CS 70 Discrete Mathematics and Probability Theory
Fall 2012 Vazirani Note 4

The Stable Marriage Problem: An Application of Proof Techniques to Analysis of Algorithms

Consider a dating agency that must match up n men and n women. Each man has an ordered *preference list* of the n women, and each woman has a similar list of the n men. Is there a good algorithm that the agency can use to determine a good pairing?

Example

Consider for example $n = 3$ men (represented by numbers 1, 2, and 3) and 3 women (A, B, and C), and the following preference lists:

Men	Women		
1	A	B	C
2	B	A	C
3	A	B	C

Women	Men		
A	2	1	3
B	1	2	3
C	1	2	3

What properties should a good pairing have? One possible criterion for a "good" pairing is one in which the number of first ranked choices is maximized. Another possibility is to minimize the number of last ranked choices. Or perhaps minimizing the sum of the ranks of the choices, which may be thought of as maximizing the average happiness. In this lecture we will focus on a very basic criterion: *stability*. A pairing is unstable if there is a man and a woman who prefer each other to their current partners. We will call such a pair a *rogue couple*. So a pairing of n men and n women is stable if it has no rogue couples.

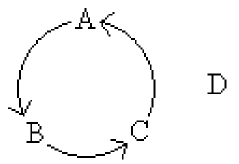
An unstable pairing from the above example is: {(1,C), (2,B), (3,A)}. The reason is that 1 and B form a rogue couple, since 1 would rather be with B than C (his current partner), and since B would rather be with 1 than 2 (her current partner). An example of a stable pairing is: {(2,A), (1,B), (3,C)}. Note that (1,A) is not a rogue couple. It is true that man 1 would rather be with woman A than his current partner. Unfortunately for him, she would rather be with her current partner than with him. Note also that both 3 and C are paired with their least favorite choice in this matching.

Before we discuss how to find a stable pairing, let us ask a more basic question: do stable pairings always exist? Surely the answer is yes, since we could start with any pairing and make it more and more stable as follows: if there is a rogue couple, modify the current pairing so that they are together. Repeat. Surely this procedure must result in a stable pairing! Unfortunately this reasoning is not sound. To demonstrate this, let us consider a slightly different scenario, the roommates problem. Here we have $2n$ people who must be paired up to be roommates (the difference being that unlike the dating scenario, a person can be paired with any of the remaining $2n-1$). The point is that nothing about the above reasoning relied on the fact that men can only be paired with women in the dating scenario, so by the same reasoning we would expect that there would be a stable pairing for the roommates problem as well. The following counter-example illustrates the fallacy in the reasoning:

Roommates Problem:

Roommates			
A	B	C	D
B	C	A	D
C	A	B	D
D	–	–	–

Visually, we have the following situation:



What is interesting about this problem is that there is no stable pairing (i.e., there is always a rogue couple). For example, the pairing {(A,B), (C,D)} contains the rogue couple B and C. Using the reasoning above, we might decide to pair B and C together, giving us the pairing: {(B,C), (A,D)}. But this pairing is also unstable because now A and C are a rogue couple. [Exercise: Verify that in this example there is *no* stable pairing!] Thus any proof that there must be a stable pairing in the dating problem must use the fact that there are two genders in an essential way.

The traditional propose & reject algorithm, which we will discuss for the rest of the lecture, is asymmetrical in the roles of men and women. We shall prove that it always finds a stable pairing and study its many interesting properties.

Traditional Propose & Reject Algorithm:

Every Morning: Each man goes to the first woman on his list not yet crossed off and proposes to her.

Every Afternoon: Each woman says "maybe" to the man she likes best among the proposals (she now has him *on a string*) and "never" to all the rest.

Every Evening: Each of the rejected suitors crosses off the woman from his list.

The above loop is repeated each successive day until there are no more rejected suitors. On this day, each woman marries the man she has on a string.

We wish to show that this algorithm always outputs a stable pairing. But how do we even know that it must terminate? Let us prove something stronger: the algorithm is guaranteed to terminate in at most n^2 days. Well, for each day (except the last), at least one woman is crossed off some man's list. Since there are n men, each starting with a list of size n, it follows that the algorithm must terminate after n^2 days.

To establish that the pairing output is stable, we need the following crucial lemma:

Improvement Lemma: If W has M on a string on the kth day, then on every subsequent day she has someone on a string whom she likes at least as much as M.

Proof: Suppose towards a contradiction that the jth day for $j > k$ is the first counterexample where W has either nobody or some M* inferior to M on a string. On day $j-1$, she has M$'$ on a string and likes M$'$ at least much as M. According to the algorithm, M$'$ still proposes to W on the jth day since she said "maybe" the previous day. So W has the choice of at least one man on the jth day; moreover, her best choice is at least as good as M$'$, and according to the algorithm she will choose him over M*. This contradicts our initial assumption. ♠

What proof techniques did we use to prove this lemma? It is really an induction on j, the number of days. Recall that there are many equivalent forms of induction — simple induction, strong induction, and the well-ordering principle. We used the well-ordering principle in the first sentence of the proof when we asserted that if the statement of the lemma is not true, then there must be a *smallest* counterexample: "the first day j ..." [Exercise: How would you restate this proof using simple induction or strong induction?]

