Chapter 2

Bayesian Decision Theory

Decision Theory

Decision

Make choice under uncertainty

Pattern Recognition

Pattern → Category

Given a test sample, its category is uncertain and a decision has to be made

In essence, PR is a decision process

Bayesian Decision Theory

Bayesian decision theory is a statistical approach to pattern recognition

The fundamentals of most PR algorithms are rooted from Bayesian decision theory

Basic Assumptions

- The decision problem is posed (formalized) in **probabilistic** terms
- All the relevant probability values are known

Key Principle

Bayes Theorem (贝叶斯定理)

Bayes Theorem

Bayes theorem $$P(H|X) = \frac{P(H)P(X|H)}{P(X)}$$

X: the observed sample (also called **evidence**; *e.g.: the length of a fish*)

H: the hypothesis (*e.g. the fish belongs to the "salmon" category*)

$P(H)$: the **prior probability** (先验概率) that H holds (*e.g. the probability of catching a salmon*)

$P(X|H)$: the **likelihood** (似然度) of observing X given that H holds (e.g. *the probability of observing a 3-inch length fish which is salmon*)

$P(X)$: the **evidence probability** that X is observed (*e.g. the probability of observing a fish with 3-inch length*)

$P(H|X)$: the **posterior probability** (后验概率) that H holds given X (e.g. *the probability of X being salmon given its length is 3-inch*)

Thomas Bayes (1702-1761)

A Specific Example

State of Nature (自然状态)

- Future events that might occur

  e.g. the next fish arriving along the conveyor belt

- State of nature is unpredictable

  e.g. it is hard to predict what type will emerge next

From statistical/probabilistic point of view, the state of nature should be favorably regarded as a random variable

e.g. let ω denote the (discrete) random variable representing the state of nature (class) of fish types

$$\omega = \omega_1 : \text{ sea bass}$$
$$\omega = \omega_2 : \text{ salmon}$$

Prior Probability

Prior Probability (先验概率)

Prior probability is the probability distribution which reflects one's prior knowledge on the random variable

Probability distribution (for discrete random variable)

Let $P(\cdot)$ be the probability distribution on the random variable ω with c possible states of nature $\{\omega_1, \omega_2, \ldots, \omega_c\}$, such that:

$$P(\omega_i) \geq 0 \ (non\text{-}negativity) \qquad \sum_{i=1}^{c} P(\omega_i) = 1 \ (normalization)$$

the catch produced as much sea bass as salmon ⇨ $P(\omega_1) = P(\omega_2) = 1/2$

the catch produced more sea bass than salmon ⇨ $P(\omega_1) = 2/3; P(\omega_2) = 1/3$

Decision Before Observation

The Problem

To make a decision on the type of fish arriving next, where 1) prior probability is known; 2) no observation is allowed

Naive Decision Rule

$$\text{Decide } \omega_1 \text{ if } P(\omega_1) > P(\omega_2); \text{ otherwise decide } \omega_2$$

- This is the *best* we can do without observation
- Fixed prior probabilities ➔ Same decisions all the time

Incorporate observations into decision!

Good when $P(\omega_1)$ *is much greater (smaller) than* $P(\omega_2)$

Poor when $P(\omega_1)$ *is close to* $P(\omega_2)$

[only 50% chance of being right if $P(\omega_1) = P(\omega_2)$]

Probability Density Function (pdf)

Probability density function (pdf)
(for continuous random variable)

Let $p(\cdot)$ be the probability density function on the continuous random variable x taking values in **R**, such that:

$$p(x) \geq 0 \ (non\text{-}negativity) \quad \int_{-\infty}^{\infty} p(x)dx = 1 \ (normalization)$$

- For continuous random variable, it no longer makes sense to talk about the probability that *x* has a particular value (almost always be zero)
- We instead talk about the probability of *x* falling into a region *R*, say *R*=(*a*,*b*), which could be computed with the pdf:

$$\Pr[x \in R] = \int_{x \in R} p(x)dx = \int_a^b p(x)dx$$

Incorporate Observations

The Problem

Suppose the fish *lightness measurement x* is observed, how could we incorporate this knowledge into usage?

Class-conditional probability density function (类条件概率密度)

- It is a probability density function (pdf) for *x* given that the state of nature (class) is ω , i.e.:

$$p(x|\omega)$$

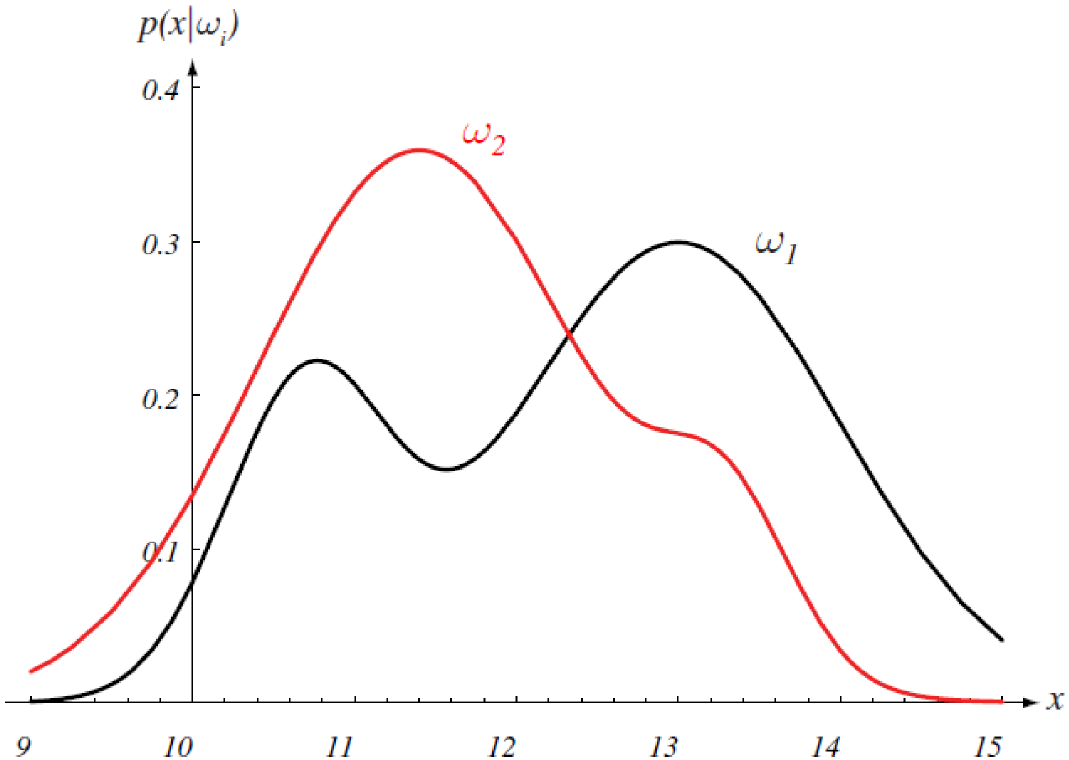
$$p(x|\omega) \geq 0 \qquad \int_{-\infty}^{\infty} p(x|\omega)dx = 1$$

