Lince Rumainum
DSA 5203
Time Series Analysis
Homework 5

Problem #1-a

For the ***MA (1) model*** where y_t is a stationary, ergodic process

$$y_t = \mu + \epsilon_t + \theta\epsilon_{t-1}$$

Where the mean, $\boldsymbol{E}(y_t) = \mu, \theta$ is the MA (1) parameter, and ϵ_t is a white noise sequence.

The properties of ϵ_t:

Mean:

$\boldsymbol{E}(\epsilon_t) = 0,$

$\boldsymbol{E}(\epsilon_t\epsilon_{t-k}) = 0$, for all $\boldsymbol{k} \geq \boldsymbol{1}$

Variance: $\boldsymbol{Var}(\epsilon_t) = \sigma^2$

Looking at the second-order properties of MA (1), we have:

Mean: $\boldsymbol{E}(y_t) = \mu$

Variance:

$$\begin{aligned}
\gamma_0 = \boldsymbol{Var}(y_t) &= \boldsymbol{E}\,[y_t - \boldsymbol{E}(y_t)]^2 \\
&= \boldsymbol{E}\,[y_t - \mu]^2 \\
&= \boldsymbol{E}\,[(\mu + \epsilon_t + \theta\epsilon_{t-1}) - \mu]^2 = \boldsymbol{E}\,[\epsilon_t + \theta\epsilon_{t-1}]^2 \\
&= \boldsymbol{E}\,[\epsilon_t{}^2 + 2\theta\epsilon_t\epsilon_{t-1} + \theta^2\epsilon_{t-1}^2] \\
&= \sigma^2 + 0 + \theta^2\sigma^2 \\
\gamma_0 &= (1 + \theta^2)\,\sigma^2
\end{aligned}$$

Autocovariance:

$$\begin{aligned}
\gamma_j &= \boldsymbol{E}\,[(y_t - \mu)(y_{t-j} - \mu)] \\
&= \boldsymbol{E}\,[((\mu + \epsilon_t + \theta\epsilon_{t-1}) - \mu)((\mu + \epsilon_{t-j} + \theta\epsilon_{t-j-1}) - \mu)] \\
&= \boldsymbol{E}\,[(\epsilon_t + \theta\epsilon_{t-1})(\epsilon_{t-j} + \theta\epsilon_{t-j-1})]
\end{aligned}$$

$$\gamma_j = \boldsymbol{E}\,[\epsilon_t\epsilon_{t-j} + \theta\epsilon_t\epsilon_{t-j-1} + \theta\epsilon_{t-1}\epsilon_{t-j} + \theta^2\epsilon_{t-1}\epsilon_{t-j-1}]$$

$\therefore$, If $j = 1$,

$$\gamma_1 = \boldsymbol{E}\,[\epsilon_t\epsilon_{t-1} + \theta\epsilon_t\epsilon_{t-2} + \theta\epsilon_{t-1}\epsilon_{t-1} + \theta^2\epsilon_{t-1}\epsilon_{t-2}]$$

Using the properties of ϵ_t, we get:

$\gamma_1 = \boldsymbol{E}\,[\theta\epsilon_{t-1}^2] = \theta\sigma^2$

Else $for\ j \geq 2\,, \gamma_j = 0$

$\therefore,\ \sum_{j=0}^{\infty} |\gamma_j| = \gamma_0 + \gamma_1 = ((1+\theta^2)\,\sigma^2) + \theta\sigma^2 = (1+\theta+\theta^2)\,\sigma^2 < \infty$

$\therefore$, we have the ***autocorrelation function (ACF) of MA (1)*** as:

$\rho_j = \frac{\gamma_j}{\gamma_0}$

For $j = 0, \rho_0 = \frac{\gamma_0}{\gamma_0} = \frac{(1+\theta^2)\,\sigma^2}{(1+\theta^2)\,\sigma^2} = 1$

For $j = 1, \rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\theta\sigma^2}{(1+\theta^2)\,\sigma^2} = \frac{\theta}{(1+\theta^2)}$

For $j \geq 2, \rho_j = 0$

For the ***MA (2) model*** where y_t is a stationary, ergodic process

$$y_t = \mu + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}$$

Where the mean, $\boldsymbol{E}(y_t) = \mu, \theta$ is the MA (2) parameter, and ϵ_t is a white noise sequence.

