Independent Model with Modified Oscillator

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1 General potential and motivation

From experiment we know the nuclear matter inside a nucleus follows a Woods-Saxon distribution so the Woods-Saxon potential is probably the best mean field for the nuclear problem. However, the harmonic oscillator potential follows all the neccesary properties for a nuclear mean field and, with some additional terms, it turns out to be pretty similar to the Woods-Saxon potential with the advantage of being easier to handle. Indeed, without the coulombian repulsive term, the Schrödinguer equation is analytically solvable, therefore the eigenfunctions and eigenvalues are well known. That is not the case of the Woods-Saxon potential which can not be solved analytically.

The general form of this potential (without a coulombian term) is:

$$V(\vec{r}) = \frac{m}{2}\left[\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right] + C\vec{l}\vec{s} + D\left(\vec{l}^2 - \langle\vec{l}^2\rangle\right) \quad . \tag{1.1}$$

where C and D are constants.

2 Spherical Symmetry

Assuming spherical symmetry means $\omega_0 \equiv \omega_x = \omega_y = \omega_z$. This assuption doesn't mean that the expected value of the quadrupole electric momentum is null because the wave function is not neccesarialy spherically symmetric.

In this case the potential can be written as:

$$V(r) = \frac{m}{2}\left(\omega_0^2 r^2\right) + C\vec{l}\vec{s} + D\left(\vec{l}^2 - \langle\vec{l}^2\rangle_N\right) \quad . \tag{2.1}$$

The single particle eigenfunction are separable in terms of the spatial coordinates r, θ and ϕ. The angular part is equal for all the potentials (spherical harmonics) and the radial part follows the radial equation:

$$\left[\frac{p_r^2}{2} + \frac{\hbar^2 l(l+1)}{2\mu r^2} + \frac{m}{2}(\omega_0^2 r^2) + C\vec{l}\vec{s} + D\left(l(l+1) - <\vec{l}^2>_N\right)\right] R_{nl}(r) = E R_{nl}(r) \quad . \tag{2.2}$$

where:

$$\langle\vec{l}^2\rangle_N = \frac{N(N+3)}{2} \tag{2.3}$$

where N is:

$$N = 2(n-1) + l \tag{2.4}$$

As we can see for this radial part there is a kinectic term $\frac{p_r^2}{2}$ which contains the radial derivatives on r and the part with $\frac{1}{r^2}$ can be consider as a effective potential. Adding the $D\vec{l}^2$ term to the harmonic oscillator provides an approximation for the Woods-Saxon potential. For very low values of r the $l(l+1)/r^2$ dominates, therefore at this region the potentials are similar. For closer values of r to the radius value the adding part in the harmonic oscillator makes both potentials quite similar. The radial eigenfunctions are:

$$R_{nl}(r) = N_{nl} r^l e^{-\frac{\alpha^2 r^2}{2}} M(-n+1, l+\frac{3}{2}, \alpha^2 r^2) \ . \tag{2.5}$$

Using spherical harmonics we have an eigenfunction for our total hamiltonian:

$$\langle \vec{r}, \sigma | njlm_j \rangle = R_{nl}(r) Y_{jlsm_j}(\theta, \phi, \sigma) \ , \tag{2.6}$$

where $Y_{jlsm_j}(\theta, \phi, \sigma)$ are the generalized spherical harmonics:

$$Y_{jlsm_j}(\theta, \phi, \sigma) = \sum_{m_l} \langle lm_l sm_s | jm_j \rangle Y_{lm_l}(\theta, \phi) X_{1/2m_s}(\sigma) \ . \tag{2.7}$$

These are the exact eigenfunctions for the MSO problem. With energy eigenvalues (see [Appendix A]):

$$E_{Nlj} = \hbar\omega_0 \left[N + \frac{3}{2} - \mu \left(l(l+1) - \frac{N(N+3)}{2} \right) - \kappa \begin{cases} l & \text{if } j = l + \frac{1}{2} \\ -(l+1) & \text{if } j = l - \frac{1}{2} \end{cases} \right]. \tag{2.8}$$

The dominating term is the part for the simple harmonic oscillator ($\hbar\omega_0(N + \frac{3}{2})$). Therefore the subshells with the same N are closer.

