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DSA 5303
HW 2

2. Wealth independence

$$U(x) = -e^{-ax}$$

W : initial wealth level

w : investment amount

where $w \leq W$

x : random payooff

Expected return if invest: $E[U(W-w+x)]$

Expected return with only initial wealth : $E(U(W))$

To be profitable $E[U(W-w+x)]$ needs to be greater than $E[U(W)]$

$$\therefore \quad E[U(W-w+x)] > E(U(W))$$

$$E(-e^{-a(W-w+x)}] > E(-e^{-aW})$$

$$E(-e^{-aW+aw-ax}) > E(-e^{-aW})$$

$$-e^{-aW} E(-e^{aw-ax}) > -e^{-aW}$$

$$E(-e^{aw-ax}) < 1$$

Since the expected return is only depend on w and x, it is independent of W.

4. Relative risk aversion

The Arrow - Pratt relative aversion coefficient is

$$\mu(x) = \frac{x\,U''(x)}{U'(x)}$$

For $U(x) = \ln(x)$

$$\mu(x) = \frac{x\left(\frac{d}{dx}\left(\frac{d}{dx}\ln(x)\right)\right)}{\frac{d}{dx}\ln(x)}$$

$$= \frac{x\left(\frac{d}{dx}\left(\frac{1}{x}\right)\right)}{\frac{1}{x}}$$

$$= x^2\left(-x^{-2}\right)$$

$$\mu(x) = -1$$

For $U(x) = \gamma x^{\gamma}$

$$\mu(x) = \frac{x\left(\frac{d}{dx}\left(\frac{d}{dx}(\gamma x^{\gamma})\right)\right)}{\frac{d}{dx}(\gamma x^{\gamma})}$$

$$= \frac{x\left(\frac{d}{dx}\left(\gamma^2 x^{\gamma-1}\right)\right)}{\gamma^2 x^{\gamma-1}}$$

$$= \frac{\cancel{x}\left({}_1\cancel{\gamma^2}(\gamma-1)\,\cancel{x^{\gamma-2}}\right)}{{}_1\cancel{\gamma^2}\,\cancel{x^{\gamma-1}}}$$

$$\mu(x) = \gamma - 1$$

6. HARA

The HARA (for hyperbolic absolute risk aversion) class of utility function is defined by

$$U(x) = \frac{1-\gamma}{\gamma}\left(\frac{ax}{1-\gamma} + b\right)^{\gamma}, \quad b > 0$$

the function are defined for those values of x where the term in parentheses is nonnegative.

a) Linear or risk neutral : $U(x) = x$

$$U(x) = \frac{1}{\gamma}\left(ax(1-\gamma)^{\frac{1-\gamma}{\gamma}} + b(1-\gamma)\right)^{\gamma}$$

$$= \lim_{\gamma \to 1} \frac{1}{\gamma}\left(ax(1-\gamma)^{\frac{1-\gamma}{\gamma}} + b(1-\gamma)\right)^{\gamma}$$

$\therefore$, if $a = 1$, $b > 0$, $\gamma = 1$ $\rightarrow$ $U(x) = x$

b) Quadratic : $U(x) = x - \frac{1}{2}cx^2$

Set $\gamma = 2$,

$$U(x) = \frac{1-2}{2}\left(\frac{ax}{1-2} + b\right)^2 = -\frac{1}{2}(-ax+b)^2$$

$$= -\frac{1}{2}a^2x^2 + abx - \frac{1}{2}b^2$$

$$= ab\,x - \frac{1}{2}(a^2x^2 - b^2)$$

$\therefore$, if $a > 0$ and $b = 1/a$ and $b > 0$ $\rightarrow$ $U(x) = x - \frac{1}{2}cx^2$

6. (continue)

c) Exponential: $U(x) = -e^{-ax}$ [Try $\gamma = -\infty$]

