DEPARTAMENTO DE
MATEMÁTICA

# RG Flow of Preactivations

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## Recap

## Deeper Layers: Accumulation of Non-Gaussianity

- Recursion
- Action
- Large-width expansion

## Marginalization Rules

- Marginalization over samples
- Marginalization over neurons
- Running couplings with partial marginalizations

## Recap

- Cálculo da distribuição condicional

$$p\left(z^{(2)}, z^{(1)} \middle| \mathcal{D}\right) = p\left(z^{(2)} \middle| z^{(1)}\right) p\left(z^{(1)} \middle| \mathcal{D}\right) \tag{4.32}$$

$$p\left(z^{(2)} \middle| z^{(1)}\right) = \frac{1}{\sqrt{\left|2\pi \hat{G}^{(2)}\right|^{n_2}}} \exp\left(-\frac{1}{2} \sum_{\alpha_1, \alpha_2 \in \mathcal{D}} \widehat{G}_{(2)}^{\alpha_1 \alpha_2} z_{\alpha_1}^{(2)} \cdot z_{\alpha_2}^{(2)}\right) \tag{4.35}$$

- Métrica estocástica da 2ª camada

$$\widehat{G}^{(2)}_{\alpha_1\alpha_2} := C_b^{(2)} + C_W^{(2)} \frac{1}{n_1} \sum_{j=1}^{n_1} \sigma^{(1)}_{j;\alpha_1} \sigma^{(1)}_{j;\alpha_2} \tag{4.36}$$

- Média da métrica da 2ª camada

$$\begin{aligned} G^{(2)}_{\alpha_1\alpha_2} :&= \mathbb{E}\left[\widehat{G}^{(2)}_{\alpha_1\alpha_2}\right] = C_b^{(2)} + C_W^{(2)} \frac{1}{n_1} \sum_{j=1}^{n_1} \mathbb{E}\left[\sigma^{(1)}_{j;\alpha_1} \sigma^{(1)}_{j;\alpha_2}\right] \\ &= C_b^{(2)} + C_W^{(2)} \left\langle \sigma_{\alpha_1} \sigma_{\alpha_2} \right\rangle_{G^{(1)}} \end{aligned} \tag{4.37}$$

- Flutuação da 2ª camada: desvio da média

$$\widehat{\Delta G}^{(2)}_{\alpha_1\alpha_2} := \widehat{G}^{(2)}_{\alpha_1\alpha_2} - G^{(2)}_{\alpha_1\alpha_2} \tag{4.38}$$

- Vértice de 4 pontos: tamanho médio da flutuação

$$\mathbb{E}\left[\widehat{G}^{(2)}_{\alpha_1\alpha_2}\widehat{G}^{(2)}_{\alpha_3\alpha_4}\right] = \frac{1}{n_1}\left(C_W^{(2)}\right)^2\left(\langle\sigma_{\alpha_1}\sigma_{\alpha_2}\sigma_{\alpha_3}\sigma_{\alpha_4}\rangle_{G^{(1)}} - \langle\sigma_{\alpha_1}\sigma_{\alpha_2}\rangle_{G^{(1)}}\langle\sigma_{\alpha_3}\sigma_{\alpha_4}\rangle_{G^{(1)}}\right) =: V^{(2)}_{(\alpha_1\alpha_2)(\alpha_3\alpha_4)} \tag{4.40}$$

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- Pré-ativação na camada $\ell + 1$ é dada por

$$z_{i;\alpha}^{(\ell+1)} = b_i^{(\ell+1)} + \sum_{j=1}^{n_\ell} W_{ij}^{(\ell+1)} \sigma_{j;\alpha}^{(\ell)}$$

com

$$\sigma_{j;\alpha}^{(\ell)} := \sigma\left(z_{i;\alpha}^{(\ell)}\right)$$

$$p\left(z^{(\ell+1)}, z^{(\ell)} \middle| \mathcal{D}\right) = p\left(z^{(\ell+1)} \middle| z^{(\ell)}\right) p\left(z^{(\ell)} \middle| \mathcal{D}\right) \tag{4.67}$$

Distribuição condicional camada $\ell + 1$

$$p\left(z^{(\ell+1)} \middle| z^{(\ell)}\right) = \frac{1}{\sqrt{\left|2\pi \hat{G}^{(\ell+1)}\right|^{n_{\ell+1}}}} \exp\left(-\frac{1}{2} \sum_{\alpha_1, \alpha_2 \in \mathcal{D}} \widehat{G}^{\alpha_1 \alpha_2}_{(\ell+1)} z^{(\ell+1)}_{\alpha_1} \cdot z^{(\ell+1)}_{\alpha_2}\right) \tag{4.69}$$

Métrica estocástica da camada $\ell + 1$

$$\widehat{G}^{(\ell+1)}_{\alpha_1 \alpha_2} := C_b^{(\ell+1)} + C_W^{(\ell+1)} \frac{1}{n_1} \sum_{j=1}^{n_1} \sigma^{(\ell)}_{j;\alpha_1} \sigma^{(\ell)}_{j;\alpha_2} \tag{4.70}$$

Média da métrica estocástica da camada $\ell + 1$

$$G^{(\ell+1)}_{\alpha_1\alpha_2} := \mathbb{E}\left[\widehat{G}^{(\ell+1)}_{\alpha_1\alpha_2}\right] = C^{(\ell+1)}_b + C^{(\ell+1)}_W \frac{1}{n_1}\sum_{j=1}^{n_\ell} \mathbb{E}\left[\sigma^{(\ell)}_{j;\alpha_1}\sigma^{(\ell)}_{j;\alpha_2}\right] \tag{4.72}$$

Essa média governa o correlator de dois pontos

$$\mathbb{E}\left[z^{(\ell+1)}_{i_1;\alpha_1} z^{(\ell+1)}_{i_2;\alpha_2}\right] = \delta_{i_1 i_2} G^{(\ell+1)}_{\alpha_1\alpha_2} \tag{4.73}$$

Flutuação da métrica

$$\widehat{\Delta G}^{(\ell+1)}_{\alpha_1\alpha_2} := \widehat{G}^{(\ell+1)}_{\alpha_1\alpha_2} - G^{(\ell+1)}_{\alpha_1\alpha_2} \tag{4.74}$$

