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HW 1

1. Shorting with margin

X_0 : deposit, initial price ~~$\bar{X}$~~ of the stock

X_1 : stock price at the end of 1 year

profit from shorting : $X_0 - X_1$

R : total return of the stock

we know

$$\text{total return} = \frac{\text{amount received}}{\text{amount invested}} \Rightarrow R = \frac{X_1}{X_0}$$

To find the total return on the short : R_s

amount received $= X_0 + (X_0 - X_1)$

$$\therefore \quad R_s = \frac{X_0 + (X_0 - X_1)}{X_0}$$

$$R_s = \frac{2X_0}{X_0} - \frac{X_1}{X_0} = 2 - R$$

where R_s is the total return on the short

4. Two stocks

Expected rates of return : $\bar{r}_1$, $\bar{r}_2$

variances + covariances : σ_1^2, σ_2^2 , σ_{12}

To minimize the total variance of the rate of return of the resulting portfolio, we will use Markowitz model, where

$$\text{minimize} \quad \frac{1}{2} \sum_{i,j=1}^{n} w_i w_j \sigma_{ij}$$

$$\text{subject to} \quad \sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}$$

$$\sum_{i=1}^{n} w_i = 1$$

$\therefore$, for $n=2$, $\quad$ minimize $\frac{1}{2}\left[w_1^2\sigma_1^2 + 2w_1w_2\sigma_{12} + w_2\sigma_2^2\right]$

$$w_1 + w_2 = 1$$

* the mean rate of return of this portfolio :

$$\bar{r} = w_1\bar{r}_1 + w_2\bar{r}_2$$

Using Lagrange multipliers λ and μ , we have

$$L = \frac{1}{2}\left(w_1^2\sigma_1^2 + w_1w_2\sigma_{12} + w_2w_1\sigma_{21} + w_2^2\sigma_2^2\right) - \lambda\left(\bar{r}_1 w_1 + \bar{r}_2 w_2 - \bar{r}\right) - \mu\left(w_1 + w_2 - 1\right)$$

$$\frac{\partial L}{\partial w_1} = \sigma_1^2 w_1 + \sigma_{12} w_2 - \lambda\bar{r}_1 - \mu = 0$$

$$\frac{\partial L}{\partial w_2} = \sigma_{21} w_1 + \sigma_2^2 w_2 - \lambda\bar{r}_2 - \mu = 0 \qquad \text{where } \sigma_{21} = \sigma_{12}$$

$$w_1 = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 - 2\sigma_{12} + \sigma_2^2}$$

* minimize the total variance

$$\text{minimize} \quad \sigma^2 = \frac{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}{\sigma_1^2 - 2\sigma_{12} + \sigma_2^2}$$

8. Tracking

r_M - random rate of return

n - # of asset

rates of return : $r_1, r_2, \dots r_n$

$$r = \alpha_1 r_1 + \alpha_2 r_2 + \dots + \alpha_n r_n \quad \text{with } \sum_{n=1}^{n} \alpha_i = 1 \text{ minimizing } var(r - r_M)$$

a) Find set of the equations for the α_i's

$$Var(r) = var\left(\sum_{i=1}^{n} \alpha_i r_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j \, cov(r_i, r_j)$$

$$Var(r - r_M) = var(r) - 2Cov(r, r_M) + var(r_M)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j \sigma_{ij} - 2\sum_{i=1}^{n} \alpha_i \sigma_{iM} + \sigma_M^2$$

Using Markowitz model to minimize the variance

minimize $var(r, r_M)$

subject to $\sum_{i=1}^{n} \alpha_i = 1$

Using the Lagrange multipliers λ and μ, we have

$$L = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j \sigma_{ij} - 2\sum_{i=1}^{n} \alpha_i \sigma_{iM} + \sigma_M^2 - \lambda\left(\sum_{i=1}^{n} \alpha_i - 1\right)$$
$$= \sum_{i=1}^{n} \alpha_i^2 \sigma_i^2 + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} \alpha_i \alpha_j \sigma_{ij} - 2\sum_{i=1}^{n} \alpha_i \sigma_{iM} + \sigma_M^2 - \lambda\left(\sum_{i=1}^{n} \alpha_i - 1\right)$$

$$\frac{\partial L}{\partial \alpha_i} = 2\alpha_i \sigma_i^2 + 2\sum_{j=1, j\neq i}^{n} \alpha_j \sigma_{ij} - 2\sigma_{iM} - \lambda = 0, \quad \text{where } \forall i = 1, 2, \dots, n$$