2.1 Coulombian repulsion

In order to construct a realistic potential it's neccesary to add a coulombian term that takes into account the fact that any proton feels a charged distribution constituted by all the other protons except itself. Assuming a spherical distribution of protons this potential will be spherically symmetric as well. But including this term in the mean hamiltonian for protons the known eigenfunctions (2.5) are not solution for the radial equation anymore. In order to determine the new eigenfunctions and the new energies eigenvalues we construct the hamiltonian matrix in the old basis (2.6), and make the diagonalization. The blocks are constituted with different values of n but the same values of l and j beacuse the lack of dependence on ϕ and θ within the potential. The new radial eigenfunction are a linear combination of the old ones. The size of the blocks to consider is abitrary, with more terms bettter is the approximation. The exact solution would be obtain considerating the entire basis which is infinity.

Let's call this coulombian term as $V_C(r)$. With this new term the old $R_{nl}(r)$ are no longer solutions for the radial equation (2.2), that is, they are not eigenfunctions for the

protonic hamiltonian. In other words $\langle n'l'j'm'_j|V_C|nljm_j\rangle$ is not diagonal but we indeed have:

$$\langle n'l'j'm'_j|V_C|nljm\rangle = \delta_{l',l}\delta_{j',j}\delta_{m'_{j'},m_j}\langle n'l'|V_C|nl\rangle \ . \tag{2.9}$$

So if we denote $|pljm_j\rangle$ ($p = 1, 2, 3, ...$) the actual eigenfunctions we can span them in the basis $|nljm_j\rangle$.

$$|pljm_j\rangle = \sum_{n=1}\langle nljm_j|pljm_j\rangle|nljm_j\rangle \ , \tag{2.10}$$

During the process of diagonalization we obtain the $\langle nljm_j|pljm_j\rangle$ coefficients but we need to set a limit. Take into account that subshells with different n values tend to be quite different and if the V_C term is not dominating we expect to have a dominant coeffcient for all eigenfunctions, the case with $n = p$.

Then our hamiltonian including V_C is divided in blocks with the same values of (l, j, m_j). So the strategy is to select the values of (l, j) neccesary to fill some subshells, those occupied in the ground state (without the V_C) and some more (possible excited states or useful states for new diagonalizations with addiotional terms in the hamiltonian). It's not neccesary to make the calculations for different values of m_j because they don't incorporate any difference if the other quantum numbers are fixed. It's not expected that the Coulomb repulsion changes the order of the subshells, only incorporates some quantitative differences.

2.1.1 Case of ^{16}O

For this nucleus we have eight protons. The subshells $|n = 1l = 0j = 1/2m_j\rangle$, $|n = 1l = 1j = 3/2m_j\rangle$ and $|n = 1l = 1j = 1/2m_j\rangle$ are filled in the ground state. And we are also interested in the $|n = 1l = 2j = 5/2m_j\rangle$, $|n = 1l = 2j = 3/2m_j\rangle$ and $|n = 2l = 0j = 1/2m_j\rangle$.

So the values that we take for our calculations are: $(l = 0, j = 1/2)$, $(l = 1, j = 3/2)$,$(l = 1, j = 1/2)$,$(l = 2, j = 3/2)$ and $(l = 2, j = 1/2)$. And using three terms for the n values $(1, 2, 3)$ we will construct all those states and see what energy eigenvalue is associated with each one.

Using a depth potential of $V_0 = -57MeV$ the results are shown in table[1] and in fig.[1]. The new energies are denoted as E'. As expected the Coulombian makes the single particle protonic energies increase their values. Due to the electric repulsion protons are less bound to the nucleus on the same subshell.

The eigenfunctions are:

$$|p = 1l = 0j = 1/2\rangle = -0.99986|101/2\rangle - 0.016704|201/2\rangle - 0.00065|301/2\rangle \tag{2.11}$$

$$|p = 1l = 1j = 3/2\rangle = 0.99981|111/2\rangle - 0.01948|211/2\rangle - 0.00145|311/2\rangle \tag{2.12}$$

$$|p = 1l = 1j = 1/2\rangle = 0.99981|113/2\rangle - 0.01948|213/2\rangle - 0.00147|313/2\rangle \tag{2.13}$$

$\|nlj> / \|plj>$	$E[MeV]$	$E'[MeV]$
$1s1/2$	-32.5937	-28.2904
$1p3/2$	-18.2754	-14.4094
$1p1/2$	-12.4178	-8.5519
$1d5/2$	-3.9569	-0.4731
$2s1/2$	-0.0520	3.4826
$1d3/2$	5.8055	9.2894

Tabla 1: The energies eigenvalues are shown for the spherically symmetric MSO. Without coulombian repulsion (E) and with it making the diagonalization (E').

$$|p = 1l = 2j = 5/2\rangle = 0.99980|125/2\rangle - 0.01991|225/2\rangle - 0.00216|325/2\rangle \tag{2.14}$$

$$|p = 2l = 0j = 1/2\rangle = 0.01668|101/2\rangle + 0.99959|201/2\rangle - 0.02323|301/2\rangle \tag{2.15}$$

$$|p = 1l = 2j = 3/2\rangle = 0.99980|123/2\rangle - 0.019905|223/2\rangle - 0.00221|323/2\rangle \tag{2.16}$$

As we can see the results are as expected,that is, the eigenfunctions are quite similar to the old ones because the strong of the coulomb potential is quite smaller than the nuclear part.

