Considere o problema

$$\begin{cases} u_t = a u_{xx} - b u_x \\ u(x,0) = \varphi(x) & 0 \leq x \leq L \\ u(0,t) = f(t) & 0 \leq t < T_f \\ u(L,t) = g(t) & 0 \leq t < T_f \end{cases}$$

Use

- método explícito no tempo
- diferenças finitas no espaço

para discretizar o problema e aplique o critério de Von Neumann para estudar estabilidade

$u_t \longrightarrow \dfrac{U_{i,j+1} - U_{i,j}}{\Delta t}$ (explícito tempo)

$u_{xx} \longrightarrow \dfrac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{(\Delta x)^2}$ (centrado espaço)

$u_x \longrightarrow \dfrac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}$

Substituindo na EDP temos

$$\frac{U_{i,j+1} - U_{ij}}{\Delta t} = a \cdot \frac{U_{i-1,j} - 2U_{ij} + U_{i+1,j}}{(\Delta x)^2} - b \cdot \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}$$

$$U_{i,j+1} - U_{ij} = \underbrace{\frac{a\Delta t}{(\Delta x)^2}}_{=\sigma}(U_{i-1,j} - 2U_{ij} + U_{i+1,j}) - \underbrace{\frac{b\,\Delta t}{2\Delta x}}_{=\mu}(U_{i+1,j} - U_{i-1,j})$$

$$U_{i,j+1} - U_{ij} = \sigma(U_{i-1,j} - 2U_{ij} + U_{i+1,j}) - \mu(U_{i+1,j} - U_{i-1,j})$$

Fazendo $U_{ij} = \xi^j e^{i\beta\Delta x i}$, substituindo na equação acima e dividindo tudo por $\xi^j e^{i\beta\Delta x i}$ temos

$$\xi - 1 = \sigma\left(e^{-i\Delta x\beta} - 2 + e^{i\Delta x\beta}\right) - \mu\left(e^{i\Delta x\beta} - e^{-i\Delta x\beta}\right)$$

Usando identidades trigonométricas temos

$$\xi - 1 = \sigma\left(2\cos(\Delta x\beta) - 2\right) - \mu\left(2i\,\mathrm{sen}(\Delta x\beta)\right)$$

$$\xi = \underbrace{1 + \sigma\left(2\cos(\Delta x\beta) - 2\right)}_{\mathrm{Re}\,\xi} + i\underbrace{\left(-2\mu\,\mathrm{sen}(\Delta x\beta)\right)}_{\mathrm{Im}\,\xi}$$

Denote por $\omega = \Delta x\beta$ e veja que

$$2\cos(\omega) - 2 = -4\,\mathrm{sen}^2(\omega/2)$$

Logo

$$\xi = 1 - 4\sigma\,\mathrm{sen}^2(\omega/2) + i\left(-2\mu\,\mathrm{sen}(\omega)\right)$$

Tomando o módulo de $|\xi|$ temos

$$|\xi|^2 = \left(1 - 4\sigma\,\mathrm{sen}^2(\omega/2)\right)^2 + \left(-2\mu\,\mathrm{sen}(\omega)\right)^2$$

$$= 1 - 8\sigma\,\mathrm{sen}^2(\omega/2) + 16\sigma^2\,\mathrm{sen}^4(\omega/2) + 4\mu\,\mathrm{sen}^2(\omega)$$

$$= 1 - 8\sigma\underbrace{\mathrm{sen}^2(\omega/2)}_{z} + 16\sigma^2\underbrace{\mathrm{sen}^4(\omega/2)}_{z^2} + 16\mu\underbrace{\mathrm{sen}^2(\omega/2)}_{z}\underbrace{\cos^2(\omega/2)}_{1-z}$$

$$= 1 - 8\sigma z + 16\sigma^2 z^2 + 16\mu z(1-z) \qquad (*)$$

Para que $|\xi| \leq 1$ devemos ter $(*) \leq 1$.

