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DSA 5203
Time Series Analysis
Midterm Exam

Problem #1

Standard error is the analog of standard deviation. So, to be able to find the standard deviation, we need to find the variance of the sample mean and the sample variance.

We let $\hat{x}(n)$ be an estimate for $\hat{x}$ of μ.

$\hat{x}(n) = \frac{1}{n}\sum_{i=1}^{n} x_i$ where x_i is a random variable and the estimator of the sample mean.

Sample mean $\boldsymbol{E}$, $\boldsymbol{E}(\hat{x}(n)) = \boldsymbol{E}\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] = \frac{1}{n}\boldsymbol{E}[\sum_{i=1}^{n} x_i] = \mu$

$\hat{x}(n)$ is unbiased and consistent.

In this case we assume the population mean, μ, is unknown and the population standard deviation, σ, is known.

We calculate the variance,

$$\boldsymbol{Var}(\hat{x}(n)) = \boldsymbol{E}(\hat{x}(n) - \mu)^2 = \boldsymbol{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) - \mu\right]^2 = \boldsymbol{E}\left[\frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)\right]^2 = \frac{1}{n^2}\sum_{i=1}^{n}\boldsymbol{E}(x_i - \mu)^2$$

$$= \frac{1}{n^2}(n\sigma^2) = \frac{\sigma^2}{n}$$

$\therefore$, standard error of the sample mean is $\sqrt{\boldsymbol{Var}(\hat{x}(n))} = \sqrt{\frac{\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}}$ **or** $\frac{s}{\sqrt{n}}$ where *s* is the sample standard deviation

Since *s* is a constant, the *standard error of the sample mean* = **O**$\left(\frac{1}{\sqrt{n}}\right)$

In the next case, we assume μ is known and σ is unknown.

We let $\hat{\sigma}^2$ be an estimate for σ^2

$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2$

Sample mean $\boldsymbol{E}$, $\boldsymbol{E}(\hat{\sigma}^2) = \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{E}(x_i - \mu)^2 = \frac{1}{n}(n\sigma^2) = \sigma^2$

$\hat{\sigma}^2$ is unbiased.

We calculate the variance,

$\boldsymbol{Var}(\hat{\sigma}^2) = \boldsymbol{E}[\hat{\sigma}^2 - \sigma^2]^2$

Let $y_i = x_i - \mu$, $\boldsymbol{E}(y_i) = 0, \boldsymbol{E}({y_i}^2) = \sigma^2$

$\boldsymbol{Var}(\hat{\sigma}^2) = \boldsymbol{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n} {y_i}^2\right) - \sigma^2\right]^2 = \boldsymbol{E}\left[\frac{1}{n}\sum_{i=1}^{n}({y_i}^2 - \sigma^2)\right]^2$

$$= \frac{1}{n^2} \boldsymbol{E}\left\{\sum_{i=1}^{n} (y_i{}^2 - \sigma^2)^2 + 2\sum_{i<j} (y_i{}^2 - \sigma^2)\left(y_j{}^2 - \sigma^2\right)\right\}$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \boldsymbol{E}(y_i{}^2 - \sigma^2)^2 + \frac{2}{n^2} \sum_{i<j} \boldsymbol{E}(y_i{}^2 - \sigma^2)\boldsymbol{E}\left(y_j{}^2 - \sigma^2\right)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \boldsymbol{E}(y_i{}^2 - \sigma^2)^2 \text{ since } y_i \text{ are } \boldsymbol{IID} \text{ and } \boldsymbol{E}(y_i{}^2 - \sigma^2) = 0$$

$$\boldsymbol{E}(y_i{}^2 - \sigma^2)^2 = \boldsymbol{E}(y_i{}^4) - 2\boldsymbol{E}(y_i{}^2)\sigma^2 + \sigma^4 = \boldsymbol{E}(y_i{}^4) - 2(\sigma^2)\sigma^2 + \sigma^4 = \boldsymbol{E}(y_i{}^4) - \sigma^4$$

Note: $\boldsymbol{E}(y_i{}^4) = 3\sigma^4 \longrightarrow$ kurtosis since $y_i \sim N(0, \sigma^2)$

$$\therefore, \boldsymbol{E}(y_i{}^2 - \sigma^2)^2 = \boldsymbol{E}(y_i{}^4) - \sigma^4 = 3\sigma^4 - \sigma^4 = 2\sigma^4$$

$$\therefore, \boldsymbol{Var}(\hat{\sigma}^2) = \frac{1}{n^2} \sum_{i=1}^{n} \boldsymbol{E}(2\sigma^4) = \frac{1}{n^2}(n2\sigma^4) = \frac{2\sigma^4}{n}$$

$\therefore$, standard error of the sample variance is $\sqrt{\boldsymbol{Var}(\hat{\sigma}^2)} = \sqrt{\frac{2\sigma^4}{n}} = \frac{\sqrt{2}\sigma^2}{\sqrt{n}}$ **or** $\frac{\sqrt{2}s^2}{\sqrt{n}}$ where *s* is the sample standard deviation

Since *s* is a constant, the *standard error of the sample variance* = $\mathbf{O}\left(\frac{1}{\sqrt{n}}\right)$

Problem 3-a:

For the ***MA (2) model*** where y_t is a stationary, ergodic process

$$y_t = \mu + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}$$

Where the mean, $\boldsymbol{E}(y_t) = \mu, \theta$ is the MA (2) parameter, and ϵ_t is a white noise sequence.

