0}^{m} b_i \frac{d^i}{dt^i} u(t)$$

is called a linear constant-coefficient ordinary differential equation.

Definition

The initial condition is a set of equations of the form

$$y(0) = A_0,\ \frac{d}{dt} y(t=0) = A_1, \cdots \frac{d^{n-1}}{dt^{n-1}} y(t=0) = A_{n-1}.$$

Free response

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A ordinary differential equation is an equality involving one or more dependent variables, one independent variable, and one or more derivatives of the dependent variables, with respect to the independent variable.
Williams et al., Feedback and Control Systems

Definition

The solution for an input $u(t) = 0$ is called the free response. The differential equation then has the form:

$$\sum_{i=0}^{n} a_i \frac{d^i}{dt^i} y(t) = 0.$$

Definition

The free response is given by a linear combination of the so-called fundamental set:

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Definition
The free response is given by a linear combination of the so-called fundamental set:

$$y(t) = \sum_{i=1}^{n} C_i y_i(t).$$

Definition
An equation of the form

$$\sum_{i=0}^{n} a_i \frac{d^i}{dt^i} y(t) = \sum_{i=0}^{m} b_i \frac{d^i}{dt^i} u(t)$$

is called a linear constant-coefficient ordinary differential equation.

Definition
The initial condition is a set of equations of the form

$$y(0) = A_0,\ \frac{d}{dt}y(t=0) = A_1, \cdots \frac{d^{n-1}}{dt^{n-1}}y(t=0) = A_{n-1}.$$

Definition
A ordinary differential equation is an equality involving one or more dependent variables, one independent variable, and one or more derivatives of the dependent variables, with respect to the independent variable.
Williams et al., Feedback and Control Systems

Properties

The coefficients C_i are determined by the initial condition

$$y(0) = A_0,\ \frac{d}{dt}y(t=0) = A_1, \cdots \frac{d^{n-1}}{dt^{n-1}}y(t=0) = A_{n-1}.$$

Free response

Properties

The coefficients C_i are determined by the initial condition

$$y(0) = A_0,\ \frac{d}{dt}y(t=0) = A_1, \cdots \frac{d^{n-1}}{dt^{n-1}}y(t=0) = A_{n-1}.$$

Definition

The solution for an input $u(t) = 0$ is called the free response. The differential equation then has the form:

$$\sum_{i=0}^{n} a_i \frac{d^i}{dt^i} y(t) = 0.$$

Definition

The free response is given by a linear combination of the so-called fundamental set:

$$y(t) = \sum_{i=1}^{n} C_i y_i(t).$$

Definition

An equation of the form

$$\sum_{i=0}^{n} a_i \frac{d^i}{dt^i} y(t) = \sum_{i=0}^{m} b_i \frac{d^i}{dt^i} u(t)$$

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Definition

The initial condition is a set of equations of the form

$$y(0) = A_0,\ \frac{d}{dt}y(t=0) = A_1, \cdots \frac{d^{n-1}}{dt^{n-1}}y(t=0) = A_{n-1}.$$

Example

$$\dot{x}(t) = C_1 x(t)$$

has the solution

$$x = A_0 e^{C_1 t}$$

in case of $x(t=0) = A_0$.

Forced response

Example

$$\dot{x}(t) = C_1 x(t)$$

has the solution

$$x = A_0 e^{C_1 t}$$

in case of $x(t=0) = A_0$.

Properties

The coefficients C_i are determined by the initial condition

$$y(0) = A_0,\ \frac{d}{dt}y(t=0) = A_1, \cdots \frac{d^{n-1}}{dt^{n-1}}y(t=0) = A_{n-1}.$$

Definition

The solution for an input $u(t) = 0$ is called the free response. The differential equation then has the form:

$$\sum_{i=0}^{n} a_i \frac{d^i}{dt^i} y(t) = 0.$$

Definition

The free response is given by a linear combination of the so-called fundamental set:

$$y(t) = \sum_{i=1}^{n} C_i y_i(t).$$

Definition

The forced response of a system described by an equation of the form

$$\sum_{i=0}^{n} a_i \frac{d^i}{dt^i} y(t) = \sum_{i=0}^{m} b_i \frac{d^i}{dt^i} u(t)$$

is the solution for an initial condition of zero:

$$y(0) = 0,\ \frac{d}{dt}y(t=0) = 0, \cdots \frac{d^{n-1}}{dt^{n-1}}y(t=0) = 0.$$

Total response

Definition

The forced response of a system described by an equation of the form

$$\sum_{i=0}^{n} a_i \frac{d^i}{dt^i} y(t) = \sum_{i=0}^{m} b_i \frac{d^i}{dt^i} u(t)$$

is the solution for an initial condition of zero:

$$y(0) = 0, \ \frac{d}{dt} y(t=0) = 0, \cdots \frac{d^{n-1}}{dt^{n-1}} y(t=0) = 0.$$

Example

$$\dot{x}(t) = C_1 x(t)$$

has the solution

$$x = A_0 e^{C_1 t}$$

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Properties

The coefficients C_i are determined by the initial condition

$$y(0) = A_0, \ \frac{d}{dt} y(t=0) = A_1, \cdots \frac{d^{n-1}}{dt^{n-1}} y(t=0) = A_{n-1}.$$

Definition

The total response of a system described by an equation of the form

$$\sum_{i=0}^{n} a_i \frac{d^i}{dt^i} y(t) = \sum_{i=0}^{m} b_i \frac{d^i}{dt^i} u(t)$$

is the sum of the free response and the forced response.

Unit step response

Definition

The total response of a system described by an equation of the form

$$\sum_{i=0}^{n} a_i \frac{d^i}{dt^i} y(t) = \sum_{i=0}^{m} b_i \frac{d^i}{dt^i} u(t)$$

is the sum of the free response and the forced response.

Definition

The forced response of a system described by an equation of the form

$$\sum_{i=0}^{n} a_i \frac{d^i}{dt^i} y(t) = \sum_{i=0}^{m} b_i \frac{d^i}{dt^i} u(t)$$

is the solution for an initial condition of zero:

$$y(0) = 0,\ \frac{d}{dt} y(t=0) = 0, \cdots \frac{d^{n-1}}{dt^{n-1}} y(t=0) = 0.$$

Example

$$\dot{x}(t) = C_1 x(t)$$

has the solution

$$x = A_0 e^{C_1 t}$$

in case of $x(t=0) = A_0$.

Definition

The unit step response of a system described by an equation of the form

$$\sum_{i=0}^{n} a_i \frac{d^i}{dt^i} y(t) = \sum_{i=0}^{m} b_i \frac{d^i}{dt^i} u(t)$$

and is the forced response in case of $u(t)$ being the step function.

Table of Contents

Mathematical foundations: Differential equations

3.2 Mathematical foundations: Differential equations
3.2.1 Definitions
3.2.2 Second-order differential equation

Second-order diff. eq.

Definition

The system described by the second-order differential equation

$$\frac{d^2}{dt^2}y(t) + 2\sigma\omega_0\frac{d}{dt}y(t) + \omega_0^2 y(t) = \omega_0^2 u(t)$$

has the unit step response

$$\begin{aligned} y(t) &= 1 - \frac{e^{-\sigma\omega_0 t}}{\sqrt{1-\sigma^2}}\sin\left(\omega_0\sqrt{1-\sigma^2}t + \phi_0\right), \\ \text{with} & \\ \phi_0 &= \tan^{-1}\left(\frac{\sqrt{1-\sigma^2}}{\sigma}\right) \end{aligned}$$

Definition

The system described by the second-order differential equation

$$\frac{d^2}{dt^2}y(t) + 2\sigma\omega_0\frac{d}{dt}y(t) + \omega_0^2 y(t) = \omega_0^2 u(t)$$

has the unit step response

$$y(t) = 1 - \frac{e^{-\sigma\omega_0 t}}{\sqrt{1-\sigma^2}}\sin\left(\omega_0\sqrt{1-\sigma^2}t + \phi_0\right),$$

with

$$\phi_0 = \tan^{-1}\left(\frac{\sqrt{1-\sigma^2}}{\sigma}\right)$$

Exercise (#3.1)

Plot the unit step response ($x(t) = u(t)$, $y(t \leq 0) = 0$) of a system described by the second-order differential equations

$$\frac{d^2}{dt^2}y(t) + 2\sigma\omega_0\frac{d}{dt}y(t) + \omega_0^2 y(t) = \omega_0^2 x(t)$$

for different values of ω_0 and σ by making of NUMPY or MATLAB. Use the equation shown on the former slide.

- ▷ What happens in case of negative σ?
- ▷ What happens if you set $\phi_0 = 0$? Does the result fulfill the requirements of a unit step response (check initial conditions)?

Exercise

Exercise (#3.1)

Plot the unit step response ($x(t) = u(t)$, $y(t \leq 0) = 0$) of a system described by the second-order differential equations

$$\frac{d^2}{dt^2}y(t) + 2\sigma\omega_0\frac{d}{dt}y(t) + \omega_0^2 y(t) = \omega_0^2 x(t)$$

for different values of ω_0 and σ by making of NUMPY or MATLAB. Use the equation shown on the former slide.

- What happens in case of negative σ?
- What happens if you set $\phi_0 = 0$? Does the result fulfill the requirements of a unit step response (check initial conditions)?

Reduction to second order

Exercise (#3.1)

Plot the unit step response ($x(t) = u(t)$, $y(t \leq 0) = 0$) of a system described by the second-order differential equations

$$\frac{d^2}{dt^2}y(t) + 2\sigma\omega_0\frac{d}{dt}y(t) + \omega_0^2 y(t) = \omega_0^2 x(t)$$

for different values of ω_0 and σ by making of NUMPY or MATLAB. Use the equation shown on the former slide.

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- What happens if you set $\phi_0 = 0$? Does the result fulfill the requirements of a unit step response (check initial conditions)?

Definition

The system described by the second-order differential equation

$$\frac{d^2}{dt^2}y(t) + 2\sigma\omega_0\frac{d}{dt}y(t) + \omega_0^2 y(t) = \omega_0^2 u(t)$$

has the unit step response

$$y(t) = 1 - \frac{e^{-\sigma\omega_0 t}}{\sqrt{1-\sigma^2}}\sin\left(\omega_0\sqrt{1-\sigma^2}t + \phi_0\right),$$

with

$$\phi_0 = \tan^{-1}\left(\frac{\sqrt{1-\sigma^2}}{\sigma}\right)$$

Spoiler

We will later discuss how to reduce higher order differential equations to first/second order.

LTI systems

3.1 Introduction
3.2 Mathematical foundations: Differential equations
3.3 LTI systems
3.4 Transfer function
3.5 Impulse response
3.6 Causality
3.7 Stability
3.8 Network of systems
3.9 Graphical representation

LTI Systems

Definition

In the following we consider LTI-systems (**L**inear **T**ime-**I**nvariant)

- One can show that if we stimulate an LTI system with an input

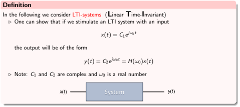
$$x(t) = C_1 e^{j\omega_0 t}$$

  the output will be of the form

$$y(t) = C_2 e^{j\omega_0 t} = H(\omega_0)x(t)$$

- Note: C_1 and C_2 are complex and ω_0 is a real number

Introduction



Multiple (additive) inputs

We will now write the signals in slightly different form:

$$
\begin{aligned}
x(t) &= C_1 e^{j\omega t} = \frac{1}{2\pi} X(\omega) e^{j\omega t} \\
y(t) &= C_2 e^{j\omega t} = \frac{1}{2\pi} X(\omega) H(\omega) e^{j\omega t}
\end{aligned}
$$

The transfer function: Relationship between input and output in the frequency domain. For N-input signals:

$$
\begin{aligned}
x(t) &= \sum_i^N \frac{1}{2\pi} X_i(\omega_i) e^{j\omega_i t} \\
y(t) &= \sum_i^N \frac{1}{2\pi} X_i(\omega_i) H(\omega_i) e^{j\omega_i t}
\end{aligned}
$$

Table of Contents

LTI systems

3.1 Introduction
3.2 Mathematical foundations: Differential equations
3.3 LTI systems
3.4 Transfer function
3.5 Impulse response
3.6 Causality
3.7 Stability
3.8 Network of systems
3.9 Graphical representation

Transfer function

In more general form:

$$x(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} X(\omega) e^{j\omega t} d\omega$$

$$y(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} X(\omega) H(\omega) e^{j\omega t} d\omega$$

It becomes obvious that the output signal is given by an inverse Fourier-Transform:

$$y(t) = \mathcal{F}^{-1}\left\{X(\omega)H(\omega)\right\}.$$

Transfer function

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It becomes obvious that the output signal is given by an inverse Fourier-Transform:

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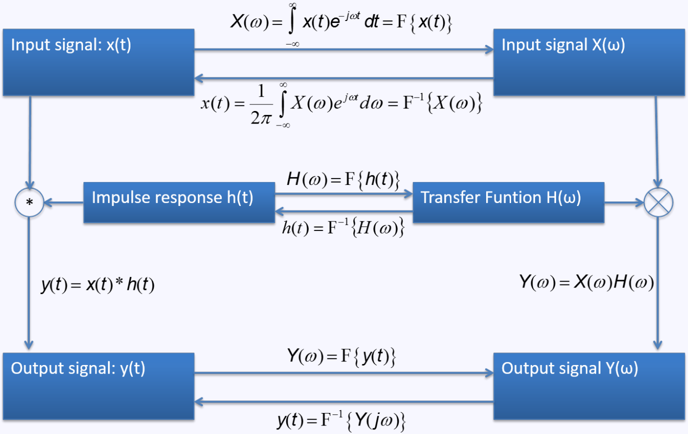
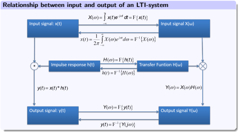
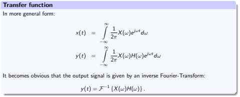
Relationship between input and output of an LTI-system



Using the Laplace tansform





Definition

Using the Laplace Transform, the output of an LTI system is given by

$$Y(s) = H(s)X(s) + Y_{init}(s),$$

with $Y_{init}(s)$ being terms due to all initial conditions. If one ignores all terms arising from initial conditions, then the transfer function is of the form

$$H(s) = \frac{Y(s)}{X(s)}.$$

Exercise

Definition

Using the Laplace Transform, the output of an LTI system is given by

$$Y(s) = H(s)X(s) + Y_{init}(s),$$

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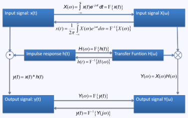
Relationship between input and output of an LTI-system



Transfer function

In more general form:

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} \frac{1}{2\pi} X(\omega)e^{j\omega t}\,d\omega \\ y(t) &= \int_{-\infty}^{\infty} \frac{1}{2\pi} X(\omega)H(\omega)e^{j\omega t}\,d\omega \end{aligned}$$

It becomes obvious that the output signal is given by an inverse Fourier-Transform:

$$y(t) = \mathcal{F}^{-1}\{X(\omega)H(\omega)\}.$$

Exercise (#3.2)

Plot the step response for a system with the transfer function

$$H(s) = \frac{s-1}{s^2 + 2s + 1}$$

using NumPy or MATLAB with and without using the built-in function `step`.

Exercise

Exercise (#3.2)

Plot the step response for a system with the transfer function

$$H(s) = \frac{s-1}{s^2+2s+1}$$

using NumPy or MATLAB with and without using the built-in function `step`.

Partial fraction

Exercise (#3.2)
Plot the step response for a system with the transfer function

$$H(s) = \frac{s-1}{s^2+2s+1}$$

using NUMPY or MATLAB with and without using the built-in function `step`.

Definition
Using the Laplace Transform, the output of an LTI system is given by

$$Y(s) = H(s)X(s) + Y_{init}(s),$$

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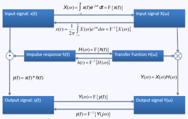
Relationship between input and output of an LTI-system



Definition

One can use the partial fraction expansion to simplify **complex**[a] functions of the form

$$A(x) = \frac{P(x)}{Q(x)}$$

as follows:

$$A(x) = \frac{P(x)}{Q(x)} = R(x) + \sum_{i=1}^{k}\sum_{j=1}^{n_i} \frac{a_{ij}}{(x-x_i)^j},$$

with n_i being the order of pole x_i. $R(x) = 0$ in case of $\deg(P) < \deg(Q)$.

[a] a complex function is a function $D \subset \mathbb{C} \to \mathbb{C}$.

Zeros

Definition

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Exercise (#3.2)

Plot the step response for a system with the transfer function

$$H(s) = \frac{s-1}{s^2 + 2s + 1}$$

using NUMPY or MATLAB with and without using the built-in function `step`.

Definition

Using the Laplace Transform, the output of an LTI system is given by

$$Y(s) = H(s)X(s) + Y_{init}(s),$$

with $Y_{init}(s)$ being terms due to all initial conditions. If one ignores all terms arising from initial conditions, then the transfer function is of the form

$$H(s) = \frac{Y(s)}{X(s)}.$$

Properties

Rational transfer functions of the form

$$\begin{aligned} H(s) &= \frac{b_0 + b_1 s + \ldots + b_M s^M}{a_0 + a_1 s + \ldots a_N s^N} \\ &= K \frac{(s - z_1)(s - z_2)\ldots(s - z_M)}{(s - p_1)(s - p_2)\ldots(s - z_N)} \end{aligned}$$

are completely characterized by the gain factor K and the poles (natural frequencies) and zeros.

Exercise

Properties

Rational transfer functions of the form

$$H(s) = \frac{b_0 + b_1 s + \ldots + b_M s^M}{a_0 + a_1 s + \ldots a_N s^N} = K\frac{(s-z_1)(s-z_2)\ldots(s-z_M)}{(s-p_1)(s-p_2)\ldots(s-z_N)}$$

are completely characterized by the gain factor K and the poles (natural frequencies) and zeros.

Definition

One can use the partial fraction expansion to simplify **complex**[a] functions of the form

$$A(x) = \frac{P(x)}{Q(x)}$$

as follows:

$$A(x) = \frac{P(x)}{Q(x)} = R(x) + \sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{a_{ij}}{(x-x_i)^j},$$

with n_i being the order of pole x_i. $R(x) = 0$ in case of $\deg(P) < \deg(Q)$.

[a] a complex function is a function $D \subset \mathbb{C} \to \mathbb{C}$.

Exercise (#3.2)

Plot the step response for a system with the transfer function

$$H(s) = \frac{s-1}{s^2+2s+1}$$

using NumPy or MATLAB with and without using the built-in function `step`.

Exercise (#3.3)

A system is described by the differential equation (initial states of 0)

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 3\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} + 2x$$

a) Calculate the transfer function $H(s)$
b) Analytically calculate the impulse response
c) Analytically calculate the (unit) step response

Exercise

Exercise (#3.3)

A system is described by the differential equation (initial states of 0)

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 3\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} + 2x$$

a) Calculate the transfer function $H(s)$
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Table of Contents

LTI systems

3.1 Introduction
3.2 Mathematical foundations: Differential equations
3.3 LTI systems
3.4 Transfer function
3.5 Impulse response
3.6 Causality
3.7 Stability
3.8 Network of systems
3.9 Graphical representation

Impulse response

Definition

The convolution is given by:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

It can be shown that

$$h(t) = \mathcal{F}^{-1}\left\{H(\omega)\right\}$$

is the reaction of the system to a $\delta(t)$ input – the so called impulse response.

Definition

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$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

It can be shown that

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is the reaction of the system to a $\delta(t)$ input – the so called impulse response.

Exercise (#3.4)

Plot the impulse response for a system with the transfer function

$$H(s) = \frac{s-1}{s^2+2s+1}$$

using NUMPY or MATLAB[a].

[a] use the *by hand* Laplace transform and built-in functions like `impulse`

Exercise

Exercise (#3.4)

Plot the impulse response for a system with the transfer function

$$H(s) = \frac{s-1}{s^2+2s+1}$$

using NUMPY or MATLAB[a].

[a] use the *by hand* Laplace transform and built-in functions like `impulse`

Table of Contents

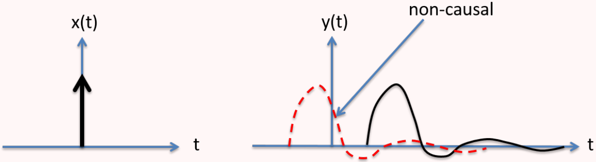
LTI systems

3.1 Introduction
3.2 Mathematical foundations: Differential equations
3.3 LTI systems
3.4 Transfer function
3.5 Impulse response
3.6 Causality
3.7 Stability
3.8 Network of systems
3.9 Graphical representation

Definition

A system is called causal if

$$h(t) = 0 \text{ for } t < 0.$$



Non-Causal means that the reaction of the system comes before the excitation

Table of Contents

LTI systems

3.1 Introduction
3.2 Mathematical foundations: Differential equations
3.3 LTI systems
3.4 Transfer function
3.5 Impulse response
3.6 Causality
3.7 Stability
3.7.1 Definition
3.7.2 Stability criterion based upon poles of the transfer function
3.7.3 Routh-Hurwitz stability criterion
3.8 Network of systems
3.9 Graphical representation

Table of Contents

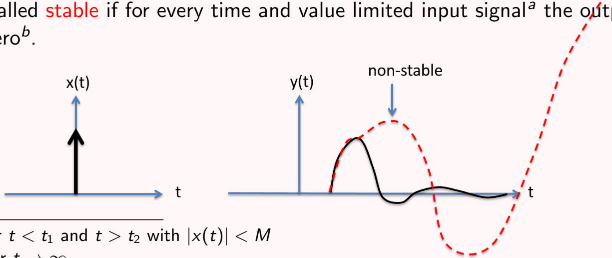
Stability

3.7 Stability
3.7.1 Definition
3.7.2 Stability criterion based upon poles of the transfer function
3.7.3 Routh-Hurwitz stability criterion

Stability

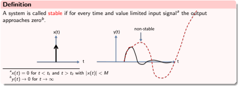
Definition

A system is called stable if for every time and value limited input signal[a] the output approaches zero[b].



[a] $x(t) = 0$ for $t < t_1$ and $t > t_2$ with $|x(t)| < M$
[b] $y(t) \rightarrow 0$ for $t \rightarrow \infty$

Criterion for Stability



Definition

A system is stable if

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

holds true.

BiBo

Definition

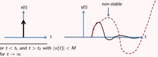
A system is stable if

$$\int_{-\infty}^{\infty} |h(t)|dt < \infty$$

holds true.

Definition

A system is called stable if for every time and value limited input signal[a] the output approaches zero[b].



[a] $x(t) = 0$ for $t < t_1$ and $t > t_2$ with $|x(t)| < M$
[b] $y(t) \to 0$ for $t \to \infty$

Definition

A causal and stable system has a bounded output in case of a bounded input (BIBO)

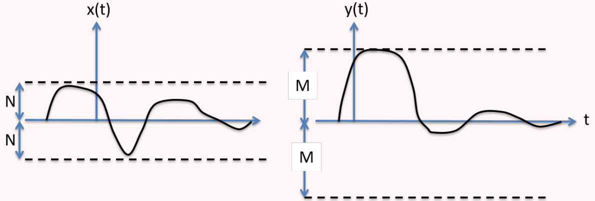


Table of Contents

Stability

3.7 Stability
3.7.1 Definition
3.7.2 Stability criterion based upon poles of the transfer function
3.7.3 Routh-Hurwitz stability criterion

Example

Consider the transfer function

$$H(s) = \frac{1}{s+a} = \mathcal{L}\left\{e^{-at}\right\} = \mathcal{L}\left\{e^{-\Re\{a\}t}e^{-i\Im\{a\}t}\right\}:$$

Obviously,

$$\lim_{t\to\infty} h(t) = 0$$

for

$$\Re\left\{a\right\} > 0.$$

Note that $H(s)$ has a pole at $s = -a$.

Stability

Example

Consider the transfer function

$$H(s) = \frac{1}{s+a} = \mathcal{L}\left\{e^{-at}\right\} = \mathcal{L}\left\{e^{-\Re\{a\}t}e^{-i\Im\{a\}t}\right\}:$$

Obviously,

$$\lim_{t\to\infty} h(t) = 0$$

for

$$\Re\{a\} > 0.$$

Note that $H(s)$ has a pole at $s = -a$.

Properties

Any causal system is absolutely stable if the poles of

$$H(s) = \frac{P(s)}{Q(s)}$$

lie in the left half space.

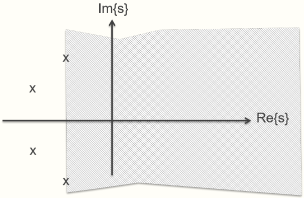


Figure 7: Poles and ROC

Note: This is only a yes/no classification.

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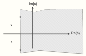

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$$H(s) = \frac{P(s)}{Q(s)}$$

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is called the characteristic equation of the system.

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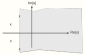

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$$H(s) \quad = \quad \frac{1}{s^2 + ps + q}$$

With poles at

$$s = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}.$$

For $p^2 - 4q < 0$ one gets poles at

$$s = -\frac{p}{2} \pm j\sqrt{q - \left(\frac{p}{2}\right)^2}.$$

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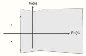

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Consider the transfer function

$$H(s) = \frac{1}{s^2+ps+q}$$

with poles at $-1 \pm j$:

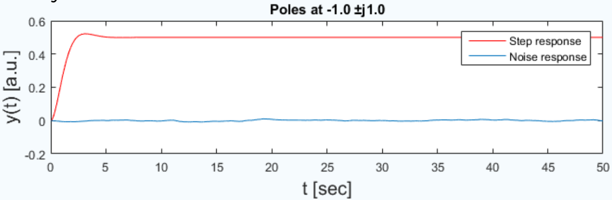

Figure 8: Step response and response to Gaussian noise (std. dev.: 0.01)

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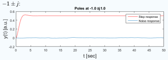

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Example

Consider the transfer function

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with poles at $-0.1 \pm j$:

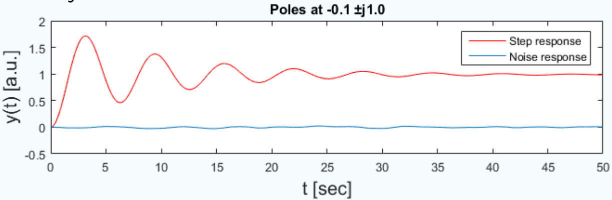

Figure 9: Step response and response to Gaussian noise (std. dev.: 0.01)

Poles

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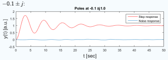

Figure 9: Step response and response to Gaussian noise (std. dev.: 0.01)

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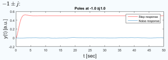

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Example

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with poles at $-1 \pm 10j$:

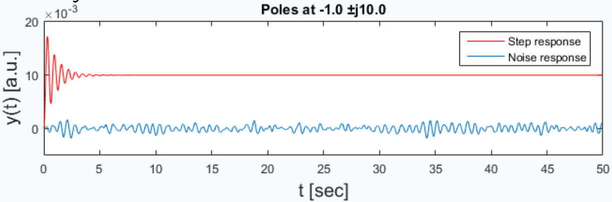

Figure 10: Step response and response to Gaussian noise (std. dev.: 0.01)

Poles

Example

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with poles at $-1 \pm 10j$:

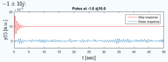

Figure 10: Step response and response to Gaussian noise (std. dev.: 0.01)

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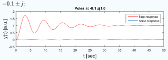

Figure 9: Step response and response to Gaussian noise (std. dev.: 0.01)

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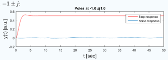

Figure 8: Step response and response to Gaussian noise (std. dev.: 0.01)

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Consider the transfer function

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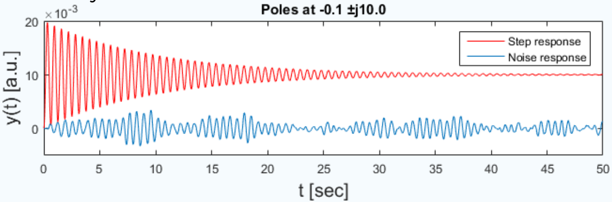
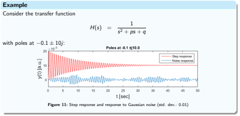
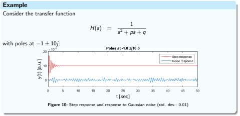
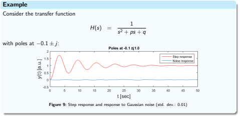


Figure 11: Step response and response to Gaussian noise (std. dev.: 0.01)







Example

Consider the transfer function

$$H(s) \quad = \quad \frac{1}{s^2 + ps + q}$$

with poles at $\pm 5j$:

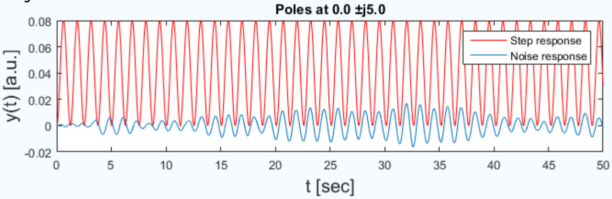
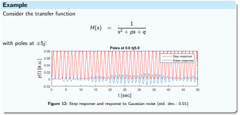
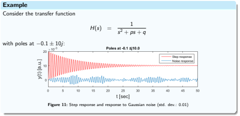
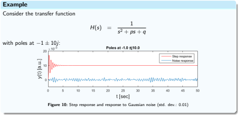


Figure 12: Step response and response to Gaussian noise (std. dev.: 0.01)







Exercise (#3.5)

Consider the transfer function

$$H(s) \quad = \quad \frac{1}{s^2 + ps + q}$$

with poles at $\alpha \pm j\beta$. Plot the step response and the response to Gaussian noise (std. dev.: 0.1) for the values of α and β given below by making use of NUMPY or MATLAB. Plot the result up to $t = 40\,\mathrm{s}$.

case	α	β
1	-10	1
2	-1	1
3	-1	10
4	1	1
5	10	1

Exercise

Exercise (#3.5)

Consider the transfer function

$$H(s) = \frac{1}{s^2 + ps + q}$$

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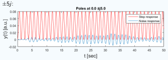

Figure 12: Step response and response to Gaussian noise (std. dev.: 0.01)

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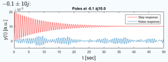

Figure 11: Step response and response to Gaussian noise (std. dev.: 0.01)

Properties

- ▷ A real pole in the left-half plane: Exponentially decaying component
- ▷ A real pole in the right-half plane: Exponentially increasing component
- ▷ A complex pole in the left-half plane: Exponentially decaying oscillatory component
- ▷ A complex pole in the right-half plane: Exponentially increasing oscillatory component
- ▷ A pole on the imaginary axis: Marginally stable
- ▷ Differential equations with real-valued coefficients yield poles that are pairwise complex conjugates

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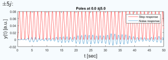

Figure 12: Step response and response to Gaussian noise (std. dev.: 0.01)

Marginally stable

Figure 13: Source: Schlecky Silberstein

Table of Contents

Stability

3.7 Stability
3.7.1 Definition
3.7.2 Stability criterion based upon poles of the transfer function
3.7.3 Routh-Hurwitz stability criterion

Routh-Hurwitz

Definition

Given is a transfer function of the form $H(s) = \frac{P(s)}{Q(s)}$, with

$$Q(s) = a_n s^n + a_{n-1}s^{n-1} + \dots a_1 s + a_0.$$

The Routh array is then of the form

$$\begin{array}{l|llll}
s^n & a_n & a_{n-2} & a_{n-4} & \dots \\
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\vdots & \vdots & \vdots & \vdots & \\
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\end{array}$$

See next slides for definition of $b_{n-1}, b_{n-3}, \dots$.

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See next slides for definition of $b_{n-1}, b_{n-3}, \ldots$.

Properties

$$\begin{aligned} b_{n-1} &= -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} \\ b_{n-3} &= -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix} \\ \vdots &= \vdots \\ c_{n-1} &= -\frac{1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix} \\ d_{n-1} &= -\frac{1}{c_{n-1}} \begin{vmatrix} b_{n-1} & b_{n-3} \\ c_{n-1} & c_{n-3} \end{vmatrix} \\ \vdots & \quad \vdots \quad \vdots \end{aligned}$$

Routh-Hurwitz

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Definition

The Routh-Hurwitz stability criterion states that the number of roots of $Q(s)$ with positive real parts is equal to the number of changes in the sign of the first column

$$\begin{bmatrix} a_n & a_{n-1} & b_{n-1} & c_{n-1} & \ldots & h_{n-1} \end{bmatrix}^T$$

in the Routh array.

Routh-Hurwitz

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The following cases need to be distinguished:

1. **There is no zero in the first column**
2. There is a zero in the first column, but at least one other element in the corresponding row is unequal to zero.
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Routh-Hurwitz stab. criterion

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Properties

$$\begin{aligned}
b_{n-1} &= -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} \\
b_{n-3} &= -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix} \\
\vdots &= \vdots \\
c_{n-1} &= -\frac{1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix} \\
d_{n-1} &= -\frac{1}{c_{n-1}} \begin{vmatrix} b_{n-1} & b_{n-3} \\ c_{n-1} & c_{n-3} \end{vmatrix} \\
\vdots & \vdots \vdots
\end{aligned}$$

Example: $Q(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0$

Routh array:

$$\begin{array}{l|ll}
s^3 & a_3 & a_1 \\
s^2 & a_2 & a_0 \\
s^1 & b_1 = \frac{a_2 a_1 - a_0 a_3}{a_2} & 0 \\
s^0 & c_1 = \frac{b_1 a_0}{b_1} = a_0 & 0
\end{array}$$

It follows that a necessary and sufficient condition for stability is that the coefficients are positive and that $a_2 a_1 > a_0 a_3$.

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Definition

The Routh-Hurwitz stability criterion states that the number of roots of $Q(s)$ with positive real parts is equal to the number of changes in the sign of the first column

$$\begin{bmatrix} a_n & a_{n-1} & b_{n-1} & c_{n-1} & \dots & h_{n-1} \end{bmatrix}^T$$

in the Routh array.

Routh-Hurwitz

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Example: $Q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$

Routh array:

s^5	1	2	11
s^3	2	4	10
s^3	$\boxed{0}$	6	0
s^2	c	10	0
s^1	d	0	0
s^0	10	0	0

The 0 in the third row can be replaced by ϵ with $0 < \epsilon \ll 1$ (see next slide)

Routh-Hurwitz

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s^2	$\frac{4\epsilon - 12}{\epsilon} < 0$	10	0
s^1	$\frac{6c - 10\varepsilon}{c}$	0	0
s^0	10	0	0

Two sign changes and thus unstable.

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Stable for $0 < K < 8$. For $K = 8$: See next slide.

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This case occurs if singularities are symmetrically located about the origin of the s-plane: $(s+\sigma)(s-\sigma)$ or $(s+j\omega)(s-j\omega)$.

Routh-Hurwitz

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Utilizing the row preceding the row of zeros to form an auxiliary equation:

$$U(s) = 2s^2 + 8s^0 = 0,$$

With $s = -j2$ and $s = j2$ the roots of $U(s)$.

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$$Q(s) = U(s)\left[1 + \frac{1}{2}s\right] = (s + j2)(s - j2)\left[s + 2\right]$$

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For the following forms of $Q(s)$ one can directly give a stability criterion:

$Q(s)$	Criterion
$s^2 + bs + 1$	$b > 0$
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For systems of the form

$$G(s)=e^{-sT}\frac{P(s)}{Q(s)}$$

with a time delay one needs to approximate e^{-sT} e.g. by making use of the Pade approximation

$$e^{-sT}\approx\frac{1-\frac{sT}{2}}{1+\frac{sT}{2}}.$$

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Exercise (#3.6)

Is the system with the transfer function

$$G(s) = \frac{3}{3s^3+12s^2+24s+36}$$

stable? Test by making use of NUMPY or MATLAB (e.g using `roots`).

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Table of Contents

LTI systems

3.1 Introduction
3.2 Mathematical foundations: Differential equations
3.3 LTI systems
3.4 Transfer function
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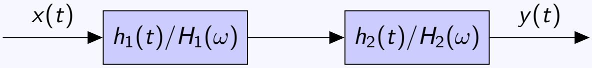
3.6 Causality
3.7 Stability
3.8 Network of systems
3.9 Graphical representation

Network of system

Cascaded systems



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H(\omega) &= H_1(\omega)H_2(\omega) \\
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Note: This is only valid, if the output of the first system is not changed by adding the second system (second system does not "load" the first system).

Network of systems

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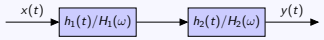

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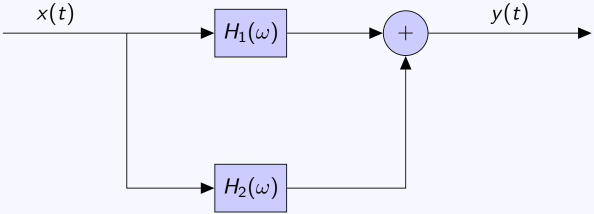

$$
\begin{aligned}
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Table of Contents

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3.1 Introduction
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3.3 LTI systems
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3.9.3 Nyquist diagrams

Table of Contents

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The transfer function can also be written as follows

$$H(\omega) = e^{-A(\omega)} e^{-jB(\omega)}.$$

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the damping of the system and

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$$H(\omega) = K \frac{(j\omega - z_1)(j\omega - z_2)\cdots(j\omega - z_m)}{(j\omega - p_1)(j\omega - p_2)\cdots(j\omega - p_n)}$$

Notes:

- In MATLAB, use *zp2tf* to convert zero-pole-gain parameters to the transfer function form
- Note $\log(a \cdot b) = \log(a) + \log(b)$

Bode diagrams

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For $k \in \mathbb{R}$:

Term	Magnitude	Phase
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$K > 0$	$20\log_{10}(K)$	0^o
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Properties

$$H(\omega) = K\frac{(j\omega - z_1)(j\omega - z_2)\cdots(j\omega - z_m)}{(j\omega - p_1)(j\omega - p_2)\cdots(j\omega - p_n)}$$

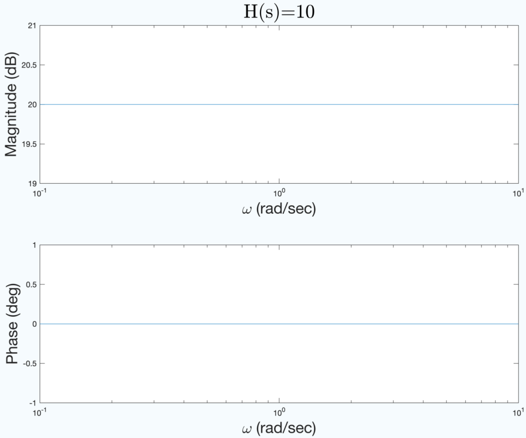
Notes:

- In MATLAB, use *zp2tf* to convert zero-pole-gain parameters to the transfer function form
- Note $\log(a \cdot b) = \log(a) + \log(b)$

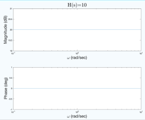
Definition

A Bode plot shows the frequency response of a system as a function of the logarithm of the frequency.