- The *class-conditional* pdf describes the difference in the distribution of observations under different classes

$$p(x|\omega_1) \text{ should be different to } p(x|\omega_2)$$

Class-Conditional PDF

An illustrative example



class-conditional pdf for *lightness*

h-axis: lightness of fish scales

v-axis: class-conditional pdf values

black curve: sea bass

red curve: salmon

- The area under each curve is 1.0 (*normalization*)
- Sea bass is somewhat brighter than salmon

Decision After Observation

Known

Unknown

Prior probability

$P(\omega_j) \ (1 \leq j \leq c)$

Class-conditional pdf

$p(x|\omega_j) \ (1 \leq j \leq c)$

Observation for test example

x^* (*e.g.: fish lightness*)

Bayes Formula

Posterior probability

$P(\omega_j|x^*) \ (1 \leq j \leq c)$

The quantity which we want to use in decision naturally (by exploiting observation information)

Convert the prior probability $P(\omega_j)$ to the posterior probability $P(\omega_j|x^*)$

Bayes Formula Revisited

Joint probability density function (联合分布) $p(\omega, x)$

Marginal distribution (边缘分布) $P(\omega)$ $p(x)$

$$P(\omega) = \int_{-\infty}^{\infty} p(\omega, x)dx \qquad p(x) = \sum_{j=1}^{c} p(\omega_j, x)$$

Law of total probability (全概率公式) [ref. pp.615]

$$p(\omega, x) = P(\omega|x) \cdot p(x)$$

$$p(\omega, x) = P(\omega) \cdot p(x|\omega)$$

$$P(\omega|x) \cdot p(x) = P(\omega) \cdot p(x|\omega)$$

$$P(\omega|x) = \frac{p(x|\omega) \cdot P(\omega)}{p(x)}$$

Bayes Formula Revisited (Cont.)

$$P(\omega_j|x) = \frac{p(x|\omega_j) \cdot P(\omega_j)}{p(x)} \quad (1 \leq j \leq c) \quad \text{(Bayes Formula)}$$

Bayes Decision Rule

$$\boxed{\text{if } P(\omega_j|x) > P(\omega_i|x),\ \forall i \neq j \implies \text{Decide } \omega_j}$$

- $P(\omega_j)$ and $p(x|\omega_j)$ are assumed to be known
- $p(x)$ is irrelevant for Bayesian decision (serving as a normalization factor, not related to any state of nature)

$$p(x) = \sum\nolimits_{j=1}^{c} p(\omega_j, x) = \sum_{j=1}^{c} p(x|\omega_j) \cdot P(\omega_j)$$

Bayes Formula Revisited (Cont.)

$$P(\omega_j|x) = \frac{p(x|\omega_j) \cdot P(\omega_j)}{p(x)} \quad \left(posterior = \frac{likelihood \times prior}{evidence} \right)$$

Special Case I: Equal prior probability

$$P(\omega_1) = P(\omega_2) = \cdots = P(\omega_c) = \frac{1}{c}$$

$\Rightarrow$ Depends on the likelihood $P(x|\omega_j)$

Special Case II: Equal likelihood

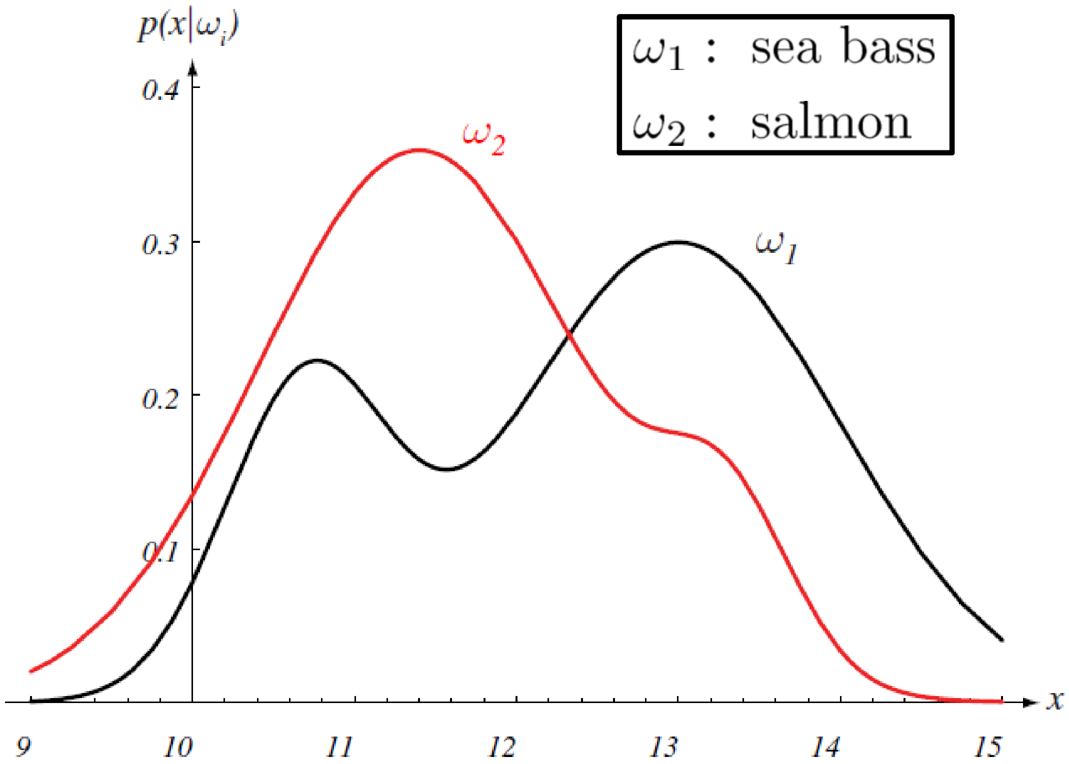
$$p(x|\omega_1) = p(x|\omega_2) = \cdots = p(x|\omega_c)$$

$\Rightarrow$ Degenerate to naive decision rule

Normally, prior probability and likelihood function together in Bayesian decision process

Bayes Formula Revisited (Cont.)

