MONASH University

Information Technology

FIT5186 Intelligent Systems

Lecture 10

Fuzzy Logic

Learning Objectives

- Understand
  - the characteristics of linguistic variables and membership functions
  - the principles of approximate reasoning in fuzzy logic
  - the construction and interpretation of fuzzy rules for fuzzy decision making

Fuzzy Logic

- **Broad Sense**

  Fuzzy set theory - a theory of classes (sets) with unsharp (fuzzy) boundaries.

- **Narrow Sense**

  A logical system which aims at emulating modes of **human reasoning** (very often dealing with the **common sense logic** only) which are approximate rather than exact.

  - Approximate reasoning

Approximate Reasoning

- Deals with imprecise data associated with human subjective judgements and preferences
- Avoids pointless precision in data required (not cost-effective or physically difficult to obtain).
- Reasons with fuzzy sets or with sets of fuzzy rules.
- Can provide precise conclusions if required.
- Fuzzy logic provides **a formal system of numerical computation for dealing with linguistic variables** whose values are characterised by fuzzy sets (fuzzy numbers).

Linguistic Variables and their Associated Linguistic Values (Terms)

- A linguistic variable takes values expressed by words or sentences in a natural or synthetic language for representing its states.
- In a specific application, a set of linguistic values or terms (**the term set**) needs to be defined to represent possible states of a linguistic variable.
- Linguistic terms are characterised by fuzzy sets.
- A fuzzy set is a set whose members belong to it to some degree.

Linguistic Variables – An Example

- Linguistic variable: Quality
- Linguistic values (terms) of “Quality”:

Excellent	Very Good	Good
Fair	Very Bad	Bad
Not Very Good	Not Bad	...

- A simple “term set” for the linguistic variable “Quality”: Good, Fair, Bad.

A Simple Application of Fuzzy Logic

- **Question:**

  Do I understand the topic we are about to discuss?

  (*Can I answer your question?*)

- **Crisp Answer:**

  Yes. - This involves risk resulting from uncertainty.

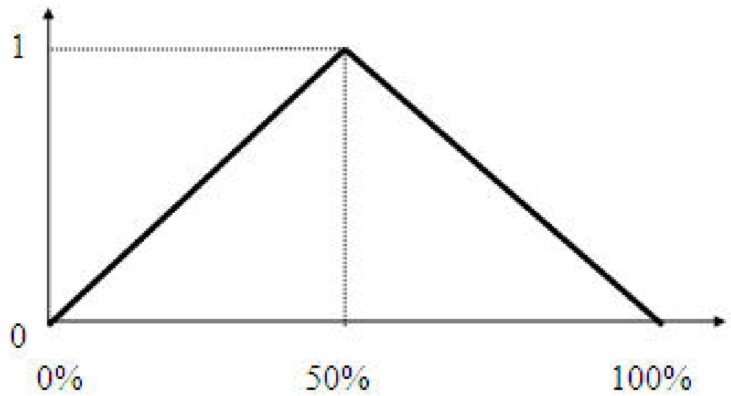
- **Fuzzy Answer:**

  Well..., it is a matter of degree.

Fuzzy Answer – An Example

- **Fuzzy Answer:**

  Well..., it is a matter of degree.



- This fuzzy answer characterised by a fuzzy set is not concerned with whether I can answer your question, but is used to indicate the degree to which my answer may satisfy you after it is given.
- The risk is understood and the uncertainty is taken into account.

Classical Logic (Two-Valued Logic)

- A Crisp Relation:

$$x = y$$

- Exact Reasoning:

$$\text{IF} \quad \underline{\boldsymbol{x = 1}} \quad \text{THEN} \quad \boldsymbol{y = 1}$$

$$\uparrow$$

(a crisp proposition)

- A crisp proposition is required to be either **true** or **false**.