3 Only axial simmetry

Breaking the spherical symmetry but keeping the axial symmetry means that the potential is now $V = V(r, \theta)$. There is still no dependence on ϕ. Now $\omega_x = \omega_y \neq \omega_z$ so it's suitable to define $\omega_\rho \equiv \omega_x = \omega_y$. And the potential takes the following form on a cylindrical basis:

$$V(\rho, z) = \frac{m}{2}[\omega_\rho^2\rho^2 + \omega_z^2 z^2] + C\vec{l}\vec{s} + D(\vec{l}^2 - <\vec{l}^2>_N) \ . \tag{3.1}$$

Now the potential is parametrized in terms of ω_ρ and ω_z. But the choice is not free, because of the incomprensibility of nuclear matter ω_ρ and ω_z are related. Therefore there is only one independent parameter, instead of using one the frecuencies we define the elongation parameter δ as was defined by Nilsson []:

$$\omega_z^2 = \omega_0^2(\delta)(1 - \frac{4}{3}\delta) \ , \tag{3.2}$$

$$\omega_\rho^2 = \omega_0^2(\delta)(1 + \frac{2}{3}\delta) \ . \tag{3.3}$$

The incomprensibility condition is:

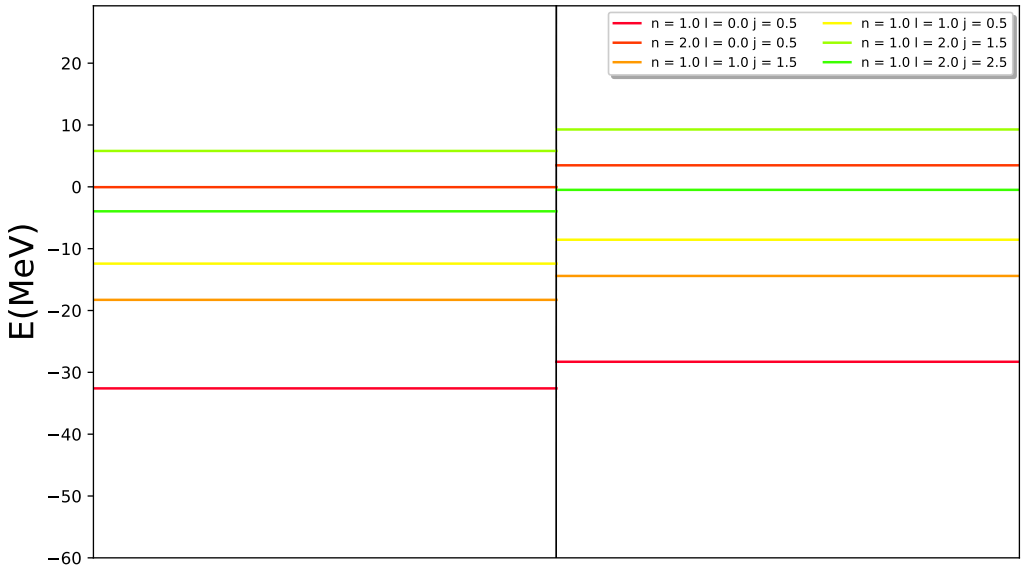

Figura 1: Protonic energies level for the ^{16}O nucleus. On the left only considering nuclear potential and on the right also considering electric interaction between protons

$$\omega_\rho^2 \omega_z = \bar{\omega}_0^3 \ , \tag{3.4}$$

where $\bar{\omega}_0$ is the frecuency that would have this nucleus if it were spherical.

So we have:

$$\omega_0(\delta) = \bar{\omega}_0 \left[\frac{1}{(1+\frac{2}{3}\delta)^2(1-\frac{4}{3}\delta)} \right]^{1/6} \ . \tag{3.5}$$

With this all the deformation in the potential is carried by the elongation parameter δ. With $\delta = 0$ we recover the spherical symmetry.