Set $\gamma = -\infty$, $b = 1$,

$$U(\gamma) = \frac{1-\gamma}{\gamma}\left(\frac{ax}{1-\gamma}+1\right)^{\gamma}$$

$$= \lim_{\gamma \to -\infty} \frac{1-\gamma}{\gamma}\left(\frac{ax}{1-\gamma}+1\right)^{\gamma}$$

$$= -1 \lim_{\gamma \to -\infty} e^{\gamma \ln\left(\frac{ax}{1-\gamma}+1\right)} = -\lim_{\gamma \to -\infty} e^{\left[\ln\left(\frac{ax}{1-\gamma}+1\right)/(1/\gamma)\right]}$$

$$U(x) = -e^{-ax}$$

d) Power: $U(x) = cx^{\gamma}$

Set $b = 0$, $U(x) = \frac{1-\gamma}{\gamma}\left(\frac{ax}{1-\gamma}\right)^{\gamma} = \frac{(1-\gamma)^{1-\gamma} a^{\gamma} x^{\gamma}}{\gamma}$

$\therefore$, if $\gamma < 1 \rightarrow U(x) = \frac{(1-\gamma)^{1-\gamma} a^{\gamma}}{\gamma} x^{\gamma} = cx^{\gamma}$

e) Logarithmic: $U(x) = \ln x$ [Try $U(x) = (1-\gamma)^{1-\gamma}((x^{\gamma}-1)/\gamma)$]

Set $a = 1$, $b = 0$:

$$U(x) = \frac{1-\gamma}{\gamma}\left(\frac{x}{1-\gamma}\right)^{\gamma} = \frac{(1-\gamma)^{1-\gamma} x^{\gamma}}{\gamma}$$

6. (continue)

Arrow-Pratt risk aversion coefficient

$$a(x) = -\frac{U''(x)}{U'(x)}$$

$$U'(x) = \frac{d}{dx}\left(\frac{1-\gamma}{\gamma}\left(\frac{ax}{1-\gamma}+b\right)^{\gamma}\right)$$

$$u = \left(\frac{ax}{1-\gamma}+b\right)$$

$$= \frac{1-\gamma}{\gamma}\ \frac{d}{du}u^{\gamma}\ \frac{d}{dx}\ \frac{ax}{1-\gamma}+b$$

$$= \frac{1-\gamma}{\gamma}\left(\gamma u^{\gamma-1}\right)\left(\frac{a}{1-\gamma}\right) = \frac{\cancel{1-\gamma}}{\cancel{\gamma}}\left(\cancel{\gamma}\left(\frac{ax}{1-\gamma}+b\right)^{\gamma-1}\right)\left(\frac{a}{\cancel{1-\gamma}}\right)$$

$$= a\left(\frac{ax}{1-\gamma}+b\right)^{\gamma-1}$$

$$U''(x) = \frac{d}{dx}\left(U'(x)\right) = \frac{d}{dx}\left(a\left(\frac{ax}{1-\gamma}+b\right)^{\gamma-1}\right)$$

$$= a\frac{d}{dx}\left(\frac{ax}{1-\gamma}+b\right)^{\gamma-1}$$

$$u = \frac{ax}{1-\gamma}+b$$

$$= a\frac{d}{du}u^{\gamma-1}\frac{d}{dx}\left(\frac{ax}{1-\gamma}+b\right) = a(\gamma-1)u^{\gamma-2}\left(\frac{a}{1-\gamma}\right) = a(\gamma-1)\left(\frac{ax}{1-\gamma}+b\right)^{\gamma-2}\left(\frac{a}{1-\gamma}\right)$$

$$= -a^2\left(\frac{ax}{1-\gamma}+b\right)^{\gamma-2}$$

$$\therefore\ a(x) = +\frac{+a^{\cancel{2}}\left(\frac{ax}{1-\gamma}+b\right)^{\gamma-2}}{\cancel{a}\left(\frac{ax}{1-\gamma}+b\right)^{\gamma-1}} = a\left(\frac{ax}{1-\gamma}+b\right)^{-1} = \frac{1}{\frac{x}{1-\gamma}+\frac{b}{a}}$$

$$\therefore\ \frac{1}{(cx+d)} = \frac{1}{\frac{1}{1-\gamma}x+\frac{b}{a}}$$ where $c = \frac{1}{1-\gamma}$ and $d = \frac{b}{a}$