Lince Rumainum

HW#3

Algorithm Analysis

2. $\log_e n! = \sum_{i=1}^{n} \log_e i \geqslant \int_1^n \log_e x \, dx$

$$\int_1^n \log_e x \, dx = \int (u\,dv - v\,du) = u \int dv - v \int du$$

$u = \log_e x$

$du = \frac{1}{x} dx$

$v = x$

$dv = dx$

$$= \log_e x \int_1^n dx - x \int_1^n \frac{1}{x} dx$$

$$= (\log_e x \cdot x)\Big|_1^n - (x\Big|_1^n)$$

$$= n \log_e n - \log_e 1 - (n-1)$$

$$= n \log_e n - n + 1$$

$\int_1^n \log_e x \, dx$ have a lower bound of $\Omega(n \log_e n)$

3. $\sum_{i=1}^{n} \frac{1}{i} \approx \int_1^n \frac{dx}{x} \rightarrow \int_1^n \frac{1}{x} dx = \log_x \Big|_1^n$

$$= \log n - \log 1$$

$$= \log n$$

$\therefore$, As $n \to \infty$, $\log(\infty) = \infty$

$\sum_{i=1}^{n} \frac{1}{i^2} \approx \int_1^n \frac{dx}{x^2} \rightarrow \int_1^n \frac{1}{x^2} dx = -\frac{1}{x}\Big|_1^n$

$$= -\frac{1}{n} - (-1)$$

$$= 1 - \frac{1}{n}$$

$\therefore$, As $n \to \infty$, $1 - \frac{1}{\infty} = 1$

4. a) $T(n) = T(n-1) + (n-1)$ where $T(2) = 1$ ... ①

@ $n = n-1$,

$T(n-1) = T(n-1-1) + (n-1-1)$

$T(n-1) = T(n-2) + (n-2)$

$\therefore$, Plug into eq. ①

$T(n) = [T(n-2) + (n-2)] + (n-1)$ ... ②

@ $n = n-2$

$T(n-2) = T(n-2-1) + (n-2-1)$

$= T(n-3) + (n-3)$

$\therefore$, Plug into eq. ②

$T(n) = [[T(n-3) + (n-3)] + (n-2)] + (n-1)$

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HW#3 Algorithm Analysis

$T(n) = T(n-3) + (n-3) + (n-2) + (n-1)$

$= T(n-3) + \sum_{i=1}^{n-3} (n-i)$

$T(n) = T(n-k) + \sum_{i=1}^{k} (n-i)$

We know $T(2) = 1$

$n-k = 2 \rightarrow k = n-2$

$T(n) = T(n-(n-2)) + \sum_{i=1}^{n-2} (n-i)$

$= T(2) + (n-1) + (n-2) + \dots + (n-(n-2))$

$= 1 + (n-1) + (n-2) + \dots + 2$

$\therefore \sum_{i=1}^{n-1} i = (n-1) + (n-2) + \dots + 1 = \frac{n(n-1)}{2}$

b) $T(n) = T\left(\lfloor \frac{n}{2} \rfloor\right) + 1$ where $T(1) = 1$ and $n = 2^k - 1$ ... ①

@ $n = \frac{n}{2}$

$T\left(\frac{n}{2}\right) = T\left(\lfloor \frac{n}{4} \rfloor\right) + 1$

$\therefore$ plug into eq ①

$T(n) = \left[T\left(\lfloor \frac{n}{4} \rfloor\right) + 1\right] + 1$

$= T\left(\lfloor \frac{n}{4} \rfloor\right) + 2$ ..... ②

@ $n = \frac{n}{4}$

$T\left(\frac{n}{4}\right) = T\left(\lfloor \frac{n}{8} \rfloor\right) + 1$

$\therefore$ plug into eq ②

$T(n) = \left[T\left(\lfloor \frac{n}{8} \rfloor\right) + 1\right] + 2$

$= T\left(\lfloor \frac{n}{8} \rfloor\right) + 3$ ..... ③

We know $T(1) = 1$ and $n = 2^k - 1$

From ①, ②, ③ we get $T(n) = T\left(\lfloor \frac{n}{2^k} \rfloor\right) + k$

$\therefore$ @ $n = 2^k$

$T(2^k) = T\left(\frac{2^k}{2^k}\right) + k = T(1) + k = 1 + k$

where $k = \log_2 n + 1$ $= 1 + \log_2 n$

$k \approx \log_2 n$