The properties of ϵ_t:

Mean:

$\boldsymbol{E}(\epsilon_t) = 0$,

$\boldsymbol{E}(\epsilon_t\epsilon_{t-k}) = 0$, for all $\boldsymbol{k} \geq \boldsymbol{1}$

Variance: $\boldsymbol{Var}(\epsilon_t) = \sigma^2$

Looking at the second-order properties of MA (2), we have:

Mean: $\boldsymbol{E}(y_t) = \mu$

Variance:

$$\begin{aligned}
\gamma_0 = \boldsymbol{Var}(y_t) &= \boldsymbol{E}\,[y_t - \boldsymbol{E}(y_t)]^2 \\
&= \boldsymbol{E}\,[y_t - \mu]^2 \\
&= \boldsymbol{E}\,[(\mu + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}) - \mu]^2 = \boldsymbol{E}\,[\,\epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}]^2 \\
&= \boldsymbol{E}\,[\,{\epsilon_t}^2 + 2\theta_1\epsilon_t\,\epsilon_{t-1} + 2\theta_2\epsilon_t\epsilon_{t-2} + \theta_1^2\epsilon_{t-1}^2 + 2\theta_1\theta_2\epsilon_{t-1}\epsilon_{t-2} + \theta_2^2\epsilon_{t-2}^2] \\
&= \sigma^2 + 0 + 0 + \theta_1^2\sigma^2 + 0 + \theta_2^2\sigma^2 \\
\gamma_0 &= (1 + \theta_1^2 + \theta_2^2)\,\sigma^2
\end{aligned}$$

Autocovariance:

$\gamma_j = \boldsymbol{E}\,[(y_t - \mu)(y_{t-j} - \mu)]$

$= \boldsymbol{E}\,[((\mu + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}) - \mu)((\mu + \epsilon_{t-j} + \theta_1\epsilon_{t-j-1} + \theta_2\epsilon_{t-j-2}) - \mu)]$

$= \boldsymbol{E}\,[(\epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2})(\epsilon_{t-j} + \theta_1\epsilon_{t-j-1} + \theta_2\epsilon_{t-j-2})]$

$\gamma_j = \boldsymbol{E}\,[\epsilon_t\epsilon_{t-j} + \theta_1\epsilon_t\epsilon_{t-j-1} + \theta_2\epsilon_t\epsilon_{t-j-2} + \theta_1\epsilon_{t-1}\epsilon_{t-j} + \theta_1^2\epsilon_{t-1}\epsilon_{t-j-1} + \theta_1\theta_2\epsilon_{t-1}\epsilon_{t-j-2} + \theta_2\epsilon_{t-2}\epsilon_{t-j} + \theta_1\theta_2\epsilon_{t-2}\epsilon_{t-j-1} + \theta_2^2\epsilon_{t-2}\epsilon_{t-j-2}]$

∴, If $\boldsymbol{j = 1}$,

$\gamma_1 = \boldsymbol{E}\,[\epsilon_t\epsilon_{t-1} + \theta_1\epsilon_t\epsilon_{t-2} + \theta_2\epsilon_t\epsilon_{t-3} + \theta_1\epsilon_{t-1}\epsilon_{t-1} + \theta_1^2\epsilon_{t-1}\epsilon_{t-2} + \theta_1\theta_2\epsilon_{t-1}\epsilon_{t-3} + \theta_2\epsilon_{t-2}\epsilon_{t-1} + \theta_1\theta_2\epsilon_{t-2}\epsilon_{t-2} + \theta_2^2\epsilon_{t-2}\epsilon_{t-3}]$

Using the properties of ϵ_t, we get:

$\gamma_1 = \boldsymbol{E}\,[\theta_1\epsilon_{t-1}^2 + \theta_1\theta_2\epsilon_{t-2}^2] = \theta_1\sigma^2 + \theta_1\theta_2\sigma^2$

$\gamma_1 = (\theta_1 + \theta_1\theta_2)\sigma^2$

∴, If $\boldsymbol{j = 2}$,

$\gamma_2 = \boldsymbol{E}\,[\epsilon_t\epsilon_{t-2} + \theta_1\epsilon_t\epsilon_{t-3} + \theta_2\epsilon_t\epsilon_{t-4} + \theta_1\epsilon_{t-1}\epsilon_{t-2} + \theta_1^2\epsilon_{t-1}\epsilon_{t-3} + \theta_1\theta_2\epsilon_{t-1}\epsilon_{t-3} + \theta_2\epsilon_{t-2}\epsilon_{t-2} + \theta_1\theta_2\epsilon_{t-2}\epsilon_{t-3} + \theta_2^2\epsilon_{t-2}\epsilon_{t-4}]$