Magnitude da flutuação

$$\frac{1}{n_\ell} V^{(\ell+1)}_{(\alpha_1\alpha_2)(\alpha_3\alpha_4)} := \mathbb{E}\left[\widehat{\Delta G}^{(\ell+1)}_{\alpha_1\alpha_2} \widehat{\Delta G}^{(\ell+1)}_{\alpha_3\alpha_4}\right] \tag{4.76}$$

$$
\begin{aligned}
\mathbb{E}\left[z_{i_1;\alpha_1}^{(\ell+1)} z_{i_2;\alpha_2}^{(\ell+1)} z_{i_3;\alpha_3}^{(\ell+1)} z_{i_4;\alpha_4}^{(\ell+1)}\right]\Big|_C = \\
= \frac{1}{n_\ell}\Big(\delta_{i_1 i_2}\delta_{i_3 i_4} V^{(\ell+1)}_{(\alpha_1\alpha_2)(\alpha_3\alpha_4)} + \delta_{i_1 i_3}\delta_{i_2 i_4} V^{(\ell+1)}_{(\alpha_1\alpha_3)(\alpha_2\alpha_4)} + \\
+ \delta_{i_1 i_4}\delta_{i_2 i_3} V^{(\ell+1)}_{(\alpha_1\alpha_4)(\alpha_2\alpha_3)}\Big) \quad (4.77)
\end{aligned}
$$

Podemos definir a distribuição na camada $\ell$ através da ação

$$p\left(z^{(\ell)}\middle|\mathcal{D}\right) = \frac{e^{-S(z^{(\ell)})}}{Z_\ell} \tag{4.78}$$

com

$$Z_\ell := \int \left[\prod_{i,\alpha} dz_{i;\alpha}^{(\ell)}\right] e^{-S(z^{(\ell)})} \tag{4.79}$$

sendo o termo de normalização.

Nosso modelo para a ação $S$ será

$$S(z^{(\ell)}) := \frac{1}{2}\sum_{\alpha_1,\alpha_2} g_{(\ell)}^{\alpha_1\alpha_2} z_{\alpha_1}\cdot z_{\alpha_2} - \frac{1}{8}\sum_{\alpha_i\in\mathcal{D}}^{1\leq i\leq 4} v_{(\ell)}^{(\alpha_1\alpha_2)(\alpha_3\alpha_4)} z_{\alpha_1}\cdot z_{\alpha_2} z_{\alpha_3}\cdot z_{\alpha_4} + \dots \tag{4.80}$$

- Esse modelo funciona para a camada 1 com

$$g_{(1)}^{\alpha_1\alpha_2} = G_{(1)}^{\alpha_1\alpha_2}, \qquad v_{(1)}^{(\alpha_1\alpha_2)(\alpha_3\alpha_4)} = 0.$$

- Funciona para a camada 2 com

$$g_{(2)}^{\alpha_1\alpha_2} = G_{(2)}^{\alpha_1\alpha_2} + O\left({}^1/_{n_1}\right), \quad v_{(2)}^{(\alpha_1\alpha_2)(\alpha_3\alpha_4)} = \frac{1}{n_1} V_{(2)}^{(\alpha_1\alpha_2)(\alpha_3\alpha_4)} + O\left({}^1/_{n_1^2}\right)$$

Por analogia, temos

$$g_{(\ell)}^{\alpha_1\alpha_2} = G_{(\ell)}^{\alpha_1\alpha_2} + \mathcal{O}(v) \tag{4.81}$$

e

$$v_{(\ell)}^{(\alpha_1\alpha_2)(\alpha_3\alpha_4)} = \frac{1}{n_{\ell-1}} V_{(\ell)}^{(\alpha_1\alpha_2)(\alpha_3\alpha_4)} + \mathcal{O}(v^2) \tag{4.82}$$

no qual o vértice invertido é dado por

$$V_{(\ell)}^{(\alpha_1\alpha_2)(\alpha_3\alpha_4)} := \sum_{\beta_i \in \mathcal{D}}^{1\leq i\leq 4} G_{(\ell)}^{\alpha_1\beta_1} G_{(\ell)}^{\alpha_2\beta_2} G_{(\ell)}^{\alpha_3\beta_3} G_{(\ell)}^{\alpha_4\beta_4} V_{(\alpha_1\alpha_2)(\alpha_3\alpha_4)}^{(\ell)} \tag{4.83}$$

- Simplificamos os cálculos fazendo

$$n_1, n_2, \ldots, n_L \sim n \gg 1 \tag{4.84}$$

**Teorema**

Se as métricas $G^{(\ell)}$ e $V^{(\ell)}$ são de ordem de grandeza constante $O(1)$, então $G^{(\ell+1)}$ e $V^{(\ell+1)}$ também são de ordem de grandeza constante.

Pela equação (4.72), temos que a métrica $G$ da camada $\ell + 1$ é dada por

$$G^{(\ell+1)}_{\alpha_1\alpha_2} = C^{(\ell+1)}_b + C^{(\ell+1)}_W \frac{1}{n_\ell} \sum_{j=1}^{n_\ell} \mathbb{E}\left[\sigma^{(\ell)}_{j;\alpha_1}\sigma^{(\ell)}_{j;\alpha_2}\right]$$

Na sessão anterior, vimos a expressão para a esperança dentro do somatório.

A equação (4.61) calculada na sessão anterior nos dá a terrível fórmula

$$\mathbb{E}\left[\sigma^{(\ell)}_{j;\alpha_1}\sigma^{(\ell)}_{j;\alpha_2}\right] = \langle \sigma_{\alpha_1}\sigma_{\alpha_2}\rangle_{G^{(\ell)}} + \frac{1}{8}\sum_{\beta_i\in\mathcal{D}}^{1\leq i\leq 4} v_{(\ell)}^{(\beta_1\beta_2)(\beta_3\beta_4)}\left(\text{𓀠}\right) + O(v^2) =$$

$$= \langle \sigma_{\alpha_1}\sigma_{\alpha_2}\rangle_{G^{(\ell)}} + \frac{1}{8}\sum_{\beta_i\in\mathcal{D}}^{1\leq i\leq 4} \frac{1}{n_\ell} V_{(\ell)}^{(\beta_1\beta_2)(\beta_3\beta_4)}\left(\text{𓀠}\right) + O(1/n_\ell^2) =$$

em que o hieróglifo 𓀠 representa a exata sensação ao ver essa expressão.

$$\begin{aligned}\text{𓀠} &= \langle \sigma_{\alpha_1}\sigma_{\alpha_2}(z_{\beta_1}z_{\beta_2} - g_{\beta_1\beta_2})(z_{\beta_3}z_{\beta_4} - g_{\beta_3\beta_4})\rangle_g \\ &\quad + 2n\,\langle \sigma_{\alpha_1}\sigma_{\alpha_2}(z_{\beta_1}z_{\beta_2} - g_{\beta_1\beta_2})\rangle_g\, g_{\beta_3\beta_4} - 2\,\langle \sigma_{\alpha_1}\sigma_{\alpha_2}\rangle_g\, g_{\beta_1\beta_3}g_{\beta_2\beta_4}\end{aligned}$$