$$\frac{\partial L}{\partial \lambda} = \sum_{i=1}^{n} \alpha_i - 1 = 0$$

b) Tracking efficient : to get the mean $\sum_{i=1}^{n} \alpha_i \bar{r}_i = \bar{r}_M$

$$L = \sum_{i=1}^{n} \alpha_i^2 \sigma_i^2 + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} \alpha_i \alpha_j \sigma_{ij} - 2\sum_{i=1}^{n} \alpha_i \sigma_{iM} + \sigma_M^2 - \lambda\left(\sum_{i=1}^{n} \alpha_i - 1\right) - \mu\left(\sum_{i=1}^{n} \alpha_i \bar{r}_i - \bar{r}_M\right)$$

$$\frac{\partial L}{\partial \alpha_i} = 2\alpha_i \sigma_i + 2\sum_{j=1, j\neq i}^{n} \alpha_j \sigma_{ij} - 2\sigma_{iM} - \lambda - \mu \bar{r}_i = 0, \quad \text{where } \forall i = 1, 2, \dots, n$$

$$\frac{\partial L}{\partial \lambda} = \sum_{i=1}^{n} \alpha_i - 1 = 0, \quad \frac{\partial L}{\partial \mu} = \sum_{i=1}^{n} \alpha_i \bar{r}_i - \bar{r}_M = 0$$

6. Simplehand

	# of shares outstanding	Price per share	Expected rate of return	Standard deviation of return
Stock A	100	\$ 1.50	15 %	15 %
Stock B	150	\$ 2.00	12 %	9 %

correlation coefficient between the returns of stocks A and B is $\rho_{AB} = 1/3$
risk-free asset, Simplehand satisfies the CAPM

a) expected rate of return of the market portfolio, $\bar{r}_M$:

$$\bar{r}_M = w_A \bar{r}_A + w_B \bar{r}_B$$

$$\bar{r}_M = \frac{(100)(1.50)}{450}(0.15) + \frac{(150)(2.00)}{450}(0.12) = 0.13$$

b) the standard deviation of the market portfolio, $\sqrt{\sigma^2}$, σ_M:

$$\sigma^2 = w_A^2 \sigma_A^2 + 2 w_A w_B \sigma_{A,B} + w_B^2 \sigma_B^2$$

$$\rho_{A,B} = \frac{\sigma_{A,B}}{\sigma_A \sigma_B} \Rightarrow \sigma_{A,B} = \rho_{A,B}\sigma_A\sigma_B = \frac{1}{3}(0.15)(0.09)$$

$$\sigma_{A,B} = 0.0045$$

$$\sigma^2 = \left(\frac{1}{3}\right)^2(0.15)^2 + 2\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)(0.0045) + \left(\frac{2}{3}\right)^2(0.09)^2 = 0.0025 + 0.002 + 0.0036$$

$$\sqrt{\sigma^2} = 0.09 \Rightarrow \sigma_M = 0.09$$

c) the beta of stock A, β_A: $\beta = \frac{cov(r, r_M)}{var(r_M)}$

$$\beta_A = \frac{\sigma_{A,M}}{\sigma_M^2}$$

$$\sigma_{A,M} = cov\left[r_A, \frac{1}{3}r_A + \frac{2}{3}r_B\right] = \frac{1}{3}cov(r_A, r_A) + \frac{2}{3}cov(r_A, r_B) = \frac{1}{3}\sigma_A^2 + \frac{2}{3}\sigma_{A,B}$$

$$\sigma_{A,M} = \frac{1}{3}(0.15)^2 + \frac{2}{3}(0.0045) = 0.0105$$

$$\beta_A = \frac{0.0105}{0.0081} = 1.296$$

d) the risk-free rate of simplehand, CAPM:

$$\bar{r}_A - r_F = \beta_A(\bar{r}_M - r_F) = \beta_A\bar{r}_M - \beta_A r_F \Rightarrow \bar{r}_A = \beta_A\bar{r}_M + r_F(1-\beta_A)$$

$$r_F = \frac{\bar{r}_A - \beta_A\bar{r}_M}{1-\beta_A} = \frac{(0.15) - (1.296)(0.13)}{1 - 1.296}$$

$$r_F = 0.0624$$