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HW 3

8 General fitting

expected returns

$$p = P\bar{r} + e$$

model P $m \times n$ matrix

$\bar{r}$ n-dimensional vector

p and e are m-dimensional vectors

vector p - set of observation values

vector e - a vector of error having zero mean

Q · error vector's covariance matrix

the best (minimum variance) estimate of $\bar{r}$

$$\hat{\bar{r}} = (P^T Q^{-1} P)^{-1} P^T Q^{-1} p$$

a) $p = \bar{r} + e$

which means $P = 1$ ∴, $P^T = 1$

$$\hat{\bar{r}} = (1\, Q^{-1}\, 1)^{-1}\, (1)\, Q^{-1} p$$

$$= Q\, Q^{-1} p$$

$$\hat{\bar{r}} = p$$

b) two uncorrelated measurements with values p_1 and p_2, having variances σ_1^2 and σ_2^2

$$Q = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \quad ∴, Q^{-1} = \frac{1}{\sigma_1^2 \sigma_2^2} \begin{bmatrix} \sigma_2^2 & 0 \\ 0 & \sigma_1^2 \end{bmatrix}$$

we let $P = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ∴, $P^T = \begin{bmatrix} 1 & 1 \end{bmatrix}$

$$\hat{\bar{r}} = \left(\begin{bmatrix} 1 & 1 \end{bmatrix} \frac{1}{\sigma_1^2\sigma_2^2} \begin{bmatrix} \sigma_2^2 & 0 \\ 0 & \sigma_1^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 \end{bmatrix} \frac{1}{\sigma_1^2\sigma_2^2} \begin{bmatrix} \sigma_2^2 & 0 \\ 0 & \sigma_1^2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

$$= \left(\frac{1}{\sigma_1^2\sigma_2^2} \begin{bmatrix} \sigma_2^2 & \sigma_1^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} \sigma_2^2 & \sigma_1^2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \frac{1}{\sigma_1^2\sigma_2^2}$$

$$= \left(\frac{1}{\sigma_1^2\sigma_2^2} (\sigma_2^2 + \sigma_1^2) \right)^{-1} \frac{1}{\sigma_1^2\sigma_2^2} (p_1 \sigma_2^2 + p_2 \sigma_1^2)$$

$$\hat{\bar{r}} = \left(\frac{p_1}{\sigma_1^2} + \frac{p_2}{\sigma_2^2} \right) \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right)^{-1}$$

2. APT Factors

Two stocks are believed to satisfy the two-factor model

$$r_1 = a_1 + 2f_1 + f_2$$
$$r_2 = a_2 + 3f_1 + 4f_2$$

$r_f = 10\%$
$\bar{r}_1 = 15\%$
$\bar{r}_2 = 20\%$

$$\bar{r}_1 = \lambda_0 + 2\lambda_1 + \lambda_2$$
$$\bar{r}_2 = \lambda_0 + 3\lambda_1 + 4\lambda_2$$

From simple APT $\bar{r}_i = \lambda_0 + \sum_{j=1}^{m} b_{ij}\lambda_j$

For $i = 1, 2, \ldots n$

$\lambda_0 = 0.10$

$$0.15 = 0.10 + 2\lambda_1 + \lambda_2 \quad \text{-- ①}$$
$$0.20 = 0.10 + 3\lambda_1 + 4\lambda_2 \quad \text{-- ②}$$

$$0.60 = 0.40 + 8\lambda_1 + 4\lambda_2$$
$$0.20 = 0.10 + 3\lambda_1 + 4\lambda_2 \quad -$$

$$0.40 = 0.30 + 5\lambda_1$$
$$0.10 = 5\lambda_1$$
$$\lambda_1 = 0.02$$

$$0.20 = 0.10 + 3(0.02) + 4\lambda_2$$
$$0.20 - 0.16 = 4\lambda_2$$
$$\lambda_2 = 0.01$$

$\therefore$, $\lambda_0 = 0.10$
$\lambda_1 = 0.02$
$\lambda_2 = 0.01$

4. Variance estimate

Let r_i, for $i = 1, 2, \ldots, n$ be independent samples of a return r of mean $\bar{r}$ and variance σ^2.

Define the estimates $\hat{\bar{r}} = \frac{1}{n} \sum_{i=1}^{n} r_i$

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (r_i - \hat{\bar{r}})^2$$

Show that $E(s^2) = \sigma^2$

$$\bar{r} = E\left(\frac{1}{n} \sum_{i=1}^{n} r_i\right)$$

$$E(s^2) = E\left(\frac{1}{n-1} \sum_{i=1}^{n} (r_i - \hat{\bar{r}})^2\right)$$

$$= E\left(\frac{1}{n-1} \sum_{i=1}^{n} \left(r_i - \frac{1}{n} \sum_{j=1}^{n} r_j\right)^2\right)$$

$$= E\left(\frac{1}{n-1} \sum_{i=1}^{n} \left((r_i - \bar{r}) - \frac{1}{n} \sum_{j=1}^{n} (r_j - \bar{r})\right)^2\right)$$

we know, $E\left[\frac{1}{n} \sum_{i=1}^{n} (r_i - \bar{r})\right]^2 = \frac{1}{n} \sigma^2$

$$= \frac{1}{n-1} E\left(\sum_{i=1}^{n} \left(\left(\frac{n-1}{n}\right)(r_i - \bar{r}) - \frac{1}{n} \sum_{j \neq i} (r_i - \bar{r})\right)^2\right)$$

$$= \frac{1}{n-1}\left(\left(\frac{n-1}{n}\right)^2 + \frac{n-1}{n^2}\right) n\sigma^2$$

$$= \frac{n\sigma^2}{n-1}\left(\frac{(n-1)^2 + (n-1)}{n^2}\right) = \frac{\cancel{n}\sigma^2}{\cancel{n-1}} \frac{\cancel{(n-1)}(n-1+1)}{n^{\cancel{2}}} = \sigma^2 \frac{\cancel{n}}{\cancel{n}}$$

$$E(s^2) = \sigma^2$$