- tem um termo de ordem $n_\ell$, que se torna de ordem constante quando dividimos por $n_\ell$.
- Esse termo de ordem constante vamos chamar de .

$$\mathbb{E}\left[\sigma^{(\ell)}_{j;\alpha_1}\sigma^{(\ell)}_{j;\alpha_2}\right] = \langle \sigma_{\alpha_1}\sigma_{\alpha_2}\rangle_{G^{(\ell)}} + \quad + O(1/n_\ell) + O(1/n_\ell^2)$$

Assim, a métrica da camada $\ell + 1$ é dada por[1]

$$
\begin{aligned}
G^{(\ell+1)}_{\alpha_1\alpha_2} &= C_b^{(\ell+1)} + C_W^{(\ell+1)} \frac{1}{n_\ell} \sum_{j=1}^{n_\ell} \left[ \langle \sigma_{\alpha_1} \sigma_{\alpha_2} \rangle_{G^{(\ell)}} + O(1/n) \right] \\
&= C_b^{(\ell+1)} + C_W^{(\ell+1)} \textcolor{red}{\langle \sigma_{\alpha_1} \sigma_{\alpha_2} \rangle_{G^{(\ell)}}} + O\left(1/n\right)
\end{aligned}
$$

Pela hipótese de indução, essa expectativa em vermelho é de ordem constante. Segue que a métrica da camada $\ell + 1$ é de ordem constante.

[1] Isso segundo os cara, eu acho que falta uma sujeirinha 🙄 aqui.

Para o vértice de quatro pontos, temos

$$\frac{1}{n_\ell} V^{(\ell+1)}_{(\alpha_1\alpha_2)(\alpha_3\alpha_4)} = \left(\frac{C_W^{(\ell+1)}}{n_\ell}\right)^2 \sum_{j,k=1}^{n_\ell} \Big\{ \mathbb{E}\left[\sigma^{(\ell)}_{j;\alpha_1}\sigma^{(\ell)}_{j;\alpha_2}\sigma^{(\ell)}_{k;\alpha_3}\sigma^{(\ell)}_{k;\alpha_4}\right] - \mathbb{E}\left[\sigma^{(\ell)}_{j;\alpha_1}\sigma^{(\ell)}_{j;\alpha_2}\right]\mathbb{E}\left[\sigma^{(\ell)}_{k;\alpha_3}\sigma^{(\ell)}_{k;\alpha_4}\right]\Big\}$$

- Vamos dar um nome para a expressão entre chaves: $\Xi^{(\ell)}_{j;k}$

Para índices iguais, a equação (4.62) nos dá o seguinte resultado:

$$\mathbb{E}\left[\sigma^{(\ell)}_{j;\alpha_1}\sigma^{(\ell)}_{j;\alpha_2}\sigma^{(\ell)}_{j;\alpha_3}\sigma^{(\ell)}_{j;\alpha_4}\right] - \mathbb{E}\left[\sigma^{(\ell)}_{j;\alpha_1}\sigma^{(\ell)}_{j;\alpha_2}\right]\mathbb{E}\left[\sigma^{(\ell)}_{j;\alpha_3}\sigma^{(\ell)}_{j;\alpha_4}\right] = \\ \left\langle \sigma_{\alpha_1}\sigma_{\alpha_2}\sigma_{\alpha_3}\sigma_{\alpha_4}\right\rangle_{G^{(\ell)}} - \left\langle \sigma_{\alpha_1}\sigma_{\alpha_2}\right\rangle_{G^{(\ell)}}\left\langle \sigma_{\alpha_3}\sigma_{\alpha_4}\right\rangle_{G^{(\ell)}} + O(1/n)$$

Para índices diferentes, a equação (4.63) nos dá o seguinte resultado:

$$\mathbb{E}\left[\sigma_{j;\alpha_1}^{(\ell)}\sigma_{j;\alpha_2}^{(\ell)}\sigma_{k;\alpha_3}^{(\ell)}\sigma_{k;\alpha_4}^{(\ell)}\right] - \mathbb{E}\left[\sigma_{j;\alpha_1}^{(\ell)}\sigma_{j;\alpha_2}^{(\ell)}\right]\mathbb{E}\left[\sigma_{k;\alpha_3}^{(\ell)}\sigma_{k;\alpha_4}^{(\ell)}\right] =$$

$$= \frac{1}{4}\sum_{\beta_i\in\mathcal{D}}^{1\leq i\leq 4} v_{(\ell)}^{(\beta_1\beta_2)(\beta_3\beta_4)}\left(\text{𓄿}\right) + O(v^2) =$$

$$= \frac{1}{4}\sum_{\beta_i\in\mathcal{D}}^{1\leq i\leq 4} \frac{1}{n_\ell} V_{(\ell)}^{(\beta_1\beta_2)(\beta_3\beta_4)}\left(\text{𓄿}\right) + O(1/n_\ell^2)$$

$$\text{𓄿} = \left\langle \sigma_{\alpha_1}\sigma_{\alpha_2}(z_{mj1}z_{\beta_2} - g_{\beta_1\beta_2})\right\rangle_g \left\langle \sigma_{\alpha_3}\sigma_{\alpha_4}(z_{mj3}z_{\beta_4} - g_{\beta_3\beta_4})\right\rangle_g$$

- O termo $\text{𓄿}$ é de ordem constante, pois só contém integrais gaussianas dependentes de $G^{(\ell)}$.

Voltando para nossa equação, separamos a soma de índices iguais e diferentes, e aplicamos $n_\ell = n$.

$$\frac{1}{n} V^{(\ell+1)}_{(\alpha_1\alpha_2)(\alpha_3\alpha_4)} = \left(\frac{C_W^{(\ell+1)}}{n}\right)^2 \left\{ \sum_{j=k}^{n} \Xi^{(\ell)}_{j;k} + \sum_{j\neq k}^{n} \Xi^{(\ell)}_{j;k} \right\} =$$

$$= \frac{{C_W^{(\ell+1)}}^2}{n^2} \left\{ \sum_{j=1}^{n} \langle \sigma_{\alpha_1}\sigma_{\alpha_2}\sigma_{\alpha_3}\sigma_{\alpha_4} \rangle_{G^{(\ell)}} - \langle \sigma_{\alpha_1}\sigma_{\alpha_2} \rangle_{G^{(\ell)}} \langle \sigma_{\alpha_3}\sigma_{\alpha_4} \rangle_{G^{(\ell)}} + O(1/n) \right.$$