Example



Example



Example: Constant

For $k \in \mathbb{R}$:

Term	Magnitude	Phase
$K > 0$	$20\log_{10}(K)$	0^o
$K < 0$	$20\log_{10}(\lvert K\rvert)$	$\pm 180^o$

Properties

$$H(\omega) = K\frac{(j\omega - z_1)(j\omega - z_2)\cdots(j\omega - z_m)}{(j\omega - p_1)(j\omega - p_2)\cdots(j\omega - p_n)}$$

Notes:

- In MATLAB, use *zp2tf* to convert zero-pole-gain parameters to the transfer function form
- Note $\log(a \cdot b) = \log(a) + \log(b)$

Example: Pole at origin

$$\frac{1}{s}$$

Magnitude	Phase
▷ -20 dB/dec ▷ 0 dB at $\omega = 1$	▷ -90^o for all ω

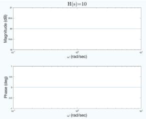
Bode diagrams

Example: Pole at origin

$$\frac{1}{s}$$

Magnitude	Phase
▷ -20 dB/dec ▷ 0 dB at $\omega = 1$	▷ $-90°$ for all ω

Example

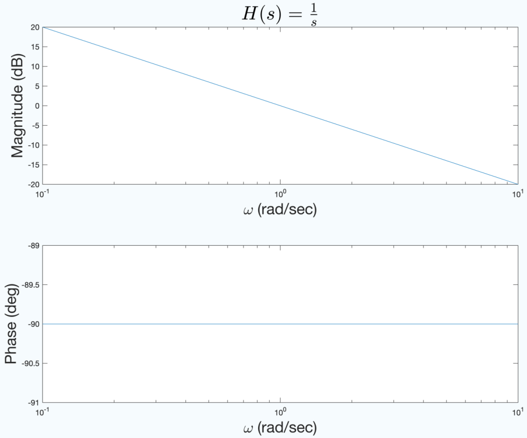


Example: Constant

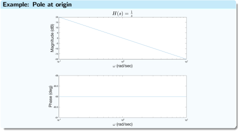
For $k \in \mathbb{R}$:

Term	Magnitude	Phase
$K > 0$	$20\log_{10}(K)$	$0°$
$K < 0$	$20\log_{10}(\lvert K \rvert)$	$\pm 180°$

Example: Pole at origin



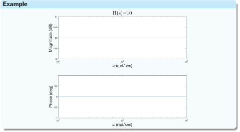
Bode diagrams



Example: Pole at origin

$\frac{1}{s}$

Magnitude	Phase
▷ -20 dB/dec ▷ 0 dB at $\omega = 1$	▷ -90^o for all ω



Example: Pole at zero

$$s$$

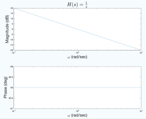
Magnitude	Phase
▷ $+20$ dB/dec ▷ 0 dB at $\omega = 1$	▷ $+90^o$ for all ω

Example: Pole at zero

s

Magnitude	Phase
▷ +20 dB/dec ▷ 0 dB at $\omega = 1$	▷ +90° for all ω

Example: Pole at origin

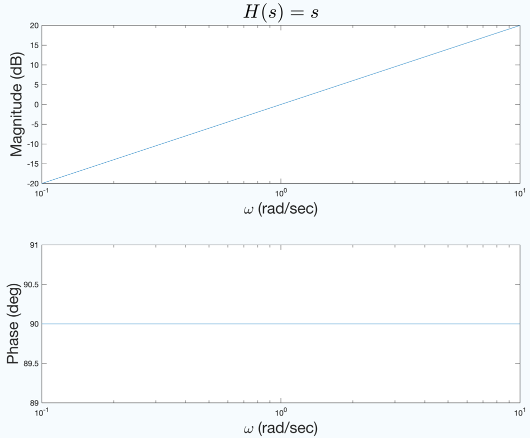


Example: Pole at origin

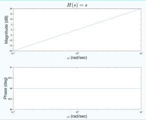
$\frac{1}{s}$

Magnitude	Phase
▷ -20 dB/dec ▷ 0 dB at $\omega = 1$	▷ −90° for all ω

Example: Pole at zero



Example: Pole at zero

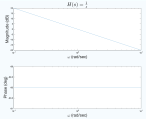


Example: Pole at zero

s

Magnitude	Phase
▷ +20 dB/dec ▷ 0 dB at $\omega = 1$	▷ $+90^o$ for all ω

Example: Pole at origin



Example: Real Pole

$$\frac{1}{\frac{s}{\omega_0} + 1}$$

▷ Low frequency asymptote at 0 dB ▷ high frequency asymptote at -20 dB/dec ▷ connect asymptotic lines at ω_0	▷ Low frequency asymptote at 0^o ▷ High frequency asymptote at -90^o ▷ connect with straight line from $0.1\,\omega_0$ to $10\,\omega_0$

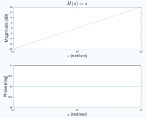
Example: Real Pole

$$\frac{1}{\frac{s}{\omega_0}+1}$$

▷ Low frequency asymptote at 0 dB ▷ high frequency asymptote at -20 dB/dec ▷ connect asymptotic lines at ω_0	▷ Low frequency asymptote at 0° ▷ High frequency asymptote at $-90°$ ▷ connect with straight line from $0.1\,\omega_0$ to $10\,\omega_0$

Example: Pole at zero

$H(s) = s$



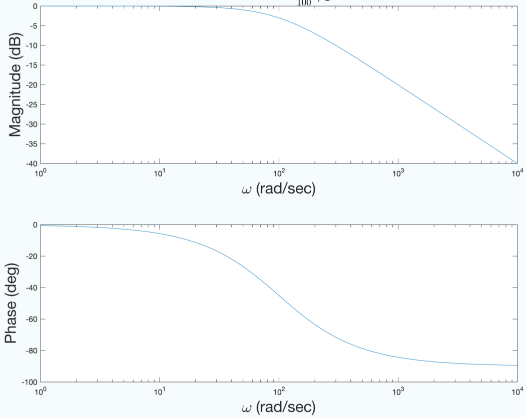
Example: Pole at zero

$$s$$

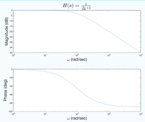
Magnitude	Phase
▷ +20 dB/dec ▷ 0 dB at $\omega = 1$	▷ +90° for all ω

Example: Real Pole

$$H(s) = \frac{1}{\frac{s}{100}+1}$$



Example: Real Pole

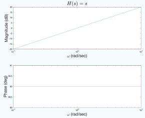


Example: Real Pole

$$\frac{1}{\frac{s}{\omega_0}+1}$$

▷ Low frequency asymptote at 0 dB ▷ high frequency asymptote at -20 dB/dec ▷ connect asymptotic lines at ω_0	▷ Low frequency asymptote at 0^o ▷ High frequency asymptote at -90^o ▷ connect with straight line from $0.1\,\omega_0$ to $10\,\omega_0$

Example: Pole at zero



Example: Real Zero

$$\frac{s}{\omega_0} + 1$$

Magnitude	Phase
▷ Low frequency asymptote at 0 dB ▷ high frequency asymptote at $+20$ dB/dec ▷ connect asymptotic lines at ω_0	▷ Low frequency asymptote at 0^o ▷ High frequency asymptote at $+90^o$ ▷ connect with straight line from $0.1\,\omega_0$ to $10\,\omega_0$

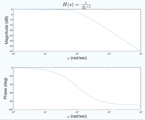
Example: Real Zero

$$\frac{s}{\omega_0}+1$$

Magnitude	Phase
▷ Low frequency asymptote at 0 dB ▷ high frequency asymptote at +20 dB/dec ▷ connect asymptotic lines at ω_0	▷ Low frequency asymptote at 0° ▷ High frequency asymptote at +90° ▷ connect with straight line from $0.1\,\omega_0$ to $10\,\omega_0$

Example: Real Pole

$$H(s) = \frac{1}{\frac{s}{10}+1}$$



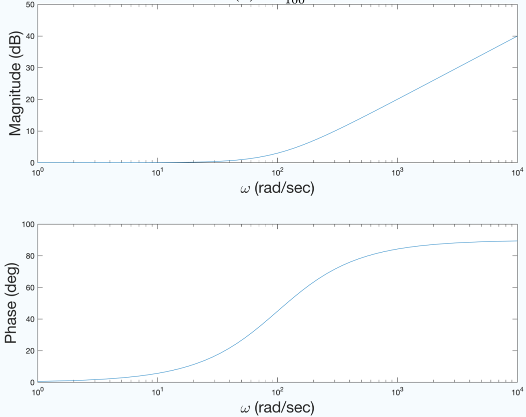
Example: Real Pole

$$\frac{1}{\frac{s}{\omega_0}+1}$$

▷ Low frequency asymptote at 0 dB ▷ high frequency asymptote at -20 dB/dec ▷ connect asymptotic lines at ω_0	▷ Low frequency asymptote at 0° ▷ High frequency asymptote at −90° ▷ connect with straight line from $0.1\,\omega_0$ to $10\,\omega_0$

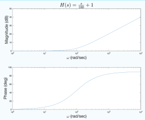
Example: Real Zero

$$H(s) = \frac{s}{100} + 1$$



Bode diagrams

Example: Real Zero

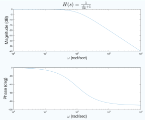


Example: Real Zero

$$\frac{s}{\omega_0}+1$$

Magnitude	Phase
▷ Low frequency asymptote at 0 dB ▷ high frequency asymptote at +20 dB/dec ▷ connect asymptotic lines at ω_0	▷ Low frequency asymptote at 0° ▷ High frequency asymptote at +90° ▷ connect with straight line from $0.1\,\omega_0$ to $10\,\omega_0$

Example: Real Pole



Example: Underdamped Pole

$$\frac{1}{\left(\frac{s}{\omega_0}\right)^2+2\sigma\left(\frac{s}{\omega_0}\right)+1}$$

▷ Low freq. asymptote at 0 dB ▷ high freq. asymptote at -40 dB/dec ▷ connect asymptotic lines at ω_0 ▷ draw peak at $\omega_p = \omega_0\sqrt{1-2\sigma^2}$ with amplitude $-20\log_{10} 2\sigma\sqrt{1-\sigma^2}$ ▷ $\sigma < \frac{1}{\sqrt{2}}$.	▷ Low freq. asymptote at 0^o ▷ High freq. asymptote at -180^o ▷ Connect with straight line from $\frac{\omega_0}{2}\log_{10}\left(\frac{2}{\sigma}\right)$ to $\frac{2\omega_0}{\log_{10}\left(\frac{2}{\sigma}\right)}$

Note: underdamped means that an oscillation takes place.

Bode diagrams

Example: Underdamped Pole

$$\frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2\sigma\left(\frac{s}{\omega_0}\right) + 1}$$

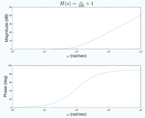
- Low freq. asymptote at 0 dB
- high freq. asymptote at -40 dB/dec
- connect asymptotic lines at ω_0
- draw peak at $\omega_p = \omega_0\sqrt{1-2\sigma^2}$ with amplitude $-20\log_{10} 2\sigma\sqrt{1-\sigma^2}$
- $\sigma < \frac{1}{\sqrt{2}}$.

- Low freq. asymptote at 0°
- High freq. asymptote at −180°
- Connect with straight line from $\frac{\omega_0}{2}\log_{10}\left(\frac{2}{\sigma}\right)$ to $\frac{2\omega_0}{\log_{10}\left(\frac{2}{\sigma}\right)}$

Note: underdamped means that an oscillation takes place.

Example: Real Zero

$H(s) = \frac{s}{100} + 1$



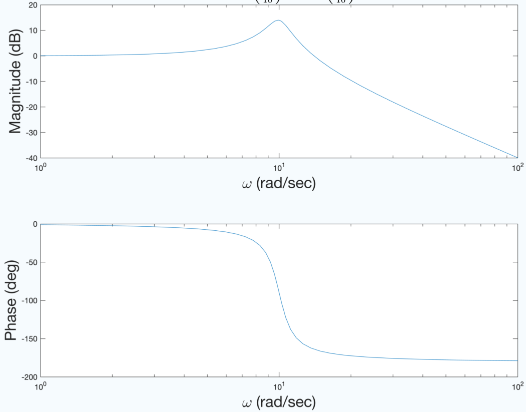
Example: Real Zero

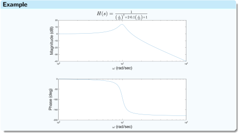
$$\frac{s}{\omega_0} + 1$$

Magnitude	Phase
▷ Low frequency asymptote at 0 dB ▷ high frequency asymptote at +20 dB/dec ▷ connect asymptotic lines at ω_0	▷ Low frequency asymptote at 0° ▷ High frequency asymptote at +90° ▷ connect with straight line from $0.1\,\omega_0$ to $10\,\omega_0$

Example

$$H(s) = \frac{1}{\left(\frac{s}{10}\right)^2 + 2\cdot 0.1\left(\frac{s}{10}\right) + 1}$$





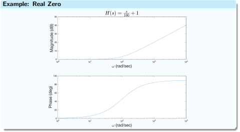
Example: Underdamped Pole

$$\frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2\sigma\left(\frac{s}{\omega_0}\right) + 1}$$

- Low freq. asymptote at 0 dB
- high freq. asymptote at -40 dB/dec
- connect asymptotic lines at ω_0
- draw peak at $\omega_p = \omega_0\sqrt{1-2\sigma^2}$ with amplitude $-20\log_{10} 2\sigma\sqrt{1-\sigma^2}$
- $\sigma < \frac{1}{\sqrt{2}}$.

- Low freq. asymptote at 0^o
- High freq. asymptote at -180^o
- Connect with straight line from $\frac{\omega_0}{2}\log_{10}\left(\frac{2}{\sigma}\right)$ to $\frac{2\omega_0}{\log_{10}\left(\frac{2}{\sigma}\right)}$

Note: underdamped means that an oscillation takes place.



Example: Underdamped Zero

$$\left(\frac{s}{\omega_0}\right)^2 + 2\sigma\left(\frac{s}{\omega_0}\right) + 1$$

- Low freq. asymptote at 0 dB
- high freq. asymptote at +40 dB/dec
- connect asymptotic lines at ω_0
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- $\sigma < \frac{1}{\sqrt{2}}$.

- Low freq. asymptote at 0^o
- High freq. asymptote at $+180^o$
- Connect with straight line from $\frac{\omega_0}{2}\log_{10}\left(\frac{2}{\sigma}\right)$ to $\frac{2\omega_0}{\log_{10}\left(\frac{2}{\sigma}\right)}$

Example: Underdamped Zero

$$\left(\frac{s}{\omega_0}\right)^2 + 2\sigma\left(\frac{s}{\omega_0}\right) + 1$$

- Low freq. asymptote at 0 dB
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- $\sigma < \frac{1}{\sqrt{2}}$.

- Low freq. asymptote at 0°
- High freq. asymptote at +180°
- Connect with straight line from $\frac{\omega_0}{2}\log_{10}\left(\frac{2}{\sigma}\right)$ to $\frac{2\omega_0}{\log_{10}\left(\frac{2}{\sigma}\right)}$

Example

$H(s) = \frac{1}{\left(\frac{s}{10}\right)^2 + 2\cdot 0.1\left(\frac{s}{10}\right)+1}$

Example: Underdamped Pole

$$\frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2\sigma\left(\frac{s}{\omega_0}\right) + 1}$$

- Low freq. asymptote at 0 dB
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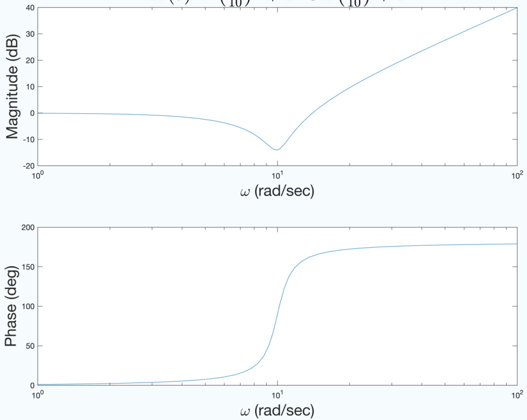
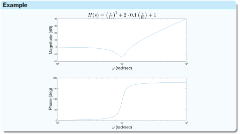
- Connect with straight line from $\frac{\omega_0}{2}\log_{10}\left(\frac{2}{\sigma}\right)$ to $\frac{2\omega_0}{\log_{10}\left(\frac{2}{\sigma}\right)}$

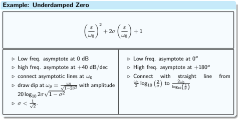
Note: underdamped means that an oscillation takes place.

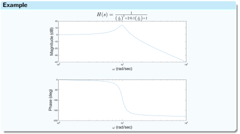
Example

$$H(s) = \left(\frac{s}{10}\right)^2 + 2\cdot 0.1\left(\frac{s}{10}\right) + 1$$









Exercise (#3.8)

Plot the Bode plot for the transfer function

$$H(s) = \frac{s-1}{s^2+2s+2}$$

using NUMPY or MATLAB with and without using the built-in function `bode`.

Exercise

Exercise (#3.8)

Plot the Bode plot for the transfer function

$$H(s) = \frac{s-1}{s^2+2s+2}$$

using NUMPY or MATLAB with and without using the built-in function bode.

Exercise (#3.8)

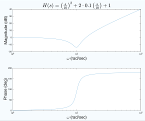
Plot the Bode plot for the transfer function

$$H(s) = \frac{s-1}{s^2+2s+2}$$

using NUMPY or MATLAB with and without using the built-in function bode.

Example

$H(s) = \left(\frac{s}{10}\right)^2 + 2\cdot 0.1\left(\frac{s}{10}\right) + 1$



Example: Underdamped Zero

$$\left(\frac{s}{\omega_0}\right)^2 + 2\sigma\left(\frac{s}{\omega_0}\right) + 1$$

- Low freq. asymptote at 0 dB
- high freq. asymptote at +40 dB/dec
- connect asymptotic lines at ω_0
- draw dip at $\omega_p = \frac{\omega_0}{\sqrt{1-2\sigma^2}}$ with amplitude $20\log_{10} 2\sigma\sqrt{1-\sigma^2}$
- $\sigma < \frac{1}{\sqrt{2}}$.

- Low freq. asymptote at 0°
- High freq. asymptote at +180°
- Connect with straight line from $\frac{\omega_0}{2}\log_{10}\left(\frac{2}{\sigma}\right)$ to $\frac{2\omega_0}{\log_{10}\left(\frac{2}{\sigma}\right)}$

Exercise (#3.9)

Sketch the Bode plot for the following transfer functions:

a)

$$H(s) = 1 + \frac{s}{10}$$

b)

$$H(s) = \frac{1}{1+s/100}$$

c)

$$H(s) = \frac{10^4(1+s)}{(10+s)(100+s)}$$

Exercise

Exercise (#3.9)

Sketch the Bode plot for the following transfer functions:

a)

$$H(s) = 1 + \frac{s}{10}$$

b)

$$H(s) = \frac{1}{1 + s/100}$$

c)

$$H(s) = \frac{10^4(1+s)}{(10+s)(100+s)}$$

Table of Contents

Graphical representation

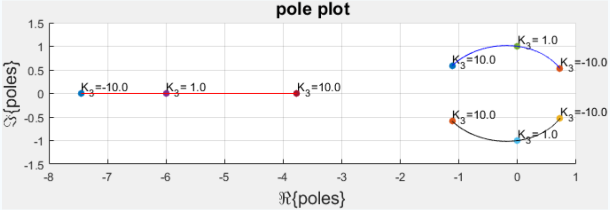
3.9 Graphical representation
3.9.1 Bode plots
3.9.2 Root locus analysis
3.9.3 Nyquist diagrams

Root locus analysis

Example

Given is a system (plant) with following transfer function:

$$H(s) = \frac{10}{s^3 + 6s^2 + K_3 s + 6}.$$

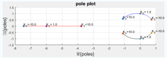


Root locus analysis

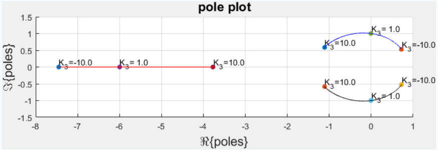
Example

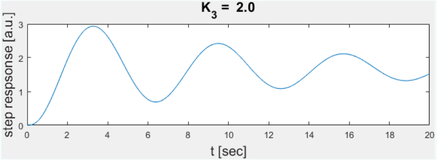
Given is a system (plant) with following transfer function:

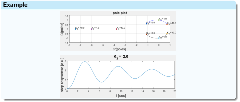
$$H(s) = \frac{10}{s^3 + 6s^2 + K_3 s + 6}.$$

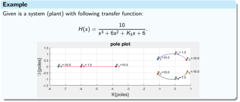


Example






Example
pole plot
K_3 = 2.0


Example
Given is a system (plant) with following transfer function:
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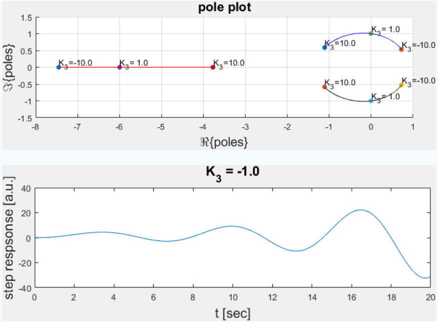
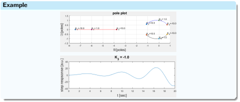
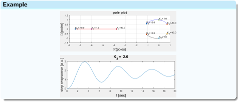
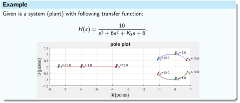
pole plot

Example

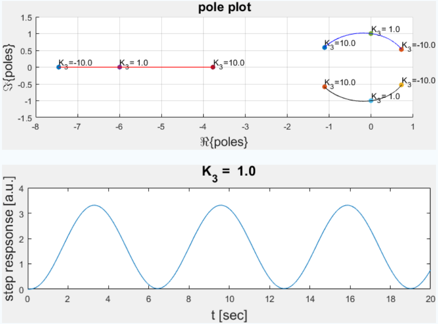

pole plot
$\Im$\{poles\}
$\Re$\{poles\}
K_3=-10.0
K_3= 1.0
K_3=10.0
K_3=-10.0
K_3= 1.0
K_3=10.0
K_3=-10.0
K_3= 1.0
K_3=10.0
K_3 = -1.0
step respsonse [a.u.]
t [sec]


Example
pole plot
K_3 = -1.0

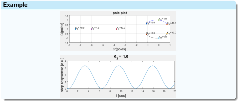

Example
pole plot
K_3 = 2.0

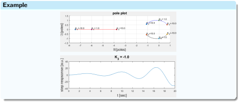

Example
Given is a system (plant) with following transfer function:
$H(s) = \frac{10}{s^3 + 6s^2 + K_3 s + 6}$
pole plot

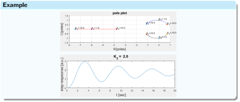
Example


pole plot
$\Im\{poles\}$
$\Re\{poles\}$
K_3=-10.0
K_3= 1.0
K_3=10.0
K_3=-10.0
K_3= 1.0
K_3=10.0
K_3=10.0
K_3=-10.0
K_3= 1.0
K_3 = 1.0
step respsonse [a.u.]
t [sec]

Root locus analysis



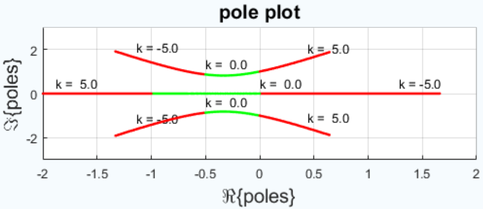




Example

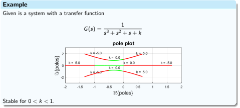
Given is a system with a transfer function

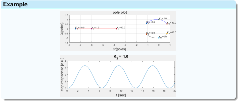
$$G(s) = \frac{1}{s^3 + s^2 + s + k}$$

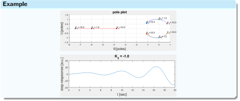


Stable for $0 < k < 1$.

Dominant pole







Definition

The slowest pole of a system is the one with the closest distance to the imaginary axis.

Definition

A pole is called dominant pole if it is much slower than the other poles and dominates the transient response.

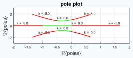
Exercise

Definition
The slowest pole of a system is the one with the closest distance to the imaginary axis.

Definition
A pole is called dominant pole if it is much slower than the other poles and dominates the transient response.

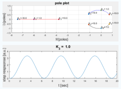
Example
Given is a system with a transfer function

$$G(s) = \frac{1}{s^3 + s^2 + s + k}$$



Stable for $0 < k < 1$.

Example



Exercise (#3.10)

Plot the step repsonses for a system with poles at $-10 \pm j2$, $-1 \pm j1$ and for a system with poles at $-1 \pm j1$. Normalize the step responses so that the final values of both become 1. Make use of NUMPY or MATLAB.

Exercise

Exercise (#3.10)

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The slowest pole of a system is the one with the closest distance to the imaginary axis.

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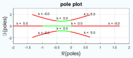
Example
Given is a system with a transfer function

$$G(s) = \frac{1}{s^3 + s^2 + s + k}$$



Stable for $0 < k < 1$.

Exercise (#3.11)

A system has the transfer function

$$H(s) = \frac{1}{(s+2-j)(s+2+j)(s^2+20s+104)}$$

- ▷ Determine the dominant pole(s).
- ▷ Determine the final value of the step response.
- ▷ Approximate the system with a second order system.
- ▷ Plot the step response of the original and approximated system using NUMPY or MATLAB.

Exercise

Exercise (#3.11)

A system has the transfer function

$$H(s) = \frac{1}{(s+2-j)(s+2+j)(s^2+20s+104)}$$

- Determine the dominant pole(s).
- Determine the final value of the step response.
- Approximate the system with a second order system.
- Plot the step response of the original and approximated system using NUMPY or MATLAB.

Graphical representation

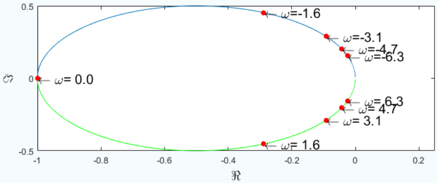
3.9 Graphical representation
3.9.1 Bode plots
3.9.2 Root locus analysis
3.9.3 Nyquist diagrams

Nyquist diagrams

Definition

The Nyquist diagram is a plot of the frequency response. Usually, the real part of the transfer function is plotted on the $x-$axis and the imaginary part is plotted on the $y-$axis.

Example: $G(s) = \frac{1}{s-1}$



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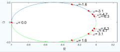
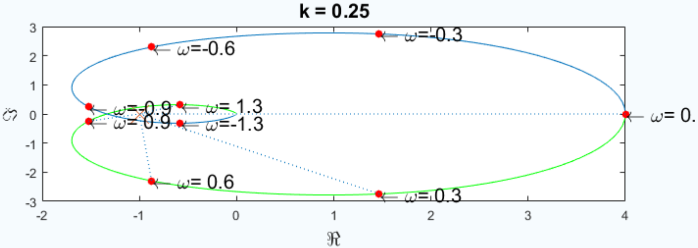


Example

Given is a system with a transfer function

$$G(s) = \frac{10}{s^3 + s^2 + s + 0.5}$$



Nyquist diagrams

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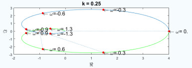

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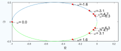
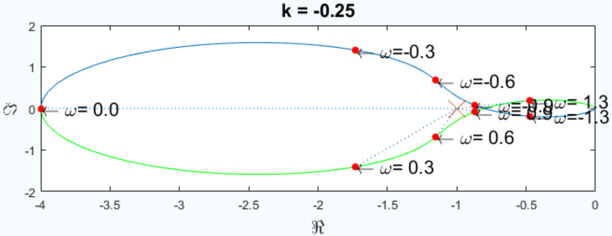


Example

Given is a system with a transfer function

$$G(s) = \frac{10}{s^3 + s^2 + s - 0.5}$$



Nyquist diagrams

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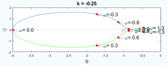
Given is a system with a transfer function

$$G(s) = \frac{10}{s^3 + s^2 + s - 0.5}$$



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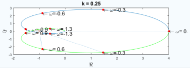

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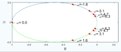


Nyquist diagrams for stability analysis

See control chapter.

Table of Contents

Signals and Systems

1. Introduction
2. Basic signals and operations
3. LTI systems
4. State variable models

Table of Contents

State variable models

4.1 Example system
4.2 Block Diagram
4.3 First order differential equation
4.4 Laplace domain
4.5 Stability of State Variable Systems
4.6 Canonical forms
4.7 Discrete time
4.8 Exercises

Example system

Mass spring damper system

- Second order Differential equation:

$$u = M\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + ky$$

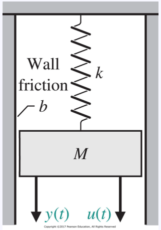
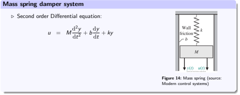


Figure 14: Mass spring (source: Modern control systems)

Reduction to first order



Introducing states

The equation

$$u = M\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + ky$$

can be reduced to a **system** of first order differential equations by using the states

$$\begin{aligned} x_1 &= y, \\ x_2 &= \frac{\mathrm{d}x_1}{\mathrm{d}t}. \end{aligned}$$

Reduction to first order

Introducing states

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Mass spring damper system

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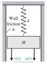

Figure 14: Mass spring (source: Modern control systems)

Matrix notation

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_2 \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -\frac{b}{M}x_2 - \frac{k}{M}x_1 + \frac{1}{M}u \end{aligned}$$

In matrix notation:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{M} \end{pmatrix} u}$$

Matrix notation

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Introducing states

The equation

$$u = M\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky$$

can be reduced to a **system** of first order differential equations by using the states

$$\begin{aligned}x_1 &= y,\\ x_2 &= \frac{dx_1}{dt}.\end{aligned}$$

Mass spring damper system

- Second order Differential equation:

$$u = M\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky$$

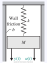


Figure 14: Mass spring (source: Modern control systems)

General form

$$\frac{d}{dt}\begin{pmatrix}x_1\\x_2\\ \vdots\\x_n\end{pmatrix} = \begin{bmatrix}a_{1,1} & \dots & a_{1,n}\\ a_{2,1} & \dots & a_{2,n}\\ \vdots & \ddots & \vdots\\ a_{n,1} & \dots & a_{n,n}\end{bmatrix}\begin{pmatrix}x_1\\x_2\\ \vdots\\x_n\end{pmatrix} + \begin{bmatrix}b_{1,1} & \dots & b_{1,m}\\ b_{2,1} & \dots & b_{2,m}\\ \vdots & \ddots & \vdots\\ b_{n,1} & \dots & b_{n,m}\end{bmatrix}\begin{pmatrix}u_1\\u_2\\ \vdots\\u_m\end{pmatrix}$$

$$\boxed{\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}}$$

- Note: $\boldsymbol{A}$ and $\boldsymbol{B}$ are constants (no function of time)
- $\boldsymbol{x}$ is called state vector.
- LTI system

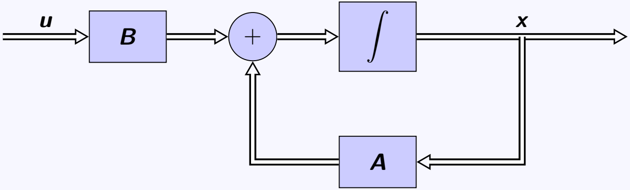
Table of Contents

State variable models

4.1 Example system
4.2 Block Diagram
4.3 First order differential equation
4.4 Laplace domain
4.5 Stability of State Variable Systems
4.6 Canonical forms
4.7 Discrete time
4.8 Exercises

Block diagram

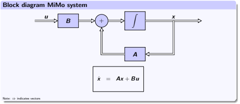
Block diagram MiMo system



$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

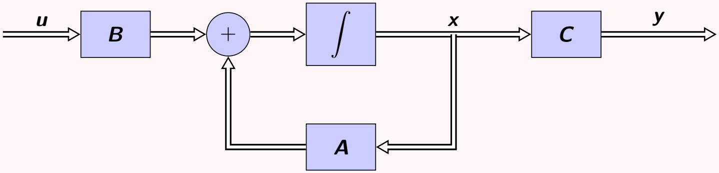
Note: $\Rightarrow$ indicates vectors

Measurement



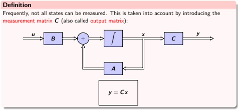
Definition

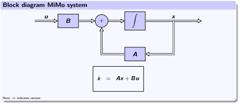
Frequently, not all states can be measured. This is taken into account by introducing the measurement matrix $\boldsymbol{C}$ (also called output matrix):



$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$$

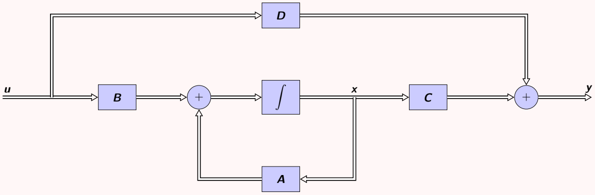
Throughput





Definition

Direct feed-through can be modeled using the feedforward matrix $\boldsymbol{D}$:



$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}$$

Definition

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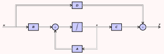

$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}$

Definition

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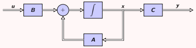
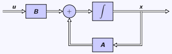


$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$

Block diagram MiMo system

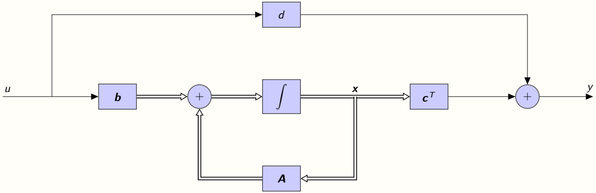


$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$

Note: ⇒ indicates vectors

Properties

In case of single (scalar) input and single (scalar) output (SiSo) the following simplifications can be used:



$$
\begin{aligned}
y &= \boldsymbol{c}^T\boldsymbol{x} + du \\
\dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u
\end{aligned}
$$

Table of Contents

State variable models

4.1 Example system
4.2 Block Diagram
4.3 First order differential equation
4.4 Laplace domain
4.5 Stability of State Variable Systems
4.6 Canonical forms
4.7 Discrete time
4.8 Exercises

Scalar first order DE

Definition

$$\dot{x} = ax + bu$$

Solution

$$\begin{aligned} x(t) &= e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)\mathrm{d}\tau \\ &= \underbrace{e^{at}k}_{\text{unforced response}} + \int_0^t e^{a(t-\tau)}bu(\tau)\mathrm{d}\tau \end{aligned}$$

Matrix first order DE

Definition

$$\dot{x} = ax + bu$$

Solution

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)\mathrm{d}\tau$$
$$= \underbrace{e^{at}k}_{\text{unforced response}} + \int_0^t e^{a(t-\tau)}bu(\tau)\mathrm{d}\tau$$

Definition

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

Unforced system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ a_{2,1} & \dots & a_{2,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$
$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$$

Matrix first order DE

Definition

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

Unforced system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ a_{2,1} & \dots & a_{2,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$
$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$$

Definition

$$\dot{x} = ax + bu$$

Solution

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)\mathrm{d}\tau$$
$$= \underbrace{e^{at}k}_{\text{unforced response}} + \int_0^t e^{a(t-\tau)}bu(\tau)\mathrm{d}\tau$$

Solution

The Ansatz

$$\boldsymbol{x}(t) = \boldsymbol{k}e^{\lambda t}$$

yields

$$\boldsymbol{k}\lambda e^{\lambda t} = \boldsymbol{A}\boldsymbol{k}e^{\lambda t}.$$

And thus the eigenvalue equations

$$\boldsymbol{k}\lambda = \boldsymbol{A}\boldsymbol{k}.$$

Solution
The Ansatz
$$x(t) = ke^{\lambda t}$$
yields
$$k\lambda e^{\lambda t} = Ake^{\lambda t}.$$
And thus the eigenvalue equations
$$k\lambda = Ak.$$

Definition
$$\dot{x} = Ax + Bu$$

Unforced system
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ a_{2,1} & \cdots & a_{2,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$
$$\dot{x} = Ax$$

Definition
$$\dot{x} = ax + bu$$

Solution
$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)\mathrm{d}\tau$$
$$= \underbrace{e^{at}k}_{\text{unforced response}} + \int_0^t e^{a(t-\tau)}bu(\tau)\mathrm{d}\tau$$

Solution

$$(\boldsymbol{A} - \lambda \boldsymbol{I}) \cdot \boldsymbol{k} = 0.$$

The non trivial solution ($\boldsymbol{k} \neq \boldsymbol{0}$) exists only if

$$\boxed{\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0.}$$

Notes:

- $\boldsymbol{I}$ is the identity matrix.

Characteristic equation

Solution

$$(\boldsymbol{A}-\lambda \boldsymbol{I}) \cdot \boldsymbol{k} = 0.$$

The non trivial solution ($\boldsymbol{k} \neq \boldsymbol{0}$) exists only if

$$\boxed{\det(\boldsymbol{A}-\lambda \boldsymbol{I}) = 0.}$$

Notes:
- $\boldsymbol{I}$ is the identity matrix.

Solution

The Ansatz

$$\boldsymbol{x}(t) = \boldsymbol{k} e^{\lambda t}$$

yields

$$\boldsymbol{k} \lambda e^{\lambda t} = \boldsymbol{A} \boldsymbol{k} e^{\lambda t}.$$

And thus the eigenvalue equations

$$\boldsymbol{k} \lambda = \boldsymbol{A} \boldsymbol{k}.$$

Definition

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

Unforced system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ a_{2,1} & \cdots & a_{2,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$
$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$$

Definition

The equation

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$$

is called characteristic equation of the system defined by

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}.$$

Matrix first order DE

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The Ansatz

$$\boldsymbol{x}(t) = \boldsymbol{k} e^{\lambda t}$$

yields

$$\boldsymbol{k} \lambda e^{\lambda t} = \boldsymbol{A}\boldsymbol{k} e^{\lambda t}.$$

And thus the eigenvalue equations

$$\boldsymbol{k}\lambda = \boldsymbol{A}\boldsymbol{k}.$$

Definition

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

Solution

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\mathrm{d}\tau.,$$

with

$$e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}t + \frac{\boldsymbol{A}^2 t^2}{2!} + \cdots \frac{\boldsymbol{A}^n t^n}{n!} + \cdots$$

Note: $\boldsymbol{I}$ is the identity matrix.

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Notes:
- $\boldsymbol{I}$ is the identity matrix.

About $\exp(\boldsymbol{A}t)$

It can be shown that for a square $(n \times n)$-matrix $\boldsymbol{A}$

$$\begin{aligned} e^{\boldsymbol{A}t} &= \boldsymbol{I} + \boldsymbol{A}t + \frac{\boldsymbol{A}^2t^2}{2!} + \cdots \frac{\boldsymbol{A}^nt^n}{n!} + \cdots = \sum_{k=0}^{\infty}\frac{1}{k!}\boldsymbol{A}^kt^k \\ &= \alpha_{n-1}(t)\boldsymbol{A}^{n-1}t^{n-1} + \alpha_{n-1}(t)\boldsymbol{A}^{n-2}t^{n-2} + \cdots \alpha_0(t)\boldsymbol{I} \\ &= \sum_{k=0}^{n-1}\alpha_k(t)\boldsymbol{A}^kt^k \end{aligned}$$

holds true. In this equation, $\alpha_0(t)\ldots\alpha_{n-1}(t)$ are functions of t which need to be determined (see next slides).

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About $\exp(\boldsymbol{A}t)$

Let $r(\lambda)$ be defined as follows:

$$r(\lambda) = \alpha_{n-1}(t)\lambda^{n-1} + \alpha_{n-2}(t)\lambda^{n-2} + \ldots + \alpha_1(t)\lambda + \alpha_0(t).$$

Then, $\alpha_k(t)$ can be determined by solving the equations shown on the next slides.

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For each eigenvalue λ_k of $\boldsymbol{A}t$ with multiplicity N, one gets N equations:

$$\begin{aligned} e^{\lambda_k} &= r(\lambda_k) \\ e^{\lambda_k} &= \left.\frac{\partial r(\lambda_k)}{\partial \lambda}\right|_{\lambda=\lambda_k} \\ \vdots \quad & \quad \vdots \\ e^{\lambda_k} &= \left.\frac{\partial^{N-1} r(\lambda_k)}{\partial \lambda^{N-1}}\right|_{\lambda=\lambda_k} \end{aligned}$$

For $N = 1$: $e^{\lambda_k} = r(\lambda_k)$.
For $N = 2$:

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Example

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 9 & 1 \end{bmatrix}.$$

$$e^{\boldsymbol{A}t} = \alpha_1 \boldsymbol{A}t + \alpha_0 \boldsymbol{I} = \begin{bmatrix} \alpha_1 t + \alpha_0 & \alpha_1 t \\ 9\alpha_1 t & \alpha_1 t + \alpha_0 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 4t$, $\lambda_2 = -2t$ and thus

$$\begin{aligned} e^{4t} &= 4t\alpha_1 + \alpha_0 \\ e^{-2t} &= -2t\alpha_1 + \alpha_0 \end{aligned}$$

About $\exp(\boldsymbol{A}t)$

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Example

Solving

$$\begin{aligned} e^{4t} &= 4t\alpha_1 + \alpha_0 \\ e^{-2t} &= -2t\alpha_1 + \alpha_0 \end{aligned}$$

leads to

$$\begin{aligned} \alpha_1 &= \frac{1}{6t}\left(e^{4t} - e^{-2t}\right) \quad \Rightarrow \quad e^{\boldsymbol{A}t} = \frac{1}{6}\begin{bmatrix} 3e^{4t} + 3e^{-2t} & e^{4t} - e^{-2t} \\ 9e^{4t} - 9e^{-2t} & 3e^{4t} + 3e^{-2t} \end{bmatrix}. \\ \alpha_0 &= \frac{1}{3}\left(e^{4t} + 2e^{-2t}\right) \end{aligned}$$

Table of Contents

State variable models

4.1 Example system
4.2 Block Diagram
4.3 First order differential equation
4.4 Laplace domain
4.5 Stability of State Variable Systems
4.6 Canonical forms
4.7 Discrete time
4.8 Exercises

Scalar first order DE

Definition and Laplace transform

$$\begin{aligned}
\dot{x} &= ax + bu \\
sX(s) - x(0) &= aX(s) + bU(s) \\
X(S) &= \frac{x(0)}{s-a} + \frac{b}{s-a}U(s)
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Solution by inverse Laplace transform

$$\begin{aligned}x(t) &= e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)\mathrm{d}\tau\\ &= \underbrace{e^{at}k}_{\text{unforced response}} + \int_0^t e^{a(t-\tau)}bu(\tau)\mathrm{d}\tau\end{aligned}$$

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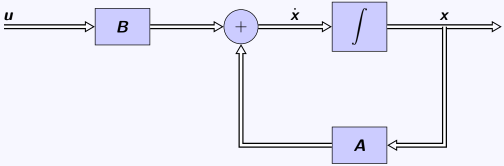
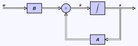


With the assumption $\boldsymbol{u}(0) = \boldsymbol{x}(0) = \boldsymbol{0}$:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\ s\boldsymbol{X}(s) &= \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{B}\boldsymbol{U}(s) \\ (s\boldsymbol{I} - \boldsymbol{A})\,\boldsymbol{X}(s) &= \boldsymbol{B}\boldsymbol{U}(s) \\ \boldsymbol{X}(s) &= [s\boldsymbol{I} - \boldsymbol{A}]^{-1}\,\boldsymbol{B}\boldsymbol{U}(s) = \boldsymbol{\Phi}(s)\boldsymbol{B}\boldsymbol{U}(s) \end{aligned}$$

Note: $\boldsymbol{X}(s)$ and $\boldsymbol{U}(s)$ are vectors (capital letters indicate frequency-domain).