With deformation and without spin-orbit term the new good quantum numbers are: n_ρ, m_l and n_z. If we include the spin-orbit term the only good quantum numbers remaining are m_j and the parity. Because of axial symmetry mj and $-m_j$ form a shell with fixed parity. We call $\Omega = |m_j|$ and π the parity. But there are more than one subshells with the same quantum numbers Ω and π. What means j_z and π do not constitute a complete set of compatible observables. We can not identify a subshell only with Ω and π therefore we may use the old basis $|nlj>$ for indentification, those are good quantum numbers for the eigenstates if $\delta = 0$.

For the diagonalization process one option is to use the $|n_\rho m_l n_z m_s>$ basis. The terms that are not digonal are the spin-orbit term and the l^2 term. Other suitable option is to use the spherical basis $|nlm_l m_s>$. To see how this basis handles the terms of the hamiltonian let's show the potential in the spherical coordinates (r, θ, ϕ).

$$V(r,\theta) = \frac{m}{2}\omega_0^2(\delta)[r^2 - \beta r^2 Y_{20}(\theta)] + C\vec{l}\vec{s} + Dl^2 \ . \tag{3.6}$$

with $\beta = \frac{4}{3}\sqrt{\frac{4\pi}{5}}\delta$.

Because subshells with the same N tends to be closer we can fix this value and diagonalize blocks characterized for N with different values of n,l,m_l and m_s conserving the value of m_j and parity (l even or odd). That's an approximation because the admixture for N are $N = N' + -2$. So for better calculation mor values of N should be added.

Because $\vec{l}\cdot\vec{s}$ is not diagonal on the basis $|nlm_lm_s>$ it is suitable to work on also spherical basis $|nljm_j>$ so the only term that is not diagonal is the r^2Y_{20} term.

This way the values that we need to evaluate are: $< l'j'm'_j|Y_{20}|ljm_j >$ and $< N'l'j'|r^2|Nlj >$. Now we focus on the first term and later we study the second term. Let's put this value in terms of $< l'm'_lm'_s|Y_{20}|lm_lm_s >$:

$$< l'j'm'_j|Y_{20}|ljm_j >= \sum_{m'_s} < l'j'm'_j|l'm'_lm'_s >< l'm'_lm'_s|Y_{20}|\sum_{m_s} lm_lm_s >< lm_lm_s|jm_jl > \quad . \tag{3.7}$$

Because Y_{20} doesn't act on $|m_s>$ and $< m'_s|m_s >= \delta_{m'_s,m_s}$ we can eliminate the second sum and there's only one m_s. So:

$$< l'j'm'_j|Y_{20}|ljm_j >= \sum_{m'_s} < l'j'm'_j|l'm'_lm'_s >< l'm'_l|Y_{20}|lm_l >< lm_lm'_s|jm_jl > \quad . \tag{3.8}$$

But $< l'm'_l|Y_{20}|lm_l >$ is zero unless $m'_l = m_l$, so:

$$< l'j'm'_j|Y_{20}|ljm_j >= \sum_{m'_s} < l'j'm'_j|l'm'_lm'_s >< l'm'_l|Y_{20}|lm'_l >< lm'_lm'_s|jm_jl > \quad . \tag{3.9}$$

And the C-B gets zero unless $mj = m'j$ because mj needs to be equal to $m'_s + m'_l$ so all terms with $m_j \neq m'_j$ are zero.

On this basis the elements of the hamiltonian are:

$$< n'l'j'mj'|H|nljmj >= \delta_{mj',mj}[\delta_{n,n'}\delta_{l,l'}\delta_{j,j'}(E_{nlj} + V_0) - \beta < n'l'j'm'_j|r^2Y_{20}|nljm_j >] \quad . \tag{3.10}$$

3.1 Blocks of fixed N and Ω

There is no dependence in the hamiltonian on j_z so there are no admixture for different values of Ω. That means:

$$< \gamma'\Omega'|H|\gamma\Omega >= \delta_{\Omega',\Omega} < \gamma'\Omega'|H|\gamma\Omega > \quad , \tag{3.11}$$

where γ represents all other quantum numbers to form a complete set of compatible observables.

The values $< \gamma'N'|H|\gamma N >$ gets zero unless $N = N'$ or $N = N' \pm 2$. However the subshells with the same N are much similar, in general, to those of the same value than those with ± 2 its value, therefore fixing the N values is a good approximation. In other words, terms $< N'|H|N >$ with different N' and N are much smaller than those with $N' = N$.