$$\left. + \sum_{j\neq k}^{n} \left( \frac{1}{4n} \sum_{\beta_i \in \mathcal{D}}^{1\le i\le 4} V_{(\ell)}^{(\beta_1\beta_2)(\beta_3\beta_4)} \left( \text{𓄿} \right) + O(1/n^2) \right) \right\} =$$

$$
\begin{aligned}
&= \frac{C_W^{(\ell+1)^2}}{n^2} \Bigg\{ n\Big[ \left\langle \sigma_{\alpha_1}\sigma_{\alpha_2}\sigma_{\alpha_3}\sigma_{\alpha_4} \right\rangle_{G^{(\ell)}} - \left\langle \sigma_{\alpha_1}\sigma_{\alpha_2} \right\rangle_{G^{(\ell)}} \left\langle \sigma_{\alpha_3}\sigma_{\alpha_4} \right\rangle_{G^{(\ell)}} + O(1/n) \Big] \\
&\qquad + (n^2 - n) \left[ \frac{1}{4n} \sum_{\beta_i \in \mathcal{D}}^{1 \le i \le 4} V_{(\ell)}^{(\beta_1\beta_2)(\beta_3\beta_4)} \left( \text{𓄿} \right) + O(1/n^2) \right] \Bigg\} = \\
&= C_W^{(\ell+1)^2} \Bigg\{ \frac{1}{n} \Big[ \left\langle \sigma_{\alpha_1}\sigma_{\alpha_2}\sigma_{\alpha_3}\sigma_{\alpha_4} \right\rangle_{G^{(\ell)}} - \left\langle \sigma_{\alpha_1}\sigma_{\alpha_2} \right\rangle_{G^{(\ell)}} \left\langle \sigma_{\alpha_3}\sigma_{\alpha_4} \right\rangle_{G^{(\ell)}} \Big] + O(1/n^2) \\
&\qquad + \frac{1}{4n} \left[ \sum_{\beta_i \in \mathcal{D}}^{1 \le i \le 4} V_{(\ell)}^{(\beta_1\beta_2)(\beta_3\beta_4)} \left( \text{𓄿} \right) \right] + O(1/n^2) \Bigg\}
\end{aligned}
$$

$$
C_W^{(\ell+1)^2} \left\{ \frac{1}{n} \left[ \left\langle \sigma_{\alpha_1}\sigma_{\alpha_2}\sigma_{\alpha_3}\sigma_{\alpha_4} \right\rangle_{G^{(\ell)}} - \left\langle \sigma_{\alpha_1}\sigma_{\alpha_2} \right\rangle_{G^{(\ell)}} \left\langle \sigma_{\alpha_3}\sigma_{\alpha_4} \right\rangle_{G^{(\ell)}} \right] \right.
$$

$$
\left. + \frac{1}{4n} \left[ \sum_{\beta_i \in \mathcal{D}}^{1 \le i \le 4} V_{(\ell)}^{(\beta_1\beta_2)(\beta_3\beta_4)} \left( \quad \right) \right] \right\} + O\left( {}^{1}\!/\!{}_{n^2} \right)
$$

Por hipótese de indução, temos que as partes em azul são de ordem constante.

$$\frac{1}{n}V^{(\ell+1)}_{(\alpha_1\alpha_2)(\alpha_3\alpha_4)} = \frac{{C_W^{(\ell+1)}}^2}{n}\Bigg\{\left[\begin{array}{l}\langle\sigma_{\alpha_1}\sigma_{\alpha_2}\sigma_{\alpha_3}\sigma_{\alpha_4}\rangle_{G^{(\ell)}}\\ -\langle\sigma_{\alpha_1}\sigma_{\alpha_2}\rangle_{G^{(\ell)}}\langle\sigma_{\alpha_3}\sigma_{\alpha_4}\rangle_{G^{(\ell)}}\end{array}\right] + \frac{1}{4}\left[\sum_{\beta_i\in\mathcal{D}}^{1\le i\le 4} V_{(\ell)}^{(\beta_1\beta_2)(\beta_3\beta_4)}\left(\text{𓄿}\right)\right]\Bigg\} + O(1/n^2)$$

Logo, segue que

$$\frac{1}{n}V^{(\ell+1)}_{(\alpha_1\alpha_2)(\alpha_3\alpha_4)} = O(1/n) \tag{4.91}$$

o que completa a indução. ∎

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- Queremos calcular o valor esperado de uma função $F(z_{\mathcal{I};\mathcal{A}})$, com $\mathcal{I} \subset \mathcal{N} =: \{1, \ldots, n_\ell\}$ e $\mathcal{A} \subset \mathcal{D}$.
- Para conjuntos $X \subset Y$, vamos usar a notação $\overline{X}$ para denotar o conjunto complementar de $X$ em $Y$, notadamente $Y \setminus X$.
- Vamos separar as variáveis de integração em dois conjuntos: $\mathcal{I} \times \mathcal{A}$ que é de interesse e o seu complementar $\overline{\mathcal{I} \times \mathcal{A}}$.

$$
\begin{aligned}
\mathbb{E}[F(z_{\mathcal{I};\mathcal{A}})] &= \int \left[ \prod_{(i,\alpha)\in\mathcal{N}\times\mathcal{D}} dz^{(\ell)}_{i;\alpha} \right] F(z_{\mathcal{I};\mathcal{A}}) p(z_{\mathcal{N};\mathcal{D}} \mid \mathcal{D}) = \\
&= \int \left[ \prod_{(i,\alpha)\in\mathcal{I}\times\mathcal{A}} dz^{(\ell)}_{i;\alpha} \prod_{(j,\beta)\in\overline{\mathcal{I}\times\mathcal{A}}} dz^{(\ell)}_{j;\beta} \right] F(z_{\mathcal{I};\mathcal{A}}) p(z_{\mathcal{N};\mathcal{D}} \mid \mathcal{D}) = \\
&= \int \left[ \prod_{(i,\alpha)\in\mathcal{I}\times\mathcal{A}} dz^{(\ell)}_{i;\alpha} \right] F(z_{\mathcal{I};\mathcal{A}}) \int \left[ \prod_{(j,\beta)\in\overline{\mathcal{I}\times\mathcal{A}}} dz^{(\ell)}_{j;\beta} \right] p(z_{\mathcal{N};\mathcal{D}} \mid \mathcal{D}) = \\
&= \int \left[ \prod_{(i,\alpha)\in\mathcal{I}\times\mathcal{A}} dz^{(\ell)}_{i;\alpha} \right] F(z_{\mathcal{I};\mathcal{A}}) p(z_{\mathcal{I};\mathcal{A}} \mid \mathcal{A}) \qquad (4.92)
\end{aligned}
$$

$$p(z_{\mathcal{I};\mathcal{A}} \mid \mathcal{A}) := \int \left[ \prod_{(j,\beta)\in\overline{\mathcal{I}\times\mathcal{A}}} dz^{(\ell)}_{j;\beta} \right] p(z_{\mathcal{N};\mathcal{D}} \mid \mathcal{D}) \tag{4.93}$$