Laplace domain



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(s\boldsymbol{I} - \boldsymbol{A})\,\boldsymbol{X}(s) &= \boldsymbol{B}\boldsymbol{U}(s) \\
\boldsymbol{X}(s) &= [s\boldsymbol{I} - \boldsymbol{A}]^{-1}\,\boldsymbol{B}\boldsymbol{U}(s) = \boldsymbol{\Phi}(s)\boldsymbol{B}\boldsymbol{U}(s)
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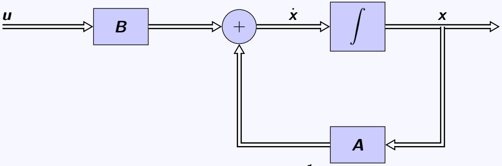
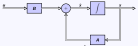


$$\begin{aligned}
\boldsymbol{X}(s) &= [s\boldsymbol{I} - \boldsymbol{A}]^{-1}\,\boldsymbol{B}\boldsymbol{U}(s) \\
\boldsymbol{X}(s) &= \boldsymbol{\Phi}(s)\boldsymbol{B}\boldsymbol{U}(s) \\
\boldsymbol{\Phi}(t) &= e^{\boldsymbol{A}t}
\end{aligned}$$

Note: Compare with

$$\mathcal{L}\left\{e^{-at}\right\} = \frac{1}{s+a}.$$

Laplace domain



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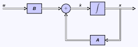

With the assumption $\boldsymbol{u}(0) = \boldsymbol{x}(0) = \boldsymbol{0}$:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\ s\boldsymbol{X}(s) &= \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{B}\boldsymbol{U}(s) \\ (s\boldsymbol{I}-\boldsymbol{A})\,\boldsymbol{X}(s) &= \boldsymbol{B}\boldsymbol{U}(s) \\ \boldsymbol{X}(s) &= [s\boldsymbol{I}-\boldsymbol{A}]^{-1}\boldsymbol{B}\boldsymbol{U}(s) = \boldsymbol{\Phi}(s)\boldsymbol{B}\boldsymbol{U}(s) \end{aligned}$$

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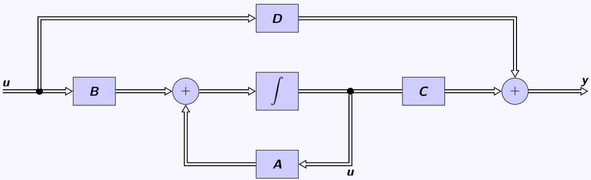
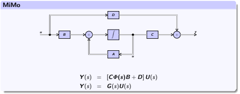
Solution by inverse Laplace transform

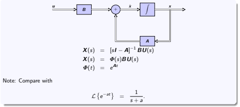
$$\begin{aligned} x(t) &= e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)\mathrm{d}\tau \\ &= \underbrace{e^{at}k}_{\text{unforced response}} + \int_0^t e^{a(t-\tau)}bu(\tau)\mathrm{d}\tau \end{aligned}$$

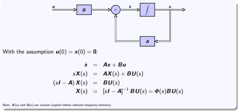
MiMo



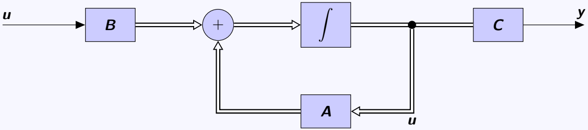
$$\begin{aligned} \boldsymbol{Y}(s) &= [\boldsymbol{C}\boldsymbol{\Phi}(s)\boldsymbol{B} + \boldsymbol{D}]\,\boldsymbol{U}(s) \\ \boldsymbol{Y}(s) &= \boldsymbol{G}(s)\boldsymbol{U}(s) \end{aligned}$$







SiSo without direct feedthrough



For SiSo systems without direct feedthrough:

$$
\begin{aligned}
Y(s) &= [\boldsymbol{C\Phi(s)B}]\, U(s) \\
Y(s) &= G(s)U(s), \\
&\text{with} \\
\boldsymbol{\Phi(s)} &= [s\boldsymbol{I} - \boldsymbol{A}]^{-1}
\end{aligned}
$$

Table of Contents

State variable models

4.1 Example system
4.2 Block Diagram
4.3 First order differential equation
4.4 Laplace domain
4.5 Stability of State Variable Systems
4.6 Canonical forms
4.7 Discrete time
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Laplace domain

Definition

The equation

$$\boldsymbol{X}(s) \quad = \quad [s\boldsymbol{I} - \boldsymbol{A}]^{-1} \boldsymbol{B}\boldsymbol{U}(s)$$

can be used to determine the stability of the system. If all of the roots of the characteristic equation

$$\det\left(s\boldsymbol{I} - \boldsymbol{A}\right) = 0$$

are located in the left half space (negative real part), then the system is stable.

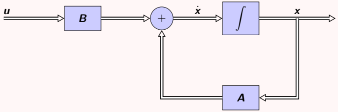

Table of Contents

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4.1 Example system
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- 4.6.1 SiSo transfer function
- 4.6.2 Phase variable canonical form
- 4.6.3 Input feedworward canonical form

4.7 Discrete time
4.8 Exercises

Table of Contents

Canonical forms

4.6 Canonical forms
4.6.1 SiSo transfer function
4.6.2 Phase variable canonical form
4.6.3 Input feedworward canonical form

Exemplary SiSo transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\left[b_0 + b_1 s^1 + b_2 s^2\right]}{\left[a_0 + a_1 s^1 + a_2 s^2 + s^3\right]}$$
$$= \frac{\left[b_0 s^{-3} + b_1 s^{-2} + b_2 s^{-1}\right]}{\left[a_0 s^{-3} + a_1 s^{-2} + a_2 s^{-1} + 1\right]}$$

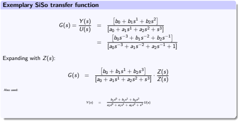
Expanding with $Z(s)$:

$$G(s) = \frac{\left[b_0 + b_1 s^1 + b_2 s^3\right]}{\left[a_0 + a_1 s^1 + a_2 s^2 + s^3\right]} \cdot \frac{Z(s)}{Z(s)}$$

Also used:

$$Y(s) = \frac{b_2 s^0 + b_1 s^1 + b_0 s^2}{a_2 s^0 + a_1 s^1 + a_0 s^2 + s^3} U(s)$$

Transfer function



SiSo transfer function

$$
\begin{aligned}
Y(s) &= \left[b_0 + b_1 s^1 + b_2 s^2\right] Z(s)\\
U(s) &= \left[a_0 + a_1 s^1 + a_2 s^2 + s^3\right] Z(s)
\end{aligned}
$$

In the time domain:

$$
\begin{aligned}
y &= b_0 z(t) + b_1 \frac{\mathrm{d}}{\mathrm{d}t} z(t) + b_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2} z(t)\\
u &= a_0 z(t) + a_1 \frac{\mathrm{d}}{\mathrm{d}t} z(t) + a_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2} z(t) + \frac{\mathrm{d}^3}{\mathrm{d}t^3} z(t)
\end{aligned}
$$

Table of Contents

Canonical forms

4.6 Canonical forms
4.6.1 SiSo transfer function
4.6.2 Phase variable canonical form
4.6.3 Input feedworward canonical form

SiSo transfer function

$$
\begin{aligned}
y &= b_0 z(t) + b_1 \frac{\mathrm{d}}{\mathrm{d}t} z(t) + b_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2} z(t) \\
u &= a_0 z(t) + a_1 \frac{\mathrm{d}}{\mathrm{d}t} z(t) + a_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2} z(t) + \frac{\mathrm{d}^3}{\mathrm{d}t^3} z(t)
\end{aligned}
$$

Introducing state variables (also called phase variables):

$$
x_1 = z(t), x_2 = \frac{\mathrm{d}}{\mathrm{d}t} z(t) \text{ and } x_3 = \frac{\mathrm{d}^2}{\mathrm{d}t^2} z(t)
$$

Note that usually the states do not represent physical values.

SiSo transfer function

$$y = b_0 z(t) + b_1 \frac{\mathrm{d}}{\mathrm{d}t} z(t) + b_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2} z(t)$$
$$u = a_0 z(t) + a_1 \frac{\mathrm{d}}{\mathrm{d}t} z(t) + a_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2} z(t) + \frac{\mathrm{d}^3}{\mathrm{d}t^3} z(t)$$

Introducing state variables (also called phase variables):

$$x_1 = z(t),\ x_2 = \frac{\mathrm{d}}{\mathrm{d}t} z(t) \text{ and } x_3 = \frac{\mathrm{d}^2}{\mathrm{d}t^2} z(t)$$

Note that usually the states do not represent physical values.

SiSo transfer function

$$
\begin{aligned}
y &= b_0 x_1 + b_1 x_2 + b_2 x_3 \\
u &= a_0 x_1 + a_1 x_2 + a_2 x_3 + \frac{\mathrm{d}}{\mathrm{d}t} x_3
\end{aligned}
$$

The resulting system of first order differential equations:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} x_1 &= x_2 \\
\frac{\mathrm{d}}{\mathrm{d}t} x_2 &= x_3 \\
\frac{\mathrm{d}}{\mathrm{d}t} x_3 &= -a_0 x_1 - a_1 x_2 - a_2 x_3 + u
\end{aligned}
$$

Transfer function

SiSo transfer function

$$\begin{aligned} y &= b_0x_1 + b_1x_2 + b_2x_3 \\ u &= a_0x_1 + a_1x_2 + a_2x_3 + \frac{\mathrm{d}}{\mathrm{d}t}x_3 \end{aligned}$$

The resulting system of first order differential equations:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}x_1 &= x_2 \\ \frac{\mathrm{d}}{\mathrm{d}t}x_2 &= x_3 \\ \frac{\mathrm{d}}{\mathrm{d}t}x_3 &= -a_0x_1 - a_1x_2 - a_2x_3 + u \end{aligned}$$

SiSo transfer function

$$\begin{aligned} y &= b_0z(t) + b_1\frac{\mathrm{d}}{\mathrm{d}t}z(t) + b_2\frac{\mathrm{d}^2}{\mathrm{d}t^2}z(t) \\ u &= a_0z(t) + a_1\frac{\mathrm{d}}{\mathrm{d}t}z(t) + a_2\frac{\mathrm{d}^2}{\mathrm{d}t^2}z(t) + \frac{\mathrm{d}^3}{\mathrm{d}t^3}z(t) \end{aligned}$$

Introducing state variables (also called phase variables):

$$x_1 = z(t), x_2 = \frac{\mathrm{d}}{\mathrm{d}t}z(t) \text{ and } x_3 = \frac{\mathrm{d}^2}{\mathrm{d}t^2}z(t)$$

Note that usually the states do not represent physical values.

As matrix equations

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \boldsymbol{C}^T\boldsymbol{x} = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

Phase variable canonical form

As matrix equations

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} &= \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\-a_0 & -a_1 & -a_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}u(t)\\ y(t) &= \boldsymbol{C}^T\boldsymbol{x} = \begin{bmatrix}b_0 & b_1 & b_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\end{aligned}$$

SiSo transfer function

$$\begin{aligned}y &= b_0x_1 + b_1x_2 + b_2x_3\\ u &= a_0x_1 + a_1x_2 + a_2x_3 + \frac{\mathrm{d}}{\mathrm{d}t}x_3\end{aligned}$$

The resulting system of first order differential equations:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}x_1 &= x_2\\ \frac{\mathrm{d}}{\mathrm{d}t}x_2 &= x_3\\ \frac{\mathrm{d}}{\mathrm{d}t}x_3 &= -a_0x_1 - a_1x_2 - a_2x_3 + u\end{aligned}$$

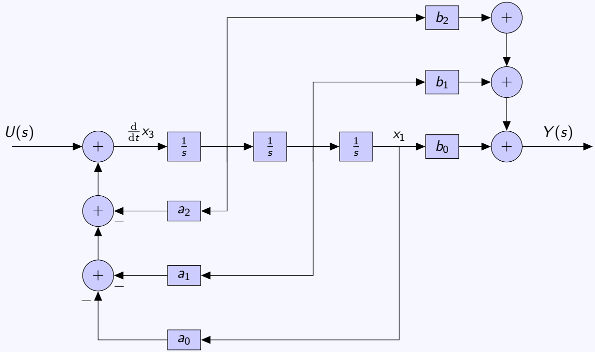
SiSo transfer function

$$\begin{aligned}y &= b_0z(t) + b_1\frac{\mathrm{d}}{\mathrm{d}t}z(t) + b_2\frac{\mathrm{d}^2}{\mathrm{d}t^2}z(t)\\ u &= a_0z(t) + a_1\frac{\mathrm{d}}{\mathrm{d}t}z(t) + a_2\frac{\mathrm{d}^2}{\mathrm{d}t^2}z(t) + \frac{\mathrm{d}^3}{\mathrm{d}t^3}z(t)\end{aligned}$$

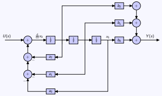
Introducing state variables (also called phase variables):

$$x_1 = z(t), x_2 = \frac{\mathrm{d}}{\mathrm{d}t}z(t) \text{ and } x_3 = \frac{\mathrm{d}^2}{\mathrm{d}t^2}z(t)$$

Note that usually the states do not represent physical values.



Phase variable canonical form



As matrix equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\-a_0 & -a_1 & -a_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}u(t)$$

$$y(t) = \boldsymbol{C}^T\boldsymbol{x} = \begin{bmatrix}b_0 & b_1 & b_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

SiSo transfer function

$$y = b_0x_1 + b_1x_2 + b_2x_3$$
$$u = a_0x_1 + a_1x_2 + a_2x_3 + \frac{\mathrm{d}}{\mathrm{d}t}x_3$$

The resulting system of first order differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}x_1 = x_2$$
$$\frac{\mathrm{d}}{\mathrm{d}t}x_2 = x_3$$
$$\frac{\mathrm{d}}{\mathrm{d}t}x_3 = -a_0x_1 - a_1x_2 - a_2x_3 + u$$

Increasing order

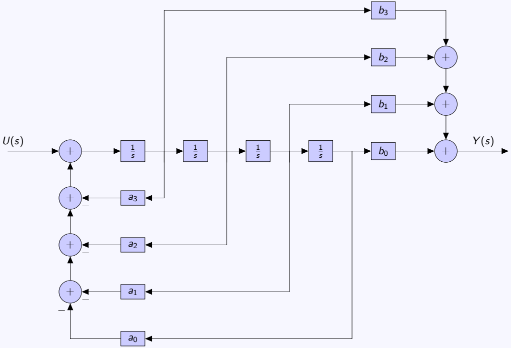


Table of Contents

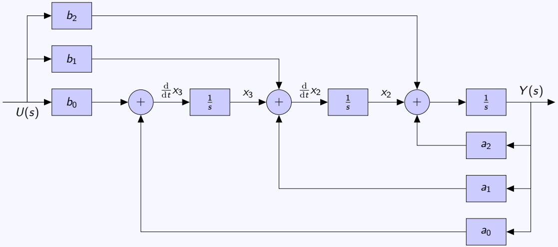
Canonical forms

4.6 Canonical forms
4.6.1 SiSo transfer function
4.6.2 Phase variable canonical form
4.6.3 Input feedworward canonical form

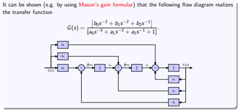
Alternative flow graph

It can be shown (e.g. by using Mason's gain formular) that the following flow diagram realizes the transfer function

$$G(s) = \frac{\left[b_0 s^{-3} + b_1 s^{-2} + b_2 s^{-1}\right]}{\left[a_0 s^{-3} + a_1 s^{-2} + a_2 s^{-1} + 1\right]}.$$



Transfer function

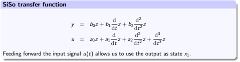


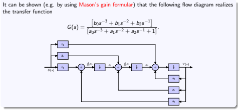
SiSo transfer function

$$
\begin{aligned}
y &= b_0 z + b_1 \frac{\mathrm{d}}{\mathrm{d}t} z + b_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2} z \\
u &= a_0 z + a_1 \frac{\mathrm{d}}{\mathrm{d}t} z + a_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2} z + \frac{\mathrm{d}^3}{\mathrm{d}t^3} z
\end{aligned}
$$

Feeding forward the input signal $u(t)$ allows us to use the output as state x_1.

Transfer function





SiSo transfer function

Introducing state variables:

$$
\begin{aligned}
x_1 &= y \\
\frac{\mathrm{d}}{\mathrm{d}t}x_1 &= -a_2 x_1 + x_3 + b_2 u \\
\frac{\mathrm{d}}{\mathrm{d}t}x_2 &= -a_1 x_1 + x_2 + b_1 u \\
\frac{\mathrm{d}}{\mathrm{d}t}x_3 &= -a_0 + b_0 u
\end{aligned}
$$

Transfer function

SiSo transfer function

Introducing state variables:

$$
\begin{aligned}
x_1 &= y \\
\frac{\mathrm{d}}{\mathrm{d}t}x_1 &= -a_2x_1 + x_3 + b_2u \\
\frac{\mathrm{d}}{\mathrm{d}t}x_2 &= -a_1x_1 + x_2 + b_1u \\
\frac{\mathrm{d}}{\mathrm{d}t}x_3 &= -a_0 + b_0u
\end{aligned}
$$

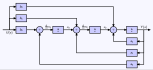
SiSo transfer function

$$
\begin{aligned}
y &= b_0z + b_1\frac{\mathrm{d}}{\mathrm{d}t}z + b_2\frac{\mathrm{d}^2}{\mathrm{d}t^2}z \\
u &= a_0z + a_1\frac{\mathrm{d}}{\mathrm{d}t}z + a_2\frac{\mathrm{d}^2}{\mathrm{d}t^2}z + \frac{\mathrm{d}^3}{\mathrm{d}t^3}z
\end{aligned}
$$

Feeding forward the input signal $u(t)$ allows us to use the output as state x_1.

It can be shown (e.g. by using Mason's gain formular) that the following flow diagram realizes the transfer function

$$G(s) = \frac{\left[b_0s^{-3} + b_1s^{-2} + b_2s^{-1}\right]}{\left[a_0s^{-3} + a_1s^{-2} + a_2s^{-1} + 1\right]}.$$



As matrix equation

$$
y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} u(t)
$$

Table of Contents

State variable models

4.1 Example system
4.2 Block Diagram
4.3 First order differential equation
4.4 Laplace domain
4.5 Stability of State Variable Systems
4.6 Canonical forms
4.7 Discrete time
4.8 Exercises

Continuous time model

Continuous time model

$$
\begin{aligned}
\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_c\boldsymbol{x}(t) + \boldsymbol{B}_c\boldsymbol{u}(t)\\
\boldsymbol{y}(t) &= \boldsymbol{C}_c\boldsymbol{x}(t) + \boldsymbol{D}_c\boldsymbol{u}(t)\\
\boldsymbol{x}(t) &= \exp\left(\boldsymbol{A}_c\left(t-t_0\right)\right)\boldsymbol{x}(t_0) + \int_{t_0}^{t}\exp\left(\boldsymbol{A}_c\left(t-\tau\right)\right)\boldsymbol{B}_c\boldsymbol{u}(\tau)d\tau
\end{aligned}
$$

Thus

$$
\begin{aligned}
\boldsymbol{x}((k+1)T) &= \exp\left(\boldsymbol{A}_c\left((k+1)T-kT\right)\right)\boldsymbol{x}(kT)\\
&\quad + \int_{kT}^{(k+1)T}\exp\left(\boldsymbol{A}_c\left(kT-\tau\right)\right)\boldsymbol{B}_c\boldsymbol{u}(\tau)d\tau\\
&= \exp\left(\boldsymbol{A}_c\left(T\right)\right)\boldsymbol{x}(kT)\\
&\quad + \int_{kT}^{(k+1)T}\exp\left(\boldsymbol{A}_c\left(kT-\tau\right)\right)\boldsymbol{B}_c\boldsymbol{u}(\tau)d\tau
\end{aligned}
$$

Discrete time

Continuous time model

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_c\boldsymbol{x}(t) + \boldsymbol{B}_c\boldsymbol{u}(t)\\ \boldsymbol{y}(t) &= \boldsymbol{C}_c\boldsymbol{x}(t) + \boldsymbol{D}_c\boldsymbol{u}(t)\\ \boldsymbol{x}(t) &= \exp\left(\boldsymbol{A}_c\left(t-t_0\right)\right)\boldsymbol{x}(t_0) + \int_{t_0}^{t}\exp\left(\boldsymbol{A}_c\left(t-\tau\right)\right)\boldsymbol{B}_c\boldsymbol{u}(\tau)d\tau\end{aligned}$$

Thus

$$\begin{aligned}\boldsymbol{x}((k+1)T) &= \exp\left(\boldsymbol{A}_c\left((k+1)T-kT\right)\right)\boldsymbol{x}(kT)\\ &\quad+\int_{kT}^{(k+1)T}\exp\left(\boldsymbol{A}_c\left(kT-\tau\right)\right)\boldsymbol{B}_c\boldsymbol{u}(\tau)d\tau\\ &= \exp\left(\boldsymbol{A}_c\left(T\right)\right)\boldsymbol{x}(kT)\\ &\quad+\int_{kT}^{(k+1)T}\exp\left(\boldsymbol{A}_c\left(kT-\tau\right)\right)\boldsymbol{B}_c\boldsymbol{u}(\tau)d\tau\end{aligned}$$

Zero-order hold assumption

$$\boldsymbol{x}((k+1)T) = \exp\left(\boldsymbol{A}_c\left(T\right)\right)\boldsymbol{x}(kT) + \int_{kT}^{(k+1)T}\exp\left(\boldsymbol{A}_c\left(kT-\tau\right)\right)\boldsymbol{B}_c\boldsymbol{u}(\tau)d\tau$$

Based on the assumption that the *analog* $\Leftrightarrow$ *digital converter* gives constant values in between time steps (*zero-order hold*) one gets

$$\begin{aligned}\boldsymbol{x}_{k+1} &= \boldsymbol{A}_d\boldsymbol{x}_k + \boldsymbol{B}_d\boldsymbol{u}_k\\ \boldsymbol{A}_d &= \exp\left(\boldsymbol{A}_c T\right)\\ \boldsymbol{B}_d &= \int_0^T\exp\left(\boldsymbol{A}_c\left(T-\tau\right)\right)\boldsymbol{B}_c d\tau\end{aligned}$$

Zero-order hold assumption

$$\boldsymbol{x}((k+1)T) = \exp(\boldsymbol{A}_c(T))\,\boldsymbol{x}(kT) + \int_{kT}^{(k+1)T} \exp(\boldsymbol{A}_c(kT-\tau))\,\boldsymbol{B}_c\boldsymbol{u}(\tau)d\tau$$

Based on the assumption that the *analog* ⇔ *digital converter* gives constant values in between time steps (*zero-order hold*) one gets

$$\begin{aligned}\boldsymbol{x}_{k+1} &= \boldsymbol{A}_d\boldsymbol{x}_k + \boldsymbol{B}_d\boldsymbol{u}_k\\ \boldsymbol{A}_d &= \exp(\boldsymbol{A}_cT)\\ \boldsymbol{B}_d &= \int_0^T \exp(\boldsymbol{A}_c(T-\tau))\,\boldsymbol{B}_c d\tau\end{aligned}$$

Continuous time model

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_c\boldsymbol{x}(t) + \boldsymbol{B}_c\boldsymbol{u}(t)\\ \boldsymbol{y}(t) &= \boldsymbol{C}_c\boldsymbol{x}(t) + \boldsymbol{D}_c\boldsymbol{u}(t)\\ \boldsymbol{x}(t) &= \exp(\boldsymbol{A}_c(t-t_0))\,\boldsymbol{x}(t_0) + \int_{t_0}^{t} \exp(\boldsymbol{A}_c(t-\tau))\,\boldsymbol{B}_c\boldsymbol{u}(\tau)d\tau\end{aligned}$$

Thus

$$\begin{aligned}\boldsymbol{x}((k+1)T) &= \exp(\boldsymbol{A}_c((k+1)T-kT))\,\boldsymbol{x}(kT)\\ &\quad + \int_{kT}^{(k+1)T} \exp(\boldsymbol{A}_c(kT-\tau))\,\boldsymbol{B}_c\boldsymbol{u}(\tau)d\tau\\ &= \exp(\boldsymbol{A}_c(T))\,\boldsymbol{x}(kT)\\ &\quad + \int_{kT}^{(k+1)T} \exp(\boldsymbol{A}_c(kT-\tau))\,\boldsymbol{B}_c\boldsymbol{u}(\tau)d\tau\end{aligned}$$

Complete description

$$\begin{aligned}\boldsymbol{x}_{k+1} &= \boldsymbol{A}_d\boldsymbol{x}_k + \boldsymbol{B}_d\boldsymbol{u}_k\\ \boldsymbol{y}_k &= \boldsymbol{C}_d\boldsymbol{x}_k + \boldsymbol{D}_d\boldsymbol{u}_k\\ \boldsymbol{A}_d &= \exp(\boldsymbol{A}_cT)\\ \boldsymbol{B}_d &= \int_0^T \exp(\boldsymbol{A}_c(T-\tau))\,\boldsymbol{B}_c d\tau\\ \boldsymbol{C}_d &= \boldsymbol{C}_c\\ \boldsymbol{D}_d &= \boldsymbol{D}_c.\end{aligned}$$

Example

Complete description

$$
\begin{aligned}
\boldsymbol{x}_{k+1} &= \boldsymbol{A}_d \boldsymbol{x}_k + \boldsymbol{B}_d \boldsymbol{u}_k \\
\boldsymbol{y}_k &= \boldsymbol{C}_d \boldsymbol{x}_k + \boldsymbol{D}_d \boldsymbol{u}_k \\
\boldsymbol{A}_d &= \exp(\boldsymbol{A}_c T) \\
\boldsymbol{B}_d &= \int_0^T \exp(\boldsymbol{A}_c (T-\tau)) \boldsymbol{B}_c d\tau \\
\boldsymbol{C}_d &= \boldsymbol{C}_c \\
\boldsymbol{D}_d &= \boldsymbol{D}_c.
\end{aligned}
$$

Zero-order hold assumption

$$
\boldsymbol{x}((k+1)T) = \exp(\boldsymbol{A}_c(T))\boldsymbol{x}(kT) + \int_{kT}^{(k+1)T} \exp(\boldsymbol{A}_c(kT-\tau))\boldsymbol{B}_c \boldsymbol{u}(\tau) d\tau
$$

Based on the assumption that the *analog* ⇔ *digital converter* gives constant values in between time steps (*zero-order hold*) one gets

$$
\begin{aligned}
\boldsymbol{x}_{k+1} &= \boldsymbol{A}_d \boldsymbol{x}_k + \boldsymbol{B}_d \boldsymbol{u}_k \\
\boldsymbol{A}_d &= \exp(\boldsymbol{A}_c T) \\
\boldsymbol{B}_d &= \int_0^T \exp(\boldsymbol{A}_c (T-\tau)) \boldsymbol{B}_c d\tau
\end{aligned}
$$

Continuous time model

$$
\begin{aligned}
\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_c \boldsymbol{x}(t) + \boldsymbol{B}_c \boldsymbol{u}(t) \\
\boldsymbol{y}(t) &= \boldsymbol{C}_c \boldsymbol{x}(t) + \boldsymbol{D}_c \boldsymbol{u}(t) \\
\boldsymbol{x}(t) &= \exp(\boldsymbol{A}_c(t-t_0))\boldsymbol{x}(t_0) + \int_{t_0}^{t} \exp(\boldsymbol{A}_c(t-\tau))\boldsymbol{B}_c \boldsymbol{u}(\tau) d\tau
\end{aligned}
$$

Thus

$$
\begin{aligned}
\boldsymbol{x}((k+1)T) &= \exp(\boldsymbol{A}_c((k+1)T-kT))\boldsymbol{x}(kT) \\
&\quad + \int_{kT}^{(k+1)T} \exp(\boldsymbol{A}_c(kT-\tau))\boldsymbol{B}_c \boldsymbol{u}(\tau) d\tau \\
&= \exp(\boldsymbol{A}_c(T))\boldsymbol{x}(kT) \\
&\quad + \int_{kT}^{(k+1)T} \exp(\boldsymbol{A}_c(kT-\tau))\boldsymbol{B}_c \boldsymbol{u}(\tau) d\tau
\end{aligned}
$$

Example

$$
T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1} + 8s^{-2} + 6s^{-3}}{1 + 8s^{-1} + 16s^{-2} + 6s^{-3}}
$$

Using phase variable state model[a] one gets:

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -16 & -8 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 6 & 8 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\end{aligned}
$$

[a] See e.g. Dorf, Bishop for relationship between transfer function and state model

Example

Example

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1}+8s^{-2}+6s^{-3}}{1+8s^{-1}+16s^{-2}+6s^{-3}}$$

Using phase variable state model[a] one gets:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -16 & -8 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 6 & 8 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

[a] See e.g. Dorf, Bishop for relationship between transfer function and state model

Complete description

$$\begin{aligned} \boldsymbol{x}_{k+1} &= \boldsymbol{A}_d \boldsymbol{x}_k + \boldsymbol{B}_d \boldsymbol{u}_k \\ \boldsymbol{y}_k &= \boldsymbol{C}_d \boldsymbol{x}_k + \boldsymbol{D}_d \boldsymbol{u}_k \\ \boldsymbol{A}_d &= \exp(\boldsymbol{A}_c T) \\ \boldsymbol{B}_d &= \int_0^T \exp(\boldsymbol{A}_c (T-\tau)) \boldsymbol{B}_c d\tau \\ \boldsymbol{C}_d &= \boldsymbol{C}_c \\ \boldsymbol{D}_d &= \boldsymbol{D}_c. \end{aligned}$$

Zero-order hold assumption

$$\boldsymbol{x}((k+1)T) = \exp(\boldsymbol{A}_c(T))\boldsymbol{x}(kT) + \int_{kT}^{(k+1)T} \exp(\boldsymbol{A}_c(kT-\tau)) \boldsymbol{B}_c \boldsymbol{u}(\tau) d\tau$$

Based on the assumption that the *analog* ⇔ *digital converter* gives constant values in between time steps (*zero-order hold*) one gets

$$\begin{aligned} \boldsymbol{x}_{k+1} &= \boldsymbol{A}_d \boldsymbol{x}_k + \boldsymbol{B}_d \boldsymbol{u}_k \\ \boldsymbol{A}_d &= \exp(\boldsymbol{A}_c T) \\ \boldsymbol{B}_d &= \int_0^T \exp(\boldsymbol{A}_c (T-\tau)) \boldsymbol{B}_c d\tau \end{aligned}$$

Example (cont.)

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1}+8s^{-2}+6s^{-3}}{1+8s^{-1}+16s^{-2}+6s^{-3}}$$

Using $T = 3$ one gets:

$$\boldsymbol{A}_d = \begin{bmatrix} 0.3214 & 0.1950 & 0.0259 \\ -0.1555 & -0.03933 & -0.0123 \\ 0.0740 & 0.0419 & 0.0054 \end{bmatrix}$$

and

$$\boldsymbol{B}_d = \begin{bmatrix} 0.1131 & 0.0259 & -0.0108 \end{bmatrix}^T$$

Example (cont.)

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1}+8s^{-2}+6s^{-3}}{1+8s^{-1}+16s^{-2}+6s^{-3}}$$

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Example

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1}+8s^{-2}+6s^{-3}}{1+8s^{-1}+16s^{-2}+6s^{-3}}$$

Using phase variable state model[a] one gets:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -16 & -8 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 6 & 8 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

[a] See e.g. Dorf, Bishop for relationship between transfer function and state model

Complete description

$$\begin{aligned} x_{k+1} &= A_d x_k + B_d u_k \\ y_k &= C_d x_k + D_d u_k \\ A_d &= \exp(A_c T) \\ B_d &= \int_0^T \exp(A_c(T-\tau)) B_c d\tau \\ C_d &= C_c \\ D_d &= D_c. \end{aligned}$$

Example (cont.)

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1}+8s^{-2}+6s^{-3}}{1+8s^{-1}+16s^{-2}+6s^{-3}}$$

Using $T=5$ one gets:

$$A_d = \begin{bmatrix} 0.1218 & 0.0741 & 0.0099 \\ -0.0592 & -0.0360 & -0.0048 \\ 0.0287 & 0.0175 & 0.0023 \end{bmatrix}$$

and

$$B_d = \begin{bmatrix} 0.1464 & 0.0098 & -0.0023 \end{bmatrix}^T$$

Example (cont.)

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1}+8s^{-2}+6s^{-3}}{1+8s^{-1}+16s^{-2}+6s^{-3}}$$

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Using phase variable state model[a] one gets:

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[a] See e.g. Dorf, Bishop for relationship between transfer function and state model

Example

Now, consider

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1}+8s^{-2}+6s^{-3}}{1+8s^{-1}+s^{-2}+s^{-3}}$$

Using phase variable state model one gets:

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Example

Now, consider

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Example (cont.)

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1} + 8s^{-2} + 6s^{-3}}{1 + 8s^{-1} + 16s^{-2} + 6s^{-3}}$$

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Example (cont.)

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1} + 8s^{-2} + 6s^{-3}}{1 + 8s^{-1} + s^{-2} + s^{-3}}$$

Using phase variable state model and $T = 3$ one gets:

$$\boldsymbol{A}_d = \begin{bmatrix} 0.5666 & 2.1239 & 0.2601 \\ -0.2601 & 0.3065 & 0.0430 \\ -0.0430 & -0.3031 & -0.037 \end{bmatrix}$$

and

$$\boldsymbol{B}_d = \begin{bmatrix} 0.4338 & 0.2602 & 0.0445 \end{bmatrix}^T$$

Example (cont.)

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1}+8s^{-2}+6s^{-3}}{1+8s^{-1}+s^{-2}+s^{-3}}$$

Using phase variable state model and $T=3$ one gets:

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Using phase variable state model one gets:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -8 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 6 & 8 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example (cont.)

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1}+8s^{-2}+6s^{-3}}{1+8s^{-1}+16s^{-2}+6s^{-3}}$$

Using $T=5$ one gets:

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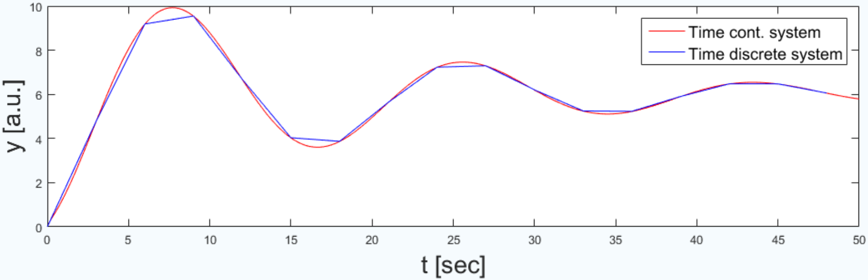
and

$$B_d = \begin{bmatrix} 0.1464 & 0.0098 & -0.0023 \end{bmatrix}^T$$

Example (cont.)

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1}+8s^{-2}+6s^{-3}}{1+8s^{-1}+s^{-2}+s^{-3}}$$



Exercise (#4.1)

Check example file `E04_07.m`.

Table of Contents

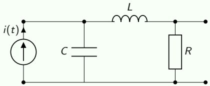
State variable models

4.1 Example system
4.2 Block Diagram
4.3 First order differential equation
4.4 Laplace domain
4.5 Stability of State Variable Systems
4.6 Canonical forms
4.7 Discrete time
4.8 Exercises

Exercise

Exercise (#4.2)

Given is the schematic shown below.



One can derive the following set of equations:

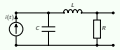
$$
\begin{aligned}
C\frac{\mathrm{d}}{\mathrm{d}t}u_c(t) &= i(t) - i_L(t) \\
L\frac{\mathrm{d}}{\mathrm{d}t}i_L(t) &= -Ri_L(t) + u_c(t) \\
u_{out}(t) &= Ri_L(t)
\end{aligned}
$$

Write this in state variable form and plot the step response making use of NumPy or MATLAB. Assume $R = 50\,\Omega$, $C = 1\,\mathrm{nF}$ and $L = 1\,\mathrm{mH}$ and zero initial state. Compare to a direct solution making use of the inverse Laplace transform.

Exercise

Exercise (#4.2)

Given is the schematic shown below.



One can derive the following set of equations:

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\begin{aligned}
C\frac{\mathrm{d}}{\mathrm{d}t}u_c(t) &= i(t) - i_L(t)\\
L\frac{\mathrm{d}}{\mathrm{d}t}i_L(t) &= -Ri_L(t) + u_c(t)\\
u_{out}(t) &= Ri_L(t)
\end{aligned}
$$

Write this in state variable form and plot the step response making use of NUMPY or MATLAB. Assume $R = 50\,\Omega$, $C = 1\,\text{nF}$ and $L = 1\,\text{mH}$ and zero initial state. Compare to a direct solution making use of the inverse Laplace transform.

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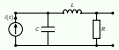
Given is the schematic shown below.



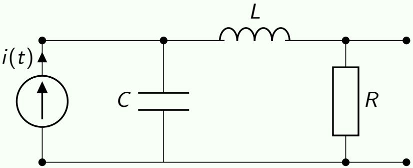
One can derive the following set of equations:

$$\begin{aligned} C\tfrac{\mathrm{d}}{\mathrm{d}t}u_c(t) &= i(t) - i_L(t) \\ L\tfrac{\mathrm{d}}{\mathrm{d}t}i_L(t) &= -Ri_L(t) + u_c(t) \\ u_{out}(t) &= Ri_L(t) \end{aligned}$$

Write this in state variable form and plot the step response making use of NUMPY or MATLAB. Assume $R = 50\,\Omega$, $C = 1\,\mathrm{nF}$ and $L = 1\,\mathrm{mH}$ and zero initial state. Compare to a direct solution making use of the inverse Laplace transform.

Exercise (#4.3)

Given is the schematic shown below.

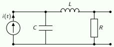


Solve the characteristic equation.

Exercise

Exercise (#4.3)

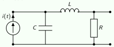
Given is the schematic shown below.



Solve the characteristic equation.

Exercise (#4.3)

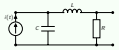
Given is the schematic shown below.



Solve the characteristic equation.

Exercise (#4.2)

Given is the schematic shown below.



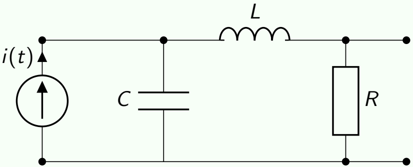
One can derive the following set of equations:

$$\begin{aligned} C\frac{d}{dt}u_c(t) &= i(t) - i_L(t) \\ L\frac{d}{dt}i_L(t) &= -Ri_L(t) + u_c(t) \\ u_{out}(t) &= Ri_L(t) \end{aligned}$$

Write this in state variable form and plot the step response making use of NUMPY or MATLAB. Assume $R = 50\,\Omega$, $C = 1\,\text{nF}$ and $L = 1\,\text{mH}$ and zero initial state. Compare to a direct solution making use of the inverse Laplace transform.

Exercise (#4.4)

Given is the schematic shown below.

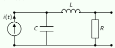


Sketch the phase variable canonical form and determine the coefficients by making use of the transfer function.

Exercise

Exercise (#4.4)

Given is the schematic shown below.



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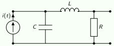

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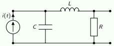
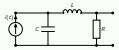


Solve the characteristic equation.

Exercise (#4.2)

Given is the schematic shown below.



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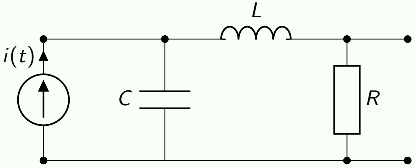
Write this in state variable form and plot the step response making use of NUMPY or MATLAB. Assume $R = 50\,\Omega$, $C = 1\,\text{nF}$ and $L = 1\,\text{mH}$ and zero initial state. Compare to a direct solution making use of the inverse Laplace transform.

Exercise (#4.5)

Given is the schematic shown below.

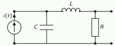


Find $H(s) = \frac{U_r(s)}{I(s)}$ using state variables.

Exercise

Exercise (#4.5)

Given is the schematic shown below.



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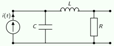
Given is the schematic shown below.



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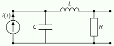
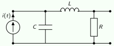
Given is the schematic shown below.



Sketch the phase variable canonical form and determine the coefficients by making use of the transfer function.

Exercise (#4.3)

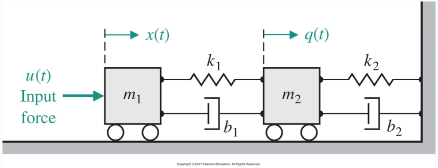
Given is the schematic shown below.



Solve the characteristic equation.

Exercise (#4.6)

Given is the system shown below.



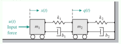
Determine a state space representation of the system taking $u(t)$ as input and $q(t)$ as output and under the assumption that friction can be neglected. Plot the step and frequency response of the system using NUMPY[a] or MATLAB. Simulate the response to a sine wave with an amplitude of 1 N and a frequency of 0.5 Hz and 0.2 Hz, respectively as well.

[a] Use $m_1 = 300\,\mathrm{g}$, $m_2 = 100\,\mathrm{g}$, $k_1 = 1\,\mathrm{N\,m^{-1}}$, $k_2 = 1\,\mathrm{N\,m^{-1}}$, $b_1 = 0.1\,\mathrm{kg\,s^{-1}}$, $b_2 = 0.1\,\mathrm{kg\,s^{-1}}$

Exercise

Exercise (#4.6)

Given is the system shown below.



Determine a state space representation of the system taking $u(t)$ as input and $q(t)$ as output and under the assumption that friction can be neglected. Plot the step and frequency response of the system using NUMPY[a] or MATLAB. Simulate the response to a sine wave with an amplitude of 1 N and a frequency of 0.5 Hz and 0.2 Hz, respectively as well.

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Table of Contents

Control

5. Feedback & Control

Table of Contents

Feedback & Control

5.1 Introduction
5.2 Open-loop systems
5.3 Closed-loop systems
5.4 Feed-forward Control
5.5 State Variable Feedback Systems
5.6 Additional Exercises
5.7 Appendix

Basic Architecture

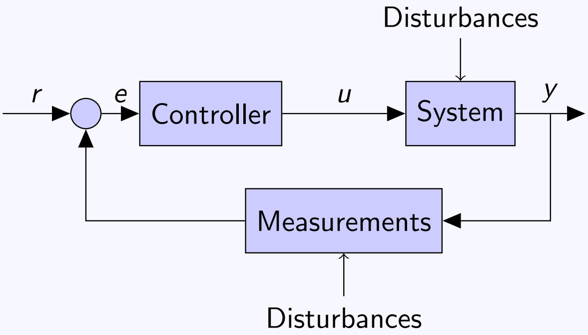


Table of Contents

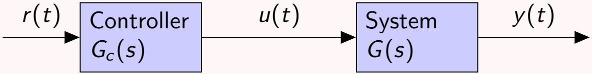
Feedback & Control

5.1 Introduction
5.2 Open-loop systems
5.3 Closed-loop systems
5.4 Feed-forward Control
5.5 State Variable Feedback Systems
5.6 Additional Exercises
5.7 Appendix

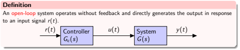
Open-loop systems

Definition

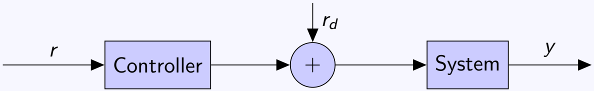
An open-loop system operates without feedback and directly generates the output in response to an input signal $r(t)$.



Tracking error



Adding disturbances



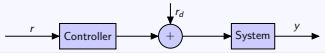
Definition

The tracking error is defined as follows:

$$\begin{aligned} E(s) &= R(s) - Y(s) \\ &= R(s) - [R(s)G_c(s) + R_d(s)]\,G(s), \end{aligned}$$

with $G_c(s)$ being the controller transfer function and $G(s)$ being the system transfer function.