3.1.1 Basis of eigenfunctions

At this point it is suitable to remark what is exactly the basis of eigenfunctions we use here for the calculations. We have a hamiltonian that we can express as: $H = H_0 - \beta r^2 Y_{20}(\theta, \phi)$. Where H_0 is:

$$H_0(\delta) = -\frac{\hbar^2}{2m}\nabla^2 + \frac{m\omega_0^2(\delta)}{2}r^2 + C\vec{l}\vec{s} + D(\vec{l}^2 - <\vec{l}^2>_N) \ . \tag{3.12}$$

Its radial eigenfunctions are:

$$R_{nl}(r,\delta) = N_{nl} r^l e^{-\frac{\alpha(\delta)^2 r^2}{2}} M(-n+1, l+\frac{3}{2}, \alpha(\delta)^2 r^2) \ . \tag{3.13}$$

where:

$$\alpha(\delta) = \frac{\sqrt{\hbar\omega_0(\delta)m}}{\hbar c} \tag{3.14}$$

The total eigenfunctions that we use are:

$$\langle r, \theta, \varphi, \sigma | nljm(\delta)\rangle = R_{nl}(r,\delta) Y_{jlm}(\theta, \varphi, \sigma) \tag{3.15}$$

Then the hamiltonian elements in this basis is:

$$\langle nljm(\delta)|H(\delta)|n'l'j'm'(\delta)\rangle = \delta_{n,n'}\delta_{l,l'}\delta_{j,j'}\delta_{m,m'_j}E_{nlj}(\delta) - \beta(\delta)\langle nl(\delta)|r^2|n'l'(\delta)\rangle\langle ljm|Y_{20}|l'j'm'\rangle \tag{3.16}$$

Why we don't use the radial part for $\delta = 0$ as our basis? Because they are not eigenfunctions of $H_0(\delta)$ so the term $\langle nljm(\delta=0)|H_0(\delta)|n'l'j'm'\rangle$ becomes incredibly cumbersome.

3.1.2 N = 0

For $N = 0$ there is only one possible value for Ω and this is $\Omega = \frac{1}{2}$. Using the notation $|\Omega nlj>$ this state is: $|\Omega = 1/21s1/2>$.

3.1.3 N = 1

For $N = 1$ there are two values for Ω, and these are $\Omega = \frac{1}{2}, \frac{3}{2}$.

For $\Omega = \frac{1}{2}$ the subspace is $2-dimensional$ constituted by the states $|\Omega = 1/21p1/2>$ and $|\Omega = 1/21p3/2>$.

For $\Omega = \frac{3}{2}$ the subspace is $1-dimensional$ constituted only by $|\Omega = 3/21p3/2>$.

3.1.4 N = 2

For $N = 2$ there are six values for Ω, and these are $\Omega = \frac{1}{2}, \frac{3}{2}$ and $\frac{5}{2}$.

For $\Omega = \frac{1}{2}$ the subspace is $3-dimensional$ constituted by the states $|\Omega = 1/22s1/2>$, $|\Omega = 1/21d3/2>$ and $|\Omega = 1/21d5/2>$.

For $\Omega = \frac{3}{2}$ the subspace is $2-dimensional$ constituted only by $|\Omega = 3/21d3/2>$ and $|\Omega = 3/21d5/2>$.

For $\Omega = \frac{5}{2}$ the subspace is $1-dimensional$ constituted only by $|\Omega = 5/21d5/2>$.

3.1.5 N = 3

For $N = 3$ there are four possible values Ω and these are $\Omega = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$.

For $\Omega = \frac{1}{2}$ the subspace is 4-dimensional and it's constituted by the states $|\Omega = 1/22p1/2 >$, $|\Omega = 1/22p3/2 >$, $|\Omega = 1/21f5/2 >$ and $|\Omega = 1/21f7/2 >$.

For $\Omega = \frac{3}{2}$ the subspace is 3-dimensional and it's constituted by the states $|\Omega = 3/22p3/2 >$, $|\Omega = 3/21f5/2 >$ and $|\Omega = 3/21f7/2 >$.

For $\Omega = \frac{5}{2}$ the subspace is 2-dimensional and it's constituted by the states $|\Omega = 5/21f5/2 >$ and $|\Omega = 5/21f7/2 >$.

For $\Omega = \frac{7}{2}$ the subspace is 1-dimensional and it's constituted by the state $|\Omega = 7/21f7/2 >$.

3.1.6 N = 4

For $N = 4$ there are five possible values Ω and these are $\Omega = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}$.