Let us now proceed to prove that at the end of the algorithm all $2n$ people are paired up. Before reading the proof, see if you can convince yourself that this is true. The proof is remarkably short and elegant and is based crucially on the Improvement Lemma:

Lemma: The algorithm terminates with a pairing.

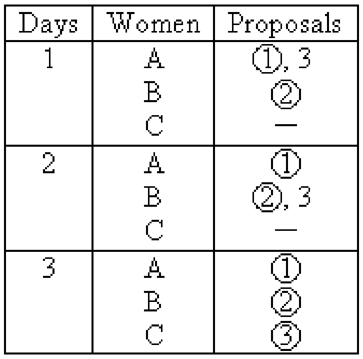
Proof: Suppose for contradiction that there is a man M who is left unpaired at the end of the algorithm. He must have proposed to every single woman on his list. By the Improvement Lemma, each of these women thereafter has someone on a string. Thus when the algorithm terminates, n women have n men on a string not including M. So there must be at least $n+1$ men. Contradiction. ♠

Now, before we prove that the output of the algorithm is a stable pairing, let us first do a sample run-through of the stable marriage algorithm. We will use the preference lists given earlier, which are duplicated here for convenience:

Men	Women		
1	A	B	C
2	B	A	C
3	A	B	C

Women	Men		
A	2	1	3
B	1	2	3
C	1	2	3

The following table shows which men propose to which women on the given day (the circled men are the ones who were told “maybe”):



Days	Women	Proposals
1	A	①, 3
	B	②
	C	—
2	A	①
	B	②, 3
	C	—
3	A	①
	B	②
	C	③

Thus, the stable pairing which the algorithm outputs is: {(1,A), (2,B), (3,C)}.

Theorem: The pairing produced by the algorithm is always stable.

Proof: We will show that no man M can be involved in a rogue couple. Consider any couple (M,W) in the pairing and suppose that M prefers some woman W* to W. We will argue that W* prefers her partner to M, so that (M,W*) cannot be a rogue couple. Since W* occurs before W in M’s list, he must have proposed to her before he proposed to W. Therefore, according to the algorithm, W* must have rejected him for somebody she prefers. By the Improvement Lemma, W* likes her final partner at least as much, and therefore prefers him to M. Thus no man M can be involved in a rogue couple, and the pairing is stable. ♠

Optimality

Consider the situation in which there are 4 men and 4 women with the following preference lists:

Men	Women			
1	A	B	C	D
2	A	D	C	B
3	A	C	B	D
4	A	B	C	D

Women	Men			
A	1	3	2	4
B	4	3	2	1
C	2	3	1	4
D	3	4	2	1

For these preference lists, there are exactly two stable pairings: $S = \{(1,A), (2,D), (3,C), (4,B)\}$ and $T = \{(1,A), (2,C), (3,D), (4,B)\}$. The fact that there is more than one stable pairing brings up an interesting question. What is the best possible partner for each person, say man 2 for example? The trivial answer is his first choice (i.e., woman A), but that is just not a realistic possibility for him. Pairing man 2 with woman A would simply not be stable, since he is so low on her preference list. And indeed there is no stable pairing in which 2 is paired with A. Examining the two stable pairings, we can see that the best possible realistic outcome for man 2 is to be matched to woman D.

Let us make some definitions to better express these ideas: we say the *optimal* woman for a man is the highest woman on his list whom he is paired with in any *stable* pairing. In other words, the optimal woman is the best that a man can do under the condition of stability. In the above example, woman D is the optimal woman for man 2. So the best that each man can hope for is to be paired with his optimal woman. But can they achieve this optimality *simultaneously*? I.e., is there a stable pairing such that each man is paired with his optimal woman? If such a pairing exists, we will call it a *male optimal* pairing. Turning to the example above, S is a male optimal pairing since A is 1's optimal woman, D is 2's optimal woman, C is 3's optimal woman, and B is 4's optimal woman. Similarly, we can define a female optimal pairing, which is the pairing in which each woman is paired with her optimal man. [Exercise: Check that T is a female optimal pairing.] We can also go in the opposite direction and define the *pessimal* woman for a man to be the lowest ranked woman whom he is ever paired with in some stable pairing. This leads naturally to the notion of a *male pessimal* pairing — can you define it, and also a female pessimal pairing?

Now, a natural question to ask is: Who is better off in the traditional propose & reject algorithm: men or women? Think about this before you read on...