Steady state error

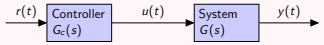

Adding disturbances

Definition

The tracking error is defined as follows:

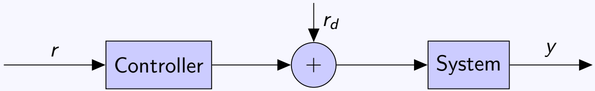
$$E(s) = R(s) - Y(s) = R(s) - [R(s)G_c(s) + R_d(s)]\,G(s),$$

with $G_c(s)$ being the controller transfer function and $G(s)$ being the system transfer function.


Definition

An open-loop system operates without feedback and directly generates the output in response to an input signal $r(t)$.

Open-loop control system



Definition

The steady state error is defined as follows:

$$\begin{aligned} e_0(\infty) &= \lim_{t\to\infty} e(t) \\ &= \lim_{s\to 0} sE(s) \\ &= \lim_{s\to 0} [sR(s) - sR(s)G_c(s)G(s) - sR_d(s)G(s)] \end{aligned}$$

Open-loop control system

r → Controller → $+$ (r_d) → System → y

Definition

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Adding disturbances

r → Controller → $+$ (r_d) → System → y

Definition

The tracking error is defined as follows:

$$\begin{aligned} E(s) &= R(s) - Y(s) \\ &= R(s) - [R(s)G_c(s) + R_d(s)]\,G(s), \end{aligned}$$

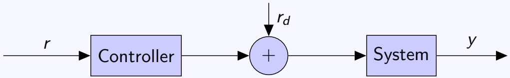
with $G_c(s)$ being the controller transfer function and $G(s)$ being the system transfer function.

Definition

An open-loop system operates without feedback and directly generates the output in response to an input signal $r(t)$.

$r(t)$ → Controller $G_c(s)$ → $u(t)$ → System $G(s)$ → $y(t)$

Open-loop control system



Properties

Using the unit step function as a comparable input, one gets:

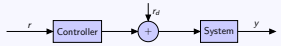
$$\begin{aligned} e_0(\infty) &= \lim_{s\to 0}\left[s\frac{1}{s} - s\left(\frac{1}{s}G_c(s) + R_d(s)\right)G(s)\right] \\ &= 1 - \lim_{s\to 0}\left[G_c(s)G(s) + sR_d(s)G(s)\right] \end{aligned}$$

For $R_d(s) = 0$ (no disturbance):

$$e_0(\infty) = 1 - \lim_{s\to 0}\left[G_c(s)G(s)\right]$$

Drawbacks

Open-loop control system



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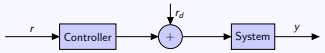
Open-loop control system



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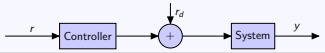
Adding disturbances



Definition

The tracking error is defined as follows:

$$\begin{aligned} E(s) &= R(s)-Y(s) \\ &= R(s)-\left[R(s)G_c(s)+R_d(s)\right]G(s). \end{aligned}$$

with $G_c(s)$ being the controller transfer function and $G(s)$ being the system transfer function.

The open-loop system

... can not react to disturbances

... may be calibrated so that $G(0)G_c(0)=1$

... but during the operation $G(s)$ might change over time

Table of Contents

Feedback & Control

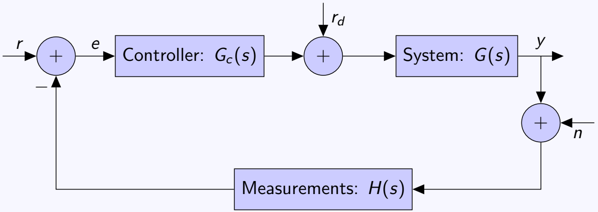
5.1 Introduction
5.2 Open-loop systems
5.3 Closed-loop systems
5.3.1 Basic structure
5.3.2 Unity feedback
5.3.3 Stability
5.3.4 Phase and Gain Margin
5.3.5 Nyquist diagrams
5.3.6 Second order systems
5.3.7 Controller performance indicators
5.3.8 Basic controller
5.3.9 Exemplary systems
5.4 Feed-forward Control
5.5 State Variable Feedback Systems
5.6 Additional Exercises
5.7 Appendix

Closed-loop systems

5.3 Closed-loop systems
5.3.1 Basic structure
5.3.2 Unity feedback
5.3.3 Stability
5.3.4 Phase and Gain Margin
5.3.5 Nyquist diagrams
5.3.6 Second order systems
5.3.7 Controller performance indicators
5.3.8 Basic controller
5.3.9 Exemplary systems

Closed-loop systems

Basic structure



- r_d: Disturbance
- n: Measurement noise

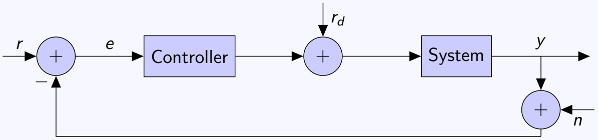
Note: Also called feedback control

Table of Contents

Closed-loop systems

5.3 Closed-loop systems
5.3.1 Basic structure
5.3.2 Unity feedback
5.3.3 Stability
5.3.4 Phase and Gain Margin
5.3.5 Nyquist diagrams
5.3.6 Second order systems
5.3.7 Controller performance indicators
5.3.8 Basic controller
5.3.9 Exemplary systems

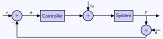
Unity feedback



For $H(s) = 1$ one gets

$$
\begin{aligned}
Y(s) &= \frac{G_c(s)G(s)}{1+G_c(s)G(s)}R(s) + \frac{G(s)}{1+G_c(s)G(s)}R_d(s) \\
&\quad - \frac{G_c(s)G(s)}{1+G_c(s)G(s)}N(s) \\
&= C(s)R(s) + \frac{G(s)}{1+G_c(s)G(s)}R_d(s) - C(s)N(s)
\end{aligned}
$$

Unity feedback



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Loop gain

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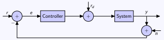
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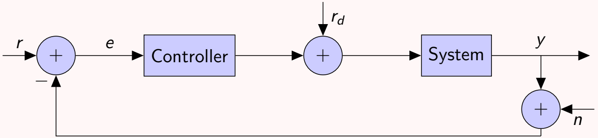
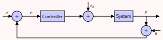
Definition



The loop gain is defined by

$$L(s) = G_c(s)G(s).$$

Definition



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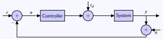
Loop gain

Sensitivity

Unity feedback

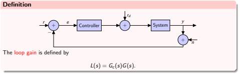


For $H(s) = 1$ one gets

$$\begin{aligned} Y(s) &= \frac{G_c(s)G(s)}{1+G_c(s)G(s)}R(s) + \frac{G(s)}{1+G_c(s)G(s)}R_d(s) \\ &\quad - \frac{G_c(s)G(s)}{1+G_c(s)G(s)}N(s) \\ &= C(s)R(s) + \frac{G(s)}{1+G_c(s)G(s)}R_d(s) - C(s)N(s) \end{aligned}$$

Sensitivity

Sensitivity



Definition

System sensitivity is the ratio of the change e.g. in the system transfer function to the change of a process transfer function (or parameter) for a small incremental change

Dorf, Bishop

Loop gain

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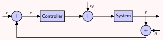

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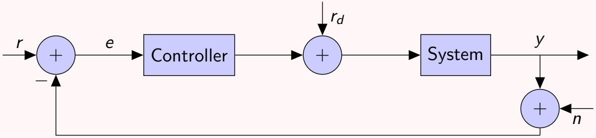


With

$$F(s) = 1 + L(s)$$

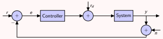
one can define the sensitivity as follows[a]

$$S(s) = \frac{G_C}{F^2(s)}.$$

[a] This describes how relative errors in the plant affect the closed loop transfer function. One can define other sensitivity functions as well. E.g. S_G^E describes the sensitivity of the tracking error to small changes in the transfer function of the plant $G(s)$: $S_G^E = \frac{-G(s)G_c(s)}{1+G_c(s)G(s)}$.

Without distortion

Definition



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Dorf, Bishop

Sensitivity

Without distortion

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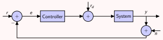

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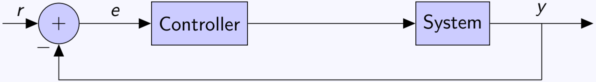
Dorf, Bishop

Tracking error

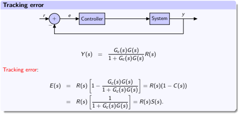


$$Y(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} R(s)$$

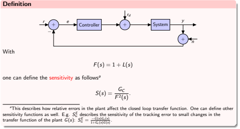
Tracking error:

$$\begin{aligned} E(s) &= R(s)\left[1 - \frac{G_c(s)G(s)}{1 + G_c(s)G(s)}\right] = R(s)(1 - C(s)) \\ &= R(s)\left[\frac{1}{1 + G_c(s)G(s)}\right] = R(s)S(s). \end{aligned}$$

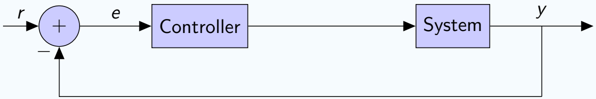
Unity feedback without disturbance



Without distortion



Example

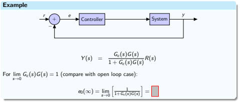


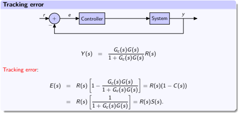
$$Y(s) = \frac{G_c(s)G(s)}{1+G_c(s)G(s)}R(s)$$

For $\lim_{s\to 0} G_c(s)G(s) = 1$ (compare with open loop case):

$$e_0(\infty) = \lim_{s\to 0}\left[\frac{1}{1+G_c(s)G(s)}\right] = \boxed{}$$

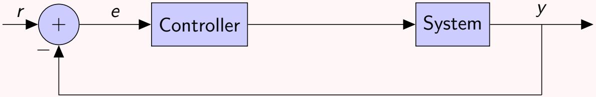
DC loop gain





Without distortion

Definition



The DC loop gain $L(0)$ is defined as follows:

$$L(0) \quad = \quad G_c(0)G(0)$$

The steady state error will be *small* if the DC loop gain is *reasonably* large:

$$e_0(\infty) \quad = \quad \frac{1}{1+L(0)}$$

Table of Contents

Closed-loop systems

5.3 Closed-loop systems
- 5.3.1 Basic structure
- 5.3.2 Unity feedback
- 5.3.3 Stability
- 5.3.4 Phase and Gain Margin
- 5.3.5 Nyquist diagrams
- 5.3.6 Second order systems
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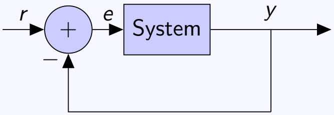
Closed-loop system

Stability criterion



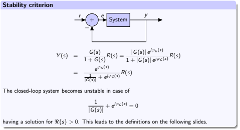
$$Y(s) = \frac{G(s)}{1+G(s)}R(s) = \frac{|G(s)|\, e^{j\varphi_G(s)}}{1+|G(s)|\, e^{j\varphi_G(s)}}R(s)$$

$$= \frac{e^{\varphi_G(s)}}{\frac{1}{|G(s)|}+e^{j\varphi_G(s)}}R(s)$$

The closed-loop system becomes unstable in case of

$$\frac{1}{|G(s)|}+e^{j\varphi_G(s)}=0$$

having a solution for $\Re\{s\} > 0$. This leads to the definitions on the following slides.



Stability criterion

The closed-loop system becomes unstable in case of

$$1 + |G(s)|\, e^{j\varphi_G(s)} = 0$$

having a solution for $\Re\{s\} > 0$.
For linear systems: The closed loop is stable if the gain of the loop transfer function is less than one for all frequencies

Complex plane

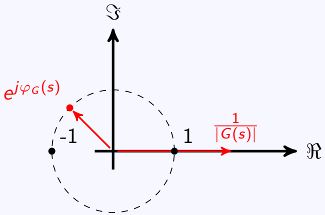


Table of Contents

Closed-loop systems

5.3 Closed-loop systems
5.3.1 Basic structure
5.3.2 Unity feedback
5.3.3 Stability
5.3.4 Phase and Gain Margin
5.3.5 Nyquist diagrams
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5.3.7 Controller performance indicators
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5.3.9 Exemplary systems

Phase margin

Margins

In practice it is not enough that the system is stable: There must also be some margins of stability. See also elective *SiCo 2*.

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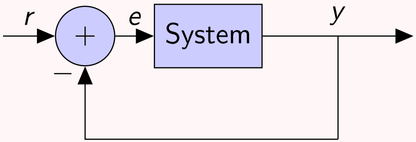
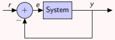


The phase margin is defined as 180° plus the phase of the open-loop transfer function at unity gain.

> *The phase margin is the amount of phase shift of $L(j\omega)$ at unity magnitude that will result in a marginally stable system with intersection of the $-1 + j0$ point on the Nyquist plot.*
>
> *Dorf, Bishop*

Phase crossover frequency

Definition



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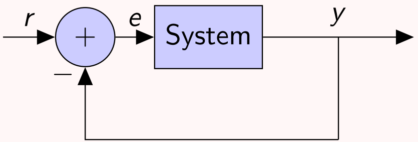
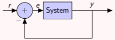


The angular frequency ω_π for which $\arg\left(G(s = j\omega_\pi)\right) = \pi$ holds true is called the phase crossover frequency.

Gain margin

Definition



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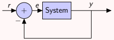

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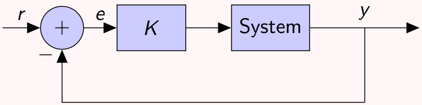


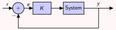
The gain margin is defined as the reciprocal of the open-loop transfer function at $s = j\omega_\pi$:

$$\mathrm{GM} = \frac{1}{|G(s = j\omega_\pi)|}.$$

The gain margin tells you how much you can increase a gain factor K until the closed loop system becomes unstable.

Gain margin

Definition



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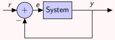

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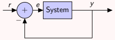
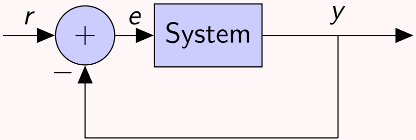


The phase margin is defined as 180° plus the phase of the open-loop transfer function at unity gain.

The phase margin is the amount of phase shift of $L(j\omega)$ at unity magnitude that will result in a marginally stable system with intersection of the $-1 + j0$ point on the Nyquist plot.

Dorf, Bishop

Definition

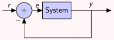


The gain margin is the increase in the system gain when phase $= -180^\circ$ that will result in a marginally stable system with intersection of the $-1 + j0$ point on the Nyquist plot.

Dorf, Bishop

Gain and phase margin

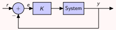
Definition



The gain margin is the increase in the system gain when phase $= -180^\circ$ that will result in a marginally stable system with intersection of the $-1 + j0$ point on the Nyquist plot.

Dorf, Bishop

Definition

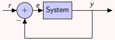


The gain margin is defined as the reciprocal of the open-loop transfer function at $s = j\omega_\pi$:

$$\mathrm{GM} = \frac{1}{|G(s = j\omega_\pi)|}.$$

The gain margin tells you how much you can increase a gain factor K until the closed loop system becomes unstable.

Definition

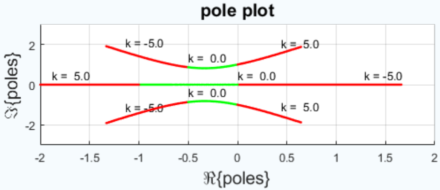


The angular frequency ω_π for which $\arg\left(G(s = j\omega_\pi)\right) = \pi$ holds true is called the phase crossover frequency.

Example

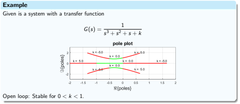
Given is a system with a transfer function

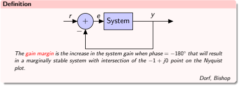
$$G(s) = \frac{1}{s^3 + s^2 + s + k}$$

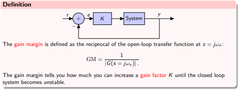


Open loop: Stable for $0 < k < 1$.

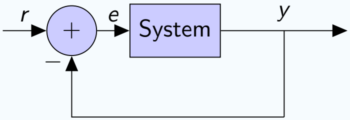
Adding feedback





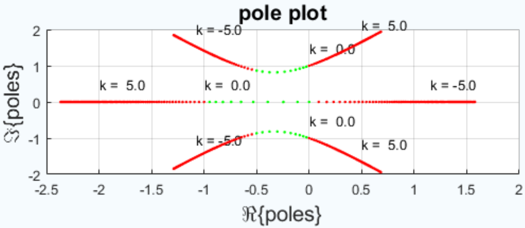


Example



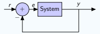
$$G(s) = \frac{1}{s^3 + s^2 + s + k}$$

Stable for $-1 < k < 0$.



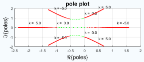
Gain and phase margin

Example



$G(s) = \frac{1}{s^3+s^2+s+k}$

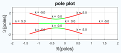
Stable for $-1 < k < 0$.



Example

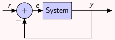
Given is a system with a transfer function

$$G(s) = \frac{1}{s^3+s^2+s+k}$$



Open loop: Stable for $0 < k < 1$.

Definition

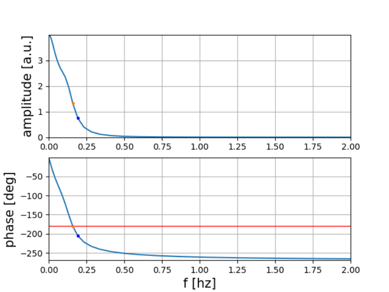


The gain margin is the increase in the system gain when phase $= -180°$ that will result in a marginally stable system with intersection of the $-1 + j0$ point on the Nyquist plot.

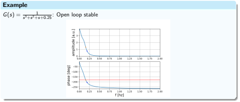
Dorf, Bishop

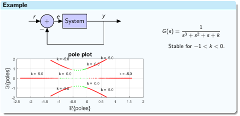
Example

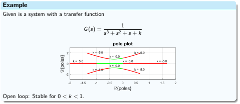
$G(s) = \frac{1}{s^3+s^2+s+0.25}$: Open loop stable



Gain and phase margin

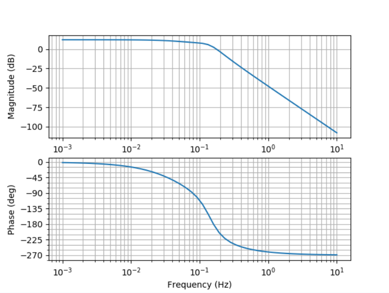




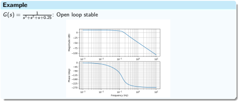


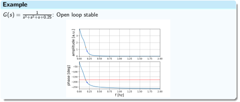
Example

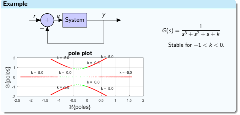
$G(s) = \frac{1}{s^3+s^2+s+0.25}$: Open loop stable



Gain and phase margin

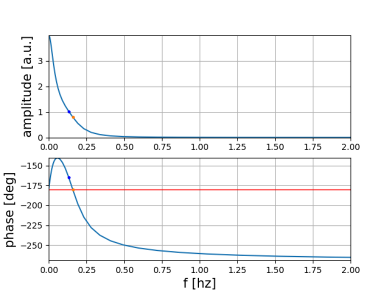




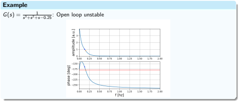


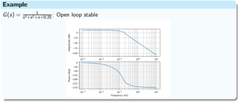
Example

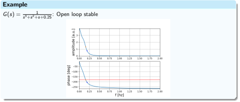
$G(s) = \frac{1}{s^3+s^2+s-0.25}$: Open loop unstable



Gain and phase margin

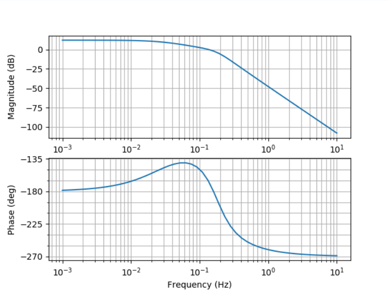




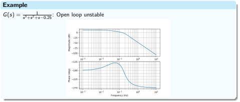


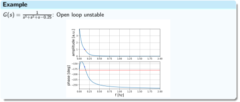
Example

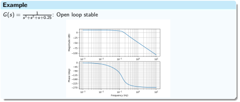
$G(s) = \frac{1}{s^3+s^2+s-0.25}$: Open loop unstable



Gain and phase margin







Properties

- ▷ Negative gain **and** phase margin: Closed loop system (with negative feedback) is unstable

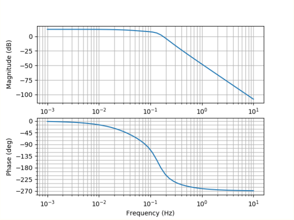


Figure 15: Unstable closed loop

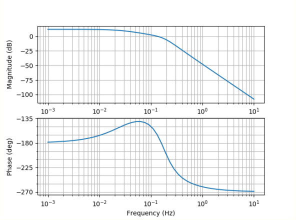
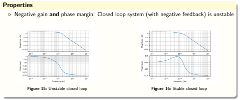
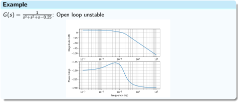
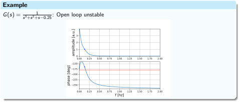


Figure 16: Stable closed loop







Exercise (#5.1)

Plot the bode diagrams and check open and closed loop (unity feedback, $G_C(s) = 1$) stability for

▷
$$G(s) = \frac{1}{s^3 + s^2 + s - 0.25}$$

▷
$$G(s) = \frac{1}{s^3 + s^2 + s + 0.25}$$

by making use of NUMPY or MATLAB. Plot the unit step responses of open and closed loop systems.

Exercise

Exercise (#5.1)

Plot the bode diagrams and check open and closed loop (unity feedback, $G_C(s) = 1$) stability for

- $$G(s) = \frac{1}{s^3 + s^2 + s - 0.25}$$
- $$G(s) = \frac{1}{s^3 + s^2 + s + 0.25}$$

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Exercise

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Plot the bode diagrams and check open and closed loop (unity feedback, $G_C(s) = 1$) stability for

- $$G(s) = \frac{1}{s^3 + s^2 + s - 0.25}$$
- $$G(s) = \frac{1}{s^3 + s^2 + s + 0.25}$$

by making use of NUMPY or MATLAB. Plot the unit step responses of open and closed loop systems.

Properties

- Negative gain **and** phase margin: Closed loop system (with negative feedback) is unstable

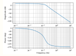


Figure 15: Unstable closed loop

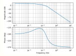
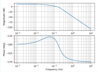


Figure 16: Stable closed loop

Example

$G(s) = \frac{1}{s^3+s^2+s-0.25}$: Open loop unstable



Exercise (#5.2)

Skech the bode diagrams and check open and closed loop (unity feedback, $G_C(s) = 1$) stability for

- $$G(s) = \frac{1}{s^2 + 2s + 3},$$
- $$G(s) = \frac{1}{s + 0.25},$$
- $$G(s) = \frac{1}{s - 0.25}.$$
- $$G(s) = \frac{1}{s^3 + 2.25s^2 + 3.5s + 0.75}$$

Sketch the unit step responses of open and closed loop systems.

Exercise

Exercise (#5.2)

Skech the bode diagrams and check open and closed loop (unity feedback, $G_C(s) = 1$) stability for

- $$G(s) = \frac{1}{s^2 + 2s + 3},$$
- $$G(s) = \frac{1}{s + 0.25},$$
- $$G(s) = \frac{1}{s - 0.25}.$$
- $$G(s) = \frac{1}{s^3 + 2.25s^2 + 3.5s + 0.75}$$

Sketch the unit step responses of open and closed loop systems.

Closed-loop systems

5.3 Closed-loop systems
5.3.1 Basic structure
5.3.2 Unity feedback
5.3.3 Stability
5.3.4 Phase and Gain Margin
5.3.5 Nyquist diagrams
5.3.6 Second order systems
5.3.7 Controller performance indicators
5.3.8 Basic controller
5.3.9 Exemplary systems

Using Nyquist diagrams to check stability: Stable plant

Checking stability for the unity feedback closed loop system with

$$C(s) = \frac{G(s)G_c(s)}{1 + G(s)G_c(s)}$$

using the Nyquist diagram in case of $G(s)$ being stable:

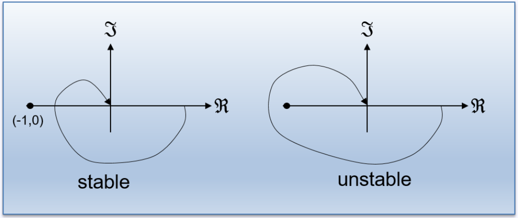
1. Plot the Nyquist diagram for the open loop system $G(s)G_c(s)$
2. Travel the Nyquist contour from $\omega = 0$ to $\omega = \infty$.
3. The closed loop system is (still) stable, if the point $(-1, 0)$ is always to the left.

Using Nyquist diagrams to check stability: Stable plant

Checking stability for the unity feedback closed loop system

$$C(s) = \frac{G(s)G_c(s)}{1 + G(s)G_c(s)}$$

using the Nyquist diagram in case of $G(s)$ being stable:



Using Nyquist diagrams to check stability: Stable plant
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Using Nyquist diagrams to check stability: Stable plant

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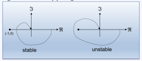

Using Nyquist diagrams to check stability: Stable plant

Checking stability for the unity feedback closed loop system with

$$C(s) = \frac{G(s)G_c(s)}{1+G(s)G_c(s)}$$

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3. The closed loop system is (still) stable, if the point $(-1,0)$ is always to the left.

Using Nyquist diagrams to check stability: Unstable plant

Checking stability for the unity feedback closed loop system with

$$C(s) = \frac{G(s)G_c(s)}{1+G(s)G_c(s)}$$

using the Nyquist diagram (general case):

1. Plot the Nyquist diagram for the **open loop** system $G(s)G_c(s)$
2. Count the number of poles r_k with a real part larger than zero
3. Count the number of poles i_k with a real part of zero.

Using Nyquist diagrams to check stability: Unstable plant

Checking stability for the unity feedback closed loop system with

$$C(s) = \frac{G(s)G_c(s)}{1 + G(s)G_c(s)}$$

using the Nyquist diagram (general case):

1. Plot the Nyquist diagram for the **open loop** system $G(s)G_c(s)$
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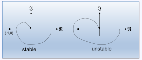
3. Count the number of poles i_k with a real part of zero.

Using Nyquist diagrams to check stability: Stable plant

Checking stability for the unity feedback closed loop system

$$C(s) = \frac{G(s)G_c(s)}{1 + G(s)G_c(s)}$$

using the Nyquist diagram in case of $G(s)$ being stable:



Using Nyquist diagrams to check stability: Stable plant

Checking stability for the unity feedback closed loop system with

$$C(s) = \frac{G(s)G_c(s)}{1 + G(s)G_c(s)}$$

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Using Nyquist diagrams to check stability: Unstable plant

Checking stability for the unity feedback closed loop system with

$$C(s) = \frac{G(s)G_c(s)}{1 + G(s)G_c(s)}$$

using the Nyquist diagram (general case):

4. Travel the Nyquist contour from $\omega = 0$ to $\omega = \infty$.
5. Sum up the angles of $\triangle\varphi$ of $G(s)G_c(s)$ encircling the point $(-1, 0)$. If the curve goes counter-clockwise: Count angles positive. Else: Count them negative. In other words: Examine the number of encirclements of the point $(-1, 0)$
6. The closed loop system is stable if $\triangle\varphi = i_k\frac{\pi}{2} + r_k\pi$.

Gain and phase margin

Using Nyquist diagrams to check stability: Unstable plant

Checking stability for the unity feedback closed loop system with

$$C(s) = \frac{G(s)G_c(s)}{1 + G(s)G_c(s)}$$

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Using Nyquist diagrams to check stability: Unstable plant

Checking stability for the unity feedback closed loop system with

$$C(s) = \frac{G(s)G_c(s)}{1 + G(s)G_c(s)}$$

using the Nyquist diagram (general case):

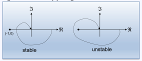
1. Plot the Nyquist diagram for the **open loop** system $G(s)G_c(s)$
2. Count the number of poles r_k with a real part larger than zero
3. Count the number of poles i_k with a real part of zero.

Using Nyquist diagrams to check stability: Stable plant

Checking stability for the unity feedback closed loop system

$$C(s) = \frac{G(s)G_c(s)}{1 + G(s)G_c(s)}$$

using the Nyquist diagram in case of $G(s)$ being stable:



Properties

- Negative gain and phase margin: Closed loop system (with negative feedback) is unstable

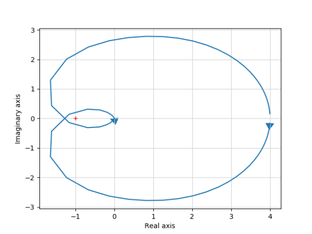


Figure 17: Transfer function $G(s)$ (unstable closed loop)

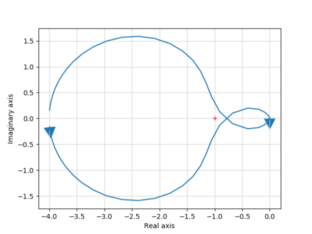


Figure 18: Transfer function $G(s)$ (stable closed loop)

Properties

- Negative gain and phase margin: Closed loop system (with negative feedback) is unstable

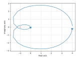


Figure 17: Transfer function $G(s)$ (unstable closed loop)

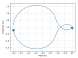


Figure 18: Transfer function $G(s)$ (stable closed loop)

Using Nyquist diagrams to check stability: Unstable plant

Checking stability for the unity feedback closed loop system with

$$C(s) = \frac{G(s)G_c(s)}{1+G(s)G_c(s)}$$

using the Nyquist diagram (general case):

4. Travel the Nyquist contour from $\omega = 0$ to $\omega = \infty$.
5. Sum up the angles of $\triangle\varphi$ of $G(s)G_c(s)$ encircling the point $(-1,0)$. If the curve goes counter-clockwise: Count angles positive. Else: Count them negative. In other words: Examine the number of encirclements of the point $(-1,0)$
6. The closed loop system is stable if $\triangle\varphi = i_k\frac{\pi}{2} + r_k\pi$.

Using Nyquist diagrams to check stability: Unstable plant

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$$C(s) = \frac{G(s)G_c(s)}{1+G(s)G_c(s)}$$

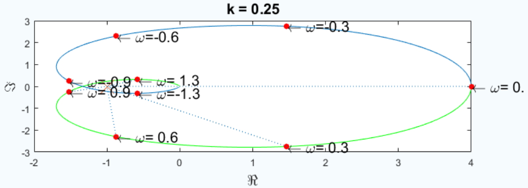
using the Nyquist diagram (general case):

1. Plot the Nyquist diagram for the **open loop** system $G(s)G_c(s)$
2. Count the number of poles r_k with a real part larger than zero
3. Count the number of poles i_k with a real part of zero.

Example: Stable open loop system

Given is a system with a transfer function

$$G(s) = \frac{1}{s^3+s^2+s+0.5}$$



Poles: $(-0.34+j0.82)$, $(-0.34-j0.82)$, $(-0.32) \rightarrow r_k = 0 = i_k = 0 \rightarrow$ closed loop is unstable.

Nyquist diagrams

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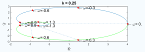

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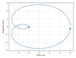

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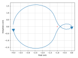

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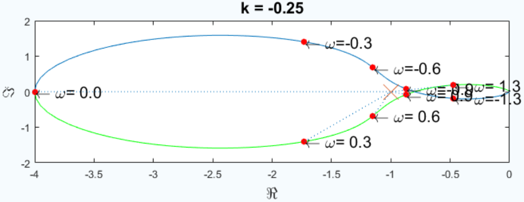
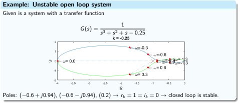
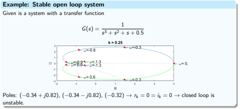
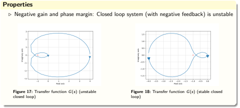


Poles: $(-0.6 + j0.94)$, $(-0.6 - j0.94)$, (0.2) $\rightarrow r_k = 1 = i_k = 0 \rightarrow$ closed loop is stable.

Nyquist diagrams







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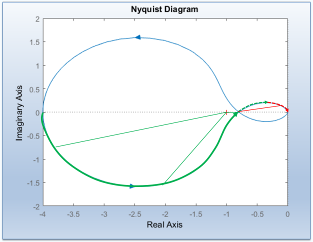

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Table of Contents

Closed-loop systems

5.3 Closed-loop systems
5.3.1 Basic structure
5.3.2 Unity feedback
5.3.3 Stability
5.3.4 Phase and Gain Margin
5.3.5 Nyquist diagrams
5.3.6 Second order systems
5.3.7 Controller performance indicators
5.3.8 Basic controller
5.3.9 Exemplary systems

Poles

Consider the transfer function

$$G(s) = \frac{1}{s^2 + ps + q}$$

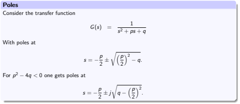
With poles at

$$s = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}.$$

For $p^2 - 4q < 0$ one gets poles at

$$s = -\frac{p}{2} \pm j\sqrt{q - \left(\frac{p}{2}\right)^2}.$$

Second-order systems



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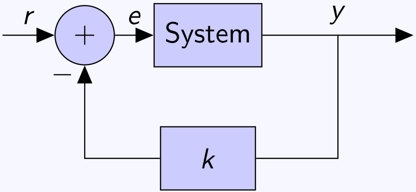
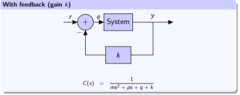
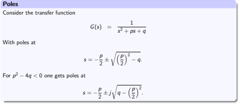
With feedback (gain k)



$$C(s) = \frac{1}{ms^2 + ps + q + k}$$

Second-order systems

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Routh array:

s^2	m	q+k
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A second-order system is stable, if all of the coefficients are either positive or negative (no sign change). See Routh-Hurwitz criterion.

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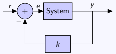

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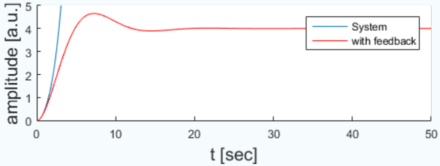
Example

$$G(s) = \frac{1}{s^2 + 0.5s - 0.75}$$

This system is stabilized by using feedback:

$$C(s) = \frac{1}{s^2 + 0.5s - 0.75 + 1.}$$



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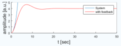

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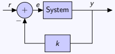

$$C(s) = \frac{1}{ms^2 + ps + q + k}$$

With feedback (gain k)

Consider

$$G(s) = \frac{\omega_0^2}{s^2 + 2\sigma\omega_0 s + \omega_0^2}$$

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$$C(s) = \frac{\omega_0^2}{s^2 + 2\sigma\omega_0 s + (k+1)\omega_0^2}$$

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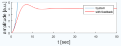

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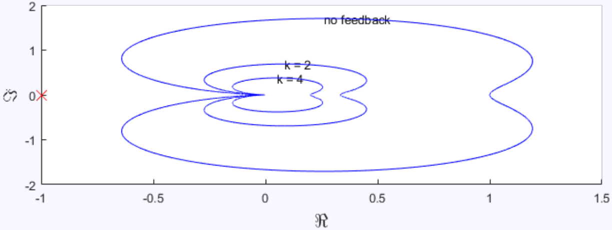
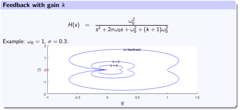
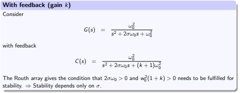
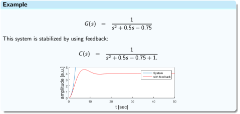


Second-order systems





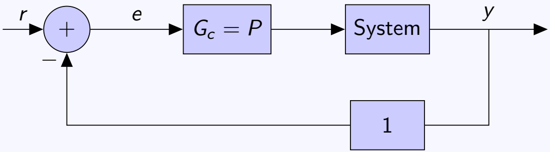


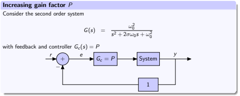
Increasing gain factor P

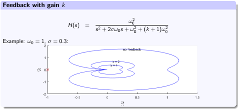
Consider the second order system

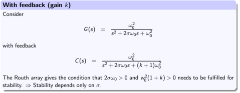
$$G(s) = \frac{\omega_0^2}{s^2+2\sigma\omega_0 s+\omega_0^2}$$

with feedback and controller $G_c(s) = P$









With feedback (gain k)

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The Routh array gives the condition that $2\sigma\omega_0 > 0$ and $w_0^2(1+k) > 0$ needs to be fulfilled for stability. $\Rightarrow$ Stability depends only on σ.

Properties: second order systems

The phase of second order systems never crosses -180 degree and thus, they never provide positive feedback. Consequently, the gain margin is infinite.

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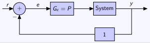

Feedback with gain k

$$H(s) = \frac{\omega_0^2}{s^2 + 2\sigma\omega_0 s + \omega_0^2 + (k+1)\omega_0^2}$$

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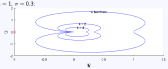

Increasing gain factor P

$$C(s) = \frac{P\omega_0^2}{s^2 + 2\sigma\omega_0 s + \omega_0^2(1+P)}$$

The Routh array gives the condition that

$$2\sigma\omega_0 > 0$$

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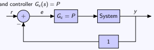

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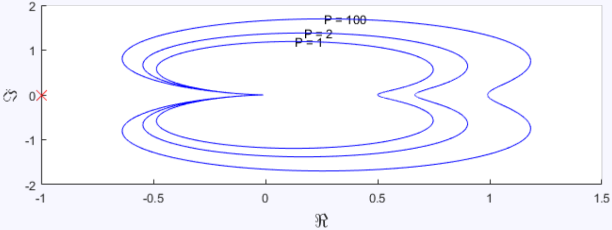

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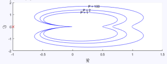

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Phase crossover frequency

$$\begin{aligned} -\pi &= \arg\left\{(j\omega_\pi)^2 + jp\omega_\pi + q\right\} \\ &= \arg\left\{-\omega_\pi^2 + jp\omega_\pi + q\right\} \end{aligned}$$

A second-order system ($p \neq 0$) does not have a phase crossover frequency and thus the gain margin is infinite.

Table of Contents

Closed-loop systems

5.3 Closed-loop systems
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5.3.4 Phase and Gain Margin
5.3.5 Nyquist diagrams
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Exemplary step response

Exemplary step response

Step response: Some properties

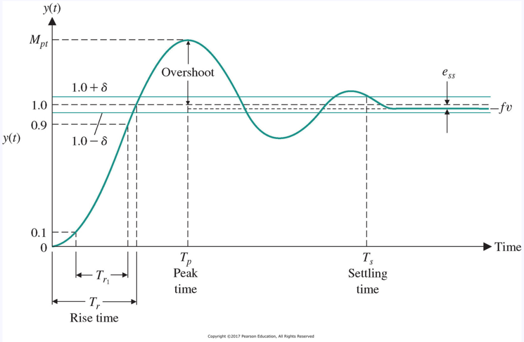
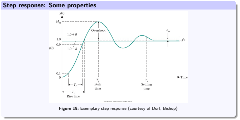


Figure 19: Exemplary step response (courtesy of Dorf, Bishop)

Steady state response



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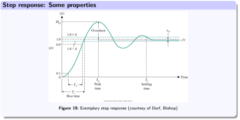

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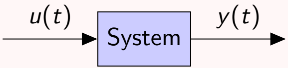
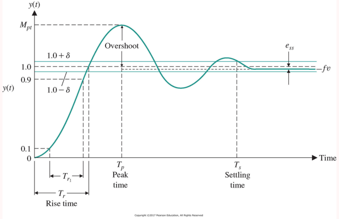
Exemplary step response

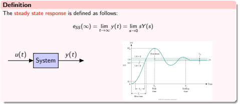
Definition

The steady state response is defined as follows:

$$e_{SS}(\infty) = \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$

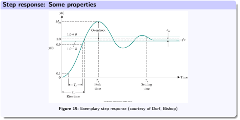






Steady state response

Overshoot



Overshoot

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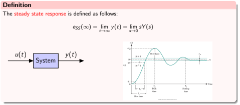
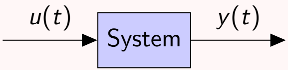
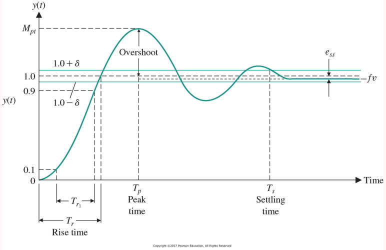


Steady state response

Definition

The overshoot is the maximum difference between the transient and steady state response towards a unit step input. It is often used as a measure for the relative stability.





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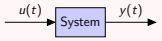
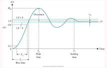




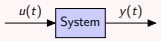
Overshoot

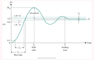
Settling time

Definition

The steady state response is defined as follows:

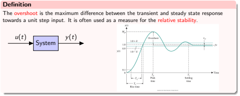
$$e_{ss}(\infty) = \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$





Settling time

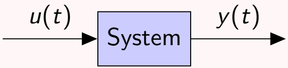
Settling time

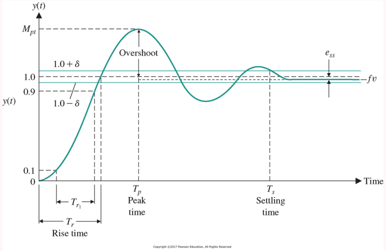


Overshoot

Definition

The settling time is usually defined as the time required for the response to a unit step input to reach and remain within a specific distance to the steady state value (e.g. 2% or 5%).





Rise time

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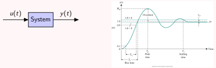

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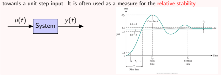
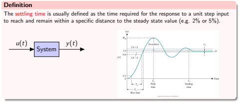


Rise time

Rise time

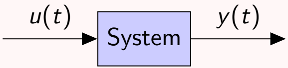
Rise time

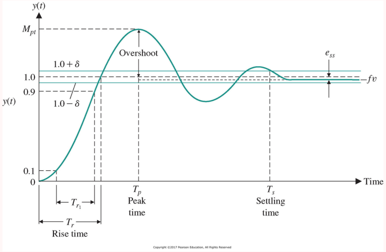


Settling time

Definition

The rise timeis frequently defined as the time required for the response to a unit step input to rise e.g. from 10% to 90% or 0% to 100% of the steady state value.