An illustrative example



class-conditional pdf for *lightness*

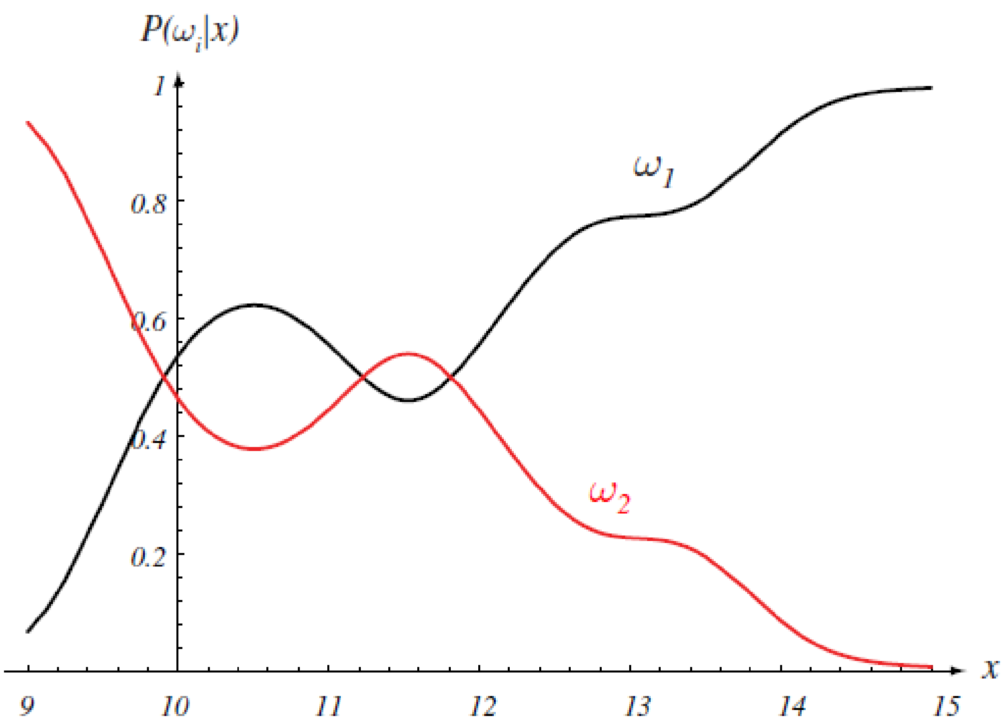
$$P(\omega_1) = \frac{2}{3}$$

$$P(\omega_2) = \frac{1}{3}$$

What will the posterior probability for either type of fish look like?

Bayes Formula Revisited (Cont.)

An illustrative example



posterior probability for either type of fish

h-axis: lightness of fish scales

v-axis: posterior probability for either type of fish

black curve: sea bass

red curve: salmon

- For each value of x, the higher curve yields the output of Bayesian decision
- For each value of x, the posteriors of either curve sum to 1.0

Another Example

Problem statement

- A new medical test is used to detect whether a patient has a certain cancer or not, whose test result is either + (*positive*) or - (*negative*)
- For patient with this cancer, the probability of returning *positive* test result is 0.98
- For patient without this cancer, the probability of returning *negative* test result is 0.97
- The probability for any person to have this cancer is 0.008

Question

If *positive* test result is returned for some person, does he/she have this kind of cancer or not?

Another Example (Cont.)

$\omega_1 : \text{ cancer} \qquad \omega_2 : \text{ no cancer} \qquad x \in \{+, -\}$

$$P(\omega_1) = 0.008 \qquad P(\omega_2) = 1 - P(\omega_1) = 0.992$$

$$P(+ \mid \omega_1) = 0.98 \qquad P(- \mid \omega_1) = 1 - P(+ \mid \omega_1) = 0.02$$

$$P(- \mid \omega_2) = 0.97 \qquad P(+ \mid \omega_2) = 1 - P(- \mid \omega_2) = 0.03$$

$$P(\omega_1 \mid +) = \frac{P(\omega_1)P(+ \mid \omega_1)}{P(+)} = \frac{P(\omega_1)P(+ \mid \omega_1)}{P(\omega_1)P(+ \mid \omega_1) + P(\omega_2)P(+ \mid \omega_2)}$$

$$= \frac{0.008 \times 0.98}{0.008 \times 0.98 + 0.992 \times 0.03} = 0.2085$$

$$P(\omega_2 \mid +) = 1 - P(\omega_1 | +) = 0.7915$$

$$\boxed{P(\omega_2 \mid +) > P(\omega_1 | +)}$$

No cancer!

Feasibility of Bayes Formula

$$P(\omega|x) = \frac{p(x|\omega) \cdot P(\omega)}{p(x)} \quad \text{(Bayes Formula)}$$

To compute posterior probability $P(\omega|x)$, we need to know:

Prior probability: $P(\omega)$ Likelihood: $p(x|\omega)$

How do we know these probabilities?

- A simple solution: Counting relative frequencies (相对频率)
- An advanced solution: Conduct density estimation (概率密度估计)

A Further Example

Problem statement

Based on the height of a car in some campus, decide whether it costs more than $50,000 or not

$$\omega_1 : \ \text{price} > \$50,000$$

$$\omega_2 : \ \text{price} \leq \$50,000$$

$$x : \ \text{height of car}$$

$$P(\omega_1|x) > P(\omega_2|x)$$

?

$$P(\omega_1|x) < P(\omega_2|x)$$

Quantities to know:

$$P(\omega_1) \quad P(\omega_2) \quad p(x|\omega_1) \quad p(x|\omega_2)$$

Counting relative frequencies via collected samples

A Further Example (Cont.)

Collecting samples

Suppose we have randomly picked 1209 cars in the campus, got prices from their owners, and measured their heights

Compute $P(\omega_1), P(\omega_2)$:

cars in ω_1 : 221

cars in ω_2 : 988

$$P(\omega_1) = \frac{221}{1209} = 0.183$$

$$P(\omega_2) = \frac{988}{1209} = 0.817$$

A Further Example (Cont.)

