APM1110: Probability and Probability Distributions
Formative Assessment 9

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Question 1: Show that the mean and variance of the gamma distribution are given by.

(a) $\mu = \beta\alpha$, (b) $\sigma^2 = \alpha\beta^2$

The gamma probability distribution function is the following:

$$f(x) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{\frac{-x}{\beta}} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

while the definition of the Gamma function is the following:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx = (\alpha - 1)!$$

Show that, for the gamma probability distribution, $\mu = \alpha\beta$, and $\sigma^2 = \alpha\beta^2$

$$\mu = E(X) = \int_0^\infty \frac{x}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{\frac{-x}{\beta}}\, dx$$

$$== \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^\alpha e^{\frac{-x}{\beta}}\, dx$$

let $u = \frac{x}{\beta}$, $x = u\beta$
then $du = \frac{1\, dx}{\beta}$, $dx = \beta\, du$
then we'll have:

$$\longrightarrow \frac{\beta^{\alpha+1}}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty (u)^\alpha e^{-u}\, du$$

using the definition of a Gamma function:

$$\frac{\beta}{\Gamma(\alpha)} \int_0^\infty (u)^{\alpha+1-1} e^{-u}\, du == \frac{\beta}{\Gamma(\alpha)} \Gamma(\alpha+1) == \frac{\beta}{\Gamma(\alpha)} \alpha\Gamma(\alpha)$$

cancelling out terms gives us:

$$\beta\alpha = E(X) = \mu$$

The definition of σ^2 is $Var(X) = E(X^2) - E(X)^2$
looking for $E(X^2)$ gives us:

$$E(X) = \int_0^\infty \frac{x^2}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{\frac{-x}{\beta}}\, dx$$

$$== \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{\alpha+1} e^{\frac{-x}{\beta}}\, dx$$

again, let $u = \frac{x}{\beta}, x = u\beta$
then $du = \frac{1\, dx}{\beta},\ dx = \beta\, du$
then we'll have:

$$\longrightarrow \frac{\beta^{\alpha+2}}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty (u)^{\alpha+1+1-1} e^{-u}\, du$$

note that we have already added 0 to the exponent of u, this makes the integral more familiar: the Gamma Function:

$$\longrightarrow \frac{\beta^2}{\Gamma(\alpha)} \Gamma(\alpha+2)$$

this gives us:

$$\longrightarrow \frac{\beta^2}{\Gamma(\alpha)} (\alpha+1)\alpha\Gamma(\alpha) = \beta^2(\alpha+1)\alpha$$

$$== \beta^2\alpha^2 + \beta^2\alpha = E(X^2)$$

solving for the variance:

$$\sigma^2 = E(X^2) - E(X)^2 = \beta^2\alpha^2 + \beta^2\alpha - (\beta\alpha)^2$$

$$== \beta^2\alpha$$

$\therefore$ for a gamma probability distribution, $\mu = \beta\alpha$ and $\sigma^2 = \beta^2\alpha$ $\square$

Question 2: Prove that the mean and variance of a binomially distributed random variable are, respectively.

$\mu = np$ and $\sigma^2 = npq$.

$$\boldsymbol{\mu = \mathbf{np}}$$

Let $U_1, \ldots, U_n$ be independent Bernoulli random variables.

$E(U_i) = p$ and $Var(U_i) = p(1-p)$, where p is the mean (μ) or the expectation and $p(1-p)$ is the variance (σ)

$X = U_1 + \ldots + U_n$, where the binomial random variable X is the sum of these n random variables.

$E(X) = E(U_1 + \ldots + U_n)$, where the expectation of our binomial random variable X denoted as E is the expectation of the sum of the n Bernoulli random variables.

Using the properties of expectation, this is simply the sum of the expectation,

$$\begin{aligned} E(X) &= E(U_1) + \ldots + E(U_n) \\ &= p + \ldots + p \\ &= np, \text{ where } n \text{ represents the number of times } (p) \text{ occurs} \quad \square. \end{aligned}$$

$$\boldsymbol{\sigma^2 = \mathbf{npq}}$$

In a binomial distribution, success and failure are complementary probabilities, let p be the probability of success and q be the probability of failure, then:

$$\textbf{q(failure) = 1 - p(success)} \tag{2}$$

Let $U_1, \ldots, U_n$ be independent Bernoulli random variables.

$E(U_i) = p$ and $Var(U_i) = p(1-p)$, where p is the mean (μ) or the expectation and p(1-p) is the variance (σ)

$X = U_1 + \ldots + U_n$, where the binomial random variable X is the sum of these n Bernoulli random variables.

$Var(X) = Var(U_1 + \ldots + U_n)$, where the variance of X is the variance of the sum of these n independent Bernoulli random variables.