For $\Omega = \frac{1}{2}$ the subspace is 5-dimensional and it's constituted by the states $|\Omega = 1/23s1/2 >$, $|\Omega = 1/22d3/2 >$, $|\Omega = 1/22d5/2 >$, $|\Omega = 1/21g7/2 >$ and $|\Omega = 1/21g9/2 >$.

For $\Omega = \frac{3}{2}$ the subspace is 4-dimensional and it's constituted by the states $|\Omega = 3/22d3/2 >$, $|\Omega = 3/22d5/2 >$, $|\Omega = 3/21g7/2 >$ and $|\Omega = 3/21g9/2 >$.

For $\Omega = \frac{5}{2}$ the subspace is 3-dimensional and it's constituted by the states $|\Omega = 5/22d5/2 >$, $|\Omega = 5/21g7/2 >$ and $|\Omega = 3/21g9/2 >$.

For $\Omega = \frac{7}{2}$ the subspace is 2-dimensional and it's constituted by the states $|\Omega = 7/21g7/2 >$ and $|\Omega = 7/21g9/2 >$.

For $\Omega = \frac{9}{2}$ the subspace is 1-dimensional and it's constituted by the state $|\Omega = 9/21g9/2 >$.

3.1.7 N = 5

For $N = 5$ there are six possible values Ω and these are $\Omega = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}$ and $\frac{11}{2}$.

For $\Omega = \frac{1}{2}$ the subspace is 6-dimensional and it's constituted by the states $|\Omega = 1/23p1/2 >$, $|\Omega = 1/23p3/2 >$, $|\Omega = 1/22f5/2 >$, $|\Omega = 1/22f7/2 >$, $|\Omega = 1/21h9/2 >$ and $|\Omega = 1/21h11/2 >$.

For $\Omega = \frac{3}{2}$ the subspace is 5-dimensional and it's constituted by the states $|\Omega = 3/23p3/2 >$, $|\Omega = 3/22f5/2 >$, $|\Omega = 3/22f7/2 >$,$|\Omega = 3/21h9/2 >$ and $|\Omega = 3/21h11/2 >$.

For $\Omega = \frac{5}{2}$ the subspace is 4-dimensional and it's constituted by the states $|\Omega = 5/22f5/2 >$, $|\Omega = 5/22f7/2 >$,$|\Omega = 5/21h9/2 >$ and $|\Omega = 5/21h11/2 >$.

For $\Omega = \frac{7}{2}$ the subspace is 3-dimensional and it's constituted by the states $|\Omega = 7/22f7/2 >$,$|\Omega = 7/21h9/2 >$ and $|\Omega = 7/21h11/2 >$.

For $\Omega = \frac{9}{2}$ the subspace is 2-dimensional and it's constituted by the state $|\Omega = 9/21h9/2 >$ and $|\Omega = 9/21h11/2 >$.

For $\Omega = \frac{11}{2}$ the subspace is 1-dimensional and it's constituted by the state $|\Omega = 11/21h11/2 >$.

3.2 Level identification

In order to identify the calculated energy eigenstates with the states $|nljm_j>$ for $\delta = 0$ we have to take into account that the relation:

$$|\gamma N\Omega >= \sum_{n',l',j'} < n'l'j'\Omega(\delta)|\gamma N\Omega > |n'l'j'\Omega(\delta) > \quad . \tag{3.17}$$

For $\delta = 0$ we have:

$$|\gamma N\Omega >= |nljm_j(\delta = 0) > \quad . \tag{3.18}$$

That is, we can find that state $|nlj >$ that serves us as a identifier. On coordinates we have:

$$1 = |nljm_j(\delta = 0) > \quad . \tag{3.19}$$

3.3 ^{24}Mg case

Now we proceed to apply the approximation of fixed N blocks to the case of ^{24}Mg. We do the calculations for different values of the deformation parameter δ and visualize the effect on the subshells. We show for this case the explicit form of the eigenfunctions on the old spherical basis $|nljm_j >$. Moreover, it's known this nucleus is deformed in the ground state, we calculate the total energy for the different deformation values and see if it is minimized for some deformation. All this calculations are made for neutrons and the depth potential selected is $V_0 = -58MeV$.

For the case $N = 0$, $\Omega = \frac{1}{2}$ we have only one possible state so the process of diagonalization is reduced to a single calculation of the expected value of the hamiltonian on this state. The energies found are shown in table [2] for different values of δ. Here we find a constant behaviour of the $1s1/2$ and, that is, it reachs its absolute minimum value for $\delta = 0$.