Compute $p(x|\omega_1), p(x|\omega_2)$ **:**

Discretize the height spectrum (say [0.5m, 2.5m]) into 20 intervals each with length 0.1m, and then count the number of cars falling into each interval for either class

Suppose

$x = 1.05$

x falls into interval I_x=[1.0m, 1.1m]

For ω_1, # cars in I_x is 46

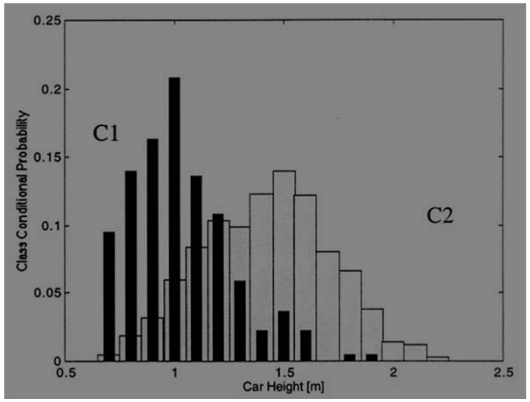
For ω_2, # cars in I_x is 59

$p(x = 1.05|\omega_1)$

$= \frac{46}{221} = 0.2081$

$p(x = 1.05|\omega_2)$

$= \frac{59}{988} = 0.0597$



A Further Example (Cont.)

Question

For a car with height 1.05m, is its price greater than $50,000?

Estimated quantities

$P(\omega_1) = 0.183$ $\quad$ $P(\omega_2) = 0.817$

$p(x = 1.05 \mid \omega_1) = 0.2081$ $\quad$ $p(x = 1.05 \mid \omega_2) = 0.0597$

$$\frac{P(\omega_2 \mid x = 1.05)}{P(\omega_1 \mid x = 1.05)} = \frac{P(\omega_2) \cdot p(x = 1.05 \mid \omega_2)}{p(x = 1.05)} \Big/ \frac{P(\omega_1) \cdot p(x = 1.05 \mid \omega_1)}{p(x = 1.05)}$$

$$= \frac{P(\omega_2) \cdot p(x = 1.05 \mid \omega_2)}{P(\omega_1) \cdot p(x = 1.05 \mid \omega_1)}$$

$$= \frac{0.817 \times 0.0597}{0.183 \times 0.2081} = 1.280$$

$$P(\omega_2 \mid x) > P(\omega_1 \mid x)$$

$$\text{price} \leq \$50,000$$

Is Bayes Decision Rule Optimal?

Bayes Decision Rule (In case of two classes)

$$\text{if } P(\omega_1|x) > P(\omega_2|x), \text{ Decide } \omega_1; \text{ Otherwise } \omega_2$$

Whenever we observe a particular *x*, the probability of error is:

$$P(error \mid x) = \begin{cases} P(\omega_1 \mid x) & \text{if we decide } \omega_2 \\ P(\omega_2 \mid x) & \text{if we decide } \omega_1 \end{cases}$$

Under Bayes decision rule, we have

$$P(error \mid x) = \min[P(\omega_1 \mid x), P(\omega_2 \mid x)]$$

For every *x*, we ensure that *P*(*error* | *x*) is as small as possible ⇒ The average probability of error over all possible *x* must be as small as possible

Bayes Decision Rule – The General Case

- By allowing to use more than one feature

  $x \in \mathbf{R} \implies \mathbf{x} \in \mathbf{R}^d$ (*d*-dimensional Euclidean space)

- By allowing more than two states of nature

  $\Omega = \{\omega_1, \omega_2, \ldots, \omega_c\}$ (finite set of *c* states of nature)

- By allowing actions other than merely deciding the state of nature

  $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots, \alpha_a\}$ (finite set of *a* possible actions)

$$\text{Note : } c \neq a$$

Bayes Decision Rule – The General Case (Cont.)

- By introducing a loss function more general than the probability of error

$$\lambda: \ \Omega \times \mathcal{A} \rightarrow \mathbf{R} \quad \text{(loss function)}$$

$\lambda(\omega_j, \alpha_i)$: the loss incurred for taking action α_i when the state of nature is ω_j

For ease of reference, usually written as:

$$\lambda(\alpha_i \mid \omega_j)$$

A simple loss function

Class \ Action	$\alpha_1 =$ "Recipe A"	$\alpha_2 =$ "Recipe B"	$\alpha_3 =$ "No Recipe"
$\omega_1 =$ "cancer"	**5**	**50**	**10,000**
$\omega_2 =$ "no cancer"	**60**	**3**	**0**

Bayes Decision Rule – The General Case (Cont.)

The problem

Given a particular **x**, we have to decide which action to take

We need to know the *loss* of taking each action $\alpha_i \ (1 \leq i \leq a)$

true state of nature is ω_j

the action being taken is α_i

incur the loss $\lambda(\alpha_i \mid \omega_j)$

However, the true state of nature is uncertain

Expected (average) loss

Bayes Decision Rule – The General Case (Cont.)

Average by enumerating over all possible states of nature!

Expected loss (期望损失)

$$R(\alpha_i \mid \mathbf{x}) = \sum_{j=1}^{c} \lambda(\alpha_i \mid \omega_j) \cdot P(\omega_j \mid \mathbf{x})$$

The incurred loss of taking action α_i in case of true state of nature being ω_j

The probability of ω_j being the true state of nature

The expected loss is also named as ***(conditional) risk*** (条件风险)

Bayes Decision Rule – The General Case (Cont.)

Suppose we have:

Action / Class	$\alpha_1 =$ "Recipe A"	$\alpha_2 =$ "Recipe B"	$\alpha_3 =$ "No Recipe"
$\omega_1 =$ "cancer"	**5**	**50**	**10,000**
$\omega_2 =$ "no cancer"	**60**	**3**	**0**

For a particular **x**:

$$P(\omega_1 \mid \mathbf{x}) = 0.01$$

$$P(\omega_2 \mid \mathbf{x}) = 0.99$$

$$R(\alpha_1 \mid \mathbf{x}) = \sum_{j=1}^{2} \lambda(\alpha_1 \mid \omega_j) \cdot P(\omega_j \mid \mathbf{x})$$

$$= \lambda(\alpha_1 \mid \omega_1) \cdot P(\omega_1 \mid \mathbf{x}) + \lambda(\alpha_1 \mid \omega_2) \cdot P(\omega_2 \mid \mathbf{x})$$

$$= 5 \times 0.01 + 60 \times 0.99 = 59.45$$

Similarly, we can get: $R(\alpha_2 \mid \mathbf{x}) = 3.47 \quad R(\alpha_3 \mid \mathbf{x}) = 100$

Bayes Decision Rule – The General Case (Cont.)