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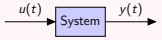

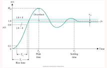

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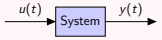
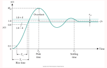
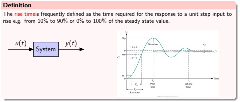




Time to peak

Time to peak

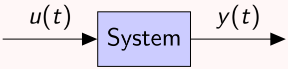
Time to peak

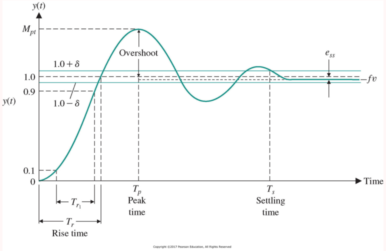


Rise time

Definition

The time to peak or peak time is the time required to reach the maximum of the response to a unit step input.





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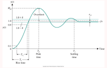
$u(t)$ → System → $y(t)$



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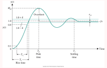
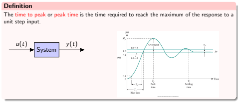


Delay-time

Delay-time

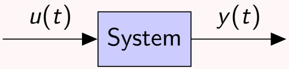
Delay-time

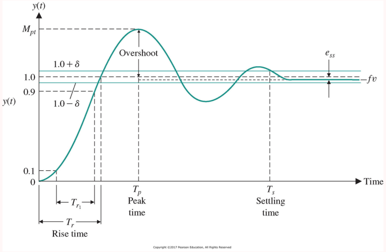


Time to peak

Definition

The delay-time is the time required for the response to reach half the final value the first time.





Resonant peak

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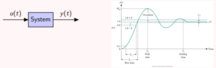

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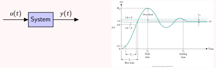
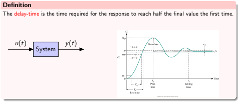


Resonant peak

Resonant peak

Resonant peak



Delay-time

Definition

The resonant peak G_r is the maximum value of $|G(j\omega)|$. The resonant frequency is the frequency at which the peak resonance G_r occurs.

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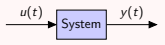

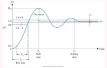

Tracking error

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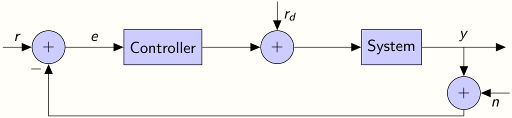
Properties

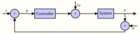


The tracking error:

$$
\begin{aligned}
E(s) &= R(S) - Y(s) \\
&= \frac{1}{1+L(s)}R(s) - \frac{G(s)}{1+L(s)}R_d(s) + \frac{L(s)}{1+L(s)}N(s) \\
&= \frac{1}{F(s)}R(s) - \frac{G(s)}{F(s)}R_d(s) + \frac{F(s)-1}{F(s)}N(s).
\end{aligned}
$$

Tracking error

Properties



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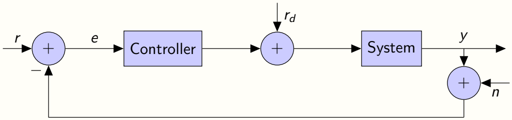


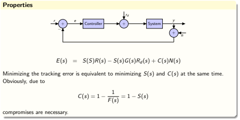
$$E(s) \quad = \quad S(S)R(s) - S(s)G(s)R_d(s) + C(s)N(s)$$

Minimizing the tracking error is equivalent to minimizing $S(s)$ and $C(s)$ at the same time. Obviously, due to

$$C(s) = 1 - \frac{1}{F(s)} = 1 - S(s)$$

compromises are necessary.

Steady state error



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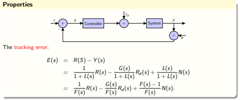

Properties

The tracking error:

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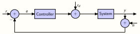
Steady state error

Tracking error

Steady-state error

Steady state error



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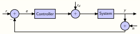

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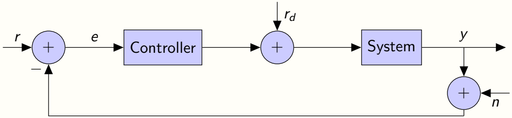
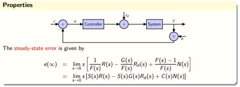
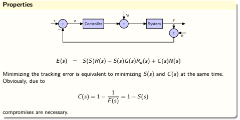


The steady-state error is given by

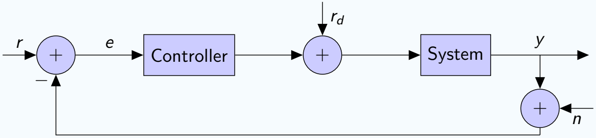
$$\begin{aligned} e(\infty) &= \lim_{s\to 0} s\left[\frac{1}{F(s)}R(s) - \frac{G(s)}{F(s)}R_d(s) + \frac{F(s)-1}{F(s)}N(s)\right] \\ &= \lim_{s\to 0} s\left[S(s)R(s) - S(s)G(s)R_d(s) + C(s)N(s)\right] \end{aligned}$$



Steady state error



Example: Unit step input



The steady-state error is given by

$$
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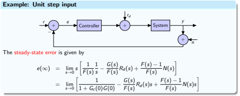
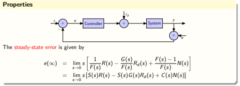
e(\infty) &= \lim_{s\to 0} s\left[\frac{1}{F(s)}\frac{1}{s} - \frac{G(s)}{F(s)}R_d(s) + \frac{F(s)-1}{F(s)}N(s)\right] \\
&= \lim_{s\to 0} \left[\frac{1}{1+G_c(0)G(0)} - \frac{G(s)}{F(s)}R_d(s)s + \frac{F(s)-1}{F(s)}N(s)s\right]
\end{aligned}
$$

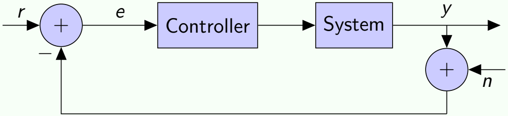




Steady state error

Exercise (#5.3)

Given is the system shown below:



$$
\begin{aligned}
E(s) &= S(S)R(s) + C(s)N(s) \\
G(s) &= \frac{10}{s+10} \\
G_c(s) &= K
\end{aligned}
$$

Plot the tracking error for the unit step response for several values of K assuming n to be white noise with standard deviation of 0, 0.1 and 1, respectively. Make use of MATLAB or NUMPY. Hint: In MATLAB make use of `lsim`.

Exercise

Exercise (#5.3)

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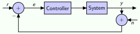

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Steady-state error

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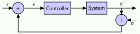

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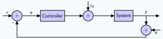
Example: Unit step input



The steady-state error is given by

$$e(\infty) = \lim_{s\to 0} s\left[\frac{1}{F(s)}\frac{1}{s} - \frac{G(s)}{F(s)}R_d(s) + \frac{F(s)-1}{F(s)}N(s)\right]$$
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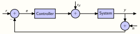
Properties



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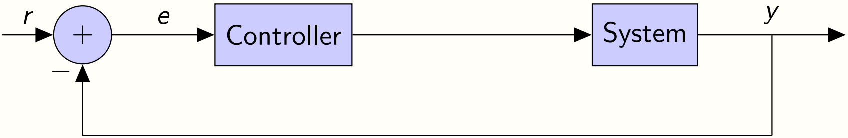

Steady state error for unit step input and absence of disturbances:

$$e_0(\infty) = \lim_{s\to 0} sE(s)$$
$$= \lim_{s\to 0}\left[\frac{1}{1+G_c(s)G(s)}\right]$$

Table of Contents

Closed-loop systems

5.3 Closed-loop systems
5.3.1 Basic structure
5.3.2 Unity feedback
5.3.3 Stability
5.3.4 Phase and Gain Margin
5.3.5 Nyquist diagrams
5.3.6 Second order systems
5.3.7 Controller performance indicators
5.3.8 Basic controller
5.3.9 Exemplary systems

Lag compensator

Lag compensator

Lag compensator

Definition

The lag compensator or phase-lag compensator has a transfer function of the form

$$
\begin{aligned}
G_c(s) &= K\frac{s+z}{s+p}, \text{ with } \alpha = z/p > 1 \text{ or } z > p \\
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Note that the choice of T_1 and T_2 will lead to lead or lag behaviour. See also Dorf, Bishop, 13th edition, page 548.

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- Provides attenuation at high frequencies
- Slows system
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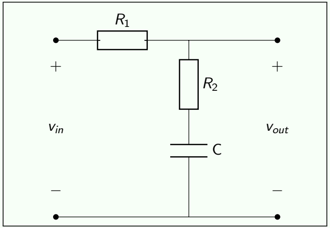

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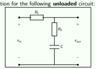

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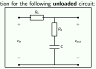

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Exercise (#5.5)

Given are

$$\begin{aligned} G(s) &= \frac{1}{(s+1-jk)(s+1+jk)} \\ &\text{and} \\ G_c(s) &= \frac{2(s+4)}{4(s+2)} \end{aligned}$$

Plot the root locus for the closed-loop systems (unity feedback, compensated/uncompensated, parameter k). Make use of NUMPY or MATLAB.

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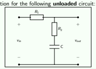

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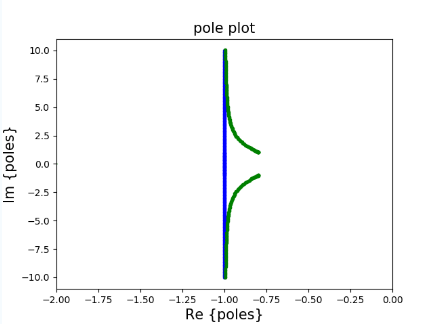

The lag compensator moves the poles to the right and increases DC gain.

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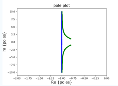

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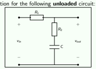

Lead compensator

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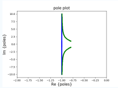

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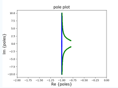

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- Adds positive phase (gain margin increases)
- System stability improves
- System responds faster
- Steady state accuracy does not increase
- Maximum phase lead occurs at $\omega_m = \sqrt{zp}$
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Lead compensator

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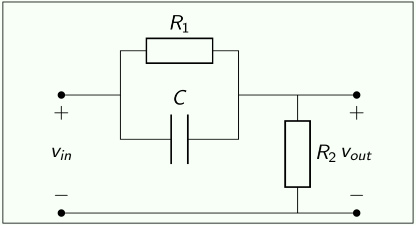

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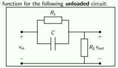

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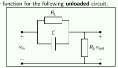

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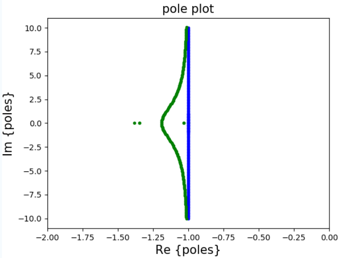

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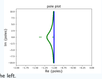

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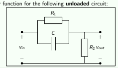

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Exercise (#5.7)

Given is a system and a compensator as follows:

$$G(s) = \frac{20}{s(s+2)}$$
$$G_c(s) = 3\frac{s+4.8}{s+14.4}$$

- Plot the Bode plots of compensated and uncompensated system.
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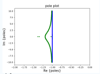

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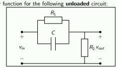

Lead lag compensator

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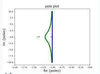

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Cascading a lead and a lag compensator yields the lead-lag compensator with transfer function of the form

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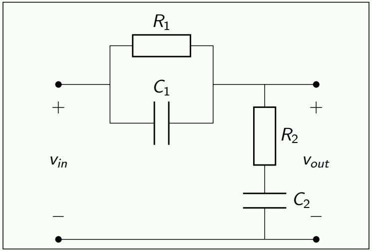

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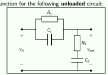

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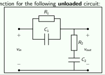

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Lead lag compensator

Exercise (#5.9)

An industrial grinding process is given by[a] the transfer function

$$G_p(s) = \frac{10}{s(s+5)}.$$

The plant shall be controlled using a digital controller based upon an analog prototype of the form

$$G_c(s) = K\frac{s+a}{s+b}.$$

Ensure a phase marging of at least 45^o and a settling time of less than 1 second (2% criterion).

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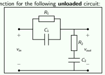

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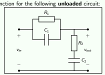

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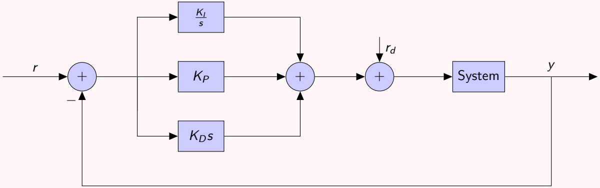

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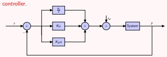

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[a] This exercise is partially based upon Dorf, Bishop. The corresponding source is: Predictive Control of a Robotic Grinding System, Journal of Engineering for Industry, ASME, November 1992, pp. 412-420. See also MATLAB

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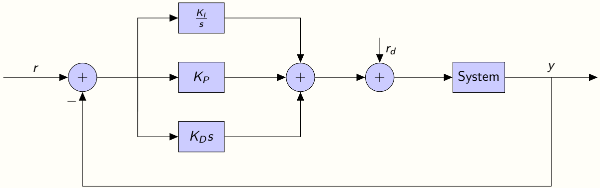
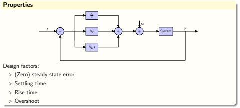
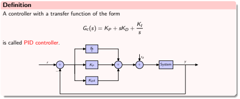


Design factors:

- (Zero) steady state error
- Settling time
- Rise time
- Overshoot

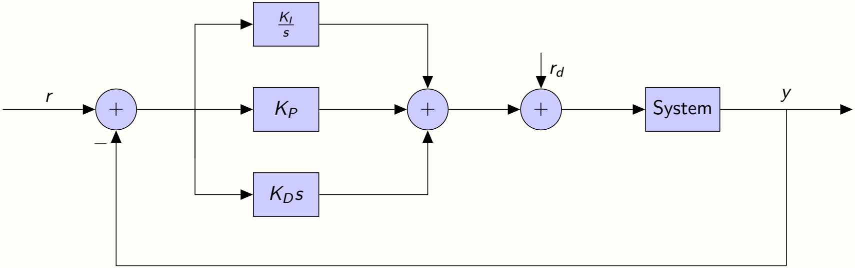
PID controller





PID controller

Properties



- ▷ P term is used to adjust the speed.
- ▷ I term provides zero error.
- ▷ D term introduces damping.

PID controller

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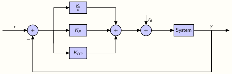

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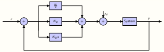

Design factors:

- (Zero) steady state error
- Settling time
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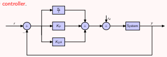
Definition

A controller with a transfer function of the form

$$G_c(s) = K_P + sK_D + \frac{K_I}{s}$$

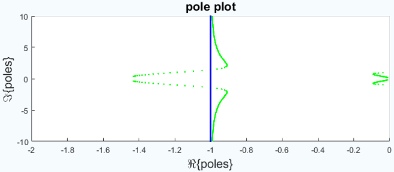
is called PID controller.

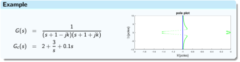


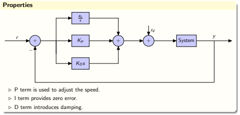
Example

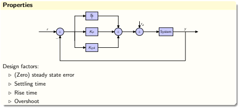
$$G(s) = \frac{1}{(s+1-jk)(s+1+jk)}$$

$$G_c(s) = 2 + \frac{3}{s} + 0.1s$$









Properties

The PID controller combines properties of the lead and the lag compensator.

- ▷ A lead compensator approximates PD control
- ▷ a lag compensator approximates a PI compensator.

PID controller

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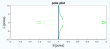

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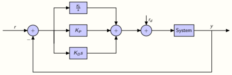

- P term is used to adjust the speed.
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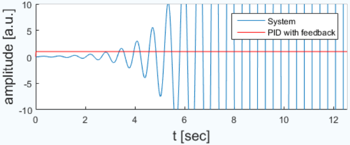
Example

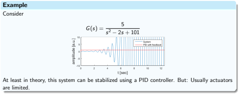
Consider

$$G(s) = \frac{5}{s^2 - 2s + 101}$$



At least in theory, this system can be stabilized using a PID controller. But: Usually actuators are limited.

Ziegler–Nichols method



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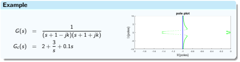

Heuristic tuning

Heuristic method of tuning a PID controller. Works for unkown plants as well! Workflow:

1. Set $K_I = K_D = 0$.
2. Increase K_P up to K_0 until the output becomes unstable
3. Measure the period T_0 of oscillation (frequency f_0)
4. Use the following table to determine K_I and K_D.

Control type	K_P	K_I/K_P	K_D/K_P
P	$0.5K_0$	–	–
PI	$0.45K_0$	$1.2f_0$	–
PD	K_0	–	0
PID tight	$0.6K_0$	$2f_0$	$0.125T_0$
Some overshoot	$0.33K_0$	$2f_0$	$0.33T_0$
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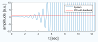

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Exercise (#5.10)

Given is a system (plant) with following transfer function:

$$H(s) = \frac{10}{s^3 + 6s^2 + 11s + 16}.$$

Plot the step response of a unity feedback system with PID controller and the following parameters:

Control type	K_P	K_I	K_D
P	5	–	–
P	5.5	–	–
P	3	–	–
PI	2.7	1.8	–
PID	2	2.22	1.2

Note: Make use of NUMPY and the functions `tf`, `step`, `series` and `feedback`.

Exercise

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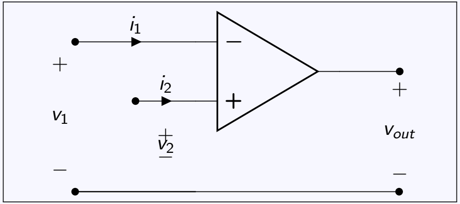
Closed-loop systems

5.3 Closed-loop systems
5.3.1 Basic structure
5.3.2 Unity feedback
5.3.3 Stability
5.3.4 Phase and Gain Margin
5.3.5 Nyquist diagrams
5.3.6 Second order systems
5.3.7 Controller performance indicators
5.3.8 Basic controller
5.3.9 Exemplary systems

OP-amps

OP-amps

Basic operation



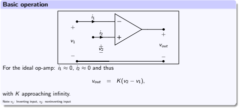
For the ideal op-amp: $i_1 \approx 0$, $i_2 \approx 0$ and thus

$$v_{out} = K(v_2 - v_1),$$

with K approaching infinity.

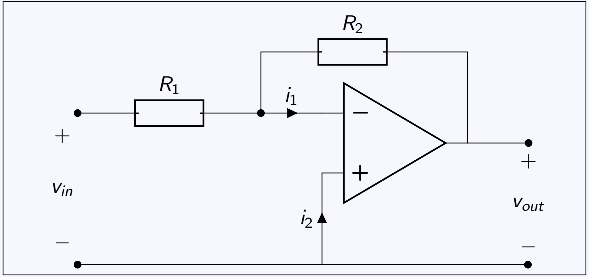
Note v_1: Inverting input, v_2: noninverting input

Inverting amplifier



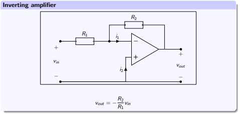
OP-amps

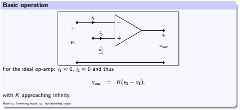
Inverting amplifier



$$v_{out} = -\frac{R_2}{R_1} v_{in}$$

Integrating circuit





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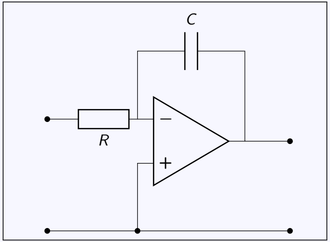
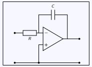


$$H(s) = -\frac{1}{RCs}$$

Differentiating circuit

Integrating circuit



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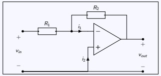

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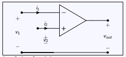

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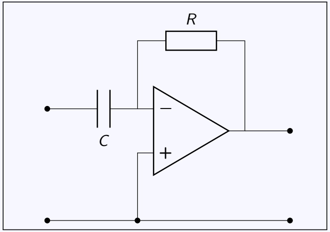
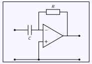


$$H(s) = -RCs$$

Differentiating circuit



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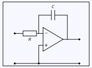

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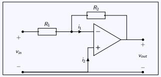

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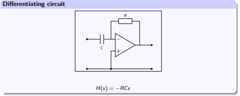
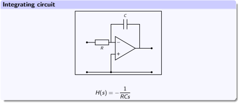
Accelerometer





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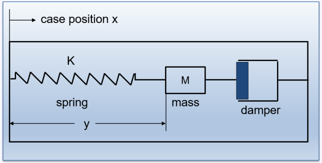
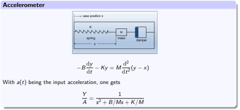


$$-B\frac{\mathrm{d}y}{\mathrm{d}t} - Ky = M\frac{\mathrm{d}^2}{\mathrm{d}t^2}(y - x)$$

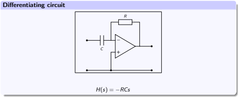
With $a(t)$ being the input acceleration, one gets

$$\frac{Y}{A} = \frac{1}{s^2 + B/Ms + K/M}$$

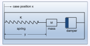


Accelerometer

DC motor



DC motor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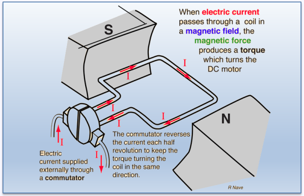
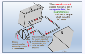


Figure 20: DC motor. Source: R. Nave, Georgia State University

Approximation for armature controlled DC motor (e.g. Dorf, Bishop, 13th edition, page 105):

$$H(s) = \frac{\phi(s)}{V(s)} = \frac{K}{s(\tau s + 1)}$$

DC motor



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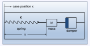

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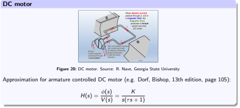

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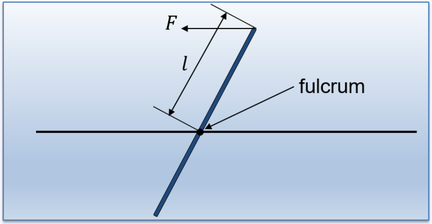
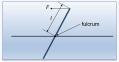


The angular momentum (torque, Drehmoment):

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F},$$

with $\boldsymbol{r}$ being the vector pointing to the fulcrum.

Lever



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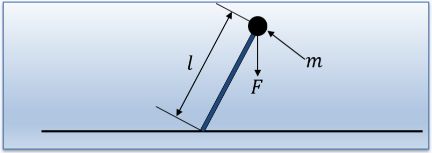


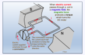
Resistance against rotation (moment of interia, Trägheitsmoment) for a simple pendulum:

$$I = ml^2.$$

Relationship between torque and rotation (shape of body **not** changing):

$$\tau = I\dot{\omega} = ml^2\dot{\omega}$$

DC motor



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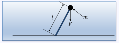
Exercise

Lever



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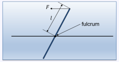

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Exercise (#5.11)

- ▷ Design a PID controler for the inverted pendulum using NUMPY.
- ▷ Use an initial value of $\phi(t) = 0$ and plot the response to $\phi_c(t) = 0$

Sources:

System modeling

PID controller desing with MATLAB

Lever

Exercise

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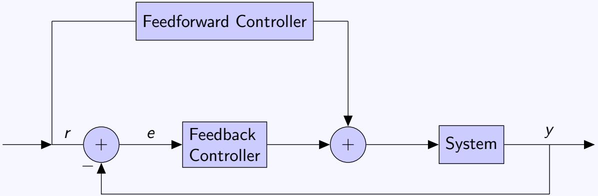
Table of Contents

Feedback & Control

5.1 Introduction
5.2 Open-loop systems
5.3 Closed-loop systems
5.4 Feed-forward Control
5.5 State Variable Feedback Systems
5.6 Additional Exercises
5.7 Appendix

Feed-forward Control

Not covered here



FF controller can be designed by using inverse model of system which leads to rapid response.

Table of Contents

Feedback & Control

5.1 Introduction
5.2 Open-loop systems
5.3 Closed-loop systems
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5.5 State Variable Feedback Systems
- 5.5.1 Introduction
- 5.5.2 Controllability
- 5.5.3 Full-State Feedback Control Design
- 5.5.4 Observable systems
- 5.5.5 Example

5.6 Additional Exercises
5.7 Appendix

Table of Contents

State Variable Feedback Systems

5.5 State Variable Feedback Systems
5.5.1 Introduction
5.5.2 Controllability
5.5.3 Full-State Feedback Control Design
5.5.4 Observable systems
5.5.5 Example

Closed-loop feedback system

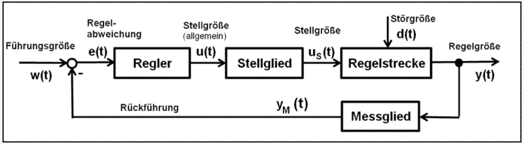


Figure 21: Source: Wikipedia (HeinrichKÜ)

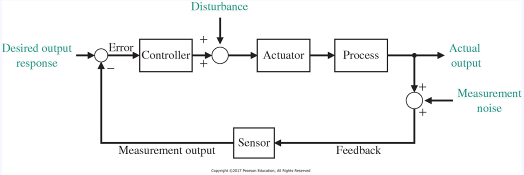


Table of Contents

State Variable Feedback Systems

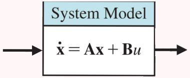
5.5 State Variable Feedback Systems
5.5.1 Introduction
5.5.2 Controllability
5.5.3 Full-State Feedback Control Design
5.5.4 Observable systems
5.5.5 Example

Definition

We consider a system fulfilling the equation

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u.$$

This system



is said to be completely controllable if for each initial state $\boldsymbol{x}(t_0)$ there exists a control signal $u(t)$ that can transfer the system state to any other desired location $\boldsymbol{x}(t_0 + T)$ in a finite time T.

Controllability

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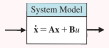

is said to be completely controllable if for each initial state $\boldsymbol{x}(t_0)$ there exists a control signal $u(t)$ that can transfer the system state to any other desired location $\boldsymbol{x}(t_0 + T)$ in a finite time T.

Definition

The $(n \times n)$ controllability matrix is defined as follows:

$$\boldsymbol{P}_c = \left[\boldsymbol{b} \;\; \boldsymbol{A}\boldsymbol{b} \;\; \boldsymbol{A}^2\boldsymbol{b} \ldots \boldsymbol{A}^{n-1}\boldsymbol{b}\right].$$

The system is completely controllable if $\boldsymbol{P_c}$ is of full rank (determinant unequal to zero). Note that controllabilty is a concept based upon unconstrained inputs.

Table of Contents

State Variable Feedback Systems

5.5 State Variable Feedback Systems
5.5.1 Introduction
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Full-state feedback

Task

In this section we deal with the control of controllable systems where all internal states area available.

Task
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Model

Full-state feedback

Model

Task
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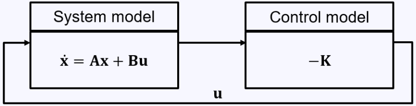
System state



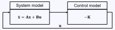
Goal: Determination of the $(1 \times n)$ matrix $\boldsymbol{K}$

- Assume that all states are available (can be measured)
- $u = -\boldsymbol{K}\boldsymbol{x}$
-

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{K}\boldsymbol{x} = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\,\boldsymbol{x}$$

Characteristic equation

System state



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Characteristic equation

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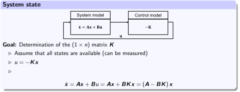
Full-state feedback

Characteristic equation

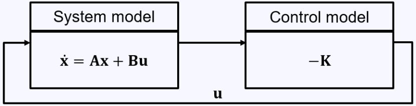


Model

Characteristic equation

The characteristic equation of the full-state feedback system shown below is given by

$$\det\left(\lambda \boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{BK})\right) = 0.$$



If all roots of the characteristic equation lie in the left half-plane, then the closed-loop is stable.

Characteristic equation

The characteristic equation of the full-state feedback system shown below is given by

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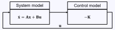

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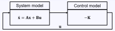
Characteristic equation

Pole placement

System state



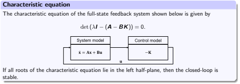
Goal: Determination of the $(1 \times n)$ matrix $\boldsymbol{K}$

- Assume that all states are available (can be measured)
- $u = -\boldsymbol{K}\boldsymbol{x}$
-

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{K}\boldsymbol{x} = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\,\boldsymbol{x}$$

Pole placement

Pole placement


Characteristic equation

The characteristic equation of the full-state feedback system shown below is given by

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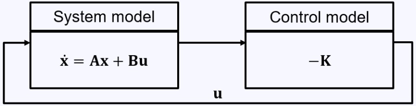
Characteristic equation

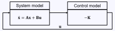
System state



If the system is completely controllable, then $\boldsymbol{K}$ can be determined to place all poles in the left half-plane, so that the transient performance meets the desired response.

Ackermann's formula

System state



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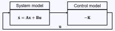

If all roots of the characteristic equation lie in the left half-plane, then the closed-loop is stable.

Definition

For a single input system with $u = -\boldsymbol{K}\boldsymbol{x}$ and a desired characteristic equation

$$q(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} \cdots + \alpha_0$$

one can calculate the gain matrix with Ackermann's formula as follows:

$$\boldsymbol{K} = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} \boldsymbol{P}_c^{-1} q(\boldsymbol{A}),$$

with

$$q(\boldsymbol{A}) = \boldsymbol{A}^n + \alpha_{n-1}\boldsymbol{A}^{n-1} + \cdots \alpha_1 \boldsymbol{A}^1 + \alpha_0 \boldsymbol{I}.$$

and $\boldsymbol{P}_c$ being the controllability matrix.

Ackermann's formula

Definition

For a single input system with $u = -\boldsymbol{Kx}$ and a desired characteristic equation

$$q(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} \cdots + \alpha_0$$

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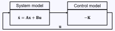
System state



If the system is completely controllable, then $\boldsymbol{K}$ can be determined to place all poles in the left half-plane, so that the transient performance meets the desired response.

Pole placement

Definition

Also used: For a single input system with $u = -\boldsymbol{Kx}$ and a desired characteristic equation

$$q(\lambda) = \lambda^n + \alpha_1\lambda^{n-1} \cdots + \alpha_n$$

one can calculate the gain matrix with Ackermann's formula as follows:

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Table of Contents

State Variable Feedback Systems

5.5 State Variable Feedback Systems
5.5.1 Introduction
5.5.2 Controllability
5.5.3 Full-State Feedback Control Design
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Full-state feedback

Task

In this section we deal with the control of controllable systems with measurement of only some internal states.

Task
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Observability

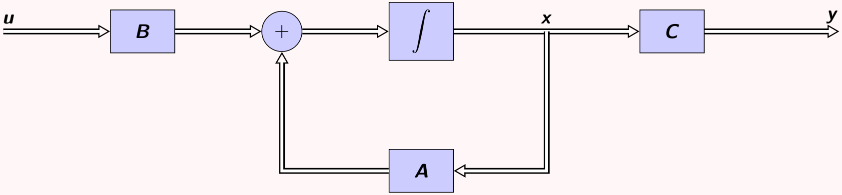
Measurement

Observability

Task
In this section we deal with the control of controllable systems with measurement of only some internal states.

Definition

Frequently, not all states can be measured. This is taken into account by introducing the measurement matrix $\boldsymbol{C}$ (also called output matrix):



$$\boxed{\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}}$$

Observalibility

Definition

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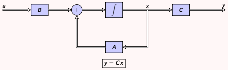


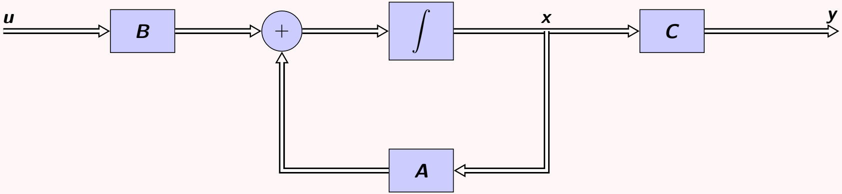
Observability

Task

In this section we deal with the control of controllable systems with measurement of only some internal states.

Definition

A system is completely observable if and only if there exists a finite time T such that the initial state $\boldsymbol{x}(0)$ can be determined from the observation history $\boldsymbol{y}(t)$ given the control $\boldsymbol{u}(t)$.[a]



[a] Dorf, Bishop

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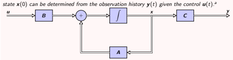

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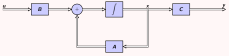

$\boxed{\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}}$

Observability

Definition

The observability matrix $\boldsymbol{P}_O$ is given by

$$\boldsymbol{P}_O = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{C}\boldsymbol{A} \\ \vdots \\ \boldsymbol{C}\boldsymbol{A}^{n-1} \end{bmatrix}.$$

The system is completely observable in case of $\boldsymbol{P}_O$ having full rank $(\det(\boldsymbol{P}_O) \neq 0)$.

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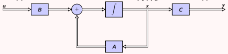

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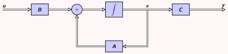

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Full-state observer

Full-state observer

Full-state observer

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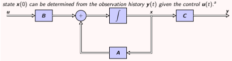

[a]Dorf, Bishop

System state determination

If the system is completely observable, then one can determine (estimate) the states that are not directly measurable. Idea: Use an estimated state $\hat{\boldsymbol{x}}$ with

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{u},$$

where $\boldsymbol{u}$ is the (known) control input. Due the fact that the initial state $\hat{\boldsymbol{x}}(t=0)$ is unknown (needs to be *guessed*), a correction is needed:

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{L}\tilde{\boldsymbol{y}},$$

where $\tilde{\boldsymbol{y}}$ is a **time dependent** correction factor.

Full-state observer

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The system is completely observable in case of $\boldsymbol{P}_O$ having full rank ($\det(\boldsymbol{P}_O) \neq 0$).

Correction factor

Making use the difference between observed state and estimated observed state:

$$\begin{aligned} \tilde{\boldsymbol{y}} &= \boldsymbol{C}\boldsymbol{x} - \boldsymbol{C}\hat{\boldsymbol{x}} \\ &= \boldsymbol{y} - \boldsymbol{C}\hat{\boldsymbol{x}} \end{aligned}$$

and thus

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{L}\left(\boldsymbol{y} - \boldsymbol{C}\hat{\boldsymbol{x}}\right),$$

Correction factor

Making use the difference between observed state and estimated observed state:

$$\begin{aligned}\tilde{y} &= Cx - C\hat{x} \\ &= y - C\hat{x}\end{aligned}$$

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$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}).$$

System state determination

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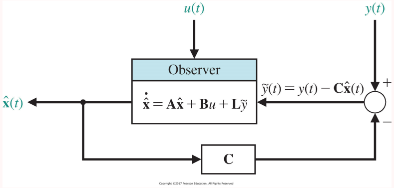
$$\boxed{\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}).}$$

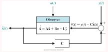


This a called an observer with L being the observer gain matrix.

Full-state observer

Definition

$$\boxed{\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})}.$$



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If the system is completely observable, then one can determine (estimate) the states that are not directly measurable. Idea: Use an estimated state $\hat{x}$ with

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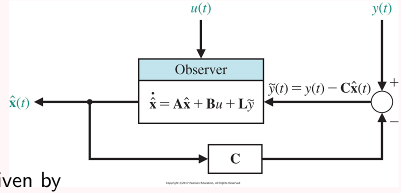
Definition



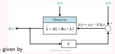
The estimation error is given by

$$\begin{aligned} \boldsymbol{e(t)} &= \boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{e}}(t) &= \boldsymbol{Ax} + \boldsymbol{Bu} - \boldsymbol{A}\hat{\boldsymbol{x}} - \boldsymbol{Bu} - \boldsymbol{L}\left(\boldsymbol{y} - \boldsymbol{C}\hat{\boldsymbol{x}}\right) \\ &= \left(\boldsymbol{A} - \boldsymbol{LC}\right)\boldsymbol{e}(t) \end{aligned}$$

Note that the error does not depend on $u(t)$.

Full-state observer

Definition



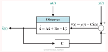
The estimation error is given by

$$\begin{aligned} \boldsymbol{e}(t) &= \boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{e}}(t) &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} - \boldsymbol{A}\hat{\boldsymbol{x}} - \boldsymbol{B}\boldsymbol{u} - \boldsymbol{L}(\boldsymbol{y} - \boldsymbol{C}\hat{\boldsymbol{x}}) \\ &= (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\,\boldsymbol{e}(t) \end{aligned}$$

Note that the error does not depend on $u(t)$.

Definition

$$\boxed{\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{L}(\boldsymbol{y} - \boldsymbol{C}\hat{\boldsymbol{x}})}.$$



This a called an observer with $\boldsymbol{L}$ being the observer gain matrix.

Correction factor

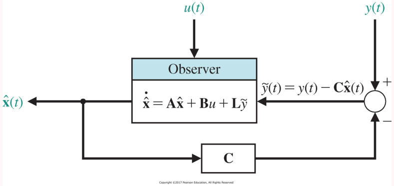
Making use the difference between observed state and estimated observed state:

$$\begin{aligned} \tilde{y} &= \boldsymbol{C}\boldsymbol{x} - \boldsymbol{C}\hat{\boldsymbol{x}} \\ &= \boldsymbol{y} - \boldsymbol{C}\hat{\boldsymbol{x}} \end{aligned}$$

and thus

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{L}(\boldsymbol{y} - \boldsymbol{C}\hat{\boldsymbol{x}}),$$

Definition

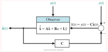


The estimation error $\boldsymbol{e}(t)$ will approach zero for $t \to \infty$ in case of

$$\det\left(\lambda \boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\right) = 0$$

having all roots in the left half-space.

Definition

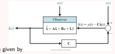


The estimation error $\boldsymbol{e}(t)$ will approach zero for $t \to \infty$ in case of

$$\det\left(\lambda \boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\right) = 0$$

having all roots in the left half-space.

Definition



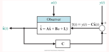
The estimation error is given by

$$\begin{aligned}
\boldsymbol{e}(\boldsymbol{t}) &= \boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t) \\
\dot{\boldsymbol{e}}(t) &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} - \boldsymbol{A}\hat{\boldsymbol{x}} - \boldsymbol{B}\boldsymbol{u} - \boldsymbol{L}\left(\boldsymbol{y} - \boldsymbol{C}\hat{\boldsymbol{x}}\right) \\
&= \left(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}\right)\boldsymbol{e}(t)
\end{aligned}$$

Note that the error does not depend on $u(t)$.

Definition

$$\boxed{\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{L}\left(\boldsymbol{y} - \boldsymbol{C}\hat{\boldsymbol{x}}\right).}$$

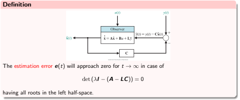


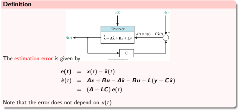
This a called an observer with $\boldsymbol{L}$ being the observer gain matrix.

Observer design

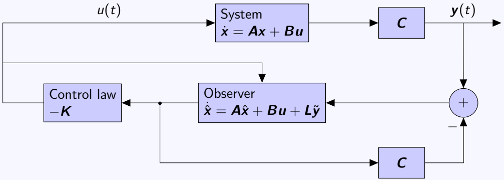
Full-state observer

Observer design





Full-state observer

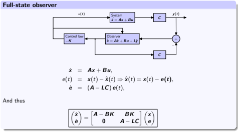


$$
\begin{aligned}
\dot{x} &= Ax + Bu, \\
e(t) &= x(t) - \hat{x}(t) \Rightarrow \hat{x}(t) = x(t) - e(t), \\
\dot{e} &= (A - LC)\, e(t),
\end{aligned}
$$

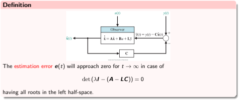
And thus

$$
\boxed{\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{pmatrix} x \\ e \end{pmatrix}}
$$

Design process



Observer design

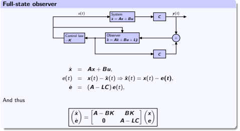


Steps

1. Determine $\boldsymbol{K}$
2. Determine $\boldsymbol{L}$
3. Connect the observer

Design process

Steps
1. Determine $\boldsymbol{K}$
2. Determine $\boldsymbol{L}$
3. Connect the observer



Observer design

Steps

1. Determine $\boldsymbol{K}$ by making use of Ackermann's formula:

$$\boldsymbol{K} = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} \boldsymbol{P}_c^{-1} q(\boldsymbol{A})$$

and the desired characteristic equation

$$\det\left(\lambda \boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\right) = \lambda^n + \alpha_{n-1}\lambda^{n-1} \cdots + \alpha_0 \quad = \quad 0,$$

with

$$q(\boldsymbol{A}) = \boldsymbol{A}^n + \alpha_{n-1}\boldsymbol{A}^{n-1} + \cdots \alpha_1 \boldsymbol{A}^1 + \alpha_0 \boldsymbol{I}.$$

and $\boldsymbol{P}_c$ being the controllability matrix.