- Ao invés de calcular $\langle \sigma_{\alpha_1} \sigma_{\alpha_2} \rangle_{G^{(\ell)}}$ como uma integral $N_{\mathcal{D}}$-dimensional, podemos calcular como uma integral dupla.
- Da mesma forma, podemos calcular $\langle \sigma_{\alpha_1} \sigma_{\alpha_2} \sigma_{\alpha_3} \sigma_{\alpha_4} \rangle_{G^{(\ell)}}$ como uma integral em no máximo 4 variáveis.
- Para o cálculo de $V^{(\ell+1)}_{(\alpha_1 \alpha_2)(\alpha_3 \alpha_4)}$, podemos usar (4.90) e somar somente sobre os índices $\mathcal{A} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, lembrando de ajustar a métrica inversa $V_{(\ell)}^{(\alpha_1 \alpha_2)(\alpha_3 \alpha_4)}$

$$-\frac{1}{8}\sum_{i,j}^{n_\ell}\sum_{\alpha_i\in\mathcal{D}}^{1\leq i\leq 4} v_{(\ell)}^{(\alpha_1\alpha_2)(\alpha_3\alpha_4)} z_{i;\alpha_1}^{(\ell)} z_{i;\alpha_2}^{(\ell)} z_{j;\alpha_3}^{(\ell)} z_{j;\alpha_4}^{(\ell)} \sim O\left(\frac{n_\ell^2}{n_\ell}\right) = O(n) \tag{4.95}$$

$$\frac{1}{2}\sum_{i}^{n_\ell}\sum_{\alpha_i\in\mathcal{D}}^{1\leq i\leq 2} g_{(\ell)}^{\alpha_1\alpha_2} z_{i;\alpha_1}^{(\ell)} z_{i;\alpha_2}^{(\ell)} \sim O(n) \tag{4.96}$$

- Como utilizamos $m_\ell \ll n_\ell$, os somatórios reduzem drasticamente o número de termos.
- $(4.95) \sim O\left(\frac{m_\ell^2}{n_\ell}\right) = O\left(\frac{1}{n}\right)$
- $(4.96) \sim O(m_\ell) = O(1)$

- Como estamos calculando apenas sobre um subconjunto de neurônios e amostras, temos que ajustar $g$ e $v$ de acordo.
- Para simplificar, vamos considerar apenas um input $x$ e vamos derrubar os índices de amostra.

$$p(z_1^{(\ell)}, \ldots, z_{m_\ell}^{(\ell)}) \propto e^{-S(z_1^{(\ell)}, \ldots, z_{m_\ell}^{(\ell)})}$$
$$= \exp\left(-\frac{g_{(\ell),m_\ell}}{2} \sum_{i=1}^{m_\ell} z_i^{(\ell)} z_i^{(\ell)} + \frac{v_{(\ell),m_\ell}}{8} \sum_{j,k=1}^{m_\ell} z_j^{(\ell)} z_j^{(\ell)} z_k^{(\ell)} z_k^{(\ell)}\right) \tag{4.97}$$

- Vamos integrar sobre os últimos $n_\ell - m_\ell$ neurônios, ignorando as constantes de normalização.

$$e^{-S(z_1^{(\ell)},\ldots,z_{m_\ell}^{(\ell)})} \propto p(z_1^{(\ell)},\ldots,z_{m_\ell}^{(\ell)}) = \int \Big[\prod_{i=m_\ell+1}^{n_\ell} dz_i^{(\ell)}\Big] p(z_1^{(\ell)},\ldots,z_{n_\ell}^{(\ell)})$$

$$\propto \int \Big[\prod_{i=m_\ell+1}^{n_\ell} dz_i^{(\ell)}\Big] \exp\left(-\frac{g_{(\ell),n_\ell}}{2}\sum_{i=1}^{n_\ell} z_i^{(\ell)} z_i^{(\ell)} + \frac{v_{(\ell)}}{8}\sum_{j,k=1}^{n_\ell} z_j^{(\ell)} z_j^{(\ell)} z_k^{(\ell)} z_k^{(\ell)}\right)$$

- Para simplificar, vamos sumir com os índices $\ell$.
- Vamos modificar a notação

$$\int \left[\prod_{i=a}^{b} dz_i\right] = \int_{i=a}^{b} dz_i$$

- Vamos lembrar que $\exp(a+b) = \exp(a)\exp(b)$.
- Vamos separar o somatório duplo

$$\sum_{j,k=1}^{n} = \sum_{j,k=1}^{m} + \sum_{j=1}^{m}\sum_{k=m+1}^{n} + \sum_{j=m+1}^{n}\sum_{k=1}^{m} + \sum_{j,k=m+1}^{n}$$

$$p(z_1^{(\ell)}, \ldots, z_{m_\ell}^{(\ell)}) \propto \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{n} z_i^2 + \frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2 - \frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right] \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right] \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$p(z_1^{(\ell)}, \ldots, z_{m_\ell}^{(\ell)}) \propto \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{n} z_i^2 + \frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2 - \frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right] \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right] \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

MTM

$$p(z_1^{(\ell)}, \ldots, z_{m_\ell}^{(\ell)}) \propto \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{n} z_i^2 + \frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2 - \frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right] \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right] \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$p(z_1^{(\ell)},\ldots,z_{m_\ell}^{(\ell)}) \propto \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{n} z_i^2 + \frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2 - \frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right] \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right] \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