The task: *find a mapping from patterns to actions*

$$\alpha : \mathbf{R}^d \rightarrow \mathcal{A} \quad (\text{decision function})$$

In other words, for every **x**, the decision function $\alpha(\mathbf{x})$ assumes one of the *a* actions $\alpha_1, \ldots, \alpha_a$

Overall risk *R*

expected loss with decision function $\alpha(\cdot)$

$$R = \int \underline{\underline{R(\alpha(\mathbf{x}) \mid \mathbf{x})}} \cdot \underline{\underline{p(\mathbf{x})}} d\mathbf{x}$$

Conditional risk for pattern **x** with action $\alpha(\mathbf{x})$

pdf for patterns

Bayes Decision Rule – The General Case (Cont.)

$$R = \int R(\alpha(\mathbf{x}) \mid \mathbf{x})) \cdot p(\mathbf{x}) d\mathbf{x} \quad (\text{overall risk})$$

For every **x**, we ensure that the conditional risk $R(\alpha(\mathbf{x}) \mid \mathbf{x})$ is as small as possible

The overall risk over all possible **x** must be as small as possible

Bayes decision rule (*General case*)

$$\alpha(\mathbf{x}) = \arg\min_{\alpha_i \in \mathcal{A}} R(\alpha_i \mid \mathbf{x})$$
$$= \arg\min_{\alpha_i \in \mathcal{A}} \sum_{j=1}^{c} \lambda(\alpha_i \mid \omega_j) \cdot P(\omega_j \mid \mathbf{x})$$

- The resulting overall risk is called the ***Bayes risk*** (denoted as R^*)
- The best performance achievable given $p(\mathbf{x})$ and loss function

Two-Category Classification

Special case

$$\begin{matrix} \lambda_{11} & \lambda_{21} \\ \lambda_{12} & \lambda_{22} \end{matrix}$$

- $\Omega = \{\omega_1, \omega_2\}$ (two states of nature)
- $\mathcal{A} = \{\alpha_1, \alpha_2\}$ ($\alpha_1 =$ decide ω_1; $\alpha_2 =$ decide ω_2)

$\lambda_{ij} = \lambda(\alpha_i \mid \omega_j)$: the loss incurred for deciding ω_i when the true state of nature is ω_j

The conditional risk:

$$R(\alpha_1 \mid \mathbf{x}) = \lambda_{11} \cdot P(\omega_1 \mid \mathbf{x}) + \lambda_{12} \cdot P(\omega_2 \mid \mathbf{x})$$

$$R(\alpha_2 \mid \mathbf{x}) = \lambda_{21} \cdot P(\omega_1 \mid \mathbf{x}) + \lambda_{22} \cdot P(\omega_2 \mid \mathbf{x})$$

$R(\alpha_1 \mid \mathbf{x}) < R(\alpha_2 \mid \mathbf{x})$

yes → decide ω_1

no → decide ω_2

Two-Category Classification (Cont.)

$$R(\alpha_1 \mid \mathbf{x}) < R(\alpha_2 \mid \mathbf{x})$$

by definition

$$\boxed{\begin{gathered}\lambda_{11} \cdot P(\omega_1 \mid \mathbf{x}) + \lambda_{12} \cdot P(\omega_2 \mid \mathbf{x}) \\ < \\ \lambda_{21} \cdot P(\omega_1 \mid \mathbf{x}) + \lambda_{22} \cdot P(\omega_2 \mid \mathbf{x})\end{gathered}}$$

by re-arrangement

$$\boxed{\begin{gathered}(\lambda_{21} - \lambda_{11})P(\omega_1 \mid \mathbf{x}) \\ > \\ (\lambda_{12} - \lambda_{22})P(\omega_2 \mid \mathbf{x})\end{gathered}}$$

by Bayes theorem

$$\boxed{\begin{gathered}(\lambda_{21} - \lambda_{11}) \cdot p(\mathbf{x} \mid \omega_1) \cdot P(\omega_1) \\ > \\ (\lambda_{12} - \lambda_{22}) \cdot p(\mathbf{x} \mid \omega_2) \cdot P(\omega_2)\end{gathered}}$$

$\lambda_{21} - \lambda_{11} > 0$

the loss for being error is ordinarily greater than the loss for being correct

$$\boxed{\underbrace{\frac{p(\mathbf{x} \mid \omega_1)}{p(\mathbf{x} \mid \omega_2)}}_{\text{likelihood ratio}} > \underbrace{\frac{\lambda_{12} - \lambda_{22}}{\lambda_{21} - \lambda_{11}} \cdot \frac{P(\omega_2)}{P(\omega_1)}}_{\text{constant } \theta \text{ independent of } \mathbf{x}}}$$

Minimum-Error-Rate Classification

Classification setting

- $\Omega = \{\omega_1, \omega_2, \ldots, \omega_c\}$ (c possible states of nature)
- $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots, \alpha_c\}$ ($\alpha_i = \text{decide } \omega_i,\ 1 \leq i \leq c$)

Zero-one (symmetrical) loss function

$$\lambda(\alpha_i \mid \omega_j) = \begin{cases} 0 & i = j \\ 1 & i \neq j \end{cases} \quad 1 \leq i, j \leq c$$

- Assign no loss (i.e. 0) to a correct decision
- Assign a unit loss (i.e. 1) to any incorrect decision (**equal cost**)

Minimum-Error-Rate Classification (Cont.)

$$R(\alpha_i \mid \mathbf{x}) = \sum_{j=1}^{c} \lambda(\alpha_i \mid \omega_j) \cdot P(\omega_j \mid \mathbf{x})$$

$$= \sum_{j \neq i} \lambda(\alpha_i \mid \omega_j) \cdot P(\omega_j \mid \mathbf{x}) \ + \ \lambda(\alpha_i \mid \omega_i) \cdot P(\omega_i \mid \mathbf{x})$$

$$= \sum_{j \neq i} P(\omega_j \mid \mathbf{x})$$

$$= 1 - P(\omega_i \mid \mathbf{x})$$

error rate (误差率/错误率)

the probability that action α_i (decide ω_i) is wrong

Minimum error rate

Decide ω_i if $P(\omega_i \mid \mathbf{x}) > P(\omega_j \mid \mathbf{x})$ for all $j \neq i$

Discriminant Function (判别函数)