2. Determine $\boldsymbol{L}$
3. Connect the observer

Design process

Steps

1. Determine $\boldsymbol{K}$ by making use of Ackermann's formula:

$$\boldsymbol{K} = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} \boldsymbol{P}_c^{-1} q(\boldsymbol{A})$$

and the desired characteristic equation

$$\det\left(\lambda \boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{BK})\right) = \lambda^n + \alpha_{n-1}\lambda^{n-1} \cdots + \alpha_0 = 0,$$

with

$$q(\boldsymbol{A}) = \boldsymbol{A}^n + \alpha_{n-1}\boldsymbol{A}^{n-1} + \cdots \alpha_1 \boldsymbol{A}^1 + \alpha_0 \boldsymbol{I}.$$

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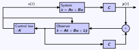
2. Determine $\boldsymbol{L}$
3. Connect the observer

Steps

1. Determine $\boldsymbol{K}$
2. Determine $\boldsymbol{L}$
3. Connect the observer

Full-state observer



$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{Ax} + \boldsymbol{Bu}, \\ \boldsymbol{e}(t) &= \boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t) \Rightarrow \hat{\boldsymbol{x}}(t) = \boldsymbol{x}(t) - \boldsymbol{e}(t), \\ \dot{\boldsymbol{e}} &= (\boldsymbol{A} - \boldsymbol{LC})\,\boldsymbol{e}(t), \end{aligned}$$

And thus

$$\boxed{\begin{pmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{e}} \end{pmatrix} = \begin{bmatrix} \boldsymbol{A} - \boldsymbol{BK} & \boldsymbol{BK} \\ \boldsymbol{0} & \boldsymbol{A} - \boldsymbol{LC} \end{bmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{e} \end{pmatrix}}$$

Steps

1. Determine $\boldsymbol{K}$
2. Determine $\boldsymbol{L}$ by making use of Ackermann's formula:

$$\boldsymbol{L} = p(\boldsymbol{A})\boldsymbol{P}_0^{-1} \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}^T$$

and the desired characteristic equation

$$\det\left(\lambda \boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{LC})\right) = \lambda^n + \beta_{n-1}\lambda^{n-1} \cdots + \beta_0 = 0$$

describing the observer dynamics and

$$q(\boldsymbol{A}) = \boldsymbol{A}^n + \beta_{n-1}\boldsymbol{A}^{n-1} + \cdots \beta_1 \boldsymbol{A}^1 + \beta_0 \boldsymbol{I},$$

where $\boldsymbol{P}_0$ is the observability matrix.

3. Connect the observer

Desing process

Steps

1. Determine $\boldsymbol{K}$
2. Determine $\boldsymbol{L}$ by making use of Ackermann's formula:

$$\boldsymbol{L} = p(\boldsymbol{A})\boldsymbol{P}_o^{-1}\begin{bmatrix}0 & 0 & \cdots & 1\end{bmatrix}^T$$

and the desired characteristic equation

$$\det\left(\lambda \boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\right) = \lambda^n + \beta_{n-1}\lambda^{n-1} \cdots + \beta_0 = 0$$

describing the observer dynamics and

$$q(\boldsymbol{A}) = \boldsymbol{A}^n + \beta_{n-1}\boldsymbol{A}^{n-1} + \cdots \beta_1\boldsymbol{A}^1 + \beta_0\boldsymbol{I}.$$

where $\boldsymbol{P}_o$ is the observability matrix.
3. Connect the observer

Steps

1. Determine $\boldsymbol{K}$ by making use of Ackermann's formula:

$$\boldsymbol{K} = \begin{bmatrix}0 & 0 & \cdots & 1\end{bmatrix}\boldsymbol{P}_c^{-1}q(\boldsymbol{A})$$

and the desired characteristic equation

$$\det\left(\lambda \boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\right) = \lambda^n + \alpha_{n-1}\lambda^{n-1} \cdots + \alpha_0 \;=\; 0,$$

with

$$q(\boldsymbol{A}) = \boldsymbol{A}^n + \alpha_{n-1}\boldsymbol{A}^{n-1} + \cdots \alpha_1\boldsymbol{A}^1 + \alpha_0\boldsymbol{I}.$$

and $\boldsymbol{P}_c$ being the controllability matrix.
2. Determine $\boldsymbol{L}$
3. Connect the observer

Steps

1. Determine $\boldsymbol{K}$
2. Determine $\boldsymbol{L}$
3. Connect the observer

Steps

1. Determine $\boldsymbol{K}$
2. Determine $\boldsymbol{L}$
3. Connect the observer by making use of

$$\boldsymbol{u}(t) = -\boldsymbol{K}\hat{\boldsymbol{x}}(t).$$

$$\begin{pmatrix}\dot{\boldsymbol{x}} \\ \dot{\boldsymbol{e}}\end{pmatrix} = \begin{bmatrix}\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K} & \boldsymbol{B}\boldsymbol{K} \\ \boldsymbol{0} & \boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}\end{bmatrix}\begin{pmatrix}\boldsymbol{x} \\ \boldsymbol{e}\end{pmatrix}$$

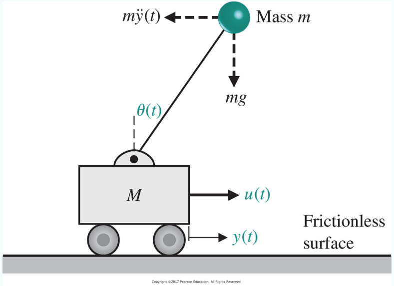
Table of Contents

State Variable Feedback Systems

5.5 State Variable Feedback Systems
5.5.1 Introduction
5.5.2 Controllability
5.5.3 Full-State Feedback Control Design
5.5.4 Observable systems
5.5.5 Example

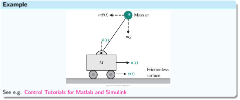
Inverted pendulum

Example

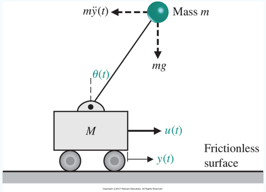


See e.g. Control Tutorials for Matlab and Simulink

Inverted pendulum



Example



Sum of forces in horizontal direction:

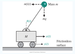
$$M\ddot{y}(t) - \cos(\pi + \theta)ml\ddot{\theta}(t) - ml\dot{\theta}^2 \sin(\pi + \theta) - u(t) = 0$$

Sum of torques at pivot point:

$$-\cos(\pi + \theta)ml\ddot{y}(t) + ml^2\ddot{\theta}(t) + \sin(\pi + \theta)mlg\theta = 0$$

Inverted pendulum

Example



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Sum of torques at pivot point:

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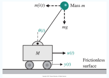
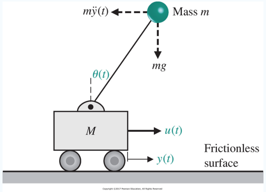
Example



See e.g. Control Tutorials for Matlab and Simulink

Example



Idea: Linearize for $\theta \approx 0$. Sum of forces in horizontal direction:

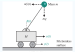
$$M\ddot{y}(t) + ml\ddot{\theta}(t) - u(t) = 0$$

Sum of torques at pivot point:

$$ml\ddot{y}(t) + ml^2\ddot{\theta}(t) - mlg\theta = 0$$

Inverted pendulum

Example



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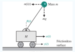
Example



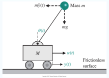
Sum of forces in horizontal direction:

$$M\ddot{y}(t) - \cos(\pi+\theta)ml\ddot{\theta}(t) - ml\dot{\theta}^2\sin(\pi+\theta) - u(t) = 0$$

Sum of torques at pivot point:

$$-\cos(\pi+\theta)ml\ddot{y}(t) + ml^2\ddot{\theta}(t) + \sin(\pi+\theta)mlg\theta = 0$$

Example



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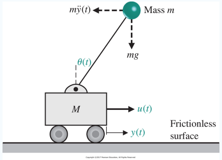
Example



Using state variables

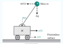
$$\boldsymbol{x}\,[y \quad \dot{y} \quad \theta \quad \dot{\theta}]^T$$

one gets

$$\begin{aligned} M\dot{x}_2(t) + ml\dot{x}_4(t) - u(t) &= 0 \\ \dot{x}_2(t) + l\dot{x}_4(t) - gx_3(t) &= 0 \end{aligned}$$

Inverted pendulum

Example



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$$x\ [y \quad \dot{y} \quad \theta \quad \dot{\theta}]^T$$

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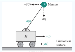
Example



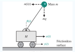
Idea: Linearize for $\theta \approx 0$. Sum of forces in horizontal direction:

$$M\ddot{y}(t) + ml\ddot{\theta}(t) - u(t) = 0$$

Sum of torques at pivot point:

$$ml\ddot{y}(t) + ml^2\ddot{\theta}(t) - mlg\theta = 0$$

Example



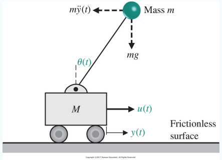
Sum of forces in horizontal direction:

$$M\ddot{y}(t) - \cos(\pi+\theta)ml\ddot{\theta}(t) - ml\dot{\theta}^2\sin(\pi+\theta) - u(t) = 0$$

Sum of torques at pivot point:

$$-\cos(\pi+\theta)ml\ddot{y}(t) + ml^2\ddot{\theta}(t) + \sin(\pi+\theta)mlg\theta = 0$$

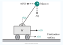
Example



Assuming $M \gg m$ one gets the state variables and systems matrices as follows:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \theta \\ \dot{\theta} \end{bmatrix} \quad \boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -mg/M & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & g/l & 0 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ 1/M \\ 0 \\ -1/(Ml) \end{bmatrix}$$

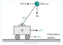
Example



Assuming $M \gg m$ one gets the state variables and systems matrices as follows:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \theta \\ \dot{\theta} \end{bmatrix} \quad \boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -mg/M & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & g/l & 0 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ 1/M \\ 0 \\ -1/(Ml) \end{bmatrix}$$

Example



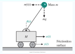
Using state variables

$$\boldsymbol{x}\,[y \;\; \dot{y} \;\; \theta \;\; \dot{\theta}]^T$$

one gets

$$\begin{aligned} M\dot{x}_2(t) + ml\dot{x}_4(t) - u(t) &= 0 \\ \dot{x}_2(t) + l\dot{x}_4(t) - gx_3(t) &= 0 \end{aligned}$$

Example



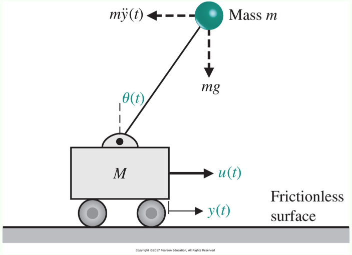
Idea: Linearize for $\theta \approx 0$. Sum of forces in horizontal direction:

$$M\ddot{y}(t) + ml\ddot{\theta}(t) - u(t) = 0$$

Sum of torques at pivot point:

$$ml\ddot{y}(t) + ml^2\ddot{\theta}(t) - mlg\theta = 0$$

Exercise (#5.12)



Given is the system shown above.

- ▷ Develop a state variable model for the system shown above and $l = 0.098\,\mathrm{m}$, $g = 9.8\,\mathrm{m\,s^{-2}}$, $m = 825\,\mathrm{g}$ and $M = 8085\,\mathrm{g}$.

Table of Contents

Feedback & Control

5.1 Introduction
5.2 Open-loop systems
5.3 Closed-loop systems
5.4 Feed-forward Control
5.5 State Variable Feedback Systems
5.6 Additional Exercises
5.7 Appendix

Exercise

Exercise (#5.13)

Given is the system shown in figure 22 with

$$H(s) = \frac{1}{s^3 + s^2 + 10s + 2}$$

and the corresponding step response for different PID controllers shown in figure 23. The coefficients of the PID controllers are given in table 1. Fill the table.

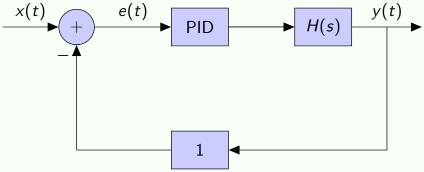


Figure 22: A system

Exercise

Exercise (#5.13)

Given is the system shown in figure 22 with

$$H(s) = \frac{1}{s^3 + s^2 + 10s + 2}$$

and the corresponding step response for different PID controllers shown in figure 23. The coefficients of the PID controllers are given in table 1. Fill the table.

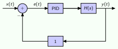


Figure 22: A system

Continued exercise (#5.13)

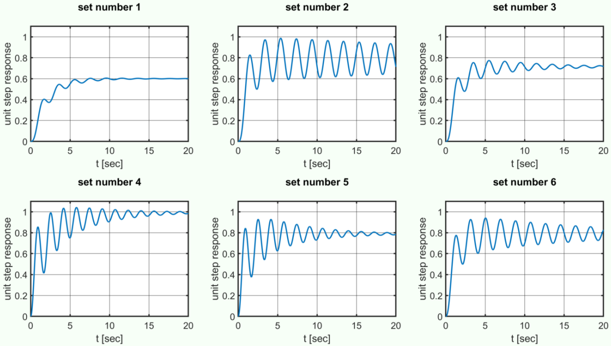
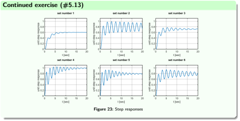
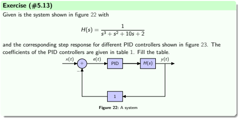


Figure 23: Step responses





Continued exercise (#5.13)

P	I	D	Set number	Justification
3	0	0		
5	0	0		
7.5	0	0		
7.5	0	1		
7.5	0	5		
7.5	1	5		

Table 1: PID coefficients

Exercise

Continued exercise (#5.13)

P	I	D	Set number	Justification
3	0	0		
5	0	0		
7.5	0	0		
7.5	0	1		
7.5	0	5		
7.5	1	5		

Table 1: PID coefficients

Continued exercise (#5.13)

P	I	D	Set number	Justification
3	0	0		
5	0	0		
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7.5	0	1		
7.5	0	5		
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Table 1: PID coefficients

Continued exercise (#5.13)

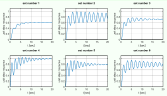

Figure 23: Step responses

Exercise (#5.13)

Given is the system shown in figure 22 with

$$H(s) = \frac{1}{s^3 + s^2 + 10s + 2}$$

and the corresponding step response for different PID controllers shown in figure 23. The coefficients of the PID controllers are given in table 1. Fill the table.

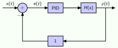

Figure 22: A system

Exercise (#5.14)

Given is a plant with the following system parameters:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 100 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ 0.1237 \\ 0 \\ -1.2621 \end{bmatrix}$$

Calculate $\boldsymbol{K}$ for poles at $s = -8 \pm 6i$ and $-0.4 \pm 0.3i$. Simulate the system for an initial state of $\boldsymbol{x} = \begin{bmatrix} 0 & 0 & 0.1 & 0 \end{bmatrix}^T$ using MATLAB and SIMULINK.
Hint: Useful functions are `acker`, `lsim`, `zp2tf` and `ss`.

Exercise

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Table of Contents

Feedback & Control

5.1 Introduction
5.2 Open-loop systems
5.3 Closed-loop systems
5.4 Feed-forward Control
5.5 State Variable Feedback Systems
5.6 Additional Exercises
5.7 Appendix

Translations

Englisch	Deutsch
Overshoot	Überschwingweite
Settling time	Ausregelzeit
Rise time	Anstiegszeit
Peak time	t_{max} − *Zeit*
Steady state error	stationäre Regeldifferenz

Source: Europa-Lehrmittel

Table of Contents

Discrete Time & Applications

6. Discrete Time
7. Filters
8. Applications and Exercises

Table of Contents

Discrete Time

6.1 Introduction
6.2 Pulse modulated signal
6.3 Discrete Fourier Transform
6.4 Z-Transform
6.5 Time discrete LTI systems
6.6 Special classes of time discrete systems
6.7 Relationship between different transforms
6.8 Exercises
6.9 Appendix

Introduction

Content

- Definition of the discrete Fourier transform
- Leakage, window function and undersampling
- $z-$transform and time discrete LTI systems including canonical forms
- Relationships between different transforms

Study goals

- You shall be able to use a discrete Fourier transform including mapping between bin and frequency
- You shall be able to describe the effect of undersampling
- You shall be able to describe the consequences of changing sampling rate and number of samples

Table of Contents

Discrete Time

6.1 Introduction
6.2 Pulse modulated signal
6.3 Discrete Fourier Transform
6.4 Z-Transform
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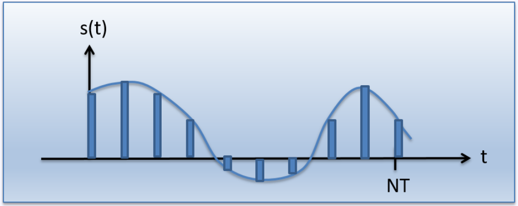
6.6 Special classes of time discrete systems
6.7 Relationship between different transforms
6.8 Exercises
6.9 Appendix

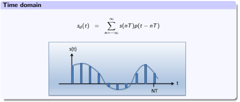
Pulse modulated signal

Time domain

$$s_d(t) = \sum_{n=-\infty}^{\infty} s(nT)p(t-nT)$$



Pulse modulated signal



Time domain

$$s_d(t) \;=\; \sum_{n=-\infty}^{\infty} s(nT)p(t-nT)$$

Properties: Dirac comb

$$\mathcal{F}\left\{\sum_{n=-\infty}^{\infty} \delta(t-nT) = \frac{1}{T}\sum_{-\infty}^{\infty} e^{j2\pi n\frac{t}{T}}\right\} = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

Frequency domain

$$\boxed{S_d(\omega) = \frac{1}{T}P(\omega)\sum_{n=-\infty}^{\infty} S\left(\omega - n\frac{2\pi}{T}\right)}$$

Time domain

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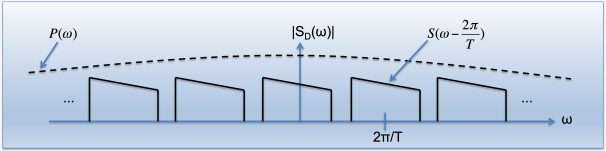

Table of Contents

Discrete Time

6.1 Introduction
6.2 Pulse modulated signal
6.3 Discrete Fourier Transform
- 6.3.1 Definition
- 6.3.2 FFTshift
- 6.3.3 Leakage
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- 6.3.5 Zero padding
- 6.3.6 Undersampling
- 6.3.7 Real signals
- 6.3.8 Noise
- 6.3.9 Discretization
- 6.3.10 Zero-Padding

6.4 Z-Transform
6.5 Time discrete LTI systems
6.6 Special classes of time discrete systems
6.7 Relationship between different transforms
6.8 Exercises
6.9 Appendix

Discrete Fourier Transform

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6.3.1 Definition
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Motivation

The spectrum $S_d(\omega) = \frac{1}{T} P(\omega) \sum_{n=-\infty}^{\infty} S\left(\omega - n\frac{2\pi}{T}\right)$ of the signal

$$s_d(t) = \sum_{n=-\infty}^{\infty} s(nT)p(t-nT) = \left[s(t) \sum_{n=-\infty}^{\infty} \delta(t-nT)\right] * p(t)$$

can be written in the following form as well:

$$S_d(\omega) = P(\omega) \sum_{n=-\infty}^{\infty} f_n e^{-j\omega nT},$$

with

$$f_n = f(nT).$$

Discrete Fourier Transform

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The spectrum $S_d(\omega) = \frac{1}{T} P(\omega) \sum_{n=-\infty}^{\infty} S\left(\omega - n\frac{2\pi}{T}\right)$ of the signal

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motivates to define the discrete Fourier transform as follows:

$$\begin{aligned} F_k &= \left.\sum_{n=0}^{N-1} f_n e^{-\frac{j2\pi nk}{N}}\right|_{k=0\ldots N-1}, \\ f_n &= \left.\sum_{k=0}^{N-1} F_k e^{\frac{j2\pi nk}{N}}\right|_{n=0\ldots N-1} \end{aligned}$$

Discrete Fourier Transform

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Properties

Mapping to frequencies: Assuming

$$f_n = f(nT),$$

one can map

$$F_k = \sum_{n=0}^{N-1} f_n e^{-\frac{j2\pi nk}{N}}$$

to the frequency

$$\omega_k = \frac{2\pi}{NT}k.$$

Note: The result of the DFT is independent of T and depends only on the values f_n.

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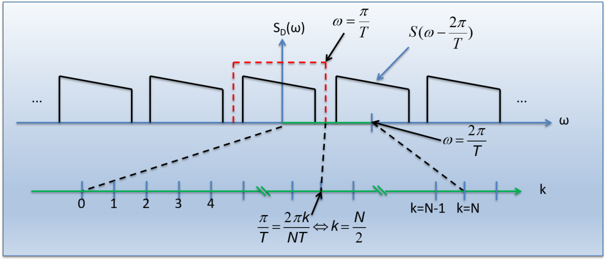
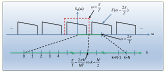


- ▷ Maximum frequency is given by sampling frequency $1/T$
- ▷ Frequency resolution is given by observation time: $df = 1/NT$

Properties



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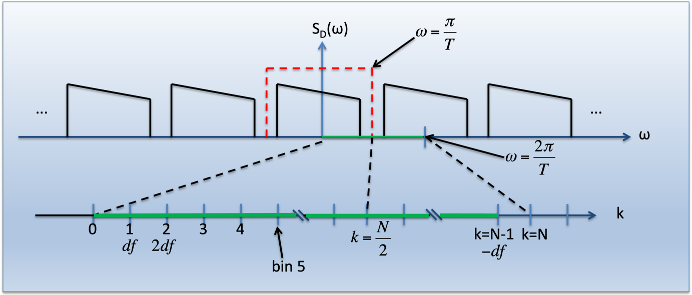
Properties

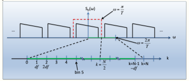
Frequency axis:



Nyquist–Shannon

Properties

Frequency axis:



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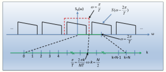

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Nyquist–Shannon sampling theorem

Sufficient condition for the minimum sample rate so that a discrete sequence of samples contains all the information from a continuous-time signal of finite bandwidth f_{bw}:

$$f_s > 2f_{bw}$$

or

$$T < \frac{1}{2f_{bw}}.$$

Exercise

Nyquist–Shannon sampling theorem

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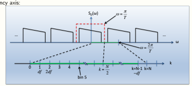

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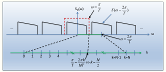

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Exercise (#6.1)

- Start NUMPY or MATLAB
- In the following you shall "sample" 5 periods of the signal $s(t) = \sin(2\pi t)$. Take 50 samples per period.
- Calculate the DFT by making use of `F=fft(s)`. Plot the result
- Determine the frequency axis to use `plot(freq,abs(F))`.
- "Sample" the sinus for 5.5 periods and compare the results. What do you observe?

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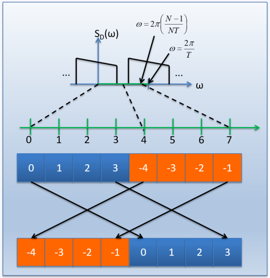
Table of Contents

Discrete Fourier Transform

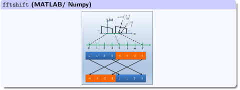
6.3 Discrete Fourier Transform
6.3.1 Definition
6.3.2 FFTshift
6.3.3 Leakage
6.3.4 Window functions
6.3.5 Zero padding
6.3.6 Undersampling
6.3.7 Real signals
6.3.8 Noise
6.3.9 Discretization
6.3.10 Zero-Padding

Discrete Fourier Transform

`fftshift` (MATLAB/ Numpy)



Exercise



Exercise (#6.2)

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Exercise (#6.2)

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Discrete Fourier Transform

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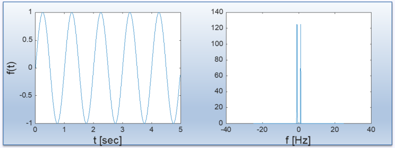
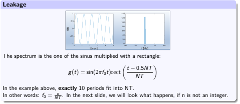
Leakage



The spectrum is the one of the sinus multiplied with a rectangle:

$$g(t) = \sin(2\pi f_0 t)\mathrm{rect}\left(\frac{t - 0.5NT}{NT}\right)$$

In the example above, **exactly** 10 periods fit into NT.
In other words: $f_0 = \frac{n}{NT}$. In the next slide, we will look what happens, if n is not an integer.



Leakage

If f(t) does not (by chance) “fit” into the observation time, then an effect called leakage occurs (more correctly: Becomes visible)

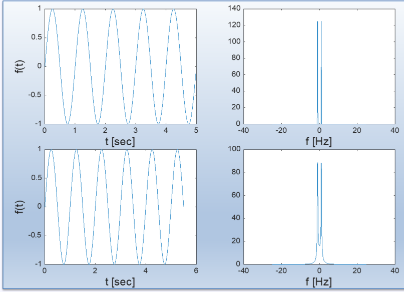
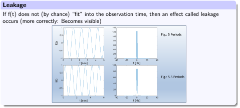
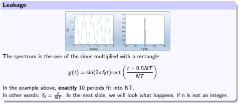


Fig.: 5 Periods

Fig.: 5.5 Periods





Leakage

This effect occurring in the application of the DFT is called leakage effect: The leakage effect is the window function (in the simplest case: the rect function) becoming visible in the DFT spectrum. This means: You will find components at $f \neq f_0$ even for a perfect cosine of frequency f_0.

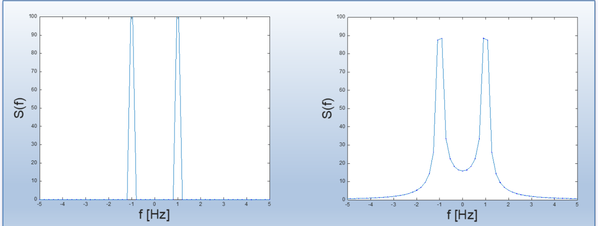


Fig.: 5 Periods

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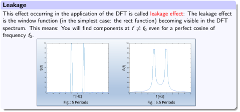
Zero of window function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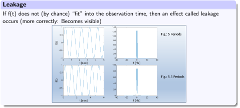
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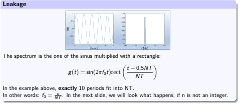
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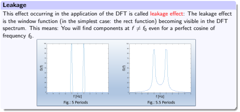
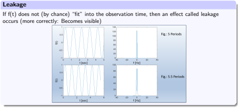
Zero of window function

Discrete Fourier Transform

Zero of window function





Leakage

The signal

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Zero of window function

Leakage

This effect occurring in the application of the DFT is called leakage effect: The leakage effect is the window function (in the simplest case: the rect function) becoming visible in the DFT spectrum. This means: You will find components at $f \neq f_0$ even for a perfect cosine of frequency f_0.

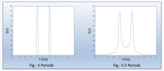

Fig.: 5 Periods Fig.: 5.5 Periods

Leakage

For $p(t) = \delta(t)$ and

$$s(t) = \sin(\omega_0 t)\mathrm{rect}\left(\frac{t-0.5TN}{NT}\right)$$

this leads to

$$S_d(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} S\left(\omega - n\frac{2\pi}{T}\right),$$

with

$$S(\omega) = NT\frac{\sin\left(\left(\omega-\omega_0\right)0.5NT\right)}{\left(\omega-\omega_0\right)0.5NT} = NT\mathrm{si}\left(\left(\omega-\omega_0\right)0.5NT\right)$$

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Leakage

The signal

$$s_d(t) = \sum_{n=-\infty}^{\infty} s(nT)p(t-nT) = \left[s(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)\right] * p(t)$$

has the spectrum

$$S_d(\omega) = \frac{1}{T}P(\omega)\sum_{n=-\infty}^{\infty} S\left(\omega - n\frac{2\pi}{T}\right)$$

Zero of window function

Leakage

$$S(\omega) = NT\mathrm{si}\left((\omega-\omega_0)\,0.5NT\right)$$

With

$$f_0 = ndf = \frac{n}{NT}$$

we get

$$S(\omega) = NT\mathrm{si}\left(\left(\omega - 2\pi\frac{n}{nT}\right)0.5NT\right)$$

Leakage

$$S(\omega) = NT\,\mathrm{si}\left(\left(\omega - \omega_0\right)0.5NT\right)$$

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$$S(\omega) = NT\,\mathrm{si}\left(\left(\omega - 2\pi\frac{n}{NT}\right)0.5NT\right)$$

The first 0 of the si-function is located at

$$\frac{\omega - \frac{2\pi n}{NT}}{2}NT = \pm\pi \Rightarrow \omega = \frac{2\pi}{NT}(n \pm 1) = (n \pm 1)2\pi df.$$

This means that the maximum appears exactly at one bin of the DFT:

$$0 = \frac{\omega_0 - \frac{2\pi n}{NT}}{2}NT \Rightarrow \omega_0 = \frac{2\pi n}{NT} = n2\pi df$$

Spectrum of periodic function

Leakage

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Spectrum of periodic function

Discrete Fourier Transform

Spectrum of periodic function

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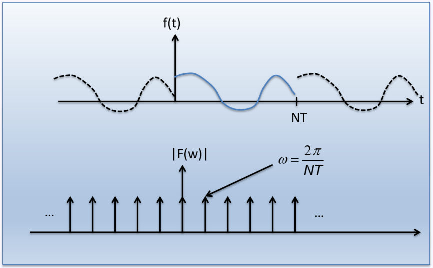
we get

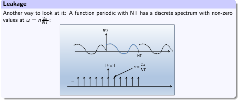
$$S(\omega) = NT\,\mathrm{si}\left(\left(\omega - 2\pi\frac{n}{nT}\right)0.5NT\right)$$

Leakage

Another way to look at it: A function periodic with NT has a discrete spectrum with non-zero values at $\omega = n\frac{2\pi}{NT}$:



Discrete Fourier Transform



Spectrum of periodic function

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Leakage

This can be shown by using the Fourier series: Each function periodic with NT can be expanded as a Fourier series:

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} F_n e^{jn\frac{2\pi}{NT}}, \\ F_n &= \int_0^T f(t) e^{jn\frac{2\pi}{NT}}\, dt. \end{aligned}$$

The corresponding Fourier transform is given by:

$$F(\omega) = \sum_{n=-\infty}^{\infty} F_n 2\pi\delta\left(\omega - n\frac{2\pi}{NT}\right)$$

In words: The spectrum of the DFT is equivalent to the Fourier series of the periodic repetition of the same signal

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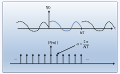

Spectrum of periodic function

Periodicity of inverse transform

Discrete Fourier Transform

Periodicity of inverse transform

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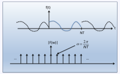

Leakage

Another way to look at it:
One can easily show that the signal calculated by the inverse transform is periodic:

$$f_{n+N} = \sum_{k=0}^{N-1} F_k e^{\frac{j2\pi(n+N)k}{N}} \Bigg|_{n=0\ldots N-1} = f_n,$$

This means the DFT spectrum is equivalent to the spectrum of periodic repetition of the signal

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Periodicity of inverse transform

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This can be shown by using the Fourier series: Each function periodic with NT can be expanded as a Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\frac{2\pi}{NT}},$$

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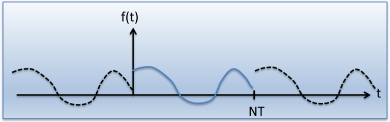
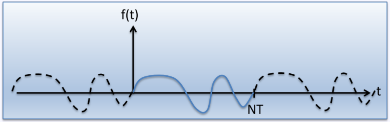


Strong leakage effect: "Jumps" of signal are seen in the spectrum



Leakage effect not visible

Discrete Fourier Transform

6.3 Discrete Fourier Transform
6.3.1 Definition
6.3.2 FFTshift
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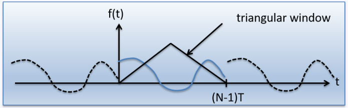
Window functions

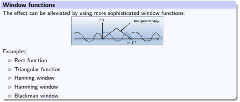
The effect can be alleviated by using more sophisticated window functions:



Examples:

- Rect function
- Triangular function
- Hanning window
- Hamming window
- Blackman window

Window functions



Properties

Applying window functions does not remove the leakage effect but windows can be chosen to either to

- ▷ get a sharp peak (resolve signals with similar strength and frequency)

or

- ▷ low sidelobes (resolve signals with dissimilar strength and frequency)

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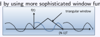

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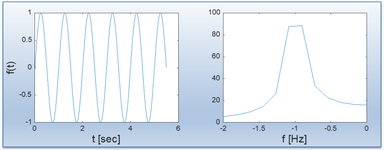
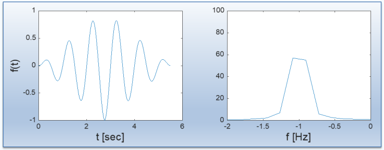


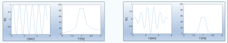
Rectangular window



Triangular window

Window functions

Example



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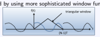

Examples:

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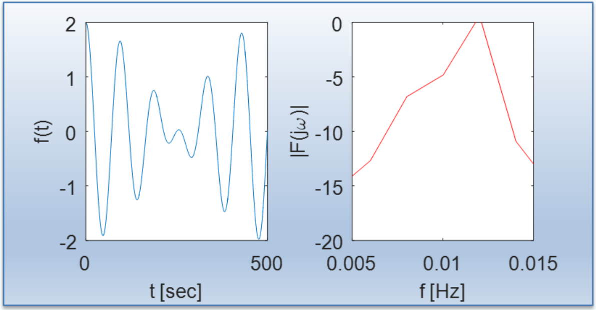
Example

Rectangular window and signal with two sinus-functions:



Window functions

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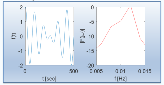

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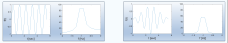

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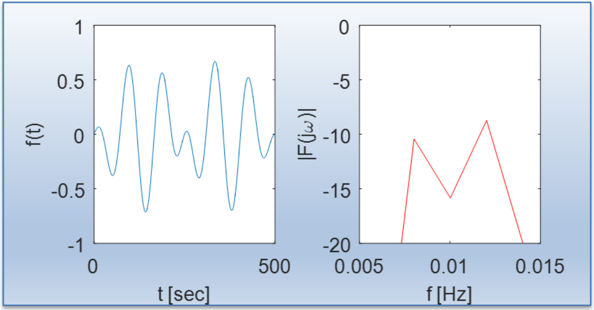

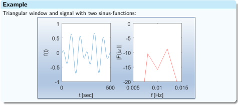

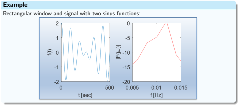

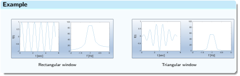

Exercise (#6.3)

Consider the function

$$s(nT) \quad = \quad \sin\left(2\pi\frac{100}{512}nT\right) + \sin\left(2\pi\frac{110}{512}nT\right)$$

Plot the DFT-spectrum for $N = 512$, $T = 0.1\,\text{sec}$ and

- ▷ using a triangular window function
- ▷ using a rectangular window function.

Use a meaningful frequency axis.

Exercise

Exercise (#6.3)

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Table of Contents

Discrete Fourier Transform

6.3 Discrete Fourier Transform
6.3.1 Definition
6.3.2 FFTshift
6.3.3 Leakage
6.3.4 Window functions
6.3.5 Zero padding
6.3.6 Undersampling
6.3.7 Real signals
6.3.8 Noise
6.3.9 Discretization
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6.3.1 Definition
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Exercise

Exercise (#6.4)

- Setup a time vector `t = [0:0.1:10]`
- Calculate $s(t) = sin(2\pi ft)$ and plot the energy spectral density
- Start with f $= 1$ Hz and increase the frequency step by step. What happens?

Exercise

Exercise (#6.4)

- Setup a time vector `t = [0:0.1:10]`
- Calculate $s(t) = sin(2\pi ft)$ and plot the energy spectral density
- Start with f = 1 Hz and increase the frequency step by step. What happens?

Undersampling

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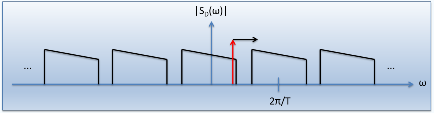
- Calculate $s(t) = sin(2\pi ft)$ and plot the energy spectral density
- Start with f = 1 Hz and increase the frequency step by step. What happens?

Example

What happens, if the signal contains a frequency above $2\pi/T$?



If the frequency of the signal becomes larger than π/T, then it is interpreted as a negative frequency!

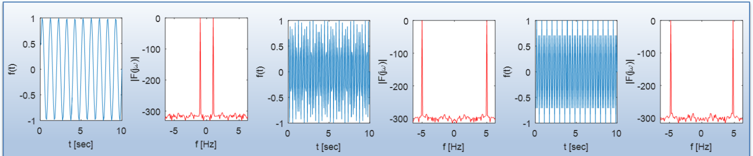


Fig.: f =1 Hz / T=78.125msec

Fig.: f =5 Hz / T=78.125msec

Fig.: f =8 Hz / T=78.125msec

Discrete Fourier Transform

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6.3.1 Definition
6.3.2 FFTshift
6.3.3 Leakage
6.3.4 Window functions
6.3.5 Zero padding
6.3.6 Undersampling
6.3.7 Real signals
6.3.8 Noise
6.3.9 Discretization
6.3.10 Zero-Padding

Exercise

Exercise (#6.5)

Given is $n = [1 \dots 512]$ and $s_n = \sin\left(2\pi \frac{n}{512} 10\right)$.

- ▷ Add some noise to the signal (use `randn`)
- ▷ Calculate the FFT in NUMPY or MATLAB.
- ▷ Compare the spectrum for positive and negative frequencies (compare $S(-\omega)$ and $S(\omega)$).

Exercise

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Table of Contents

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6.3.1 Definition
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6.3.3 Leakage
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6.3.5 Zero padding
6.3.6 Undersampling
6.3.7 Real signals
6.3.8 Noise
6.3.9 Discretization
6.3.10 Zero-Padding

Exercise

Exercise (#6.6)

Given is $n = [1 \dots 512]$ and $s_n = \sin\left(2\pi \frac{n}{512} 10\right)$.

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- Calculate the FFT in NumPy or MATLAB.
- Plot the magnitude of the spectrum in dB.
- Compare signal to noise ratio for different noise values.

Exercise

Exercise (#6.6)

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FFT Gain

Exercise (#6.6)

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- ▷ Add some noise to the signal (use `randn`).
- ▷ Calculate the FFT in NUMPY or MATLAB.
- ▷ Plot the magnitude of the spectrum in dB.
- ▷ Compare signal to noise ratio for different noise values.

Properties

- ▷ The FFT offers some sort of *gain*. This effect is often called FFT-gain.
- ▷ Rule of thumb: In case of increasing the number of samples by two, then the amplitude of the FFT increases by 6 dB (signal) and 3 dB (noise), respectively.

Table of Contents

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6.3.6 Undersampling
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Exercise (#6.7)

Given is $n = [1 \dots 512]$ and $s_n = \sin\left(\frac{n}{512}10\right)$.

- Simulate the effect of discretization by making use of `round`[a] and different accuracies.
- Calculate the FFT in NUMPY or MATLAB.
- Plot the magnitude of the spectrum in dB.
- Change the frequency of the signal.

[a] e.g. `x = round(x*10)/10`

Exercise

Exercise (#6.7)

Given is $n = [1 \dots 512]$ and $s_n = \sin\left(\frac{n}{512}10\right)$.

- ▷ Simulate the effect of discretization by making use of `round`[a] and different accuracies.
- ▷ Calculate the FFT in NumPy or MATLAB.
- ▷ Plot the magnitude of the spectrum in dB.
- ▷ Change the frequency of the signal.

[a] e.g. `x = round(x*10)/10`

Discretization

Exercise (#6.7)

Given is $n = [1 \dots 512]$ and $s_n = \sin\left(\frac{n}{512}10\right)$.

- ▷ Simulate the effect of discretization by making use of `round`[a] and different accuracies.
- ▷ Calculate the FFT in NUMPY or MATLAB.
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Example

Numerical example using MATLAB.

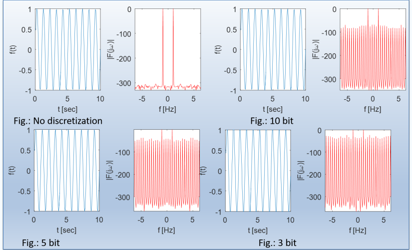

Fig.: No discretization

Fig.: 10 bit

Fig.: 5 bit

Fig.: 3 bit

Table of Contents

Discrete Fourier Transform

6.3 Discrete Fourier Transform
6.3.1 Definition
6.3.2 FFTshift
6.3.3 Leakage
6.3.4 Window functions
6.3.5 Zero padding
6.3.6 Undersampling
6.3.7 Real signals
6.3.8 Noise
6.3.9 Discretization
6.3.10 Zero-Padding

Table of Contents

Discrete Time

6.1 Introduction
6.2 Pulse modulated signal
6.3 Discrete Fourier Transform
6.4 Z-Transform
6.4.1 Definition
6.4.2 Convergence
6.4.3 Properties
6.4.4 Unilateral transform
6.4.5 Mapping to a frequency
6.4.6 Inverse transform
6.4.7 Inverse transform using Fourier Transform
6.5 Time discrete LTI systems
6.6 Special classes of time discrete systems
6.7 Relationship between different transforms
6.8 Exercises
6.9 Appendix

Z-Transform

6.4 Z-Transform
6.4.1 Definition
6.4.2 Convergence
6.4.3 Properties
6.4.4 Unilateral transform
6.4.5 Mapping to a frequency
6.4.6 Inverse transform
6.4.7 Inverse transform using Fourier Transform

Discrete Fourier transform

Discrete Fourier Transform

$$F_k = \sum_{n=0}^{N-1} f_n e^{-\frac{j2\pi nk}{N}},$$

with a mapping to *real* frequencies as follows: $\omega_k = \frac{2\pi k}{NT}$.