Using the properties of ϵ_t, we get:

$\gamma_2 = \boldsymbol{E}\,[\theta_2\epsilon_{t-2}^2]$

$\gamma_2 = \theta_2\sigma^2$

∴, Else $for\ \boldsymbol{j \geq 3}$, $\gamma_j = 0$

∴, $\sum_{j=0}^{\infty} |\gamma_j| = \gamma_0 + \gamma_1 + \gamma_2 = ((1 + \theta_1^2 + \theta_2^2)\,\sigma^2) + ((\theta_1 + \theta_1\theta_2)\sigma^2) + (\theta_2\sigma^2)$

$= (1 + \theta_1 + \theta_2 + \theta_1\theta_2 + +\theta_1^2 + \theta_2^2)\,\sigma^2 < \infty$

∴, we have the ***autocorrelation function (ACF) of MA (2)*** as:

$\rho_j = \frac{\gamma_j}{\gamma_0}$

For $j = 0, \rho_0 = \frac{\gamma_0}{\gamma_0} = \frac{(1+\theta_1^2+\theta_2^2)\,\sigma^2}{(1+\theta_1^2+\theta_2^2)\,\sigma^2} = 1$

For $j = 1, \rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{(\theta_1+\theta_1\theta_2)\sigma^2}{(1+\theta_1^2+\theta_2^2)\,\sigma^2} = \frac{(\theta_1+\theta_1\theta_2)}{(1+\theta_1^2+\theta_2^2)}$

For $j = 2, \rho_1 = \frac{\gamma_2}{\gamma_0} = \frac{\theta_2\sigma^2}{(1+\theta_1^2+\theta_2^2)\,\sigma^2} = \frac{\theta_2}{(1+\theta_1^2+\theta_2^2)}$

For $j \geq 3, \rho_j = 0$

Problem #1-b

For the ***AR (1) model*** where y_t is a stationary, ergodic process

$$y_t = \mathrm{c} + \phi \mathrm{y}_{t-1} + \epsilon_t\,, \qquad \text{for } |\phi| < 1$$

Where the mean, $\boldsymbol{E}(y_t) = \mu$ for all t, ϕ is the AR (1) parameter, and ϵ_t is a white noise sequence.

The properties of ϵ_t:

Mean:

$\boldsymbol{E}(\epsilon_t) = 0,$

$\boldsymbol{Cov}(\epsilon_{\mathrm{i}}, \epsilon_t) = 0$, for all $\boldsymbol{i} \neq \boldsymbol{t}$

Variance: $\boldsymbol{Var}(\epsilon_t) = \sigma^2$

Looking at the second-order properties of AR (1), we have:

Mean: $\boldsymbol{E}(y_t) = \mu$

To find μ, we can use the y_t equation:

$\mu = \boldsymbol{E}(y_t) = \mathrm{c} + \phi \boldsymbol{E}(\mathrm{y}_{t-1}) + \boldsymbol{E}(\epsilon_t) = \mathrm{c} + \phi\mu + 0$

$\therefore,\ \mu = \frac{\mathrm{c}}{1-\phi}$

Variance:

Substituting c from where we find μ,

$y_t = (\mu - \phi\mu) + \phi \mathrm{y}_{t-1} + \epsilon_t$

$y_t - \mu = \phi\, (\mathrm{y}_{t-1} - \mu) + \epsilon_t$

$$\boldsymbol{\gamma_0} = \boldsymbol{Var}(y_t) = \boldsymbol{E}\,[y_t - \boldsymbol{E}(y_t)]^2$$

$$= \boldsymbol{E}\,[y_t - \mu]^2$$

$$= \boldsymbol{E}\,[\phi\, (\mathrm{y}_{t-1} - \mu) + \epsilon_t]^2 = \boldsymbol{E}\,[\,\phi^2\, (\mathrm{y}_{t-1} - \mu) + {\epsilon_t}^2 + 2\phi\, (\mathrm{y}_{t-1} - \mu)]^2$$