Classification

Pattern ➔ Category

actions ⟷ **decide categories**

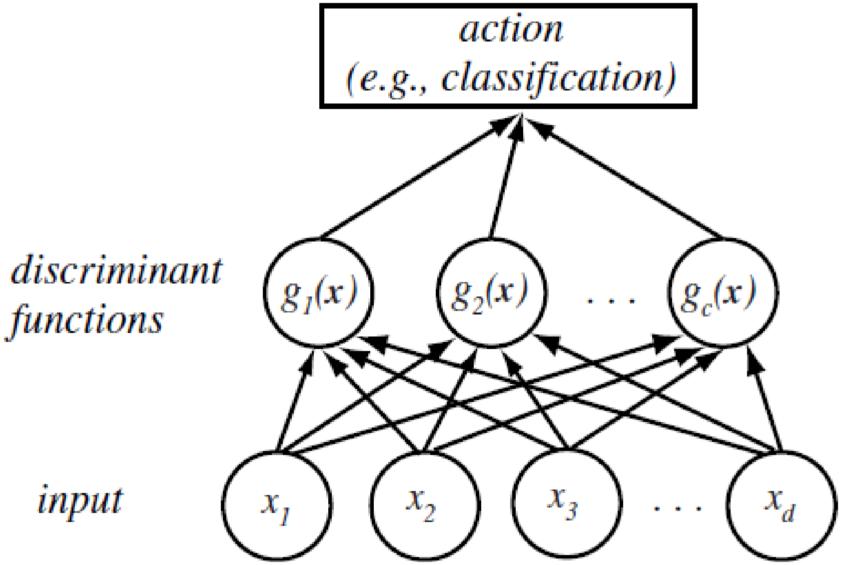
Discriminant functions

$g_i : \mathbf{R}^d \rightarrow \mathbf{R} \quad (1 \leq i \leq c)$

- *Useful way to represent classifiers*
- *One function per category*

Decide ω_i

if $g_i(\mathbf{x}) > g_j(\mathbf{x})$ for all $j \neq i$



Discriminant Function (Cont.)

Minimum risk: $g_i(\mathbf{x}) = -R(\alpha_i \mid \mathbf{x}) \quad (1 \le i \le c)$

Minimum-error-rate: $g_i(\mathbf{x}) = P(\omega_i|\mathbf{x}) \quad (1 \le i \le c)$

Various discriminant functions ⇒ **Identical classification results**

$f(\cdot)$ is a *monotonically increasing function* (单调递增函数)

$$f(g_i(\mathbf{x})) \Longleftrightarrow g_i(\mathbf{x})$$ (*i.e. equivalent in decision*)

e.g.:

$f(x) = k \cdot x \ (k > 0)$ ⇒ $f(g_i(\mathbf{x})) = k \cdot g_i(\mathbf{x}) \ (1 \le i \le c)$

$f(x) = \ln x$ ⇒ $f(g_i(\mathbf{x})) = \ln g_i(\mathbf{x}) \ (1 \le i \le c)$

Discriminant Function (Cont.)

Decision region (决策区域)

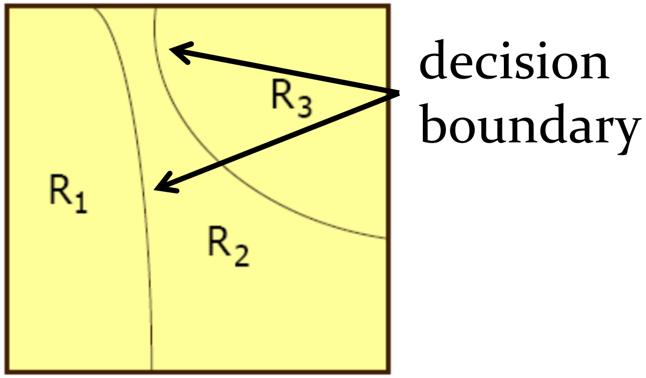
c discriminant functions $g_i(\cdot)\ (1 \leq i \leq c)$ → c decision regions $\mathcal{R}_i \subset \mathbf{R}^d\ (1 \leq i \leq c)$

$$\mathcal{R}_i = \{\mathbf{x} \mid \mathbf{x} \in \mathbf{R}^d : g_i(\mathbf{x}) > g_j(\mathbf{x}) \quad \forall j \neq i\}$$

where $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset\ (i \neq j)$ and $\bigcup_{i=1}^{c} \mathcal{R}_i = \mathbf{R}^d$

Decision boundary (决策边界)

surface in feature space where ties occur among several largest discriminant functions



Expected Value

Expected value (数学期望), a.k.a. *expectation*, *mean* or *average* of a random variable *x*

Discrete case

$x \in \mathcal{X} = \{x_1, x_2, \ldots, x_c\}$

$x \sim P(\cdot)$

($\sim$: "has the distribution")

$$\mathcal{E}[x] = \sum_{x \in \mathcal{X}} x \cdot P(x) = \sum_{i=1}^{c} x_i \cdot P(x_i)$$

Continuous case

Notation: $\mu = \mathcal{E}[x]$

$x \in \mathbf{R}$

$x \sim p(\cdot)$

$$\mathcal{E}[x] = \int_{-\infty}^{\infty} x \cdot p(x)\, dx$$

Expected Value (Cont.)

Given random variable *x* and function $f(\cdot)$, what is the expected value of $f(x)$?

Discrete case: $\mathcal{E}[f(x)] = \sum_{x \in \mathcal{X}} f(x) \cdot P(x) = \sum_{i=1}^{c} f(x_i) \cdot P(x_i)$

Continuous case: $\mathcal{E}[f(x)] = \int_{-\infty}^{\infty} f(x) \cdot p(x) \, dx$

Variance (方差) $\mathrm{Var}[x] = \mathcal{E}[(x - \mathcal{E}[x])^2]$ (i.e. $f(x) = (x - \mu)^2$)

Discrete case: $\mathrm{Var}[x] = \sum_{i=1}^{c} (x_i - \mu)^2 \cdot P(x_i)$

Continuous case: $\mathrm{Var}[x] = \int_{-\infty}^{\infty} (x - \mu)^2 \cdot p(x) \, dx$

Notation: $\sigma^2 = \mathrm{Var}[x]$ (σ : *standard deviation* (标准偏差))

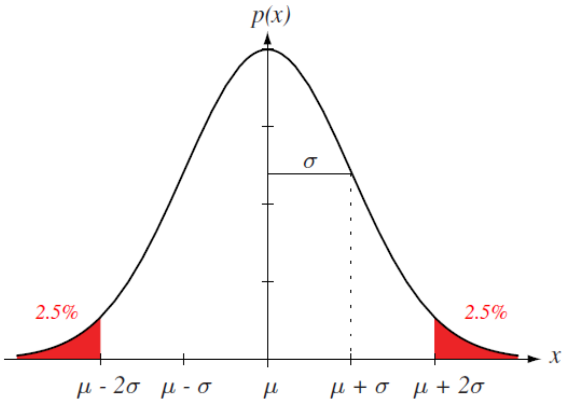
Gaussian Density – Univariate Case

Gaussian density (高斯密度函数), a.k.a. *normal density* (正态密度函数), for continuous random variable