Inverse transform

$$f_n = \sum_{k=0}^{N-1} F_k e^{\frac{j2\pi nk}{N}}$$

Z transform

Discrete Fourier Transform

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Discrete Fourier Transform

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The z transform is defined as follows:

$$\mathcal{Z}\left\{x_n\right\} = X(z) = \sum_{n=-\infty}^{\infty} x_n z^{-n}$$

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Inverse transform

$$f_n = \sum_{k=0}^{N-1} F_k e^{\frac{j2\pi nk}{N}}$$

Properties

$$\mathcal{Z}\{x_n\} = X(z) = \sum_{n=-\infty}^{\infty} x_n z^{-n}$$

- z is a complex number!
- This series above does not necessarily converge for all z (see next slides)
- The (inverse) transform is only unique when specifying the region of convergence (ROC)

Table of Contents

Z-Transform

6.4 Z-Transform
6.4.1 Definition
6.4.2 Convergence
6.4.3 Properties
6.4.4 Unilateral transform
6.4.5 Mapping to a frequency
6.4.6 Inverse transform
6.4.7 Inverse transform using Fourier Transform

Convergence

The following behaviours can be observed:

1. $X(z)$ converges for all z with $|z| < \infty$
2. $X(z)$ converges for all z with $|z| < R_0$
3. **$X(z)$ converges for all z with $|z| > R_0$**
4. $X(z)$ converges for all z with $R_0 < |z| < R_1$
5. $X(z)$ converges only for $z = 0$

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Exercise (#6.8)

Consider the two transforms

$$\mathcal{Z}\left\{f_n = a^{n-1} u_{n-1}\right\}$$

and

$$\mathcal{Z}\left\{g_n = -a^{n-1} u_{-n}\right\}$$

- Sketch both series
- Calculate the Z transform

Note: Use $\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$

Exercise

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Importance of the region of convergence

Both sequences have the same Z transform:

$$\mathcal{Z}\left\{f_n = a^{n-1}u_{n-1}\right\} = \frac{1}{z-a}, \text{ with ROC: } |z| > |a|$$

and

$$\mathcal{Z}\left\{g_n = -a^{n-1}u_{-n}\right\} = \frac{1}{z-a}, \text{ with ROC: } |z| < |a|$$

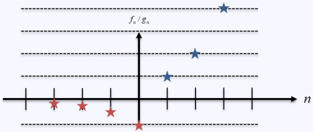

Figure 24: Both series for $a = 2$

Z transform

Importance of the region of convergence

Both sequences have the same Z transform:

$$\mathcal{Z}\left\{f_n = a^{n-1}u_{n-1}\right\} = \frac{1}{z-a}, \text{ with ROC: } |z| > |a|$$

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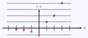

Figure 24: Both series for $a = 2$

Exercise (#6.8)

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- ▷ Sketch both series
- ▷ Calculate the Z transform

Note: Use $\sum\limits_{k=0}^{\infty} q^k = \frac{1}{1-q}$

Convergence

The following behaviours can be observed:

1. $X(z)$ converges for all z with $|z| < \infty$
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4. $X(z)$ converges for all z with $R_0 < |z| < R_1$
5. $X(z)$ converges only for $z = 0$

Convergence

In the following we will primarily consider causal signals (and systems): $x_n = 0 \; \forall \; n < 0$

$$\mathcal{Z}\left\{x_n\right\} = X(z) = \sum_{n=0}^{\infty} x_n z^{-n}.$$

For such signals: If the series converges for any radius r_0, then the series will converge for $r_1 > r_0$ as well.

Table of Contents

Z-Transform

6.4 Z-Transform
6.4.1 Definition
6.4.2 Convergence
6.4.3 Properties
6.4.4 Unilateral transform
6.4.5 Mapping to a frequency
6.4.6 Inverse transform
6.4.7 Inverse transform using Fourier Transform

Z transform

Properties

$$
\begin{aligned}
X(z) &= \mathcal{Z}\{x_n\} = \sum_{n=-\infty}^{\infty} x_n z^{-n} \\
\mathcal{Z}\{af_n + bg_n\} &= a\mathcal{Z}\{f_n\} + b\mathcal{Z}\{g_n\} \\
\mathcal{Z}\{a^n f_n\} &= F\left(\frac{z}{a}\right) \\
\mathcal{Z}\{f_{n-1}\} &= \mathcal{Z}\{f_n\} z^{-1} \\
\mathcal{Z}\{nf_n\} &= -z\frac{\partial F(z)}{\partial z}\mathcal{Z}\{f_n\} \\
\mathcal{Z}\{f_{-n}\} &= F\left(z^{-1}\right) \\
\mathcal{Z}\{f_n^*\} &= F^*\left(z^*\right)
\end{aligned}
$$

Properties

$$
\begin{aligned}
X(z) &= \mathcal{Z}\left\{x_n\right\} = \sum_{n=-\infty}^{\infty} x_n z^{-n} \\
\mathcal{Z}\left\{af_n + bg_n\right\} &= a\mathcal{Z}\left\{f_n\right\} + b\mathcal{Z}\left\{g_n\right\} \\
\mathcal{Z}\left\{a^n f_n\right\} &= F\left(\frac{z}{a}\right) \\
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\mathcal{Z}\left\{nf_n\right\} &= -z\frac{\partial F(z)}{\partial z}\mathcal{Z}\left\{f_n\right\} \\
\mathcal{Z}\left\{f_{-n}\right\} &= F\left(z^{-1}\right) \\
\mathcal{Z}\left\{f_n^*\right\} &= F^*\left(z^*\right)
\end{aligned}
$$

Properties

$$
\mathcal{Z}\left\{f_n * g_n\right\} = \mathcal{Z}\left\{\sum_{l=-\infty}^{\infty} f_l g_{n-l}\right\} = F(z)G(z)
$$

Table of Contents

Z-Transform

6.4 Z-Transform
6.4.1 Definition
6.4.2 Convergence
6.4.3 Properties
6.4.4 Unilateral transform
6.4.5 Mapping to a frequency
6.4.6 Inverse transform
6.4.7 Inverse transform using Fourier Transform

Unilateral transform

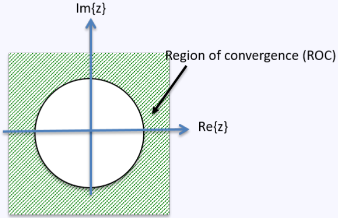
Definition

The z-Transform of an infinite series f_n (with $n \geq 0$) is given as follows:

$$\mathcal{Z}\left\{f_n\right\} = \sum_{k=0}^{\infty} f_k z^{-k}$$

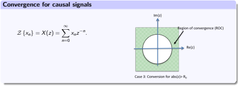
Convergence for causal signals

$$\mathcal{Z}\left\{x_n\right\} = X(z) = \sum_{n=0}^{\infty} x_n z^{-n}.$$



Case 3: Conversion for abs(z)> R_0

Z transform



Transform pairs

Time domain	$z-$domain
u_n	$\frac{z}{z-1}$
$u_n a^n$	$\frac{z}{z-a}$

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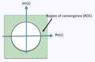

Case 3: Conversion for abs(z)> R_0

Transform pairs

Time domain	Laplace Domain	$z-$domain
$\delta(t)$	1	
Unit step	$\frac{1}{s}$	$\frac{z}{z-1}$
t	$\frac{1}{s^2}$	$\frac{T \cdot z}{(z-1)^2}$

Unilateral transform

Properties

Table 13.2 Properties of the *z*-Transform

	$x(t)$	$X(z)$
1.	$kx(t)$	$kX(z)$
2.	$x_1(t) + x_2(t)$	$X_1(z) + X_2(z)$
3.	$x(t + T)$	$zX(z) - zx(0)$
4.	$tx(t)$	$-Tz\dfrac{dX(z)}{dz}$
5.	$e^{-at}x(t)$	$X(ze^{aT})$
6.	$x(0)$, initial value	$\lim_{z\to\infty} X(z)$ if the limit exists
7.	$x(\infty)$, final value	$\lim_{z\to 1}(z-1)X(z)$ if the limit exists and the system is stable; that is, if all poles of $(z-1)X(z)$ are inside the unit circle $\lvert z \rvert = 1$ on z-plane.

Unilateral transform

Properties

Table 13.1 z-Transforms

$x(t)$	$X(s)$	$X(z)$
$\delta(t) = \begin{cases} \frac{1}{\epsilon}, & t < \epsilon, \epsilon \to 0 \\ 0 & \text{otherwise} \end{cases}$	1	—
$\delta(t-a) = \begin{cases} \frac{1}{\epsilon}, & a < t < a+\epsilon, \epsilon \to 0 \\ 0 & \text{otherwise} \end{cases}$	e^{-as}	—
$\delta_o(t) = \begin{cases} 1 & t = 0, \\ 0 & t = kT, k \neq 0 \end{cases}$	—	1
$\delta_o(t-kT) = \begin{cases} 1 & t = kT, \\ 0 & t \neq kT \end{cases}$	—	z^{-k}

Unilateral transform

Properties

Table 13.1 z-Transforms

$x(t)$	$X(s)$	$X(z)$
$u(t)$, unit step	$1/s$	$\frac{z}{z-1}$
t	$1/s^2$	$\frac{Tz}{(z-1)^2}$
e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$
$1-e^{-at}$	$\frac{1}{s(s+a)}$	$\frac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})}$

Unilateral transform

Properties

Table 13.1 z-Transforms

$x(t)$	$X(s)$	$X(z)$
$\sin(\omega t)$	$\dfrac{\omega}{s^2+\omega^2}$	$\dfrac{z\sin(\omega T)}{z^2-2z\cos(\omega T)+1}$
$\cos(\omega t)$	$\dfrac{s}{s^2+\omega^2}$	$\dfrac{z(z-\cos(\omega T))}{z^2-2z\cos(\omega T)+1}$
$e^{-at}\sin(\omega t)$	$\dfrac{\omega}{(s+a)^2+\omega^2}$	$\dfrac{(ze^{-aT}\sin(\omega T))}{z^2-2ze^{-aT}\cos(\omega T)+e^{-2aT}}$
$e^{-at}\cos(\omega t)$	$\dfrac{s+a}{(s+a)^2+\omega^2}$	$\dfrac{z^2-ze^{-aT}\cos(\omega T)}{z^2-2ze^{-aT}\cos(\omega T)+e^{-2aT}}$

Table of Contents

Z-Transform

6.4 Z-Transform
- 6.4.1 Definition
- 6.4.2 Convergence
- 6.4.3 Properties
- 6.4.4 Unilateral transform
- 6.4.5 Mapping to a frequency
- 6.4.6 Inverse transform
- 6.4.7 Inverse transform using Fourier Transform

Mapping to a frequency

$$\mathcal{Z}\left\{x_n\right\} = X(z) = \sum_{n=-\infty}^{\infty} x_n z^{-n}$$

The following subtitution maps to a *normalized* frequency:

$$z = e^{j\omega}$$

The following subtitution maps to a frequency (sampling period T):

$$z = e^{j\omega T}$$

Mapping to a frequency

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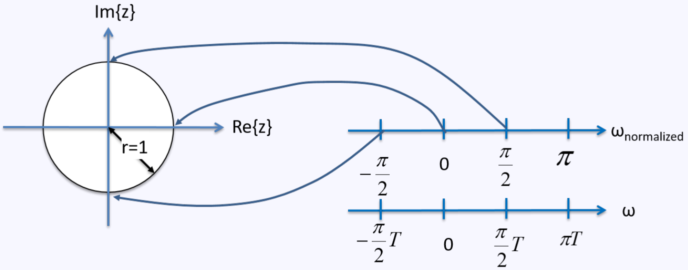
Mapping to a frequency



Note: In case of undersampling, signal components with different frequencies map to the same position on the circle.

Table of Contents

Z-Transform

6.4 Z-Transform
6.4.1 Definition
6.4.2 Convergence
6.4.3 Properties
6.4.4 Unilateral transform
6.4.5 Mapping to a frequency
6.4.6 Inverse transform
6.4.7 Inverse transform using Fourier Transform

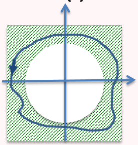
Inverse Z transform

Definition

The inverse transform is defined as follows:

$$f_n = \frac{1}{2\pi j} \oint F(z) z^{n-1} dz.$$

This is a counterclockwise integral with a path inside the region of convergence (independent of the integration path, can be solved with Cauchy's residue theorem). This means that the inverse transform only depends on the poles of $F(z)$.



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Properties

- We will use the Z-transform for the analysis of discrete-time LTI systems.
- Thus, we will use
  - Z-transform tables,
  - partial fraction decomposition,
  - polynominal long division,

  and other techniques of signal decomposition for transformations
- Using well-known structures (see next block) will give you the ability to implement e.g. filters without applying the inverse transformation.

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Polynomial long division

Bring

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots}{a_0 + a_1 z^{-1} + \cdots}$$

into the form

$$\begin{aligned} H(z) &= c_0 + c_1 z^{-1} + c_2 z^{-2} + \cdots \\ y_n &= c_0 x_n + c_1 x_{n-1} + c_2 x_{n-2} + \cdots = x_n * h_n \\ &= \sum_{l=0}^{\infty} h_l x_{n-l} \end{aligned}$$

Inverse Z transform

Polynomial long division

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$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots}{a_0 + a_1 z^{-1} + \cdots}$$

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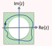

Example

$$\begin{aligned} H(z) &= \frac{3z^{-1} + z^{-2} + 2}{z^{-1} + 1} \\ &= 2 + z^{-1} \Rightarrow h_0 = 2, h_1 = 1 \end{aligned}$$

Note: The polyomial division gives an infinite result in case of an IIR filter (see next chapter)

Table of Contents

Z-Transform

6.4 Z-Transform
6.4.1 Definition
6.4.2 Convergence
6.4.3 Properties
6.4.4 Unilateral transform
6.4.5 Mapping to a frequency
6.4.6 Inverse transform
6.4.7 Inverse transform using Fourier Transform

Inverse Z transform

Properties

If the unit circle belongs to the region of convergence, then one can integrate on the unit circle (substitution):

$$
\begin{aligned}
f_n &= \frac{1}{2\pi j}\oint F(z) z^{n-1} dz \\
&= \frac{1}{2\pi j}\int_{-\pi}^{\pi} F\left(z = e^{j\omega}\right) e^{j\omega(n-1)} j e^{j\omega} d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} F\left(z = e^{j\omega}\right) e^{j\omega n} d\omega
\end{aligned}
$$

Inverse Z transform

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Properties

Unit circle belonging to the region of convergence:

$$f_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} F\left(z=e^{j\omega}\right) e^{j\omega n} d\omega$$

This is equivalent to the inverse Fourier transform:

$$\mathcal{F}^{-1}\left\{F_d(\omega)\mathrm{rect}\left(\frac{\omega}{2\pi}\right)\right\}(t=nT) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_d(\omega) e^{j\omega nT} d\omega,$$

with $F_d(\omega)$ being the Fourier transform of the signal $f_d(t) = \sum_{n=-\infty}^{\infty} f_n \delta(t-nT)$.

Inverse Z transform

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Unit circle belonging to the region of convergence:

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$$\begin{aligned} f_n &= \frac{1}{2\pi j}\oint F(z)z^{n-1}dz \\ &= \frac{1}{2\pi j}\int_{-\pi}^{\pi} F\left(z = e^{j\omega}\right) e^{j\omega(n-1)} j e^{j\omega} d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} F\left(z = e^{j\omega}\right) e^{j\omega n} d\omega \end{aligned}$$

Properties

$$\begin{aligned} f_d(t) &= \sum_{n=-\infty}^{\infty} f_n \delta(t - nT) \\ F_d(\omega) &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f_n \delta(t - nT) e^{-j\omega t} dt \\ &= \sum_{n=-\infty}^{\infty} f_n e^{-j\omega nT} \int_{-\infty}^{\infty} \delta(t - nT) dt = \underbrace{\sum_{n=-\infty}^{\infty} f_n e^{-j\omega nT}}_{\text{discrete Fourier transform}} \end{aligned}$$

Table of Contents

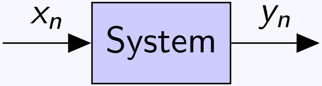
Discrete Time

6.1 Introduction
6.2 Pulse modulated signal
6.3 Discrete Fourier Transform
6.4 Z-Transform
6.5 Time discrete LTI systems
6.5.1 Stability
6.5.2 FIR and IIR
6.5.3 Example system
6.5.4 Exercises
6.6 Special classes of time discrete systems
6.7 Relationship between different transforms
6.8 Exercises
6.9 Appendix

Inverse transform using Fourier

LTI

We now consider the LTI system shown below:



With the transfer function $H(z)$ and impulse response h_n the following relations hold true:

$$
\begin{aligned}
y_n &= x_n * h_n = h_n * x_n = \sum_{n=-\infty}^{\infty} x_l h_{n-l} \\
H(z) &= \mathcal{Z}\left\{h_n\right\} \\
Y(z) &= X(z)H(z)
\end{aligned}
$$

Frequency response

$$\begin{aligned} H(z) &= \mathcal{Z}\{h_n\} \\ Y(z) &= X(z)H(z) \end{aligned}$$

The frequency response can be calculated by letting

$$z = e^{j\omega T}.$$

Note: In the literature T is most times omitted (set to 1) for a "normalized" frequency:

$$z = e^{j\omega}.$$

Table of Contents

Time discrete LTI systems

6.5 Time discrete LTI systems
6.5.1 Stability
6.5.2 FIR and IIR
6.5.3 Example system
6.5.4 Exercises

Stability

Stability

The system is BIBO stable (bounded input, bounded output) if and only if

$$\sum_{n=0}^{\infty} |h_n| < \infty$$

Stability

Stability

Stability can also be checked in the z-domain. Usually H(Z) can be written in the following form:

$$H(z) = \frac{N(z)}{D(z)}.$$

The system is stable, if the poles of H(z) lie inside the unity circle:

$$D(z) \neq 0 \text{ for } |z| \leq 1$$

Time discrete LTI systems

6.5 Time discrete LTI systems
6.5.1 Stability
6.5.2 FIR and IIR
6.5.3 Example system
6.5.4 Exercises

Classification

Finite Impulse Response (FIR)

- h_n has only an endless number of values unequal to 0
- non-recursive
- always stable

Infinite Impulse Response (IIR)

- h_n has an unlimited number of values unequal to 0
- recursive

Classification

Finite Impulse Response (FIR)

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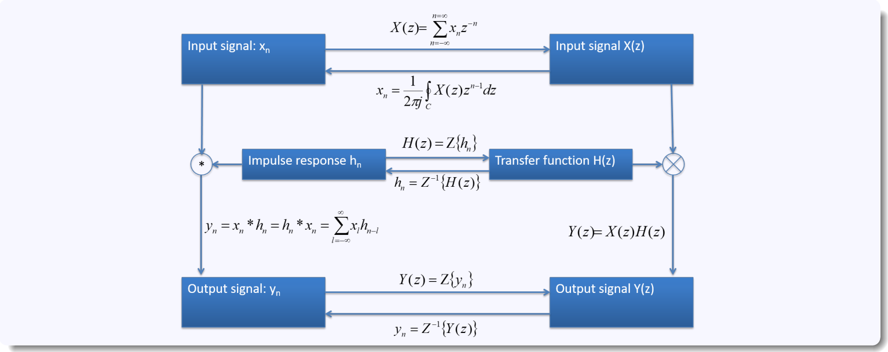
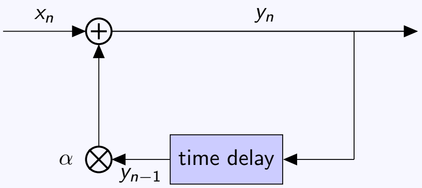


Time discrete LTI systems

6.5 Time discrete LTI systems
6.5.1 Stability
6.5.2 FIR and IIR
6.5.3 Example system
6.5.4 Exercises

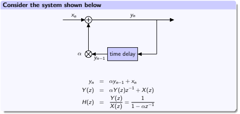
Example system

Consider the system shown below

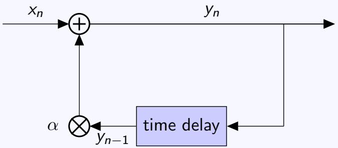


$$
\begin{aligned}
y_n &= \alpha y_{n-1} + x_n \\
Y(z) &= \alpha Y(z) z^{-1} + X(z) \\
H(z) &= \frac{Y(z)}{X(z)} = \frac{1}{1-\alpha z^{-1}}
\end{aligned}
$$

Example system



Finding the poles



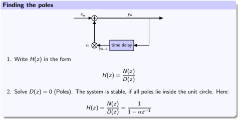
1. Write $H(z)$ in the form

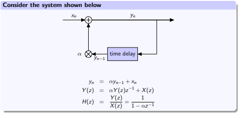
$$H(z) = \frac{N(z)}{D(z)}$$

2. Solve $D(z) = 0$ (Poles). The system is stable, if all poles lie inside the unit circle. Here:

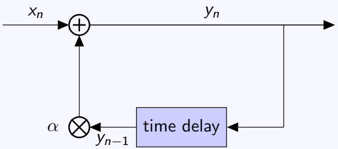
$$H(z) = \frac{N(z)}{D(z)} = \frac{1}{1 - \alpha z^{-1}}$$

Example system





Finding the poles



2. Solve $D(z) = 0$ (Poles).

$$
\begin{aligned}
0 &= 1 - \alpha z^{-1} \\
z &= \alpha z
\end{aligned}
$$

This system is stable if

$$|\alpha| < 1$$

Table of Contents

Time discrete LTI systems

6.5 Time discrete LTI systems
6.5.1 Stability
6.5.2 FIR and IIR
6.5.3 Example system
6.5.4 Exercises

Exercise (#6.9)

Given is a system with the transfer function

$$H(z) = \frac{z}{z^2 - \frac{1}{4}z - \frac{3}{8}}.$$

Calculate the impulse response.

Exercise

Exercise (#6.9)

Given is a system with the transfer function

$$H(z) = \frac{z}{z^2 - \frac{1}{4}z - \frac{3}{8}}.$$

Calculate the impulse response.

Exercise (#6.9)
Given is a system with the transfer function

$$H(z) = \frac{z}{z^2 - \frac{1}{4}z - \frac{1}{8}}.$$

Calculate the impulse response.

Exercise (#6.10)

Given is a system with the transfer function

$$H(z) = \frac{z-1}{z+\frac{1}{2}}.$$

Sketch the impulse response.

Exercise

Exercise (#6.10)

Given is a system with the transfer function

$$H(z) = \frac{z-1}{z+\frac{1}{2}}.$$

Sketch the impulse response.

Table of Contents

Discrete Time

6.1 Introduction
6.2 Pulse modulated signal
6.3 Discrete Fourier Transform
6.4 Z-Transform
6.5 Time discrete LTI systems
6.6 Special classes of time discrete systems
 6.6.1 Introduction
 6.6.2 Direct form I
 6.6.3 Direct form II
6.7 Relationship between different transforms
6.8 Exercises
6.9 Appendix

Table of Contents

Special classes of time discrete systems

6.6 Special classes of time discrete systems
6.6.1 Introduction
6.6.2 Direct form I
6.6.3 Direct form II

Standard form

We will now consider systems of the form

$$y_n = \sum_{k=0}^{N} a_k x_{n-k} - \sum_{k=1}^{L} b_k y_{n-k},$$

with the transfer function of the form

$$H(z) = \frac{a_0 z^0 + a_1 z^{-1} + \cdots a_N z^{-N}}{b_0 z^0 + b_1 z^{-1} + \cdots b_L z^{-L}} = \frac{\sum\limits_{k=0}^{N} a_n z^{-k}}{\sum\limits_{k=0}^{L} b_n z^{-k}}$$

Poles and Zeros (holes)

Standard form

We will now consider systems of the form

$$y_n = \sum_{k=0}^{N} a_k x_{n-k} - \sum_{k=1}^{L} b_k y_{n-k},$$

with the transfer function of the form

$$H(z) = \frac{a_0 z^0 + a_1 z^{-1} + \cdots a_N z^{-N}}{b_0 z^0 + b_1 z^{-1} + \cdots b_L z^{-L}} = \frac{\sum_{k=0}^{N} a_n z^{-k}}{\sum_{k=0}^{L} b_n z^{-k}}$$

The transfer function of the form

$$H(z) = \frac{\sum_{k=0}^{N} a_n z^{-k}}{\sum_{k=0}^{L} b_n z^{-k}} = \frac{N(z)}{D(z)}$$

can be modified using the fundamental theorem of algebra as follows:

$$H(z) = \frac{a_N \prod_{k=0}^{N} (z - h_k)}{b_L \prod_{k=0}^{L} (z - p_k)}$$

Partial fraction decomposition

The transfer function of the form

$$H(z)=\frac{\sum\limits_{k=0}^{N} a_n z^{-k}}{\sum\limits_{k=0}^{L} b_n z^{-k}}=\frac{N(z)}{D(z)}$$

can be modified using the fundamental theorem of algebra as follows:

$$H(z)=\frac{a_N\prod\limits_{k=0}^{N}(z-h_k)}{b_L\prod\limits_{k=0}^{L}(z-p_k)}$$

Standard form

We will now consider systems of the form

$$y_n=\sum_{k=0}^{N} a_k x_{n-k}-\sum_{k=1}^{L} b_k y_{n-k},$$

with the transfer function of the form

$$H(z)=\frac{a_0 z^0+a_1 z^{-1}+\cdots a_N z^{-N}}{b_0 z^0+b_1 z^{-1}+\cdots b_L z^{-L}}=\frac{\sum\limits_{k=0}^{N} a_n z^{-k}}{\sum\limits_{k=0}^{L} b_n z^{-k}}$$

Properties

In case of $N(z)$ being a polynom with smaller grade than $D(z)$ and O_k the order of the pole k one can use the partial fraction composition to derive

$$H(z)=\sum_{k=1}^{N}\sum_{l=1}^{O_k}\frac{a_{k,l}}{(z-p_k)^l}.$$

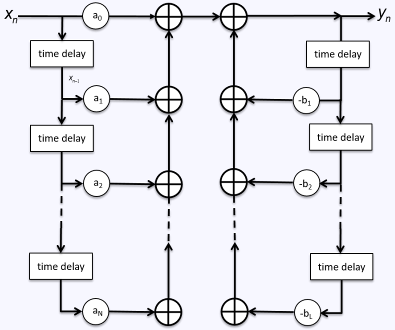
Table of Contents

Special classes of time discrete systems

6.6 Special classes of time discrete systems
6.6.1 Introduction
6.6.2 Direct form I
6.6.3 Direct form II

Direct form I

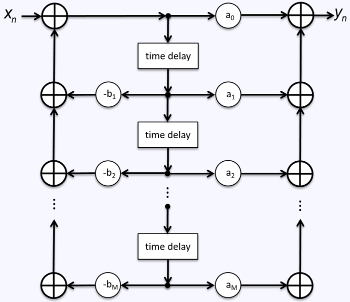
Direct form I



Special classes of time discrete systems

6.6 Special classes of time discrete systems
6.6.1 Introduction
6.6.2 Direct form I
6.6.3 Direct form II

Direct form II



$$m = \max(N, L)$$

Table of Contents

Discrete Time

6.1 Introduction
6.2 Pulse modulated signal
6.3 Discrete Fourier Transform
6.4 Z-Transform
6.5 Time discrete LTI systems
6.6 Special classes of time discrete systems
6.7 Relationship between different transforms
- 6.7.1 Laplace and Fourier transform
- 6.7.2 Laplace and Z-transform

6.8 Exercises
6.9 Appendix

Table of Contents

Relationship between different transforms

6.7 Relationship between different transforms
6.7.1 Laplace and Fourier transform
6.7.2 Laplace and Z-transform

Laplace and Fourier

Properties

If

$f(t < 0) = 0$

and if

the imaginary axis belongs to the ROC,

then

$$\mathcal{L}\left\{f(t)\right\}(s = j\omega) = \mathcal{F}\left\{f(t)\right\}$$

Note: Frequently, the Fourier transform is written as follows:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}$$

Table of Contents

Relationship between different transforms

6.7 Relationship between different transforms
6.7.1 Laplace and Fourier transform
6.7.2 Laplace and Z-transform

Laplace and Z-transform

	continuous time	**discrete time**
Variable	t	n
Transformation	One-sided Laplace	Unilateral Z-transform[a]
Frequency domain variable	s	z
Area of convergence	Half spaces	$r > r_0$
Frequency response	$s = j\omega$	$z = e^{j\omega T}$
Stability condition for poles	$Re\{s_p\} < 0$	$\lvert z_p\rvert < 1$
Stability condition in time domain	$\int\limits_0^{\infty} \lvert h(t)\rvert dt < \infty$	$\sum\limits_{n=0}^{\infty} \lvert h_n\rvert < \infty$

[a] In the following we will only consider the unilateral transform. For properties of the bilateral Z-transform: See the according chapter

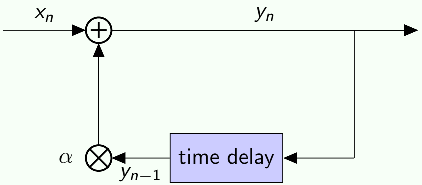
Table of Contents

Discrete Time

6.1 Introduction
6.2 Pulse modulated signal
6.3 Discrete Fourier Transform
6.4 Z-Transform
6.5 Time discrete LTI systems
6.6 Special classes of time discrete systems
6.7 Relationship between different transforms
6.8 Exercises
6.9 Appendix

Exercise

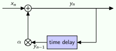
Exercise (#6.11)



- Numerically calculate the impulse response with NUMPY or MATLAB for $n = 0 \ldots 1000$ (w/o using the function `filter`)
- Plot the result for different values of α

Exercise

Exercise (#6.11)



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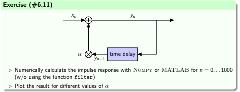

Exercise (#6.12)

Given is a system with the transfer function

$$H(z) = 1 - z^{-1}$$

- Find the impulse response h_n
- Write y_n as a function of x_n
- Sketch the magnitude of the transfer function as a function of the frequency ω.
- Find an analog circuit with a *similar* transfer function.

Exercise

Exercise (#6.12)

Given is a system with the transfer function

$$H(z) = 1 - z^{-1}$$

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- ▷ Write y_n as a function of x_n
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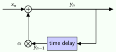
Exercise (#6.11)



- Numerically calculate the impulse response with NUMPY or MATLAB for $n = 0 \dots 1000$ (w/o using the function `filter`)
- Plot the result for different values of α

Exercise (#6.13)

Given is a system with the transfer function

$$H(z) = \frac{z^2 + z}{z^3 - 2z^2 + z - 0.5}$$

- Find the impulse response h_n
- Find the impulse response making use of NUMPY or MATLAB and the function `impz`.

Exercise

Exercise (#6.13)

Given is a system with the transfer function

$$H(z) = \frac{z^2 + z}{z^3 - 2z^2 + z - 0.5}$$

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- Find the impulse response making use of NUMPY or MATLAB and the function `impz`.

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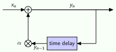
Exercise (#6.11)



- ▷ Numerically calculate the impulse response with NUMPY or MATLAB for $n = 0 \ldots 1000$ (w/o using the function `filter`)
- ▷ Plot the result for different values of α

Exercise (#6.14)

Given is the file `B6E14.csv` which contains a heartbeat signal (first column: time in m sec).

- ▷ Plot the spectrum of the signal using meaningful axis.
- ▷ Interpret the result.

Exercise

Exercise (#6.14)

Given is the file `B6E14.csv` which contains a heartbeat signal (first column: time in m sec).

- ▷ Plot the spectrum of the signal using meaningful axis.
- ▷ Interpret the result.

Table of Contents

Discrete Time

6.1 Introduction
6.2 Pulse modulated signal
6.3 Discrete Fourier Transform
6.4 Z-Transform
6.5 Time discrete LTI systems
6.6 Special classes of time discrete systems
6.7 Relationship between different transforms
6.8 Exercises
6.9 Appendix

Table of Contents

Discrete Time & Applications

6. Discrete Time
7. Filters
8. Applications and Exercises

Table of Contents

Filters

7.1 Introduction
7.2 Filter properties
7.3 Filter structures
7.4 Bandform transformations
7.5 Digital filters
7.6 Excursion: Control theory
7.7 Exercises

Study goals

Study goals

- Classify basic types of filters
- Name limitations
- Transform from continuous time to discrete time and vice versa

Study goals
- Classify basic types of filters
- Name limitations
- Transform from continuous time to discrete time and vice versa

Definition

Referring to LTI systems, the most common meaning of filter is a device that removes/amplifies/attenuates parts of the spectrum. An example is shown below:

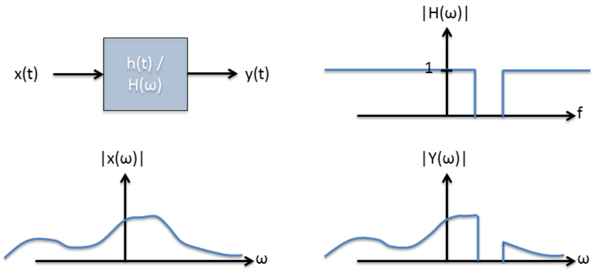


Figure 25: Example filter

Applications

Definition

Referring to LTI systems, the most common meaning of filter is a device that removes/amplifies/attenuates parts of the spectrum. An example is shown below:

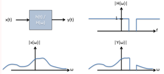


Figure 25: Example filter

Study goals

- Classify basic types of filters
- Name limitations
- Transform from continuous time to discrete time and vice versa

Examples

Figure 26: Equalizer

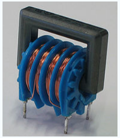


Figure 27: Choke

Figure 28: Audio crossover

Figure 29: Waveguide filter

Figure 30: AC line filter

Note: All pictures from Wikipedia

Classification

Examples

Figure 26: Equalizer
Figure 27: Choke
Figure 28: Audio crossover
Figure 29: Waveguide filter
Figure 30: AC line filter

Note: All pictures from Wikipedia

Definition

Referring to LTI systems, the most common meaning of filter is a device that removes/amplifies/attenuates parts of the spectrum. An example is shown below:

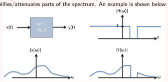


Figure 25: Example filter

Study goals

- ▷ Classify basic types of filters
- ▷ Name limitations
- ▷ Transform from continuous time to discrete time and vice versa

Examples

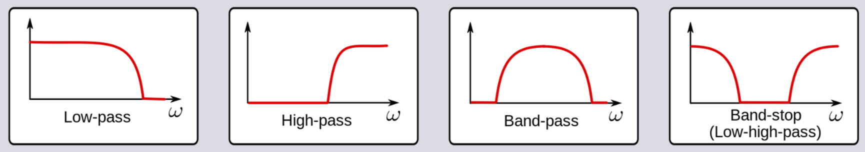


Figure 31: Filter types

Other types:

- ▷ Notch filter: Band-stop filter with a narrow stopband
- ▷ Comb filter: Multiple equally spaced passbands

Table of Contents

Filters

7.1 Introduction

7.2 Filter properties

- 7.2.1 Time versus frequency limitation
- 7.2.2 Phase delay
- 7.2.3 Group delay
- 7.2.4 Cascading

7.3 Filter structures

7.4 Bandform transformations

7.5 Digital filters

7.6 Excursion: Control theory

7.7 Exercises

Table of Contents

Filter properties

7.2 Filter properties
7.2.1 Time versus frequency limitation
7.2.2 Phase delay
7.2.3 Group delay
7.2.4 Cascading

Ideal low-pass

Transfer function

$$H(\omega) \quad = \quad \begin{cases} 1 & \text{for } |\omega| < 0.5\Delta\omega \\ 0 & \text{else} \end{cases}$$

Impulse response

$$\begin{aligned} h(t) \quad &= \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{j\omega t}d\omega \\ &= \quad \frac{\sin \pi t}{\pi t} t \end{aligned}$$

Properties

This is a non-causal filter with unlimited impulse response

Filter properties

7.2 Filter properties
7.2.1 Time versus frequency limitation
7.2.2 Phase delay
7.2.3 Group delay
7.2.4 Cascading

Phase delay

Properties

Phase is the integral of the frequency. Example:

$$\omega(t) = \omega_0 + \Delta\omega t,$$

then

$$\varphi(t) = \omega_0 t + 0.5\Delta\omega t^2.$$

Definition

The phase delay is the time delay of the phase:

$$\tau_{ph}(\omega) = -\frac{\varphi(\omega)}{\omega} = -\frac{\arg\{H(\omega)\}}{\omega}$$

This is equivalent to the time a signal with fixed frequency f needs to pass the system:

$$x(t) = \sin(2\pi f_0 t) \rightarrow y(t) \sim \sin(2\pi f_0 (t - \tau_{ph}))$$

Filter properties

7.2 Filter properties
7.2.1 Time versus frequency limitation
7.2.2 Phase delay
7.2.3 Group delay
7.2.4 Cascading

Group delay

Definition

Assuming a signal $a(t)$ with a small bandwidth and

$$x(t) = a(t)\cos(\omega_0 t),$$

then one can derive the output of an LTI system as follows:

$$y(t) = |H(\omega_0)|\, a(t - \tau_g)\cos\left(\omega_0(t - \tau_{ph})\right).$$

The group delay is the time delay of the amplitude envelope:

$$\tau_g(\omega) = -\frac{\mathrm{d}}{\mathrm{d}\omega}\varphi(\omega)$$

Filter properties

7.2 Filter properties
7.2.1 Time versus frequency limitation
7.2.2 Phase delay
7.2.3 Group delay
7.2.4 Cascading

Cascading

Example

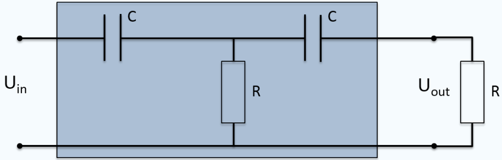
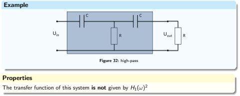


Figure 32: high-pass

Properties

The transfer function of this system **is not** given by $H_1(\omega)^2$



Example

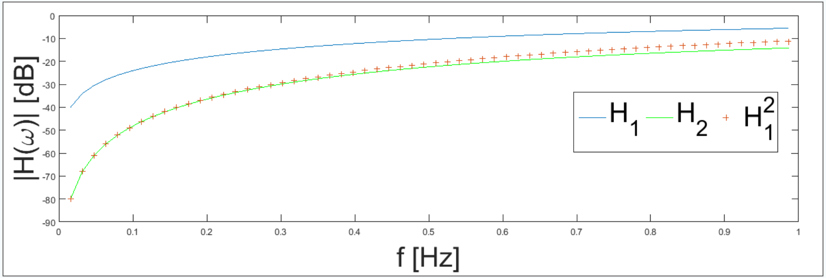


Figure 33: high-pass (R $= 1\text{k}\Omega$, C $= 100\ \mu$F)

Cascading

Example

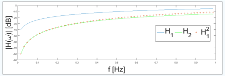


Figure 33: high-pass (R = 1kΩ, C = 100 μF)

Example

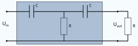


Figure 32: high-pass

Properties

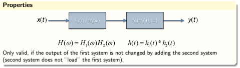
The transfer function of this system **is not** given by $H_1(\omega)^2$

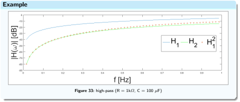
Properties

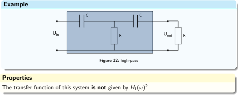
$$H(\omega) = H_1(\omega) H_2(\omega) \qquad h(t) = h_1(t) * h_2(t)$$

Only valid, if the output of the first system is not changed by adding the second system (second system does not "load" the first system).