Fuzzy Logic (Multi-valued Logic)

- A Fuzzy Relation:

$$\boldsymbol{x} \approx \boldsymbol{y} \qquad (x \text{ is approximately equal to } y)$$

- Approximate Reasoning:

$$\text{IF} \quad \underline{\boldsymbol{x} \ \ \textbf{is} \ \ \boldsymbol{small}} \quad \text{THEN} \quad \boldsymbol{y} \ \ \textbf{is} \ \ \textbf{?}$$

$$\uparrow$$

(a fuzzy proposition)

- The **truth** of a fuzzy proposition **is a matter of degree**.

Defining Linguistic Terms Used in Fuzzy Propositions as Fuzzy Sets

- IF x **is *small*** THEN ***y*** **is ?**

  $\uparrow$ (a given fuzzy set) $\uparrow$ (the consequent fuzzy set)

- Let the linguistic term "***small***" for all elements in $A = \{1, 2, 3, 4\}$ be defined by the **membership function**

$$\mu_{small}(\boldsymbol{x}) = \frac{1}{1} + \frac{0.6}{2} + \frac{0.2}{3} + \frac{0}{4}$$

  which is a (discrete) fuzzy set indicating the degree (grade) of membership of each element ($\boldsymbol{x} \in A$) in the linguistic term "***small***".

- A membership function can be interpreted as a *possibility distribution* over the real line.

Membership Function Notation

- $\mu_A(x)$ stands for the degree of membership of the element x in the fuzzy set A.

- For example, $\mu_{Young}(40) = 0.5$ means that the degree of membership of age 40 belonging to the linguistic term "*Young*" is 0.5.

- In general terms, the possibility of age 40 being young is denoted as $\mu_{Young}(40)$, whose value reflects the degree to which age 40 matches one's perception of "*Young*" in a specific context.

Fuzzy Relations

- A (binary) fuzzy relation between two sets ***A*** and ***B*,** specifies the degree by which ***A*** is related to ***B***, which is defined as the fuzzy set of all ordered pairs of elements from ***A*** and ***B***.
- Let $A = B = \{1, 2, 3, 4\}$.
- The fuzzy relation R = **"*x* is approximately equal to *y*"** between A and B (where $x \in A$ and $y \in B$) is a fuzzy subset of their Cartesian product ($A \times B$), whose membership function may be given as
- $\mu_{\boldsymbol{R}}(x, y) = \frac{1}{(1,1)} + \frac{1}{(2,2)} + \frac{1}{(3,3)} + \frac{1}{(4,4)} + \frac{0.5}{(1,2)} + \frac{0.5}{(2,1)} + \frac{0.5}{(2,3)} + \frac{0.5}{(3,2)} + \frac{0.5}{(3,4)} + \frac{0.5}{(4,3)}$

Fuzzy Relations - Membership Matrix

- Fuzzy Relations

$$\mu_R(x, y) = \frac{1}{(1,1)} + \frac{1}{(2,2)} + \frac{1}{(3,3)} + \frac{1}{(4,4)} + \frac{0.5}{(1,2)} + \frac{0.5}{(2,1)} + \frac{0.5}{(2,3)} + \frac{0.5}{(3,2)} + \frac{0.5}{(3,4)} + \frac{0.5}{(4,3)}$$

can be expressed by the membership matrix:

$$\mu_R(x, y) = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0.5 & 1 & 0.5 & 0 \\ 0 & 0.5 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{bmatrix} \end{array}$$

Approximate Reasoning (Fuzzy Reasoning)

- Given the fuzzy relation R = "*x is approximately equal to y*", defined as

$$\mu_R(x, y) = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0.5 & 1 & 0.5 & 0 \\ 0 & 0.5 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{bmatrix} \end{array}$$

- IF **x is *small*** THEN ***y* is *?***

$$\mu_{small}(x) \qquad\qquad \mu_{?}(y)$$

- $\mu_{small}(\mathbf{x}) = \frac{1}{1} + \frac{0.6}{2} + \frac{0.2}{3} + \frac{0}{4}$

