Lince Rumainum
Time Series Analysis

Midterm - II

Estimate parameters a and b in the model:

$$y = a + e^{-bx}$$

from a set of m samples (y_i, x_i), $1 \le i \le m$.

Let

$$error_i = y_i - (a + e^{-bx_i})$$ denotes the error in the i^{th} relation

If $error = (error_1, error_2, \ldots, error_m)^T \in \mathbb{R}^m$, then

$$error = (y - X\beta)$$

We have the sum of squared error

$$f(\beta) = \sum_{i=1}^{m} error_i^2 = \sum_{i=1}^{m} \left[y_i - (a + e^{-bx_i}) \right]^2$$

We consider

$$J(a,b) = \frac{1}{m} \sum_{i=1}^{m} \left[y_i - (a + e^{-bx_i}) \right]^2$$

We then differentiate with respect to a and b and set it equal to zero

$$\frac{\partial J}{\partial a} = 2\frac{1}{m} \sum_{i=1}^{m} \left[y_i - (a + e^{-bx_i}) \right] (-1) = -\frac{2}{m} \sum_{i=1}^{m} \left[y_i - (a + e^{-bx_i}) \right] = 0 \quad \cdots ①$$

$$\frac{\partial J}{\partial b} = 2\frac{1}{m} \sum_{i=1}^{m} \left[y_i - (a + e^{-bx_i}) \right] (-e^{-bx_i} \cdot -x_i) = \frac{2}{m} \sum_{i=1}^{m} \left[e^{-bx_i} x_i \left(y_i - (a + e^{-bx_i}) \right) \right] = 0 \quad \cdots ②$$

From eq. 1

$$\left[\sum_{i=1}^{m} y_i = ma + \sum_{i=1}^{m} e^{-bx_i} \right] \cdot \frac{1}{m} \quad \cdots ③$$

From eq. 2

$$\left[\sum_{i=1}^{m} e^{-bx_i} x_i y_i = \left(\sum_{i=1}^{m} e^{-bx_i} x_i \right) a + \sum_{i=1}^{m} e^{-bx_i} \cdot e^{-bx_i} \cdot x_i \right] \cdot \frac{1}{m} \quad \cdots ④$$

Let $\bar{y} = \frac{1}{m} \sum_{i=1}^{m} y_i$, eq. 3 becomes:

$$\bar{y} = a + \frac{1}{m} \sum_{i=1}^{m} e^{-bx_i}$$, then we solve for a

$$a = \bar{y} - \frac{1}{m} \sum_{i=1}^{m} e^{-bx_i} \quad \cdots ⑤$$

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From eq. 4, we solve for a

$$\frac{1}{m}\left[\sum_{i=1}^{m} e^{-bx_i} x_i y_i - \sum_{i=1}^{m} e^{-2bx_i} x_i\right] = \frac{1}{m}\left[\sum_{i=1}^{m} e^{-bx_i} x_i\right] a$$

$$a = \frac{\sum_{i=1}^{m} e^{-bx_i} x_i y_i - \sum_{i=1}^{m} e^{-2bx_i} x_i}{\sum_{i=1}^{m} e^{-bx_i} x_i} \quad \ldots (6)$$

Set eq. 5 and eq. 6 equal to each other we got:

$$\bar{y} - \frac{1}{m}\sum_{i=1}^{m} e^{-bx_i} = \frac{\sum_{i=1}^{m} e^{-bx_i} x_i y_i - \sum_{i=1}^{m} e^{-2bx_i} x_i}{\sum_{i=1}^{m} e^{-bx_i} x_i} \quad \ldots (7)$$

Eq. 7 is used to find the optimal estimates of the slopes "b"
(x_i, y_i, and m assume to be known values)

And the optimal estimates of intercept a can be found using

$$\hat{a} = \bar{y} - \frac{1}{m}\sum_{i=1}^{m} e^{-\hat{b}x_i} \quad \ldots (8)$$

where $\hat{b}$ is the optimal estimates of the slop found in eq. 7 and

$$\bar{y} = \frac{1}{m}\sum_{i=1}^{m} y_i$$