Cascading







Example

Using a buffer amplifier:

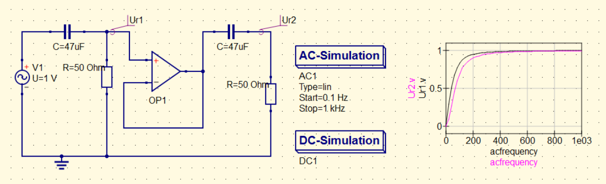


Figure 34: Cascading using a unity gain buffer amplifier

Note: QUCS[a] model can be found on ILIAS

[a] http://qucs.sourceforge.net/

Table of Contents

Filters

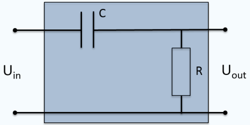
7.1 Introduction
7.2 Filter properties
7.3 Filter structures
7.3.1 A simple high-pass
7.3.2 A simple band-pass
7.3.3 Butterworth filter
7.3.4 Bessel filter
7.3.5 Chebyshev filter
7.3.6 Elliptical filter
7.3.7 Comparison
7.4 Bandform transformations
7.5 Digital filters
7.6 Excursion: Control theory
7.7 Exercises

Filter structures

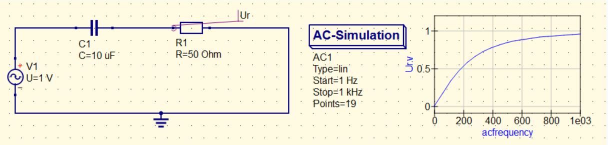
7.3 Filter structures
7.3.1 A simple high-pass
7.3.2 A simple band-pass
7.3.3 Butterworth filter
7.3.4 Bessel filter
7.3.5 Chebyshev filter
7.3.6 Elliptical filter
7.3.7 Comparison

A simple high-pass

Example



Simulation with QUCS[a]:



[a] http://qucs.sourceforge.net/

Table of Contents

Filter structures

7.3 Filter structures
7.3.1 A simple high-pass
7.3.2 A simple band-pass
7.3.3 Butterworth filter
7.3.4 Bessel filter
7.3.5 Chebyshev filter
7.3.6 Elliptical filter
7.3.7 Comparison

A simple band-pass

Example

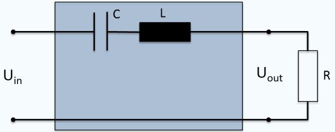


Figure 35: band-pass

$$H(\omega) = \frac{j\omega RC}{1 + j\omega RC + \omega^2 LC}$$

Table of Contents

Filter structures

7.3 Filter structures
7.3.1 A simple high-pass
7.3.2 A simple band-pass
7.3.3 Butterworth filter
7.3.4 Bessel filter
7.3.5 Chebyshev filter
7.3.6 Elliptical filter
7.3.7 Comparison

Butterworth filter

Properties

- a low pass filter that shows no ripples in the pass band
- also known as maximally flat amplitude filter
- normalized frequency response for a filter with order n:

$$|H(\omega)| = \sqrt{\frac{1}{1+\omega^{2n}}}$$

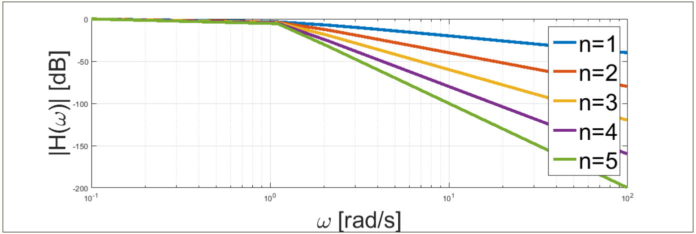
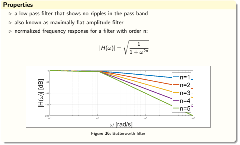


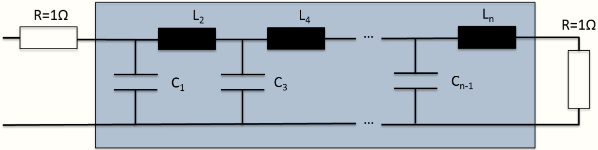
Figure 36: Butterworth filter

Butterworth filter



Properties

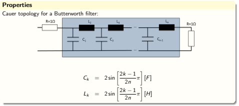
Cauer topology for a Butterworth filter:

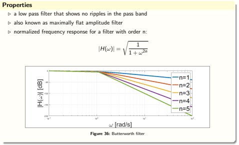


$$C_k = 2\sin\left[\frac{2k-1}{2n}\pi\right]\ [F]$$

$$L_k = 2\sin\left[\frac{2k-1}{2n}\pi\right]\ [H]$$

Butterworth filter





Design process

Consider a lowpass filter with

- Passband: $1 - \delta_p \leq |H(\omega)| \leq 1 + \delta_p$ for $|\omega| \leq \omega_p$
- Stopband $|H(\omega)| \leq \delta_s$ for $\omega > \omega_s$

Steps:

1. Determine order N
2. Determine transfer function
3. Map frequencies

Design process

Consider a lowpass filter with

- Passband: $1-\delta_p \le |H(\omega)| \le 1+\delta_p$ for $|\omega| \le \omega_p$
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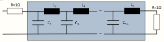

$$C_k = 2\sin\left[\frac{2k-1}{2n}\pi\right] [F]$$
$$L_k = 2\sin\left[\frac{2k-1}{2n}\pi\right] [H]$$

Properties

- a low pass filter that shows no ripples in the pass band
- also known as maximally flat amplitude filter
- normalized frequency response for a filter with order n:

$$|H(\omega)| = \sqrt{\frac{1}{1+\omega^{2n}}}$$

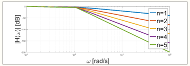

Figure 36: Butterworth filter

Design process

Steps:

1. Determine order N

$$N = \frac{1}{2}\frac{\ln(G_p/G_s)}{\ln(\omega_p/\omega_s)},$$

   with $G_p = \frac{1}{(1-\delta_p)^2} - 1,\ G_s = \frac{1}{(\delta_s)^2} - 1$

2. Determine transfer function
3. Map frequencies

Design process

Steps:

1. Determine order N
2. Determine transfer function $H(s) = \frac{1}{D(s)}$, with

Order	Denominator $D(s)$
1	$(s+1)$
2	$(s^2 + 1.414214s + 1)$
3	$(s+1)(s^2 + s + 1)$
4	$(s^2 + 0.765367s + 1)(s^2 + 1.847759s + 1)$
5	$(s+1)(s^2 + 0.618034s + 1)(s^2 + 1.618034s + 1)$
6	$s^2 + 0.517638s + 1)(s^2 + 1.414214s + 1)(s^2 + 1.931852s + 1)$
7	$(s+1)(s^2 + 0.445042s + 1)(s^2 + 1.246980s + 1)(s^2 + 1.801938s + 1)$

3. Map frequencies

Design process

Steps:

1. Determine order N

$$N = \frac{1}{2}\frac{\ln(G_p/G_s)}{\ln(\omega_p/\omega_s)},$$

with $G_p = \frac{1}{(1-\delta_p)^2} - 1$, $G_s = \frac{1}{(\delta_s)^2} - 1$

2. Determine transfer function
3. Map frequencies

Design process

Consider a lowpass filter with

- Passband: $1 - \delta_p \leq |H(\omega)| \leq 1 + \delta_p$ for $|\omega| \leq \omega_p$
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Steps:

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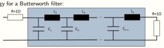
2. Determine transfer function
3. Map frequencies

Properties

Cauer topology for a Butterworth filter:



$$C_k = 2\sin\left[\frac{2k-1}{2n}\pi\right] [F]$$
$$L_k = 2\sin\left[\frac{2k-1}{2n}\pi\right] [H]$$

Design process

Steps:

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Order	Denominator $D(s)$
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3. Map frequencies

Design process

Steps:

1. Determine order N

$$N = \frac{1}{2}\frac{\ln(G_p/G_s)}{\ln(\omega_p/\omega_s)},$$

with $G_p = \frac{1}{(1-\delta_p)^2} - 1$, $G_s = \frac{1}{(\delta_s)^2} - 1$

2. Determine transfer function
3. Map frequencies

Design process

Consider a lowpass filter with

- Passband: $1 - \delta_p \leq |H(\omega)| \leq 1 + \delta_p$ for $|\omega| \leq \omega_p$
- Stopband $|H(\omega)| \leq \delta_s$ for $\omega > \omega_s$

Steps:

1. Determine order N
2. Determine transfer function
3. Map frequencies

Design process

Steps:

1. Determine order N
2. Determine transfer function
3. Map frequencies

$$H(s) = H(S)|_{S=s/\omega_c},$$

with

$$\omega_c = \frac{\omega_p}{(G_p)^{0.5/N}} \text{ (passband contraint) or (stopband constraint) } \omega_c = \frac{\omega_s}{(G_s)^{0.5/N}}$$

and the gain terms given $G_p = \frac{1}{\delta_p^2} - 1$ and $G_s = \frac{1}{\delta_s^2} - 1$, where δ_p and δ_s are the linear limits for the pass and stop band.

Table of Contents

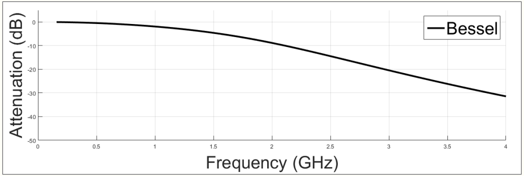
Filter structures

7.3 Filter structures
7.3.1 A simple high-pass
7.3.2 A simple band-pass
7.3.3 Butterworth filter
7.3.4 Bessel filter
7.3.5 Chebyshev filter
7.3.6 Elliptical filter
7.3.7 Comparison

Bessel filter

Properties

- ▷ Maximally linear phase response and thus prevents the wave shape of filtered signals in the passband.
- ▷ Not as steep as Butterworth



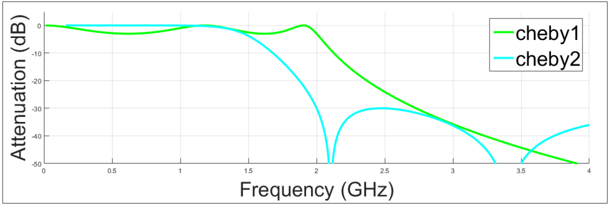
Filter structures

7.3 Filter structures
7.3.1 A simple high-pass
7.3.2 A simple band-pass
7.3.3 Butterworth filter
7.3.4 Bessel filter
7.3.5 Chebyshev filter
7.3.6 Elliptical filter
7.3.7 Comparison

Tschebyscheff (Chebyshev) filter

Properties

- steeper roll-off and passband ripples (type 1) or stopband ripples (type 2).
- Digital implementation available in closed form[a]



[a] See e.g. Xiao, Fast Design of IIR Digital Filters With a General Chebyshev Characteristic

Filter structures

7.3 Filter structures
7.3.1 A simple high-pass
7.3.2 A simple band-pass
7.3.3 Butterworth filter
7.3.4 Bessel filter
7.3.5 Chebyshev filter
7.3.6 Elliptical filter
7.3.7 Comparison

Cauer filter (elliptical filter)

Properties

- very steep roll-off
- Ripples in both bands

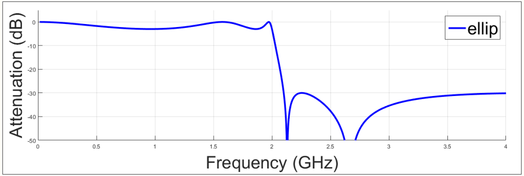


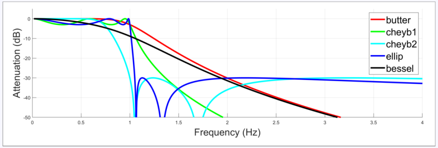
Table of Contents

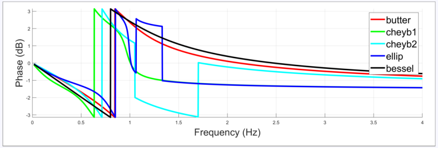
Filter structures

7.3 Filter structures
7.3.1 A simple high-pass
7.3.2 A simple band-pass
7.3.3 Butterworth filter
7.3.4 Bessel filter
7.3.5 Chebyshev filter
7.3.6 Elliptical filter
7.3.7 Comparison

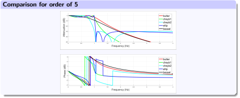
Comparision

Comparison for order of 5





Comparision



Properties

- ▷ Narrower transition band → more ripples

Type	passband	stopband
Butterworth	flat	flat
Bessel	flat	flat
Chebyshev 1	ripples	flat
Chebyshev 2	flat	ripples
Cauer	ripples	ripples

Table of Contents

Filters

7.1 Introduction
7.2 Filter properties
7.3 Filter structures
7.4 Bandform transformations
7.5 Digital filters
7.6 Excursion: Control theory
7.7 Exercises

Bandform transformation

Definition

The following bandform transformations can be used:

- Normalized lowpass to lowpass
$$j\omega' = \frac{j\omega}{j\omega_c}$$
- Lowpass to highpass
$$j\omega' = \frac{1}{j\omega}$$
- Lowpass to bandpass
$$j\omega' = \frac{1}{2\pi B}\left(1 + \frac{1}{j\omega}\right),$$
with B being the bandwidth.

Table of Contents

Filters

7.1 Introduction
7.2 Filter properties
7.3 Filter structures
7.4 Bandform transformations
7.5 Digital filters
7.5.1 Overview
7.5.2 Direct discretization
7.5.3 Impulse invariance methode
7.5.4 Bilinear transform
7.5.5 Window method
7.5.6 Examples
7.5.7 Summary
7.6 Excursion: Control theory
7.7 Exercises

Table of Contents

Digital filters

7.5 Digital filters
7.5.1 Overview
7.5.2 Direct discretization
7.5.3 Impulse invariance methode
7.5.4 Bilinear transform
7.5.5 Window method
7.5.6 Examples
7.5.7 Summary

Ways to design digital filters

- Analog prototypes
  - Direct discretization
  - Impulse invariance method
  - Bilinear transformation
- Window method

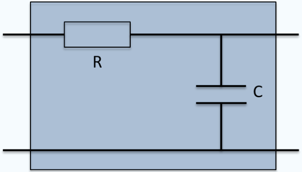
Digital filters

7.5 Digital filters
7.5.1 Overview
7.5.2 Direct discretization
7.5.3 Impulse invariance methode
7.5.4 Bilinear transform
7.5.5 Window method
7.5.6 Examples
7.5.7 Summary

Direct discretization

Example

Consider the low-pass shown below:

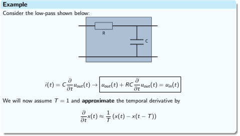


$$i(t) = C\frac{\partial}{\partial t}u_{out}(t) \rightarrow \boxed{u_{out}(t) + RC\frac{\partial}{\partial t}u_{out}(t) = u_{in}(t)}$$

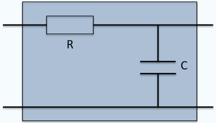
We will now assume $T = 1$ and **approximate** the temporal derivative by

$$\frac{\partial}{\partial t}x(t) \approx \frac{1}{T}\left(x(t) - x(t-T)\right)$$

Direct discretization



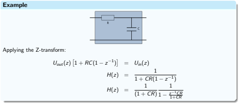
Example

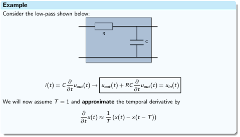


Applying the Z-transform:

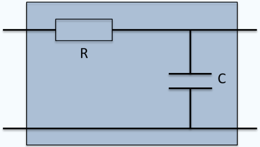
$$
\begin{aligned}
U_{out}(z)\left[1 + RC(1 - z^{-1})\right] &= U_{in}(z) \\
H(z) &= \frac{1}{1 + CR(1 - z^{-1})} \\
H(z) &= \frac{1}{(1 + CR)}\frac{1}{1 - \frac{z^{-1}CR}{1+CR}}
\end{aligned}
$$

Direct discretization





Example



- Transfer function

$$\boxed{H(z) = \frac{1}{(1+CR)} \frac{1}{1 - z^{-1}\frac{CR}{1+CR}}}$$

- Compare this with standard form

$$H_d(z) = \frac{a_0 z^0 + a_1 z^{-1} + \ldots}{1 + b_1 z^{-1} + \ldots}$$

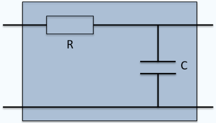
Table of Contents

Digital filters

7.5 Digital filters
7.5.1 Overview
7.5.2 Direct discretization
7.5.3 Impulse invariance methode
7.5.4 Bilinear transform
7.5.5 Window method
7.5.6 Examples
7.5.7 Summary

Solution in frequency domain

Example



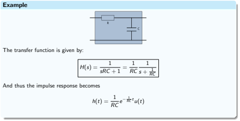
The transfer function is given by:

$$\boxed{H(s) = \frac{1}{sRC+1} = \frac{1}{RC}\frac{1}{s+\frac{1}{RC}}}$$

And thus the impulse response becomes

$$h(t) = \frac{1}{RC}e^{-\frac{1}{RC}t}u(t)$$

Impulse invariance method



Definition

Application of impulse invariance method: Derive sampled impulse response

$$h_n \quad = \quad T h_a(nT)$$

and the spectrum of the sampled impulse response:

$$H_d(\omega) \quad = \quad \sum_{k=-\infty}^{k=\infty} H_a\left(\omega - \frac{2\pi k}{T}\right).$$

Impulse invariance method

Definition

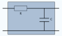
Application of impulse invariance method: Derive sampled impulse response

$$h_n = Th_a(nT)$$

and the spectrum of the sampled impulse response:

$$H_d(\omega) = \sum_{k=-\infty}^{k=\infty} H_a\left(\omega - \frac{2\pi k}{T}\right).$$

Example



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And thus the impulse response becomes

$$h(t) = \frac{1}{RC}e^{-\frac{1}{RC}t}u(t)$$

Properties

- **Note:** Aliasing may occur
- Impulse response shows IIR behaviour
- **But:** Number N of poles defines necessary elements for standard form implemenation

Impulse invariance method

Properties

- ▹ **Note:** Aliasing may occur
- ▹ Impulse response shows IIR behaviour
- ▹ **But:** Number N of poles defines necessary elements for standard form implemenation

Definition

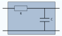
Application of impulse invariance method: Derive sampled impulse response

$$h_n = Th_a(nT)$$

and the spectrum of the sampled impulse response:

$$H_d(\omega) = \sum_{k=-\infty}^{k=\infty} H_a\left(\omega - \frac{2\pi k}{T}\right).$$

Example



The transfer function is given by:

$$\boxed{H(s) = \frac{1}{sRC+1} = \frac{1}{RC}\frac{1}{s+\frac{1}{RC}}}$$

And thus the impulse response becomes

$$h(t) = \frac{1}{RC}e^{-\frac{1}{RC}t}u(t)$$

Properties

▹ Assuming that all N poles have an order/multiplicity of one and applying partial fraction expansion leads to:

$$H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} \Leftrightarrow h_a(t) = \sum_{k=1}^{N} A_k e^{s_k t} u(t).$$

▹ Sampled impulse response will become

$$h_n = Th_a(nT) = \sum_{k=1}^{N} TA_k e^{s_k nT} u(nT)$$

Impulse invariance method

Properties

- Assuming that all N poles have an order/multiplicity of one and applying partial fraction expansion leads to:

$$H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} \Leftrightarrow h_a(t) = \sum_{k=1}^{N} A_k e^{s_k t} u(t).$$

- Sampled impulse response will become

$$h_n = T h_a(nT) = \sum_{k=1}^{N} T A_k e^{s_k n T} u(nT)$$

Properties

- **Note:** Aliasing may occur
- Impulse response shows IIR behaviour
- **But:** Number N of poles defines necessary elements for standard form implemenation

Definition

Application of impulse invariance method: Derive sampled impulse response

$$h_n = T h_a(nT)$$

and the spectrum of the sampled impulse response:

$$H_d(\omega) = \sum_{k=-\infty}^{k=\infty} H_a\left(\omega - \frac{2\pi k}{T}\right).$$

Properties

- Sampled impulse response

$$h_n = T \sum_{k=1}^{N} A_k p_k^n u_n$$

- Z transform

$$H_d(z) = \sum_{k=1}^{N} \frac{T A_k}{1 - p_k z^{-1}}$$

- Bring it to a common denominator to derive the standard form

$$\boxed{H_d(z) = \frac{a_0 z^0 + a_1 z^{-1} + \ldots a_{M-1} z^{-(M-1)}}{1 + b_1 z^{-1} + \ldots b_N z^{-N}}}$$

Impulse invariance method

Properties

- Sampled impulse response

$$h_n = T \sum_{k=1}^{N} A_k p_k^n u_n$$

- Z transform

$$H_d(z) = \sum_{k=1}^{N} \frac{TA_k}{1 - p_k z^{-1}}$$

- Bring it to a common denominator to derive the standard form

$$\boxed{H_d(z) = \frac{a_0 z^0 + a_1 z^{-1} + \ldots a_{M-1} z^{-(M-1)}}{1 + b_1 z^{-1} + \ldots b_N z^{-N}}}$$

Properties

- Assuming that all N poles have an order/multiplicity of one and applying partial fraction expansion leads to:

$$H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} \Leftrightarrow h_a(t) = \sum_{k=1}^{N} A_k e^{s_k t} u(t).$$

- Sampled impulse response will become

$$h_n = Th_a(nT) = \sum_{k=1}^{N} TA_k e^{s_k nT} u(nT)$$

Properties

- **Note:** Aliasing may occur
- Impulse response shows IIR behaviour
- **But:** Number N of poles defines necessary elements for standard form implementation

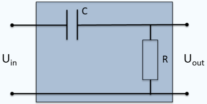
Example

Re-consider the low-pass ($T = 1$ for simplicity):

$$H(s) = \frac{1}{RC} \frac{1}{s + \frac{1}{RC}}$$

$$h(t) = \frac{1}{RC} e^{-\frac{1}{RC} t} u(t) \rightarrow h_n = \frac{1}{RC} e^{-\frac{1}{RC} n} u_n \rightarrow H(z) = \frac{1}{RC} \frac{1}{1 - e^{-\frac{1}{RC}} z^{-1}}$$

And thus $a_0 = \frac{1}{RC}$ and $b_1 = e^{-\frac{1}{RC}}$.



Impulse invariance method

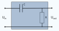
Example

Re-consider the low-pass ($T = 1$ for simplicity):

$$H(s) = \frac{1}{RC}\frac{1}{s+\frac{1}{RC}}$$

$$h(t) = \frac{1}{RC}e^{-\frac{1}{RC}t}u(t) \to h_n = \frac{1}{RC}e^{-\frac{1}{RC}n}u_n \to H(z) = \frac{1}{RC}\frac{1}{1-e^{-\frac{1}{RC}}z^{-1}}$$

And thus $a_0 = \frac{1}{RC}$ and $b_1 = e^{-\frac{1}{RC}}$.



Properties

- Sampled impulse response

$$h_n = T\sum_{k=1}^{N} A_k p_k^n u_n$$

- Z transform

$$H_d(z) = \sum_{k=1}^{N}\frac{TA_k}{1-p_k z^{-1}}$$

- Bring it to a common denominator to derive the standard form

$$\boxed{H_d(z) = \frac{a_0 z^0 + a_1 z^{-1} + \dots a_{M-1} z^{-(M-1)}}{1 + b_1 z^{-1} + \dots b_N z^{-N}}}$$

Properties

- Assuming that all N poles have an order/multiplicity of one and applying partial fraction expansion leads to:

$$H_a(s) = \sum_{k=1}^{N}\frac{A_k}{s-s_k} \Leftrightarrow h_a(t) = \sum_{k=1}^{N} A_k e^{s_k t}u(t).$$

- Sampled impulse response will become

$$h_n = Th_a(nT) = \sum_{k=1}^{N} TA_k e^{s_k nT}u(nT)$$

Some identities

In case of poles of higher order, on can make use of the following identities:

$H(s)$	$h(t)$	h_n	$H(z)$
1	$\delta(t)$	$T\delta[k]$	T
$\frac{1}{s}$	$\mathrm{u}(t)$	1	$\frac{Tz}{z-1}$
$\frac{1}{s+\alpha}$	$e^{-\alpha t}u(t)$	$Te^{-\alpha kT}u_n$	$\frac{Tz}{z-e^{-\alpha T}}$
$\frac{1}{s^2}$	$tu(t)$	$nT^2u(k)$	$\frac{T^2z}{(z-1)^2}$
$\frac{1}{(s+\alpha)^2}$	$te^{-\alpha t}u(t)$	$nT^2e^{-\alpha kT}u_n$	$\frac{T^2e^{-\alpha T}z}{[z-e^{-\alpha T}]^2}$
$\frac{s+\alpha}{(s+\alpha)^2+\beta^2}$	$e^{-\alpha t}\cos(\beta t)u(t)$	$Te^{-\alpha nT}\cos(\beta kT)u_n$	$\frac{Tz[z-e^{-\alpha T}\cos(\beta T)]}{z^2-2e^{-\alpha T}\cos(\beta T)z+e^{-2\alpha T}}$
$\frac{\beta}{(s+\alpha)^2+\beta^2}$	$e^{-\alpha t}\sin(\beta t)u(t)$	$Te^{-\alpha nT}\sin(\beta kT)u_n$	$\frac{Tze^{-\alpha T}\sin(\beta T)}{z^2-2e^{-\alpha T}\cos(\beta T)z+e^{-2\alpha T}}$

Table of Contents

Digital filters

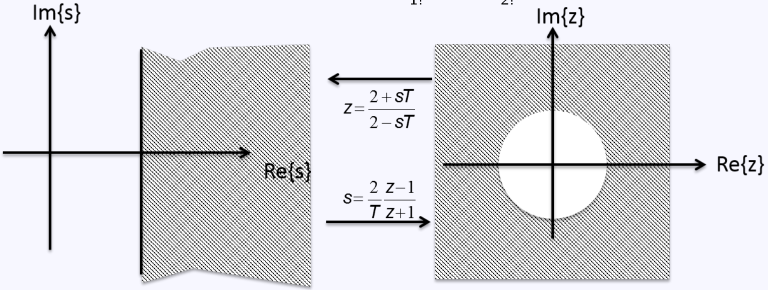
7.5 Digital filters
7.5.1 Overview
7.5.2 Direct discretization
7.5.3 Impulse invariance methode
7.5.4 Bilinear transform
7.5.5 Window method
7.5.6 Examples
7.5.7 Summary

Bilinear transform

Using the Taylor series

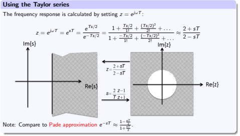
The frequency response is calculated by setting $z = e^{j\omega T}$:

$$z = e^{j\omega T} = e^{sT} = \frac{e^{Ts/2}}{e^{-Ts/2}} = \frac{1 + \frac{Ts/2}{1!} + \frac{(Ts/2)^2}{2!} + \ldots}{1 + \frac{-Ts/2}{1!} + \frac{(-Ts/2)^2}{2!} + \ldots} \approx \frac{2+sT}{2-sT}$$



Note: Compare to Pade approximation $e^{-sT} \approx \frac{1-\frac{sT}{2}}{1+\frac{sT}{2}}$.

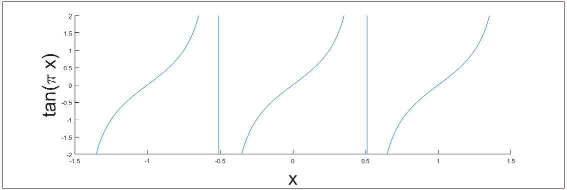
Bilinear transform



Definition

Frequency warping describes the effect of the bilinear transform onto the frequency response[a]:

$$\omega_a = \frac{2}{T}\tan\left(\omega_d \frac{T}{2}\right) \Leftrightarrow \omega_d = \frac{2}{T}\tan\left(\omega_a \frac{T}{2}\right)$$



[a] $s_a = j\omega_a = \frac{2}{T}\frac{1-e^{-j\omega_d T}}{1+e^{-j\omega_d T}} = \frac{2}{T}\frac{e^{j\frac{1}{2}\omega_d T}-e^{-j\frac{1}{2}\omega_d T}}{e^{j\frac{1}{2}\omega_d T}e^{-j\frac{1}{2}\omega_d T}} = j\frac{2}{T}\tan\left(\omega_d \frac{T}{2}\right)$

Table of Contents

Digital filters

7.5 Digital filters
7.5.1 Overview
7.5.2 Direct discretization
7.5.3 Impulse invariance methode
7.5.4 Bilinear transform
7.5.5 Window method
7.5.6 Examples
7.5.7 Summary

Window method

Definition

The window method as applied as follows:

- ▷ Define a filter in the spectral domain
- ▷ Use inverse Fourier transform to get the impulse response
- ▷ Take a subset of the impulse repsonse
- ▷ Shift the impulse response in time to derive a causal system
- ▷ FIR filter coefficients are given by the impulse response
- ▷ Use a window function to remove ripples

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Example

Continuous impulse response $h(t)$ for an ideal low-pass with cut-off frequency w_c:

$$h(t) = \frac{\sin(\omega_c t)}{\pi t}$$

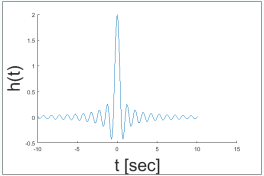


Figure 37: $h(t)$

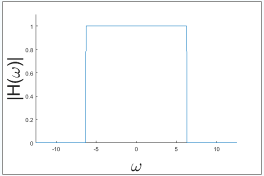


Figure 38: $H(\omega)$

Window method

Example

Continuous impulse response $h(t)$ for an ideal low-pass with cut-off frequency w_c:

$$h(t) = \frac{\sin(\omega_c t)}{\pi t}$$

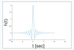
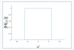


Figure 37: $h(t)$



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Properties

- ▷ Non-causal
- ▷ Infinite number of coefficients

Window method

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- ▷ Non-causal
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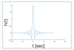

Figure 37: $h(t)$

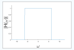

Figure 38: $H(\omega)$

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- ▷ Shift the impulse response in time to derive a causal system
- ▷ FIR filter coefficients are given by the impulse response
- ▷ Use a window function to remove ripples

Solution

- ▷ Shift all coefficients by a reasonable number k (e.g. $k = 10$)
- ▷ Take only the first $2k$ values

$$h'(n) = \sum_{i=0}^{2k} a_i \delta_i$$

$$G(z) = \sum_{i=0}^{2k} a_i z^{-i}$$

Window method

Solution

- Shift all coefficients by a reasonable number k (e.g. $k = 10$)
- Take only the first $2k$ values

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Example

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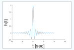

Figure 37: $h(t)$

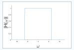

Figure 38: $H(\omega)$

Digital filter with 20 coefficients

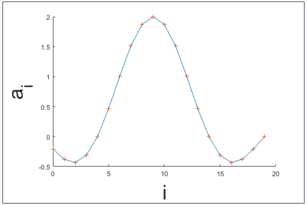

Figure 39: Coefficients

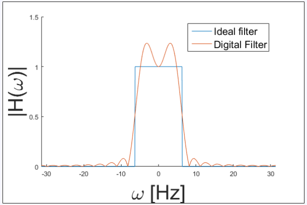

Figure 40: Spectrum

- The truncation of the impulse response is equivalent to multiplying it with a rectangle
- Overshoot and ripples (Gibb's phenomenon)

Window method

Digital filter with 20 coefficients

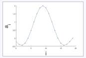

Figure 39: Coefficients

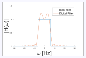

Figure 40: Spectrum

- ▷ The truncation of the impulse response is equivalent to multiplying it with a rectangle
- ▷ Overshoot and ripples (Gibb's phenomenon)

Solution

- ▷ Shift all coefficients by a reasonable number k (e.g. $k = 10$)
- ▷ Take only the first $2k$ values

$$h'(n) = \sum_{i=0}^{2k} a_i \delta_i$$

$$G(z) = \sum_{i=0}^{2k} a_i z^{-i}$$

Properties

- ▷ Non-causal
- ▷ Infinite number of coefficients

Example

Digital filter with $2k$ coefficients

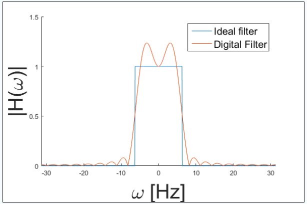

Figure 41: 20 taps

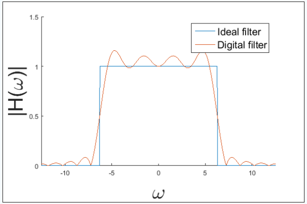

Figure 42: 40 taps

- ▷ Amplitude of ripples does not decrease with the number of coefficients

Window method

Example

Digital filter with $2k$ coefficients

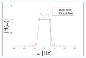

Figure 41: 20 taps

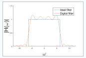

Figure 42: 40 taps

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Digital filter with 20 coefficients

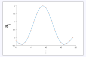


Figure 39: Coefficients

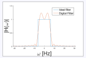


Figure 40: Spectrum

▷ The truncation of the impulse response is equivalent to multiplying it with a rectangle

▷ Overshoot and ripples (Gibb's phenomenon)

Solution

▷ Shift all coefficients by a reasonable number k (e.g. $k = 10$)

▷ Take only the first $2k$ values

$$h'(n) = \sum_{i=0}^{2k} a_i \delta_i$$

$$G(z) = \sum_{i=0}^{2k} a_i z^{-i}$$

Example

Weighting with a window function:

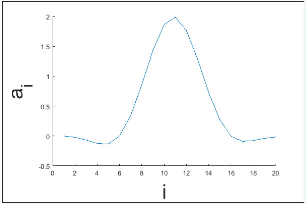


Figure 43: 20 coefficients multiplied with a Hamming window

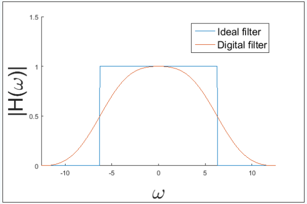


Figure 44: Spectrum

▷ The use of a window function can reduce ripples

▷ Usually, this is for the sake of getting a less steep cut-off slope

Table of Contents

Digital filters

7.5 Digital filters
7.5.1 Overview
7.5.2 Direct discretization
7.5.3 Impulse invariance methode
7.5.4 Bilinear transform
7.5.5 Window method
7.5.6 Examples
7.5.7 Summary

First order FIR filter

Example

$$\begin{aligned} y_n &= x_n + \alpha x_{n-1} \\ Y(z) &= X(z) - \alpha Z^{-1} X(z) \\ H(z) &= 1 - \alpha z^{-1} \end{aligned}$$

The choice of α determines if the filter shows a low pass or high pass behavior.

First order FIR filter

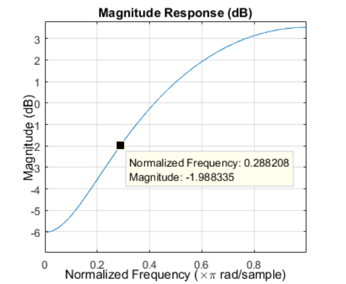
Example

$$\begin{aligned} y_n &= x_n + \alpha x_{n-1} \\ Y(z) &= X(z) - \alpha Z^{-1} X(z) \\ H(z) &= 1 - \alpha z^{-1} \end{aligned}$$

The choice of α determines if the filter shows a low pass or high pass behavior.

Example

```
clear;close all;clc;
s = sin((1:512)*2*pi*10/512);
myzeros = 0.5;
poles = 0;
gain = 1;
[NUM,DEN] = zp2tf(myzeros,poles,
    gain);
s1 = filter(NUM,DEN,s);
plot(abs(fft(s)),'red');
hold on
plot(abs(fft(s1)),'green');
fvtool(NUM,DEN);
```



DC blocker

Example

```
clear;close all;clc;
s = sin((1:512)*2*pi*10/512);
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gain = 1;
[NUM,DEN] = zp2tf(myzeros,poles,
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```
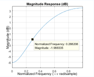



Example

$$
\begin{aligned}
y_n &= x_n + \alpha x_{n-1} \\
Y(z) &= X(z) - \alpha Z^{-1} X(z) \\
H(z) &= 1 - \alpha z^{-1}
\end{aligned}
$$

The choice of α determines if the filter shows a low pass or high pass behavior.

Example

DC-Blocker:

$$
\begin{aligned}
y_n &= x_n - x_{n-1} + \alpha y_{n-1} \\
Y(z) &= X(z) - Z^{-1} X(z) + \alpha Z^{-1} Y(z) \\
H(z) &= \frac{1 - z^{-1}}{1 - \alpha z^{-1}}
\end{aligned}
$$

A typical choice for α is $0.9 < \alpha < 1$.

Parametric equalizer

Example

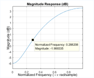
DC-Blocker:

$$\begin{aligned} y_n &= x_n - x_{n-1} + \alpha y_{n-1} \\ Y(z) &= X(z) - Z^{-1}X(z) + \alpha Z^{-1}Y(z) \\ H(z) &= \frac{1 - z^{-1}}{1 - \alpha z^{-1}} \end{aligned}$$

A typical choice for α is $0.9 < \alpha < 1$.

Example

```
clear;close all;clc;
s = sin((1:512)*2*pi*10/512);
myzeros = 0.5;
poles = 0;
gain = 1;
[NUM,DEN] = zp2tf(myzeros,poles,
    gain);
s1 = filter(NUM,DEN,s);
plot(abs(fft(s)),'red');
hold on
plot(abs(fft(s1)),'green');
fvtool(NUM,DEN);
```



Example

$$\begin{aligned} y_n &= x_n + \alpha x_{n-1} \\ Y(z) &= X(z) - \alpha Z^{-1}X(z) \\ H(z) &= 1 - \alpha z^{-1} \end{aligned}$$

The choice of α determines if the filter shows a low pass or high pass behavior.

Example

Equalizer with gain g and bandwidth B:

$$\begin{aligned} H(s) &= 1 + (g-1)\frac{Bs}{s^2 + Bs + 1} \\ &= \frac{s^2 + gBs + 1}{s^2 + Bs + 1} \end{aligned}$$

Table of Contents

Digital filters

7.5 Digital filters
7.5.1 Overview
7.5.2 Direct discretization
7.5.3 Impulse invariance methode
7.5.4 Bilinear transform
7.5.5 Window method
7.5.6 Examples
7.5.7 Summary

Linear-phase filters

Properties

- Linear-phase filters have a symmetric impulse response:

$$h(n) = h(N-1-n),$$

  with $n = 0 \cdots N-1$.
- Linear-phase filters must be FIR-filters.

FIR vs. IIR

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▷ No feedback (only zeros) $\rightarrow$ always stable ▷ Can have linear phase	▷ Feedback (poles and zeros) $\rightarrow$ may be unstable ▷ Phase is difficult to control
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- Use an analog prototype
  - Impulse invariance
  - Bilinear transform
- Window method

Table of Contents

Filters

7.1 Introduction
7.2 Filter properties
7.3 Filter structures
7.4 Bandform transformations
7.5 Digital filters
7.6 Excursion: Control theory
  7.6.1 Prefilter
  7.6.2 Digital control
7.7 Exercises

Table of Contents

Excursion: Control theory

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Table of Contents

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- passband $(0 \leq |\omega| \leq 0.25\,\pi \times \text{rad/sec})$, with $0.8 \leq |H(\omega)| \leq 1.2$
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Given is the circuit shown in figure 45.

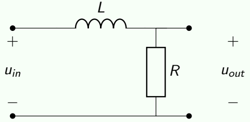

Figure 45: Circuit

- ▷ Derive the transfer function and discuss the frequency behavior of this circuit.
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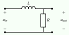

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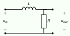

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- ▷ Sketch a standard implementation of the filter and determine the corresponding coefficients $a_0, a_1, \ldots$ and $b_1, b_2, \ldots$.
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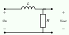

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Assume that the sampling frequency is $f_s = 1\,\mathrm{Hz}$.

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Table of Contents

Discrete Time & Applications

6. Discrete Time
7. Filters
8. Applications and Exercises

Table of Contents

Appendix

9. Appendix
10. MATLAB

Table of Contents

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10. MATLAB

Table of Contents

MATLAB

10.1 Introduction
10.2 Collection of MATLAB-commands
10.3 Index

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- ▷ `xcorr`: Cross-correlation function estimates

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- `zpk`: Create zero-pole-gain model; convert to zero-pole-gain model
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System identification toolbox

- ▷ `nyquistplot`: Draws the Nyquist plot
- ▷ `nyquist`: Nyquist plot of frequency response

Table of Contents

MATLAB

10.1 Introduction
10.2 Collection of MATLAB-commands
10.3 Index

index

ACKERMANN'S FORMULA, **461**, 462, 462, 478, 479, 480
ANALYTIC SIGNAL, 73, 74, 75, 76
BAND-LIMITED, 108, 109, 110, 110, 111, 111
BANDFORM TRANSFORMATIONS, 676
BANDWIDTH, 108, 109, 110, 111
BIBO, 162
BODE PLOT, 203, 204, 205, 206, 779, 780, 781
CAUSAL, 157
CHARACTERISTIC EQUATION, 166, 167, 168, 169, 255, 256, 257, 258, 270
COMPLETELY CONTROLLABLE, 452
CONTROLLABILITY MATRIX, 452, 461, 462, 462, 478, 479, 480
CONTROLLABLE, 451, 452
CONVOLUTION, 24, 25, 26
CROSS-CORRELATION, 29, 30, 31, 57, 58, 59, 60, 61, 62, 63, 64
DAMPING, 202, 203, 204, 205
DC LOOP GAIN, 329
DELAY-TIME, 385, 386, 387, 388
DIRAC COMB, 45, 46, 47, 64, 65, 67, 69
DIRECT FEED-THROUGH, 248, 249
DOMINANT POLE, 229, 230, 232, 232, 233
ENERGY, 105, 106
ESTIMATION ERROR, 473, 474, 474, 475, 475, 476, 476, 477
FFT-GAIN, 553
FILTER, 639, 640, 641
FORCED RESPONSE, 131, 132, 133
FOURIER TRANSFORM, 52, 53, 54, 55
FREE RESPONSE, 128, 129, 130, 131
FREQUENCY WARPING, 694
FUNDAMENTAL SET, 128, 129, 130, 131
GAIN FACTOR, 149, 150, 337, 338, 339, 340
GAIN MARGIN, 337, 338, 338, 339, 339, 340, 340, 341, 370
GROUP DELAY, 648
HEAVISIDE STEP FUNCTION, 42, 43, 44, 45
HILBERT TRANSFORM, 34, 35
IMPULSE INVARIANCE METHOD, 686, 687, 688, 689
IMPULSE RESPONSE, 153, 154
INITIAL CONDITION, 127, 128, 129, 130
INVERSE FOURIER TRANSFORM, 52, 53, 54, 55
INVERSE LAPLACE TRANSFORM, 87, 88, 89, 90
LAG COMPENSATOR, 399, 400, 401, 403
LAPLACE TRANSFORM, 86, 87, 88, 89, 145, 146, 148, 149
LEAD COMPENSATOR, 407, 408, 409, 411
LEAD-LAG COMPENSATOR, 415, 416, 418, 420
LEAKAGE EFFECT, 522, 523, 524, 525
LINEAR CONSTANT-COEFFICIENT ORDINARY DIFFERENTIAL EQUATION, 127, 128, 129, 130
LOOP GAIN, 322, 323, 324, 325
LTI-SYSTEMS, 140, 141
MASON'S GAIN FORMULAR, 282, 283, 284, 285
MEAN SQUARE ERROR, 114
MEASUREMENT MATRIX, 247, 248, 249, 466, 467, 468, 469
NATURAL FREQUENCIES, 149, 150
NYQUIST DIAGRAM, 235, 236, 237, 238
OBSERVABILITY MATRIX, 468, 469, 470, 471
OBSERVABLE, 467, 468, 469, 470
OBSERVER, 472, 473, 474, 475
OBSERVER GAIN MATRIX, 472, 473, 474, 475
OPEN-LOOP, 311, 312, 313, 314
ORDINARY DIFFERENTIAL EQUATION, 126, 127, 128, 129
OUTPUT MATRIX, 247, 248, 249, 466, 467, 468, 469
OVERSHOOT, 377, 378, 379, 380
PADE APPROXIMATION, 192, 193, 195, 693, 694
PADE-APPROXIMATION, 760, 761, 762, 764, 766
PARTIAL FRACTION EXPANSION, 148, 149, 150
PEAK TIME, 383, 384, 385, 386
PHASE, 202, 203, 204, 205
PHASE CROSSOVER FREQUENCY, 336, 337, 338, 339, 370
PHASE DELAY, 646
PHASE MARGIN, 335, 336, 337, 338
PHASE-LAG, 399, 400, 401, 403
PID CONTROLLER, 421, 422, 423, 424
RAMP FUNCTION, 41, 42, 43, 44
RELATIVE STABILITY, 377, 378, 379, 380
RESONANT FREQUENCY, 387, 388, 389, 390
RESONANT PEAK, 387, 388, 389, 390
RISE TIME, 381, 382, 383, 384
ROUTH ARRAY, 178, 179, 180, 181
ROUTH-HURWITZ CRITERION, 362, 363, 364, 365
ROUTH-HURWITZ STABILITY CRITERION, 180, 181, 182, 183
SENSITIVITY, 325, 326, 327, 328
SETTLING TIME, 379, 380, 381, 382
SISO, 249, 268
SLOWEST POLE, 229, 230, 232
STABLE, 160, 161, 162, 270
STATE VECTOR, 244
STEADY STATE ERROR, 313, 314, 315, 329, 396
STEADY STATE RESPONSE, 375, 376, 377, 378
STEADY-STATE ERROR, 392, 393, 393, 394, 394, 396, 396
STEP RESPONSE, 778, 779, 780, 781
TIME TO PEAK, 383, 384, 385, 386
TIME-LIMITED, 110, 111
TOTAL RESPONSE, 132, 133
TRACKING ERROR, 312, 313, 314, 315, 327, 328, 329, 389, 390, 391, 392
TRANSFER FUNCTION, 145, 146, 148, 149
TRANSIENT RESPONSE, 229, 230, 232
UNDERDAMPED, 215, 216, 217, 218
UNIT STEP RESPONSE, 133
WINDOW METHOD, 696, 697, 698, 699