MTM

$$
p(z_1^{(\ell)}, \ldots, z_{m_\ell}^{(\ell)}) \propto \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{n} z_i^2 + \frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =
$$

$$
= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =
$$

$$
= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2 - \frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =
$$

$$
= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right] \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =
$$

$$
= \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right] \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =
$$

$$p(z_1^{(\ell)},\ldots,z_{m_\ell}^{(\ell)}) \propto \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{n} z_i^2 + \frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2 - \frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right] \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right] \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$p(z_1^{(\ell)}, \ldots, z_{m_\ell}^{(\ell)}) \propto \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{n} z_i^2 + \frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2 - \frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right] \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right] \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$p(z_1^{(\ell)}, \ldots, z_{m_\ell}^{(\ell)}) \propto \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{n} z_i^2 + \frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2 - \frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right] \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right] \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$p(z_1^{(\ell)}, \ldots, z_{m_\ell}^{(\ell)}) \propto \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{n} z_i^2 + \frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2 - \frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right] \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$= \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right] \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{n} z_j^2 z_k^2\right] =$$

$$
\begin{aligned}
&= \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right]\int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right]\exp\left[\frac{v}{8}\sum_{j,k=1}^{m} z_j^2 z_k^2 + \right.\\
&\left. +\frac{v}{8}\sum_{j=1}^{m}\sum_{k=m+1}^{n} z_j^2 z_k^2 + \frac{v}{8}\sum_{j=m+1}^{n}\sum_{k=1}^{n} z_j^2 z_k^2 + \frac{v}{8}\sum_{j,k=m+1}^{n} z_j^2 z_k^2\right] = \\
&= \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right]\exp\left[\frac{v}{8}\sum_{j,k=1}^{m} z_j^2 z_k^2\right]\times \\
&\times \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right]\exp\left[\frac{v}{8}\left(2\sum_{j=1}^{m}\sum_{k=m+1}^{n} z_j^2 z_k^2 + \sum_{j,k=m+1}^{n} z_j^2 z_k^2\right)\right]
\end{aligned}
$$

$$= \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right]\int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right]\exp\left[\frac{v}{8}\sum_{j,k=1}^{m} z_j^2 z_k^2 + \right.$$

$$\left. + \frac{v}{8}\sum_{j=1}^{m}\sum_{k=m+1}^{n} z_j^2 z_k^2 + \frac{v}{8}\sum_{j=m+1}^{n}\sum_{k=1}^{n} z_j^2 z_k^2 + \frac{v}{8}\sum_{j,k=m+1}^{n} z_j^2 z_k^2\right] =$$

$$= \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right]\exp\left[\frac{v}{8}\sum_{j,k=1}^{m} z_j^2 z_k^2\right]\times$$

$$\times\int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right]\exp\left[\frac{v}{8}\left(2\sum_{j=1}^{m}\sum_{k=m+1}^{n} z_j^2 z_k^2 + \sum_{j,k=m+1}^{n} z_j^2 z_k^2\right)\right]$$

$$= \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right]\int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right]\exp\left[\frac{v}{8}\sum_{j,k=1}^{m} z_j^2 z_k^2 + \right.$$

$$\left. + \frac{v}{8}\sum_{j=1}^{m}\sum_{k=m+1}^{n} z_j^2 z_k^2 + \frac{v}{8}\sum_{j=m+1}^{n}\sum_{k=1}^{n} z_j^2 z_k^2 + \frac{v}{8}\sum_{j,k=m+1}^{n} z_j^2 z_k^2\right] =$$

$$= \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right]\exp\boxed{\left[\frac{v}{8}\sum_{j,k=1}^{m} z_j^2 z_k^2\right]}\times$$

$$\times\int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right]\exp\left[\frac{v}{8}\left(2\sum_{j=1}^{m}\sum_{k=m+1}^{n} z_j^2 z_k^2 + \sum_{j,k=m+1}^{n} z_j^2 z_k^2\right)\right]$$

$$= \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right]\int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right]\exp\left[\frac{v}{8}\sum_{j,k=1}^{m} z_j^2 z_k^2 +\right.$$

$$\left. +\frac{v}{8}\sum_{j=1}^{m}\sum_{k=m+1}^{n} z_j^2 z_k^2 + \frac{v}{8}\sum_{j=m+1}^{n}\sum_{k=1}^{n} z_j^2 z_k^2 + \frac{v}{8}\sum_{j,k=m+1}^{n} z_j^2 z_k^2\right] =$$

$$= \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{m} z_j^2 z_k^2\right] \times$$

$$\times \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right]\exp\left[\frac{v}{8}\left(2\sum_{j=1}^{m}\sum_{k=m+1}^{n} z_j^2 z_k^2 + \sum_{j,k=m+1}^{n} z_j^2 z_k^2\right)\right]$$

$$= \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right] \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{m} z_j^2 z_k^2 + \right.$$

$$\left. + \frac{v}{8}\sum_{j=1}^{m}\sum_{k=m+1}^{n} z_j^2 z_k^2 + \frac{v}{8}\sum_{j=m+1}^{n}\sum_{k=1}^{n} z_j^2 z_k^2 + \frac{v}{8}\sum_{j,k=m+1}^{n} z_j^2 z_k^2\right] =$$

$$= \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2\right] \exp\left[\frac{v}{8}\sum_{j,k=1}^{m} z_j^2 z_k^2\right] \times$$

$$\times \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \exp\left[\frac{v}{8}\left(2\sum_{j=1}^{m}\sum_{k=m+1}^{n} z_j^2 z_k^2 + \sum_{j,k=m+1}^{n} z_j^2 z_k^2\right)\right]$$

- Agora usamos $\exp(v\Sigma) \approx 1 + v\Sigma + O(v^2)$ para trocar a expoencial

$$\exp\left[\frac{v}{8}\left(2\sum_{j=1}^{m}\sum_{k=m+1}^{n} z_j^2 z_k^2 + \sum_{j,k=m+1}^{n} z_j^2 z_k^2\right)\right] \approx$$

$$\approx 1 + \frac{2v}{8}\sum_{j=1}^{m}\sum_{k=m+1}^{n} z_j^2 z_k^2 + \frac{v}{8}\sum_{j,k=m+1}^{n} z_j^2 z_k^2 + O(v^2)$$

$$p(z_1^{(\ell)}, \ldots, z_{m_\ell}^{(\ell)}) \propto \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2 + \frac{v}{8}\sum_{j,k=1}^{m} z_j^2 z_k^2 + \right] \times$$

$$\times \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right]\left(1 + \underbrace{\frac{2v}{8}\sum_{j=1}^{m}\sum_{k=m+1}^{n} z_j^2 z_k^2}_{(II)} + \underbrace{\frac{v}{8}\sum_{j,k=m+1}^{n} z_j^2 z_k^2}_{(III)} + O(v^2)\right)$$

Vamos aplicar a propriedade distribuitiva e resolver as integrais $(II)$ e $(III)$ separadamente.

$$(II) = \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \frac{2v}{8}\sum_{j=1}^{m}\sum_{k=m+1}^{n} z_j^2 z_k^2 =$$

$$\frac{v}{4}\int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right]\left[\sum_{j=1}^{m} z_j^2\right]\left[\sum_{k=m+1}^{n} z_k^2\right]$$

$$\frac{v}{4}\left[\sum_{j=1}^{m} z_j^2\right]\underbrace{\int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right]\left[\sum_{k=m+1}^{n} z_k^2\right]}_{(ii)}$$