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \qquad x \sim N(\mu, \sigma^2)$$

$$\int_{-\infty}^{\infty} p(x)dx = 1$$

$$\mathcal{E}[x] = \int_{-\infty}^{\infty} x \cdot p(x) = \mu$$



$$\mathrm{Var}[x] = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot p(x) = \sigma^2$$

Vector Random Variables (随机向量)

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix}$$

$\mathbf{x} \sim p(\mathbf{x}) = p(x_1, x_2, \ldots, x_d)$ **(joint pdf)**

$p(\mathbf{x}_1) = \int p(\mathbf{x}_1, \mathbf{x}_2)\, d\mathbf{x}_2$ **(marginal pdf)**

$(\mathbf{x}_1 \cap \mathbf{x}_2 = \emptyset;\ \mathbf{x}_1 \cup \mathbf{x}_2 = \mathbf{x})$

Expected vector

$$\mathcal{E}[\mathbf{x}] = \begin{pmatrix} \mathcal{E}[x_1] \\ \mathcal{E}[x_2] \\ \vdots \\ \mathcal{E}[x_d] \end{pmatrix}$$

$$\mathcal{E}[x_i] = \int_{-\infty}^{\infty} x_i \cdot \underline{\underline{p(x_i)}}\, dx_i \quad (1 \leq i \leq d)$$

marginal pdf on the *i*-th component

Notation:

$\boldsymbol{\mu} = \mathcal{E}[\mathbf{x}];\ \mu_i = \mathcal{E}[x_i]\ (1 \leq i \leq d)$

Vector Random Variables (Cont.)

Covariance matrix (协方差矩阵)

$$\mathbf{\Sigma} = [\sigma_{ij}]_{1 \leq i,j \leq d} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1d} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{d1} & \sigma_{d2} & \cdots & \sigma_{dd} \end{pmatrix}$$

Properties of $\mathbf{\Sigma}$

- symmetric (对称矩阵)
- Positive semidefinite (半正定矩阵)

$$\sigma_{ij} = \sigma_{ji} = \mathcal{E}[(x_i - \mu_i)(x_j - \mu_j)]$$

Appendix A.4.9 [pp.617]

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_i - \mu_i)(x_j - \mu_j) \cdot p(x_i, x_j)\, dx_i dx_j$$

$$\sigma_{ii} = \mathrm{Var}[x_i] = \sigma_i^2$$

marginal pdf on a pair of random variables (x_i, x_j)

Gaussian Density – Multivariate Case

$\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

$$\mu_i = \mathcal{E}[x_i] \qquad \sigma_{ij} = \sigma_{ji} = \mathcal{E}[(x_i - \mu_i)(x_j - \mu_j)]$$

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^t \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})\right]$$

$\mathbf{x} = (x_1, x_2, \ldots, x_d)^t$: *d*-dimensional *column vector*

$\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_d)^t$: *d*-dimensional *mean vector*

$$\boldsymbol{\Sigma} = [\sigma_{ij}]_{1 \leq i,j \leq d} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1d} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{d1} & \sigma_{d2} & \cdots & \sigma_{dd} \end{pmatrix}$$

$d \times d$ *covariance matrix*

$|\boldsymbol{\Sigma}|$: determinant

$\boldsymbol{\Sigma}^{-1}$: inverse

Gaussian Density – Multivariate Case (Cont.)

$\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}):\quad p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^t \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]$

$(\mathbf{x}-\boldsymbol{\mu})^t : 1 \times d$ matrix

$\boldsymbol{\Sigma}^{-1} : d \times d$ matrix

$(\mathbf{x}-\boldsymbol{\mu}) : d \times 1$ matrix

$\Rightarrow$ $(\mathbf{x}-\boldsymbol{\mu})^t \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})$

scalar $(1 \times 1$ matrix$)$

$\boldsymbol{\Sigma}$: positive definite $\Rightarrow$ $\boldsymbol{\Sigma}^{-1}$: positive definite

$\Downarrow$

$(\mathbf{x}-\boldsymbol{\mu})^t \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) \geq 0$ $\Rightarrow$ $-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^t \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) \leq 0$

Discriminant Functions for Gaussian Density

Minimum-error-rate classification

$$g_i(\mathbf{x}) = P(\omega_i|\mathbf{x}) \quad (1 \leq i \leq c)$$

$$g_i(\mathbf{x}) = P(\omega_i|\mathbf{x}) \iff g_i(\mathbf{x}) = \ln P(\omega_i|\mathbf{x})$$

$$g_i(\mathbf{x}) = \ln p(\mathbf{x}|\omega_i) + \ln P(\omega_i)$$

$$p(\mathbf{x}|\omega_i) \sim N(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$$

Constant, could be ignored

$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_i)^t\boldsymbol{\Sigma}_i^{-1}(\mathbf{x}-\boldsymbol{\mu}_i) - \frac{d}{2}\ln 2\pi - \frac{1}{2}\ln|\boldsymbol{\Sigma}_i| + \ln P(\omega_i)$$

Case I: $\boldsymbol{\Sigma}_i = \sigma^2 \mathbf{I}$

$$p(\mathbf{x}|\omega_i) \sim N(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$$

$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_i)^t \boldsymbol{\Sigma}_i^{-1}(\mathbf{x}-\boldsymbol{\mu}_i) - \frac{1}{2}\ln|\boldsymbol{\Sigma}_i| + \ln P(\omega_i)$$

Covariance matrix: σ^2 times the identity matrix **I**

$$\boldsymbol{\Sigma}_i = \sigma^2 \cdot \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} \sigma^2 & & & \\ & \sigma^2 & & \\ & & \ddots & \\ & & & \sigma^2 \end{pmatrix} \Rightarrow \begin{array}{l} |\boldsymbol{\Sigma}_i| = \sigma^{2d} \\ \boldsymbol{\Sigma}_i^{-1} = (1/\sigma^2)\,\mathbf{I} \end{array}$$

$$g_i(\mathbf{x}) = -\frac{||\mathbf{x}-\boldsymbol{\mu}_i||^2}{2\sigma^2} + \ln P(\omega_i)$$