The Compositional Rule of Inference

- **?** **=** *small* $\circ$ R

  where $\circ$ is the **max-min** compositional operator, defined as:

$$\mu_{?}(y) = \max_{x \in A} \{\min(\mu_{small}(x), \mu_R(x, y))\}$$

$$= \frac{1}{1} + \frac{0.6}{2} + \frac{0.5}{3} + \frac{0.2}{4} \quad \Rightarrow \text{ consequent fuzzy set}$$

Interpretation of the Consequent Fuzzy Set - Crisp Conclusion

- IF *x* **is** ***small*** THEN ***y*** **is ?**

$$\mu_{?}(y) = \frac{1}{1} + \frac{0.6}{2} + \frac{0.5}{3} + \frac{0.2}{4}$$

Two methods are available:

(1) Arithmetic Defuzzification for Crisp Conclusion

- aims at extracting a single crisp value which most accurately represents the consequent fuzzy set.

- "***y* is *1***" is the crisp answer, as the number "***1***" has the highest possibility of 1.

Interpretation of the Consequent Fuzzy Set - Fuzzy Conclusion

(2) Linguistic Approximation for Fuzzy Conclusion

- aims at translating the consequent fuzzy set into a corresponding linguistic term.

- "***y* is *rather small***" is the fuzzy answer, as the fuzzy set for the linguistic term "***rather small***" is the **closest** to the consequent fuzzy set among all linguistic terms available.

$$\mu_{small}(y) = \frac{1}{1} + \frac{0.6}{2} + \frac{0.2}{3} + \frac{0}{4} \qquad \mu_{very\ small}(y) = \frac{1}{1} + \frac{0.3}{2} + \frac{0.1}{3} + \frac{0}{4}$$

$$\mu_{rather\ small}(y) = \frac{1}{1} + \frac{0.7}{2} + \frac{0.4}{3} + \frac{0}{4} \qquad \mu_{a\ bit\ small}(y) = \frac{1}{1} + \frac{0.8}{2} + \frac{0.6}{3} + \frac{0}{4}$$

Reasoning with Fuzzy Rules in Fuzzy Systems (Expert Systems, Control Systems)

- Knowledge used in fuzzy systems is often represented by a set of conditional fuzzy rules.
- A fuzzy rule takes the form (the simplest one):

IF *X is A* THEN *Y is B*

<condition> ***<action>***

where

- *X* is a linguistic variable taking values in the universe of discourse *U*
- *Y* is a linguistic variable taking values in the universe of discourse *V*
- *A* is a fuzzy set defined on *U*
- *B* is a fuzzy set defined on *V*.

Fuzzy Rule and Fuzzy Relation

- A fuzzy rule can be represented by means of **a fuzzy relation R** between ***A*** and ***B*** from *U* to *V*, expressed by

$$\mu_R (x, y) = \max \{ 1-\mu_A (x), \mu_B (y)\}, \quad \forall (x, y) \in U \times V$$

- In other words, the fuzzy relation *R* specifies the relationship between input and output of a fuzzy system.

- In logic, the implication $A \rightarrow B$ (*if A then B*) defines the relationship between *A* and *B*, which is equivalent to

$\sim A \vee B$ (*either* ***not A*** *or* ***B***).

- Hence $A \rightarrow B$ is false *only* when *A* is true and *B* is false.

What Does Logic Do?

- The main concern in **logic** is not whether a conclusion is in fact accurate, but whether the process by which it is derived from a set of initial assumptions (propositions) (conditions, antecedents or premises) is correct.

- **Logic** is not concerned with examining the truth of the initial assumptions; it is concerned with the form and structure of arguments, not their contents.

$$A \rightarrow B \quad = \quad \sim A \vee B$$

(*if A then B*) (*either not A or B*).

The Compositional Rule of Inference for Fuzzy Rules in Fuzzy Systems

- Given
  (1) the fuzzy relation ***R*** implied by the fuzzy rule
  IF *X is A* THEN *Y is B* and
  (2) the fuzzy set ***A,***
  the fuzzy set ***B*** is determined by
  $$\boldsymbol{B = A \circ R}$$
  where $\circ$ is the max-min compositional operator, defined as
  $$\mu_B(y) = \max_{x \in U} \{ \min (\mu_A(x), \mu_R(x, y))\}$$
- In actual applications, a given fact matches a fuzzy proposition in a fuzzy rule to a certain degree.
- In other words, the **truth** of a fuzzy proposition **is a matter of degree**.