$$= \boldsymbol{E}\,[\,\phi^2\, (\mathrm{y}_{t-1} - \mu)^2 + 2\phi\, (\mathrm{y}_{t-1} - \mu)\epsilon_t + {\epsilon_t}^2]$$

$$= \phi^2 \boldsymbol{E}\,[\,(\mathrm{y}_{t-1} - \mu)^2] + 0 + \boldsymbol{E}\,[{\epsilon_t}^2]$$

$$\gamma_0 = \phi^2 \gamma_0 + \sigma^2$$

$$\gamma_0 = \frac{\sigma^2}{1-\phi^2}$$

Autocovariance:

$$\gamma_j = \boldsymbol{E}\,[(y_t - \mu)(y_{t-j} - \mu)] = \boldsymbol{E}[(\,\phi\, (\mathrm{y}_{t-1} - \mu) + \epsilon_t)(\,y_{t-j} - \mu)]$$

$$= \boldsymbol{E}\,[\phi\, (\mathrm{y}_{t-1} - \mu)(\mathrm{y}_{t-\mathrm{j}} - \mu) + \epsilon_t(\mathrm{y}_{t-\mathrm{j}} - \mu)]$$

$= \phi \boldsymbol{E} [(y_{t-1} - \mu)(y_{t-j} - \mu) + 0]$

$= \phi\, \gamma_{j-1}$

Iterating γ_j,

$\gamma_j = \phi^j \gamma_0 = \phi^j \left(\frac{\sigma^2}{1-\phi^2}\right)$

∴, we have the ***autocorrelation function (ACF) of AR (1)*** as:

$\rho_j = \frac{\gamma_j}{\gamma_0} = \frac{\phi^j \gamma_0}{\gamma_0}$

$\rho_j = \phi^j$

For the ***AR (2) model*** where y_t is a stationary process

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t , \qquad \text{for } |\lambda| < 1$$

Where the mean, $\boldsymbol{E}(y_t) = \mu$ for all t, ϕ is the AR (1) parameter, and ϵ_t is a white noise sequence.

For y_t to be stable, the roots of the characteristic equation lie within the unit circle in the complex plane. Using λ, we can describe the characteristic equation for y_t as follow:

$\lambda^2 - \phi_1 \lambda - \phi_2 = 0$

$\lambda = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2}$, where $|\lambda| < 1$ for stationary process.

The properties of ϵ_t:

Mean:

$\boldsymbol{E}(\epsilon_t) = 0$,

$\boldsymbol{Cov}(\epsilon_i, \epsilon_t) = 0$, for all $\boldsymbol{i} \neq \boldsymbol{t}$

Variance: $\boldsymbol{Var}(\epsilon_t) = \sigma^2$

Looking at the second-order properties of AR (1), we have:

Mean: $\boldsymbol{E}(y_t) = \mu$

To find μ, we can use the y_t equation:

$\mu = \boldsymbol{E}(y_t) = c + \phi_1 \boldsymbol{E}(y_{t-1}) + \phi_2 \boldsymbol{E}(y_{t-2}) + \boldsymbol{E}(\epsilon_t) = c + \phi_1 \mu + \phi_2 \mu + 0$

∴, $\mu = \frac{c}{1-\phi_1-\phi_2}$

Variance:

Substituting c from where we find μ,

$$y_t = (\mu - \phi_1\mu - \phi_2\mu) + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t$$

$$y_t - \mu = \phi_1 (y_{t-1} - \mu) + \phi_2 (y_{t-2} - \mu) + \epsilon_t$$

$$\boldsymbol{\gamma_0} = \boldsymbol{Var}(y_t) = \boldsymbol{E}[y_t - \boldsymbol{E}(y_t)]^2$$

$$= \boldsymbol{E}[y_t - \mu]^2$$

$$= \boldsymbol{E}[(\phi_1 (y_{t-1} - \mu) + \phi_2 (y_{t-2} - \mu) + \epsilon_t)(y_t - \mu)]$$

$$= \boldsymbol{E}[(\phi_1 (y_t - \mu)(y_{t-1} - \mu) + \phi_2 (y_t - \mu)(y_{t-2} - \mu) + (y_t - \mu)\epsilon_t)]$$

$$= \boldsymbol{E}[(\phi_1 (y_t - \mu)(y_{t-1} - \mu)] + \boldsymbol{E}[\phi_2 (y_t - \mu)(y_{t-2} - \mu)] + \boldsymbol{E}[(y_t - \mu)\epsilon_t)]$$