States	$\delta = -0.4$	$\delta = -0.2$	$\delta = 0$	$\delta = 0.2$	$\delta = 0.4$
$\|N = 0\Omega = 1/21s1/2 >$	-35.9827	-36.5013	-36.6791	-36.4654	-35.6256

Tabla 2: The energies eigenvalues $[MeV]$ are shown for the deformed axially symmetric MSO for different deformation parameter values.

For the case $N = 1$, $\Omega = \frac{1}{2}$ we have two states and therefore, a hamiltonian block of two dimensions. The results are shown in table [3]. We have $|N = 1\Omega = 1/21p\ 1/2 >= -0.984873|1p1/2 > +0.173277|1p3/2 >$ for $\delta = -0.4$. That is, 97 of the state belongs to the $|1p1/2 >$. But remember this is a deformed state, not the state $|1p1/2 >$ for the case $\delta = 0$. So it's difficult to quantify the effect on the state with respect the case without deformation. Anyway even for a high value of the deformation parameter the calculated state is much more associated with the quantum number it would have at $\delta = 0$.

For the case $N = 1$, $\Omega = \frac{3}{2}$ again we have only one state so all the work is a expected value of the hamiltonian on $|1p3/2m_j = |3/2| >$. The results are show in table[4].

States	$\delta = -0.4$	$\delta = -0.2$	$\delta = 0$	$\delta = 0.2$	$\delta = 0.4$
$\|N = 1\Omega = 1/21p1/2 >$	-16.4043	-18.2330	-18.9130	-18.3301	-16.3365
$\|N = 1\Omega = 1/21p3/2 >$	-22.6372	-23.4492	-24.3117	-25.0988	-25.0554

Tabla 3: The energies eigenvalues $[MeV]$ are shown for the deformed axially symmetric MSO for different deformation parameter values.

States	$\delta = -0.4$	$\delta = -0.2$	$\delta = 0$	$\delta = 0.2$	$\delta = 0.4$
$\|N = 1\Omega = 3/21p3/2 >$	-24.8718	-24.8247	-24.1709	-22.8983	-20.7365

Tabla 4: The energies eigenvalues $[MeV]$ are shown for the deformed axially symmetric MSO for different deformation parameter values.

At the moment we have eight different states for the neutrons. For completing the ground state we need four more. With $N = 2$ we got enough states for that and some more for possible excited states. It's not expected a subshell with $N = 3$ has an asscoiated energy to be taken into account.

All the results for $N = 2$ are shown in table [5] for $\Omega = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$. Now we have greater mix between states. For example, for $\delta = -0.4$ we have $|N = 2\Omega = 1/22s1/2 >= -0.903645|2s1/2 > +0.187952|1d3/2 > +0.384836|1d5/2 >$. The reason is these subshells are closer to each other.

States	$\delta = -0.4$	$\delta = -0.2$	$\delta = 0$	$\delta = 0.2$	$\delta = 0.4$
$\|N = 2\Omega = 1/21d3/2 >$	2.7743	-0.86496	-3.0955	-3.2127	-0.4006
$\|N = 2\Omega = 1/22s1/2 >$	-4.4395	-6.88478	-8.1052	-7.5332	-6.0415
$\|N = 2\Omega = 1/21d5/2 >$	-11.2054	-11.2557	-11.8474	-13.5623	-14.1874
$\|N = 2\Omega = 3/21d3/2 >$	-3.0969	-3.7976	-3.1134	-1.1620	2.2645
$\|N = 2\Omega = 3/21d5/2 >$	-10.1429	-11.0837	-11.6835	-11.6964	-10.3812
$\|N = 2\Omega = 5/21d5/2 >$	-13.6466	-13.1326	-11.6626	-9.3497	-6.0133

Tabla 5: The energies eigenvalues $[MeV]$ are shown for the deformed axially symmetric MSO for different deformation parameter values.

In fig.[2] you can see the single particle energy levels for different values of the deformation parameter. As you can see the values vary differently for different states. These changes lead to a deformation in the ground state because the ground state could be found at $\delta = 0$.