When random variables are independent, the variance of the sum is the sum of the variances.

$$\begin{aligned} Var(X) &= Var(U_1) + \ldots + Var(U_n) \\ &= p(1-p) + \ldots + p(1-p) \\ &= np(1-p), \text{ where } n \text{ represents the number of times } p(1-p) \text{ occurs} \end{aligned}$$

Now, recall from equation ((2)) that $q = 1 - p$

Substituting $q = 1 - p$ into $np(1-p)$, we get npq. Therefore,

$$Var(X) = npq \qquad \square.$$

Question 3: Establish the validity of the Poisson approximation to the binomial distribution.

The PMF of the Binomial Distribution

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \tag{3}$$

The PMF of the Poisson Distribution

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} \tag{4}$$

We know that the mean (μ) of the binomial random variable is np and the μ of the poisson random variable is lambda (λ). If we set $np = \lambda$ and through algebraic manipulation, $p = \frac{\lambda}{n}$,

We can substitute p into the binomial PMF:

$$P(X = x) = \binom{n}{x} \left(\frac{\lambda}{n}\right)^x (1 - \frac{\lambda}{n})^{n-x}$$

Using the binomial coefficient properties,

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

Substituting this value into the equation, we get:

$$\frac{n!}{x!(n-x)!} \left(\frac{\lambda}{n}\right)^x (1 - \frac{\lambda}{n})^{n-x}$$

Taking λ^x to the front so that it will resemble the PMF poisson formula from equation (4) ,

$$\frac{\lambda^x}{x!} \frac{n!}{(n-x)!} \frac{1}{n^x} (1 - \frac{\lambda}{n})^{n-x}$$

Now all we need to show is that the,

$$\frac{n!}{(n-x)!} \frac{1}{n^x} (1 - \frac{\lambda}{n})^{n-x} = e^{-}\lambda, \text{of the PMF poisson formula as } n \to \infty$$

Let's simplify first this term, $\frac{n!}{(n-x)!} \frac{1}{n^x}$

$$\begin{aligned}
&= \frac{n(n-1)(n-2)\dots(n-x+1)(n-x)!}{(n-x)!n^x} \\
&= \frac{n(n-1)(n-2)\dots(n-x+1)}{n^x} \\
&= \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \dots \frac{n-x+1}{n} \\
&= 1(1 - \frac{1}{n})(1 - \frac{2}{n}) \dots (1 - \frac{x+1}{n})
\end{aligned}$$

We now have,

$$\frac{\lambda^x}{x!}(1 - \frac{1}{n})(1 - \frac{2}{n}) \dots (1 - \frac{x+1}{n})(1 - \frac{\lambda}{n})^{n-x}$$

Simplifying the last term, $(1 - \frac{\lambda}{n})^{n-x}$ using the properties of exponent, we have:

$$\frac{\lambda^x}{x!}(1 - \frac{1}{n})(1 - \frac{2}{n}) \dots (1 - \frac{x+1}{n})(1 - \frac{\lambda}{n})^n (1 - \frac{\lambda}{n})^{-x}$$

Now, we have rewritten the PMF binomial formula into this:

$$\binom{n}{x}p^x(1-p)^{n-x} = \frac{\lambda^x}{x!}(1-\frac{1}{n})(1-\frac{2}{n})\dots(1-\frac{x+1}{n})(1-\frac{\lambda}{n})^n(1-\frac{\lambda}{n})^{-x}$$

We'll restructure this a bit to make it easier to get the $\lim_{n\to\infty}$

$$\begin{aligned}
\lim_{n\to\infty}\binom{n}{x}p^x(1-p)^{n-x} = {} & \frac{\lambda^x}{x!} \\
& \times \lim_{n\to\infty}(1-\frac{1}{n})(1-\frac{2}{n})\dots(1-\frac{x+1}{n}) \\
& \times \lim_{n\to\infty}(1-\frac{\lambda}{n})^n \\
& \times \lim_{n\to\infty}(1-\frac{\lambda}{n})^{-x}
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty}\binom{n}{x}p^x(1-p)^{n-x} = {} & \frac{\lambda^x}{x!} \\
& \times \lim_{n\to\infty}(1-0)(1-0)\dots(1-0) = 1 \\
& \times \lim_{n\to\infty}(1-\frac{\lambda}{n})^n = e^-\lambda \\
& \times \lim_{n\to\infty}(1-0)^{-x} = 1
\end{aligned}$$

As $n \to \infty$, $p \to 0$ and $\lambda = np$ stays constant:

$$\begin{aligned}
\lim_{n\to\infty}\binom{n}{x}p^x(1-p)^{n-x} &= \frac{\lambda^x}{x!} \times 1 \times e^-\lambda \times 1 \\
&= \frac{\lambda^x e^-\lambda}{x!} \qquad \square.
\end{aligned}$$