Theorem: The pairing output by the traditional propose & reject algorithm is male optimal.

Proof: Suppose for the sake of contradiction that the pairing is *not* male optimal. Consider the first day when some man got rejected by his optimal woman, and let this be the kth day. On this day, M was rejected by W* (his optimal mate) in favor of M* who proposed to her. By the definition of optimal woman, there must exist a stable pairing T in which M and W* are paired together; suppose this looks like this: $\{\dots$, (M, W*), $\dots$, (M*, W$'$), $\dots\}$. We will argue that (M*,W*) is a rogue couple in T, thus contradicting stability. Since day k is the first day when some man got rejected by his optimal woman, on day k M* has not yet been rejected by his optimal woman. This implies that M* likes W* at least as much as his optimal woman. So by the definition of optimal woman, M* prefers W* to his partner W$'$ in the stable matching T. But we already said that W* prefers M* to M (because she rejected M in favor of M*). Thus (M*, W*) form a rogue couple in T, and so T is not stable. Contradiction. Thus, our assumption was wrong, and the pairing is male optimal. ♠

What proof techniques did we use to prove this theorem? Again it is a proof by induction, structured as an application of the well-ordering principle. How do we see it as a regular induction proof? This is a bit subtle to figure out. See if you can do so before reading on... the proof is really showing by induction on k the following statement: for every k, no man gets rejected by his optimal woman on the kth day. [Exercise: Can you complete the induction?]

So men appear to fare very well by following this algorithm. How about the women? The following theorem reveals the sad truth:

Theorem: If a pairing is male optimal, then it is also female pessimal.

Proof: Let T = $\{\dots$, (M, W), $\dots\}$ be the male optimal pairing (which we know is output by the algorithm). Suppose for the sake of contradiction that there exists a stable pairing S = $\{\dots$, (M*, W), $\dots$, (M, W$'$), $\dots\}$ such that M* is lower on W's list than M (i.e., M is not her pessimal man). We will argue that S cannot possibly be stable by showing that (M,W) is a rogue couple in S. By assumption, W prefers M to M* since

M* is lower on her list. And M prefers W to his partner W′ in S because W is his partner in the male optimal pairing T. Contradiction. Therefore, the male optimal pairing is female pessimal. ♠

All this seems a bit unfair to the women! Are there any lessons to be learned from this? Make the first move!

The Residency Match

Perhaps the most well-known application of the stable marriage algorithm is the residency match program, which pairs medical school graduates and residency slots (internships) at teaching hospitals. Graduates and hospitals submit their ordered preference lists, and the stable pairing produced by a computer matches students with residency programs.

The road to the residency match program was long and twisted. Medical residency programs were first introduced about a century ago. Since interns offered a source of cheap labor for hospitals, soon the number of residency slots exceeded the number of medical graduates, resulting in fierce competition. Hospitals tried to outdo each other by making their residency offers earlier and earlier. By the mid-40s, offers for residency were being made by the beginning of junior year of medical school, and some hospitals were contemplating even earlier offers — to sophomores! The American Medical Association finally stepped in and prohibited medical schools from releasing student transcripts and reference letters until their senior year. This sparked a new problem, with hospitals now making "short fuse" offers to make sure that if their offer was rejected they could still find alternate interns to fill the slot. Once again the competition between hospitals led to an unacceptable situation, with students being given only a few hours to decide whether they would accept an offer.

Finally, in the early 50s, this unsustainable situation led to the centralized system called the National Residency Matching Program (N.R.M.P.) in which the hospitals ranked the residents and the residents ranked the hospitals. The N.R.M.P. then produced a pairing between the applicants and the hospitals, though at first this pairing was not stable. It was not until 1952 that the N.R.M.P. switched to the traditional propose & reject algorithm, resulting in a stable pairing. Until recently the algorithm was run with the hospitals doing the proposing, and so the pairings produced were hospital optimal. Only recently were the roles reversed such that the medical students were proposing to the hospitals. In the 1990’s, there were other improvements made to the algorithm which the N.R.M.P. used. For example, the pairing takes into account preferences for married students for positions at the same or nearby hospitals.

Further reading (optional!)

Though it was in use 10 years earlier, the propose-and-reject algorithm was first properly analyzed by Gale and Shapley, in a famous paper dating back to 1962 that still stands as one of the great achievements in the analysis of algorithms. The full reference is:

D. Gale and L.S. Shapley, “College Admissions and the Stability of Marriage,” *American Mathematical Monthly* **69** (1962), pp. 9–14.

Stable marriage and its numerous variants remains an active topic of research in computer science. Although it is by now twenty years old, the following very readable book covers many of the interesting developments since Gale and Shapley’s algorithm:

D. Gusfield and R.W. Irving, *The Stable Marriage Problem: Structure and Algorithms*, MIT Press, 1989.