Interpolative Reasoning of Fuzzy Rules (Fuzzy Interpolation)

- Fuzzy Rule:

  IF ***X is A*** **THEN** ***Y is B***

  <*condition*> **<*action*>**

- Given Fact: ***X is A'***

- By the compositional rule of inference

$$B' = A' \circ R$$

- Inferred Conclusion: ***Y is B'***

Input to a Fuzzy System

- Given Fact: ***X is A'***
- ***A´*** represents an input to a rule-based fuzzy system, which can be a crisp value or a fuzzy set.
- ***A´*** is usually different from ***A***, i.e. no exact match is required.
- The given fact determines the degree of truth of the antecedent (the condition) in the fuzzy rule, which limits the degree of truth of the consequent (the action) in the same fuzzy rule to no more than the same degree.
- Classical logic is a special case of fuzzy logic where ***A´ = A.***

Fuzzy Decision Making Using Fuzzy Rules with Discrete Problem Space

- A fuzzy rule:

  IF ***Price* is *Low*** AND ***Quality* is *Good***

  THEN ***Buy More***

- See **additional material for Lecture 10, pp. 1-2.**

Fuzzy Decision Making Using Fuzzy Rules with Continuous Problem Space

Three fuzzy rules:

- *Rule 1:*
  IF ***Price*** **is** ***Low*** AND ***Quality*** **is** ***Good***
  THEN ***Buy More***
- *Rule 2:*
  IF ***Price*** **is** ***Medium*** AND ***Quality*** **is** ***Fair***
  THEN ***Buy Moderate***
- *Rule 3:*
  IF ***Price*** **is** ***High*** AND ***Quality*** **is** ***Bad***
  THEN ***Buy Less***

- See **additional material for Lecture 10, pp. 3-8.**

Similarity-Based Fuzzy Reasoning

- The construction of a **fuzzy relation** between *A* and *B* for the following fuzzy rule is **not required**.

  IF *X is A* THEN *Y is B*

  Fact: ***X is A′***

  Conclusion: ***Y is B′***

- The fuzzy inference process is based on the **degree of similarity** between ***A*** and ***A′***, which is measured by the distance between the two fuzzy sets ***A*** and ***A′***, $d(A, A')$.

- If $\boldsymbol{d(A, A')}$ is greater than or equal to a specified threshold value, ***B′*** is obtained by modifying ***B*** with a modification function, which may include μ_A, $\mu_{A'}$, $d(A, A')$ and other parameters.

Similarity-Based Fuzzy Reasoning

- Different methods have been proposed using different distance measures and modification functions.
- While the **compositional rule of inference** is simple in computation, this inference process may be desirable if

(a) The matrix operations on the membership functions are not straightforward.

(b) The outcome is not desirable, e.g. $A = A' \rightarrow B \neq B'$.

Fuzzy Logic Application Research Examples

- Chang, Y.-H., Yeh, C.-H., and Cheng, J.-H. (1998). Decision support for bus operations under uncertainty: a fuzzy expert system approach. *Omega - The International Journal of Management Science,* 26(3), 367-380.
  - See additional material for Lecture 10, pp. 9-10.
- Yeh, C.-H., Willis, R.J., Deng, H. and Pan, H. (1999). Task oriented weighting in multi-criteria analysis. *European Journal of Operational Research*, 119(1), 130-146.
- Lin, Y.-C., Lai, H.-H. and Yeh, C.-H. (2007). Consumer oriented product form design based on fuzzy logic: A case study of mobile phones. *International Journal of Industrial Ergonomics*, 37(6), 531-543.

Conclusion

- No theory is good unless one uses it to go beyond.

  - Andre Gide

- AI has not been able to attain the objectives it set for itself. And I place the blame on its commitment to classical logic.

  - Lotfi Zadeh