$|| \cdot || :$ *Euclidean norm*

$$||\mathbf{x}-\boldsymbol{\mu}_i||^2 = (\mathbf{x}-\boldsymbol{\mu}_i)^t(\mathbf{x}-\boldsymbol{\mu}_i)$$

Case I: $\boldsymbol{\Sigma}_i = \sigma^2 \mathbf{I}$ (Cont.)

$$g_i(\mathbf{x}) = -\frac{||\mathbf{x} - \boldsymbol{\mu}_i||^2}{2\sigma^2} + \ln P(\omega_i)$$

the same for all *states of nature*,
could be ignored

$$g_i(\mathbf{x}) = -\frac{1}{2\sigma^2}[\mathbf{x}^t\mathbf{x} - 2\boldsymbol{\mu}_i^t\mathbf{x} + \boldsymbol{\mu}_i^t\boldsymbol{\mu}_i] + \ln P(\omega_i)$$

Linear discriminant functions (线性判别函数)

$$g_i(\mathbf{x}) = \mathbf{w}_i^t\mathbf{x} + w_{i0}$$

$$\mathbf{w}_i = \frac{1}{\sigma^2}\boldsymbol{\mu}_i$$ *weight vector* (权值向量)

$$w_{i0} = -\frac{1}{2\sigma^2}\boldsymbol{\mu}_i^t\boldsymbol{\mu}_i + \ln P(\omega_i)$$ *threshold/bias* (阈值/偏置)

Case II: $\boldsymbol{\Sigma}_i = \boldsymbol{\Sigma}$

$$p(\mathbf{x}|\omega_i) \sim N(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$$

$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_i)^t\boldsymbol{\Sigma}_i^{-1}(\mathbf{x}-\boldsymbol{\mu}_i) - \frac{1}{2}\ln|\boldsymbol{\Sigma}_i| + \ln P(\omega_i)$$

Covariance matrix: *identical* for all classes

$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_i)^t\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}_i) + \ln P(\omega_i)$$

$(\mathbf{x}-\boldsymbol{\mu}_i)^t\,\boldsymbol{\Sigma}^{-1}\,(\mathbf{x}-\boldsymbol{\mu}_i)$: squared *Mahalanobis distance* (马氏距离)

$\boldsymbol{\Sigma} = \mathbf{I}$ $\Rightarrow$ reduces to *Euclidean distance*

P. C. Mahalanobis (1893-1972)

Case II: $\boldsymbol{\Sigma}_i = \boldsymbol{\Sigma}$ (Cont.)

$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_i)^t\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}_i) + \ln P(\omega_i)$$

the same for all *states of nature*,
could be ignored

$$g_i(\mathbf{x}) = -\frac{1}{2}[\mathbf{x}^t\boldsymbol{\Sigma}^{-1}\mathbf{x} - 2\boldsymbol{\mu}_i^t\boldsymbol{\Sigma}^{-1}\mathbf{x} + \boldsymbol{\mu}_i^t\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_i] + \ln P(\omega_i)$$

Linear discriminant functions

$$g_i(\mathbf{x}) = \mathbf{w}_i^t\mathbf{x} + w_{i0}$$

$\mathbf{w}_i = \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_i$ *weight vector*

$w_{i0} = -\frac{1}{2}\boldsymbol{\mu}_i^t\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_i + \ln P(\omega_i)$ *threshold/bias*

Case III: $\boldsymbol{\Sigma}_i = \textbf{arbitrary}$

$p(\mathbf{x}|\omega_i) \sim N(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$

$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_i)^t \boldsymbol{\Sigma}_i^{-1}(\mathbf{x}-\boldsymbol{\mu}_i) - \frac{1}{2}\ln|\boldsymbol{\Sigma}_i| + \ln P(\omega_i)$$

quadratic discriminant functions (二次判别函数)

$$g_i(\mathbf{x}) = \mathbf{x}^t \mathbf{W}_i \mathbf{x} + \mathbf{w}_i^t \mathbf{x} + w_{i0}$$

$\mathbf{W}_i = -\frac{1}{2}\boldsymbol{\Sigma}_i^{-1}$ *quadratic matrix*

$\mathbf{w}_i = \boldsymbol{\Sigma}_i^{-1}\boldsymbol{\mu}_i$ *weight vector*

$w_{i0} = -\frac{1}{2}\boldsymbol{\mu}_i^t \boldsymbol{\Sigma}_i^{-1}\boldsymbol{\mu}_i - \frac{1}{2}\ln|\boldsymbol{\Sigma}_i| + \ln P(\omega_i)$ *threshold/bias*

Summary

- Bayesian Decision Theory
  - PR: essentially a decision process
  - Basic concepts
    - States of nature
    - Probability distribution, probability density function (pdf)
    - Class-conditional pdf
    - Joint pdf, marginal distribution, law of total probability
  - Bayes theorem
    - Prior + likelihood + observation ➔ Posterior probability
  - Bayes decision rule
    - Decide the state of nature with maximum posterior

Summary (Cont.)

- Feasibility of Bayes decision rule
  - Prior probability + likelihood
  - Solution I: counting relative frequencies
  - Solution II: conduct density estimation (chapters 3,4)
- Bayes decision rule: The general scenario
  - Allowing more than one feature
  - Allowing more than two states of nature
  - Allowing actions than merely deciding state of nature
  - Loss function: $\lambda : \ \Omega \times \mathcal{A} \rightarrow \mathbf{R}$

Summary (Cont.)

- Expected loss (*conditional risk*)

$$R(\alpha_i \mid \mathbf{x}) = \sum_{j=1}^{c} \lambda(\alpha_i \mid \omega_j) \cdot P(\omega_j \mid \mathbf{x})$$

Average by enumerating over all possible states of nature

- General Bayes decision rule
  - Decide the action with minimum expected loss
- Minimum-error-rate classification
  - Actions ←→ Decide states of nature
  - Zero-one loss function
    - Assign *no loss*/*unit loss* for *correct*/*incorrect* decisions

Summary (Cont.)

- Discriminant functions
  - General way to represent classifiers
  - One function per category
  - Induce *decision regions* and *decision boundaries*

- Gaussian/Normal density

$$\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) : \quad p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^t \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]$$

- Discriminant functions for Gaussian pdf

$\boldsymbol{\Sigma}_i = \sigma^2 \mathbf{I}$, $\boldsymbol{\Sigma}_i = \boldsymbol{\Sigma}$: linear discriminant function

$\boldsymbol{\Sigma}_i = \textbf{arbitrary}$: quadratic discriminant function