$$= \phi_1\gamma_1 + \phi_2\gamma_2 + \boldsymbol{E}[(\phi_1 (y_{t-1} - \mu) + \phi_2 (y_{t-2} - \mu) + \epsilon_t)\,\epsilon_t)]$$

$$= \phi_1\gamma_1 + \phi_2\gamma_2 + \boldsymbol{E}[(\phi_1 (y_{t-1} - \mu)\epsilon_t + \phi_2 (y_{t-2} - \mu)\epsilon_t + \epsilon_t\epsilon_t)]$$

$$= \phi_1\gamma_1 + \phi_2\gamma_2 + 0 + 0 + \sigma^2$$

$$\boldsymbol{\gamma_0} = \phi_1\gamma_1 + \phi_2\gamma_2 + \sigma^2$$

Autocovariance:

$$\gamma_j = \boldsymbol{E}[\phi_1 (y_t - \mu)(y_{t-j} - \mu)] = \boldsymbol{E}[(\phi_1 (y_{t-1} - \mu) + \phi_2 (y_{t-2} - \mu) + \epsilon_t)(y_{t-j} - \mu)]$$

$$= \phi_1 \boldsymbol{E}[(y_{t-1} - \mu)(y_{t-j} - \mu) + \phi_2 \boldsymbol{E}[(y_{t-2} - \mu)(y_{t-j} - \mu)] + \boldsymbol{E}[\epsilon_t (y_{t-j} - \mu)]$$

$\boldsymbol{\gamma_j} = \phi_1 \gamma_{j-1} + \phi_2 \gamma_{j-2}$, for $j \geq 1$

∴, we have the ***autocorrelation function (ACF) of AR (2)*** as:

$$\rho_j = \frac{\gamma_j}{\gamma_0} = \frac{\phi_1 \gamma_{j-1} + \phi_2 \gamma_{j-2}}{\gamma_0}$$

$$= \phi_1 \frac{\gamma_{j-1}}{\gamma_0} + \phi_2 \frac{\gamma_{j-2}}{\gamma_0}$$

$\rho_j = \phi_1 \rho_{j-1} + \phi_2 \rho_{j-2}$, for $j \geq 1$

For $j = 0, \rho_0 = \frac{\gamma_0}{\gamma_0} = \frac{\phi_1\gamma_1 + \phi_2\gamma_2 + \sigma^2}{\phi_1\gamma_1 + \phi_2\gamma_2 + \sigma^2} = 1$

Using $\rho_k = \rho_{-k}$

For $j = 1$,

$$\rho_1 = \phi_1 \rho_{1-1} + \phi_2 \rho_{1-2} = \phi_1 \rho_0 + \phi_2 \rho_1$$

$$\rho_1 = \frac{\phi_1}{1 - \phi_2}$$

For $j = 2, \rho_2 = \phi_1 \rho_{2-1} + \phi_2 \rho_{2-2} = \phi_1 \rho_1 + \phi_2$

For $j \geq 3, \rho_j = \phi_1 \rho_{j-1} + \phi_2 \rho_{j-2}$

Note: for $\boldsymbol{\gamma_0} = \boldsymbol{Var}(y_t) = \phi_1\gamma_1 + \phi_2\gamma_2 + \sigma^2$, since we now know $\rho_1$and ρ_2, we can find γ_1 and γ_2.

$$\boldsymbol{\gamma_0} = \phi_1\left(\frac{\phi_1}{1-\phi_2}\right)\gamma_0 + \phi_2(\phi_1\rho_1 + \phi_2)\gamma_0 + \sigma^2$$

$$\boldsymbol{\gamma_0}\left(1 - \phi_1\left(\frac{\phi_1}{1-\phi_2}\right) - \phi_2\left(\phi_1\left(\frac{\phi_1}{1-\phi_2}\right) + \phi_2\right)\right) = \sigma^2$$

$$\boldsymbol{\gamma_0}\left(\frac{(1-\phi_2) - {\phi_1}^2 - \phi_2\,{\phi_1}^2 - {\phi_2}^2(1-\phi_2)}{(1-\phi_2)}\right) = \sigma^2$$

$$\boldsymbol{\gamma_0} = \frac{(1-\phi_2)\,\sigma^2}{(1+\phi_2)[(1-\phi_2)^2 - \phi_1^2]}$$