$$1 - 8\sigma z + 16\sigma^2 z^2 + 16\mu z(1-z) \leq 1$$

$$-8\sigma z + 16\sigma^2 z^2 + 16\mu z(1-z) \leq 1$$

$$-\sigma z + 2\sigma^2 z^2 + 2\mu z(1-z) \leq 0$$

$$\sigma z(2\sigma z - 1) + 2\mu z(1-z) \leq 0$$

Note que $z = \text{sen}^2(\omega/2) \in [0,1]$. Se $z = 0$ a desigualdade vale. Se $z > 0$, dividimos tudo por z e obtemos

$$\sigma(2\sigma z - 1) + 2\mu(1-z) \leq 0$$

$$2\sigma^2 z - \sigma + 2\mu - 2\mu z \leq 0$$

$$z(2\sigma^2 - 2\mu) + 2\mu - \sigma \leq 0$$

$$\text{sen}^2\left(\frac{\beta\Delta x}{2}\right) \leq \frac{a\Delta t - b\Delta t\Delta x}{2a^2(\Delta t)^2 - b\Delta t\Delta x}$$

$$z \leq \frac{\sigma - 2\mu}{2\sigma^2 - 2\mu} = \frac{1}{2}\,\frac{\sigma - 2\mu}{\sigma^2 - \mu}$$

$$\frac{\frac{a\Delta t}{(\Delta x)^2} - b\frac{\Delta t}{\Delta x}}{\frac{2a^2(\Delta t)^2}{(\Delta x)^2} - b\frac{\Delta t}{\Delta x}}$$

Precisamos então garantir que

(1) $$\frac{\sigma - 2\mu}{\sigma^2 - \mu} \leq 2$$

supondo denominador positivo

$$\sigma - 2\mu \leq 2\sigma^2 - 2\mu$$

$$\sigma - 2\sigma^2 \leq 0$$

$$\sigma(1 - 2\sigma) \leq 0$$

$$1 - 2\sigma \leq 0$$

$$\frac{1}{2} \leq \sigma$$

(2) $$\frac{\sigma - 2\mu}{\sigma^2 - \mu} \leq 2$$

supondo denominador negativo

$$\sigma - 2\mu \geq 2\sigma^2 - 2\mu$$

$$\sigma - 2\sigma^2 \geq 0$$

$$\sigma(1 - 2\sigma) \geq 0$$

$$1 - 2\sigma \geq 0$$

$$\sigma \leq \frac{1}{2}$$

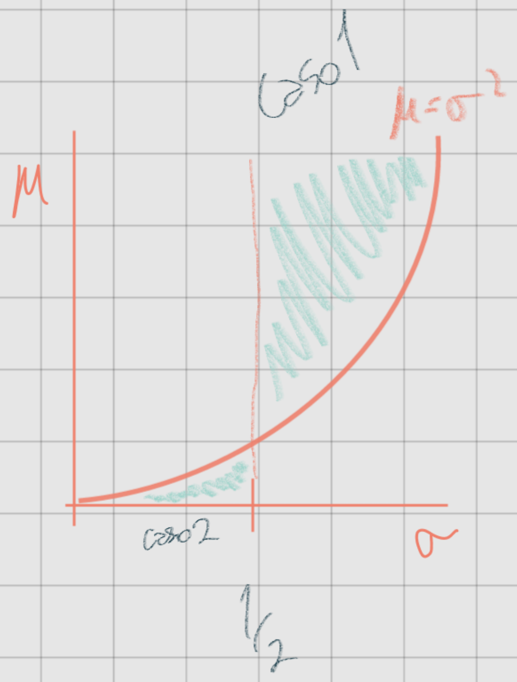
Podemos fazer então

$$\frac{\sigma - 2\mu}{\sigma^2 - \mu} = 2$$

$$\sigma - 2\mu = 2\sigma^2 - 2\mu$$

$$\sigma = 2\sigma^2$$

$$\sigma = \tfrac{1}{2}$$



Caso (1) nos dá $\sigma^2 \geq \mu$ e $\sigma \geq \frac{1}{2}$

Caso (2) nos dá $\sigma^2 \leq \mu$ e $\sigma \leq \frac{1}{2}$