Para resolver $(ii)$, vamos usar a linearidade da exponencial:

$$(ii) = \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right]\left[\sum_{k=m+1}^{n} z_k^2\right] =$$

$$\sum_{k=m+1}^{n}\int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] z_k^2 =: \sum_{k=m+1}^{n} I_k$$

Vamos usar a propriedade da exponencial e o teorema de Fubini para calcular $I_k$.

$$I_k = \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] z_k^2 = \int_{i=m+1}^{n} dz_i \prod_{i=m+1}^{n} \exp\left[-\frac{g}{2} z_i^2\right] z_k^2 =$$

$$= \int dz_k \;\; z_k^2 \exp\left[-\frac{g}{2} z_k^2\right] \prod_{\substack{i=m+1 \\ i\neq k}}^{n} \int dz_i \exp\left[-\frac{g}{2} z_i^2\right] =$$

$$\frac{\sqrt{\pi}}{2g^{3/2}} \cdot \text{Constante} = \frac{1}{g}\frac{C_1}{\sqrt{g}}$$

Voltando para $(ii)$, temos que

$$(ii) = \sum_{k=m+1}^{n} I_k = (n-m)I_k = \frac{(n-m)}{g}\frac{C_1}{\sqrt{g}}$$

Voltando para $(II)$, temos que

$$(II) = \frac{v}{4}\left[\sum_{j=1}^{m} z_j^2\right]\left[\sum_{k=m+1}^{n} I_k\right] = \frac{v}{4}\left[\sum_{j=1}^{m} z_j^2\right]\left[\frac{(n-m)}{g}\frac{C_1}{\sqrt{g}}\right] =$$

$$= \frac{(n-m)}{4}\frac{v}{h}\frac{C_1}{\sqrt{g}}\left[\sum_{j=1}^{m} z_j^2\right]$$

Vamos calcular $(III)$ agora, separando o somatório de índices iguais e diferentes.

$$
\begin{aligned}
(III) &= \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \frac{v}{8}\sum_{j,k=m+1}^{n} z_j^2 z_k^2 = \\
&= \frac{v}{8}\int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] \left[\sum_{\nu=m+1}^{n} z_\nu^4 + \sum_{\substack{\mu,\nu=m+1\\ \mu\neq\nu}}^{n} z_\nu^2 z_\mu^2\right] \\
&= \frac{v}{8}\left[\sum_{\nu=m+1}^{n} \underbrace{\int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] z_\nu^4}_{(iii.1)} + \right. \\
&\qquad \left. \sum_{\substack{\mu,\nu=m+1\\ \mu\neq\nu}}^{n} \underbrace{\int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] z_\nu^2 z_\mu^2}_{(iii.2)}\right]
\end{aligned}
$$

Para resolver $(iii.1)$, vamos usar a propriedade da exponencial e o teorema de Fubini:

$$(iii.1) = \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] z_\nu^4 = \int_{i=m+1}^{n} dz_i \prod_{i=m+1}^{n} \exp\left[-\frac{g}{2} z_i^2\right] z_\nu^4 =$$

$$= \int dz_\nu z_\nu^4 \exp\left[-\frac{g}{2} z_\nu^2\right] \prod_{\substack{i=m+1 \\ i\neq\nu}}^{n} \int dz_i \exp\left[-\frac{g}{2} z_i^2\right] =$$

$$= \frac{3\sqrt{\pi}}{4g^{5/2}} \cdot \text{Constante} = \frac{1}{g^2}\frac{C_2}{\sqrt{g}}$$

Vamos calcular $(iii.2)$ agora.

Para resolver $(iii.2)$, vamos usar a propriedade da exponencial e o teorema de Fubini:

$$(iii.2) = \int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right] z_\nu^2 z_\mu^2 = \int_{i=m+1}^{n} dz_i \prod_{i=m+1}^{n} \exp\left[-\frac{g}{2} z_i^2\right] z_\nu^2 z$$

$$= \int dz_\nu z_\nu^2 \exp\left[-\frac{g}{2} z_\nu^2\right] \int dz_\mu z_\mu^2 \exp\left[-\frac{g}{2} z_\mu^2\right] \prod_{\substack{i=m+1 \\ i\neq \nu,\mu}}^{n} \int dz_i \exp\left[-\frac{g}{2} z_i^2\right] =$$

$$= \frac{\sqrt{\pi}}{2g^{3/2}} \frac{\sqrt{\pi}}{2g^{3/2}} \cdot \text{Constante} = \frac{1}{g^3} C_3$$

Vamos voltar para $(III)$ agora.

Voltando para $(III)$, temos que

$$
\begin{aligned}
(III) &= \frac{v}{8}\left[\sum_{\nu=m+1}^{n}(iii.1)\right] + \frac{v}{8}\left[\sum_{\substack{\mu,\nu=m+1\\ \mu\neq\nu}}^{n}(iii.2)\right] = \\
&= \frac{v}{8}\left[\sum_{\nu=m+1}^{n}\frac{1}{g^2}\frac{C_2}{\sqrt{g}}\right] + \frac{v}{8}\left[\sum_{\substack{\mu,\nu=m+1\\ \mu\neq\nu}}^{n}\frac{1}{g^3}C_3\right] = \\
&= \frac{v}{8}\left[\frac{(n-m)}{g^2}\frac{C_2}{\sqrt{g}} + \frac{(n-m)^2-(n-m)}{g^3}C_3\right]
\end{aligned}
$$

Agora, vamos juntar os resultados de $(II)$ e $(III)$.

$$
\exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2 + \frac{v}{8}\sum_{j,k=1}^{m} z_j^2 z_k^2 + \right]\times
$$

$$
\times\int_{i=m+1}^{n} dz_i \exp\left[-\frac{g}{2}\sum_{i=m+1}^{n} z_i^2\right]\left(1+\frac{2v}{8}\sum_{j=1}^{m}\sum_{k=m+1}^{n} z_j^2 z_k^2 + \frac{v}{8}\sum_{j,k=m+1}^{n} z_j^2 z_k^2 + O(v^2)\right)
$$

$$
= \exp\left[-\frac{g}{2}\sum_{i=1}^{m} z_i^2 + \frac{v}{8}\sum_{j,k=1}^{m} z_j^2 z_k^2 + \right]\times
$$

$$
\times\left\{1+\frac{(n-m)}{4}\frac{v}{h}\frac{C_1}{\sqrt{g}}\left[\sum_{j=1}^{m} z_j^2\right]+\frac{v}{8}\left[\frac{(n-m)}{g^2}\frac{C_2}{\sqrt{g}}+\frac{(n-m)^2-(n-m)}{g^3}C_3\right]+O(v^2\right.
$$

$$
\times\left\{1+\frac{(n-m)}{4}\frac{v}{g}\left[\sum_{j=1}^{m} z_j^2\right]+\frac{v}{8g^2}\left[(n-m)^2+2(n-m)\right]+O(v^2)\right\}
$$