Appendix A MSO eigenfunctions

The radial equation for the SO promblem($V(r) = \mu\frac{\omega^2}{2}r^2$) has the following shape:

$$\left[-\frac{p_r^2}{2} - \frac{\hbar^2 l(l+1)}{2\mu r^2} + E_{nl}^0 - \frac{\hbar^2}{2\mu}\alpha^4 r^2\right] R_{nl}(r) = 0 \qquad \text{(Appendix A.1)}$$

where:

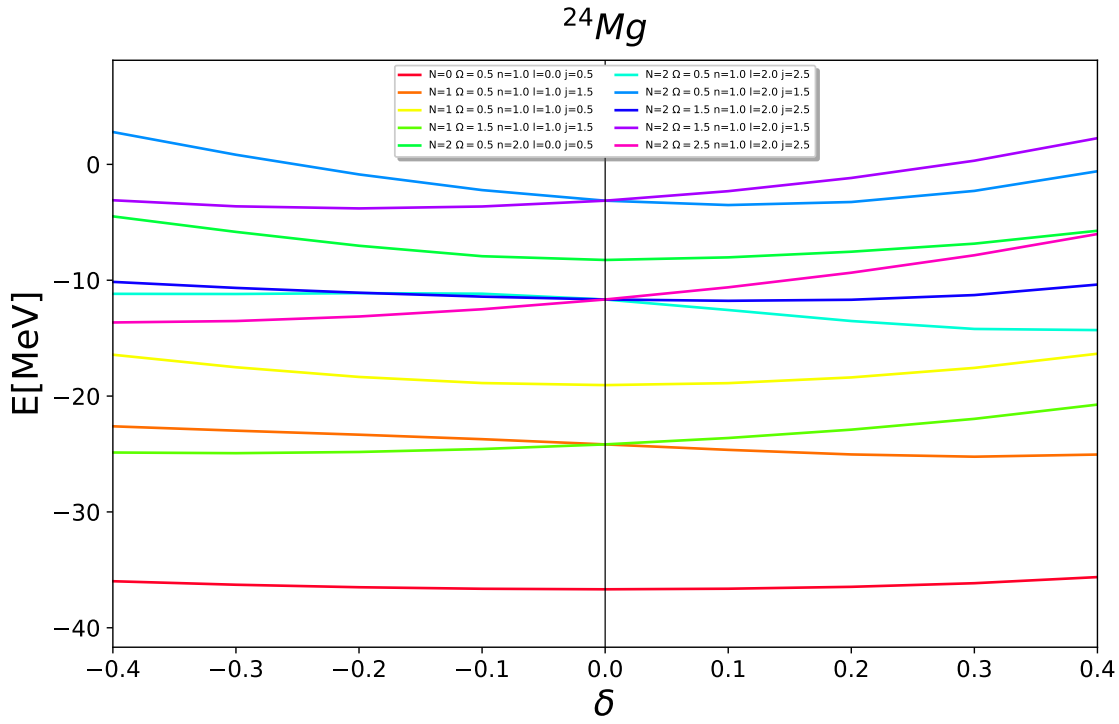

Figura 2: Neutronic energies level for the ^{24}Mg nucleus. Some single particle energy levels are displayed depending on the deformation parameter δ

$$\alpha = \sqrt{\frac{\mu\omega}{\hbar}} \qquad \text{(Appendix A.2)}$$

The eigenfunctions $R_n l(r)$ are known to be:

$$R_{nl}(r) = N_{nl} r^l e^{-\frac{\alpha^2 r^2}{2}} M(-n+1, l+\frac{3}{2}, \alpha^2 r^2) \ . \qquad \text{(Appendix A.3)}$$

and the eigenvalues E_{nl} are:

$$E^0_{nl} = \hbar\omega \left[2(n-1) + l + \frac{3}{2}\right] \qquad \text{(Appendix A.4)}$$

For MSO proble we have:

$$\left[-\frac{p_r^2}{2} - \frac{\hbar^2 l(l+1)}{2\mu r^2} + E_{nlj} - \frac{\hbar^2}{2\mu}\alpha^4 r^2 + C\frac{\hbar^2}{2}\left[j(j+1) + l(l+1) - 3/4\right] + D\hbar^2\left[l(l+1) - \frac{N(N+3)}{2}\right]\right] R_{nl}($$

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Sustituying we have:

$$\left[E_{nlj} - E^0_{nl} + C\frac{\hbar^2}{2}\left[j(j+1) + l(l+1) - 3/4\right] + D\hbar^2\left[l(l+1) - \frac{N(N+3)}{2}\right]\right] R_{nl}(r) = 0$$

(Appendix A.6)

Therefore the eigenvalues are:

$$E_{nlj} = E^0_{nl} - C\frac{\hbar^2}{2}\left[j(j+1) + l(l+1) - 3/4\right] - D\hbar^2\left[l(l+1) - \frac{N(N+3)}{2}\right] \quad \text{(Appendix A.7)}$$

Making the following change of parameters we get the same shape of the equations:

$$C = \frac{2\kappa\omega_0}{\hbar} \qquad D = \frac{\mu\omega_0}{\hbar} \quad \text{(Appendix A.8)}$$