Utilizando umas aproximações bem aproximadas, podemos escrever

$$
\begin{aligned}
&\exp\left[-\frac{g_{(\ell),n_\ell}}{2}\sum_{i=1}^{m} z_i^2 + \frac{v_{(\ell)}}{8}\sum_{j,k=1}^{m} z_j^2 z_k^2\right]\times \\
&\times\left\{1+\frac{(n_\ell - m_\ell)}{4}\frac{v_{(\ell)}}{g_{(\ell),n_\ell}}\left[\sum_{j=1}^{m} z_j^2\right]+\frac{v_{(\ell)}}{8g^2_{(\ell),n_\ell}}\left[(n-m)^2+2(n-m)\right]+O(v^2)\right\} \\
&\approx \exp\left[-\frac{g_{(\ell),n_\ell}}{2}\sum_{i=1}^{m} z_i^2+\dots\right]\times\left\{1+\frac{(n_\ell - m_\ell)}{4}\frac{v_{(\ell)}}{g_{(\ell),n_\ell}}\left[\sum_{j=1}^{m} z_j^2\right]+\dots\right\} \\
&\approx \exp\left[-\frac{g_{(\ell),n_\ell}}{2}\sum_{i=1}^{m} z_i^2\dots\right]\exp\left[\frac{(n_\ell - m_\ell)}{4}\frac{v_{(\ell)}}{g_{(\ell),n_\ell}}\left[\sum_{j=1}^{m} z_j^2\right]\right] \\
&= \exp\left[\left(-\frac{g_{(\ell),n_\ell}}{2}+\frac{(n_\ell - m_\ell)}{4}\frac{v_{(\ell)}}{g_{(\ell),n_\ell}}\right)\sum_{j=1}^{m} z_j^2\right]
\end{aligned}
$$

Utilizando umas aproximações bem aproximadas, podemos escrever

$$
\begin{aligned}
&\exp\left[-\frac{g_{(\ell),n_\ell}}{2}\sum_{i=1}^{m} z_i^2 + \frac{v_{(\ell)}}{8}\sum_{j,k=1}^{m} z_j^2 z_k^2\right] \times \\
&\times\left\{1 + \frac{(n_\ell - m_\ell)}{4}\frac{v_{(\ell)}}{g_{(\ell),n_\ell}}\left[\sum_{j=1}^{m} z_j^2\right] + \frac{v_{(\ell)}}{8g^2_{(\ell),n_\ell}}\left[(n-m)^2 + 2(n-m)\right] + O(v^2)\right\} \\
&\approx \exp\left[-\frac{g_{(\ell),n_\ell}}{2}\sum_{i=1}^{m} z_i^2 + \dots\right] \times \left\{1 + \frac{(n_\ell - m_\ell)}{4}\frac{v_{(\ell)}}{g_{(\ell),n_\ell}}\left[\sum_{j=1}^{m} z_j^2\right] + \dots\right\} \\
&\approx \exp\left[-\frac{g_{(\ell),n_\ell}}{2}\sum_{i=1}^{m} z_i^2 \dots\right] \exp\left[\frac{(n_\ell - m_\ell)}{4}\frac{v_{(\ell)}}{g_{(\ell),n_\ell}}\left[\sum_{j=1}^{m} z_j^2\right]\right] \\
&= \exp\left[\left(-\frac{g_{(\ell),n_\ell}}{2} + \frac{(n_\ell - m_\ell)}{4}\frac{v_{(\ell)}}{g_{(\ell),n_\ell}}\right)\sum_{j=1}^{m} z_j^2\right]
\end{aligned}
$$

Comparando o termo da expressão acima que multiplica $\sum_{j=1}^{m} z_j^2$ com o mesmo termo equação (4.97), temos que

$$-\frac{g_{(\ell),n_\ell}}{2} + \frac{(n_\ell - m_\ell)}{4} \frac{v_{(\ell)}}{g_{(\ell),n_\ell}} = -\frac{g_{(\ell),m_\ell}}{2}$$

o que nos leva a

$$g_{(\ell),m_\ell} = g_{(\ell),n_\ell} - \frac{(n_\ell - m_\ell)}{4} \frac{v_{(\ell)}}{g_{(\ell),n_\ell}} + O(v^2) \qquad (4.100)$$

Outra maneira de obter essa relação é utilizando a equação (4.47).

Especializando a equação (4.47) para $i_1 = i_2$ e somando sombre $m_\ell$ índices, temos que

$$\mathbb{E}[z_{i_1;\alpha_1} z_{i_2;\alpha_2}] = \delta_{i_1 i_2} \times$$
$$\times \left[ g_{\alpha_1\alpha_2} + \frac{1}{2} \sum_{\beta_i \in \mathcal{D}}^{1\leq i \leq 4} v^{(\beta_1\beta_2)(\beta_3\beta_4)} (n g_{\alpha_1\beta_1} g_{\alpha_2\beta_2} g_{\beta_3\beta_4} + 2 g_{\alpha_1\beta_1} g_{\alpha_2\beta_3} g_{\beta_2\beta_4}) \right] \tag{4.47}$$

$$\mathbb{E}[z_i z_i] = g^{(\ell),m_\ell} + \frac{1}{2} v^{(\ell)} (m_\ell + 2)(g^{(\ell),m_\ell})^3 = \delta_{ii} G^{(\ell)} \Leftrightarrow$$
$$G^{(\ell)} = \frac{1}{g_{(\ell),m_\ell}} + \frac{v^{(\ell)}(m_\ell + 2)}{2(g^{(\ell),m_\ell})^3} + O\left(v^2\right) \tag{4.101}$$

Isolando $g_{(\ell),m_\ell}$ temos a equação (4.102):

$$\frac{1}{g_{(\ell),m_\ell}} = G^{(\ell)} - \frac{(m_\ell + 2)}{2} \frac{V^{(\ell)}}{n_{\ell-1}(G^{(\ell)})^4} + O\left(\frac{1}{n^2}\right) \qquad (4.102)$$

na qual utilizamos relação

$$v_{(\ell)} = \frac{V^{(\ell)}}{n_{\ell-1}(G^{(\ell)})^4} + O\left(\frac{1}{n^2}\right) \qquad (4.103)$$

obtida de

$$g_{(\ell)} = G_{(\ell)} + O(v) \ (4.82) \qquad V_{(\ell)} = V_{(\ell)}/n_{\ell-1} + O(v^2) \ (4.83)$$


That's all Folks!