The properties of ϵ_t:

Mean:

$\boldsymbol{E}(\epsilon_t) = 0,$

$\boldsymbol{E}(\epsilon_t\epsilon_{t-k}) = 0$, for all $\boldsymbol{k \geq 1}$

Variance: $\boldsymbol{Var}(\epsilon_t) = \sigma^2$

Looking at the second-order properties of MA (2), we have:

Mean: $\boldsymbol{E}(y_t) = \mu$

Variance:

$$\gamma_0 = \boldsymbol{Var}(y_t) = \boldsymbol{E}\,[y_t - \boldsymbol{E}(y_t)]^2$$

$$= \boldsymbol{E}\,[y_t - \mu]^2$$

$$= \boldsymbol{E}\,[(\mu + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}) - \mu]^2 = \boldsymbol{E}\,[\epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}]^2$$

$$= \boldsymbol{E}\,[\epsilon_t{}^2 + 2\theta_1\epsilon_t\,\epsilon_{t-1} + 2\theta_2\epsilon_t\epsilon_{t-2} + \theta_1^2\epsilon_{t-1}^2 + 2\theta_1\theta_2\epsilon_{t-1}\epsilon_{t-2} + \theta_2^2\epsilon_{t-2}^2]$$

$$= \sigma^2 + 0 + 0 + \theta_1^2\sigma^2 + 0 + \theta_2^2\sigma^2$$

$$\gamma_0 = (1 + \theta_1^2 + \theta_2^2)\,\sigma^2$$

Autocovariance:

$$\gamma_j = \boldsymbol{E}\,[(y_t - \mu)(y_{t-j} - \mu)]$$

$$= \boldsymbol{E}\,[((\mu + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}) - \mu)((\mu + \epsilon_{t-j} + \theta_1\epsilon_{t-j-1} + \theta_2\epsilon_{t-j-2}) - \mu)]$$

$$= \boldsymbol{E}\,[(\epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2})(\epsilon_{t-j} + \theta_1\epsilon_{t-j-1} + \theta_2\epsilon_{t-j-2})]$$

$$\gamma_j = \boldsymbol{E}\,[\epsilon_t\epsilon_{t-j} + \theta_1\epsilon_t\epsilon_{t-j-1} + \theta_2\epsilon_t\epsilon_{t-j-2} + \theta_1\epsilon_{t-1}\epsilon_{t-j} + \theta_1^2\epsilon_{t-1}\epsilon_{t-j-1} + \theta_1\theta_2\epsilon_{t-1}\epsilon_{t-j-2} + \theta_2\epsilon_{t-2}\epsilon_{t-j} + \theta_1\theta_2\epsilon_{t-2}\epsilon_{t-j-1} + \theta_2^2\epsilon_{t-2}\epsilon_{t-j-2}]$$

$\therefore$, If $\boldsymbol{j = 1}$,

$$\gamma_1 = \boldsymbol{E}\,[\epsilon_t\epsilon_{t-1} + \theta_1\epsilon_t\epsilon_{t-2} + \theta_2\epsilon_t\epsilon_{t-3} + \theta_1\epsilon_{t-1}\epsilon_{t-1} + \theta_1^2\epsilon_{t-1}\epsilon_{t-2} + \theta_1\theta_2\epsilon_{t-1}\epsilon_{t-3} + \theta_2\epsilon_{t-2}\epsilon_{t-1} + \theta_1\theta_2\epsilon_{t-2}\epsilon_{t-2} + \theta_2^2\epsilon_{t-2}\epsilon_{t-3}]$$

Using the properties of ϵ_t, we get:

$$\gamma_1 = \boldsymbol{E}\,[\theta_1\epsilon_{t-1}^2 + \theta_1\theta_2\epsilon_{t-2}^2] = \theta_1\sigma^2 + \theta_1\theta_2\sigma^2$$

$\gamma_1 = (\theta_1 + \theta_1\theta_2)\sigma^2$

$\therefore$, If $\boldsymbol{j} = \mathbf{2}$,

$\gamma_2 = \boldsymbol{E}\,[\epsilon_t\epsilon_{t-2} + \theta_1\epsilon_t\epsilon_{t-3} + \theta_2\epsilon_t\epsilon_{t-4} + \theta_1\epsilon_{t-1}\epsilon_{t-2} + \theta_1^2\epsilon_{t-1}\epsilon_{t-3} + \theta_1\theta_2\epsilon_{t-1}\epsilon_{t-3} + \theta_2\epsilon_{t-2}\epsilon_{t-2} + \theta_1\theta_2\epsilon_{t-2}\epsilon_{t-3} + \theta_2^2\epsilon_{t-2}\epsilon_{t-4}]$

Using the properties of ϵ_t, we get:

$\gamma_2 = \boldsymbol{E}\,[\theta_2\epsilon_{t-2}^2]$

$\gamma_2 = \theta_2\sigma^2$

$\therefore$, Else $for\ \boldsymbol{j} \geq \mathbf{3}\,, \gamma_j = 0$

$\therefore$, $\sum_{j=0}^{\infty} |\gamma_j| = \gamma_0 + \gamma_1 + \gamma_2 = ((1 + \theta_1^2 + \theta_2^2)\,\sigma^2) + ((\theta_1 + \theta_1\theta_2)\sigma^2) + (\theta_2\sigma^2)$

$= (1 + \theta_1 + \theta_2 + \theta_1\theta_2 + +\theta_1^2 + \theta_2^2)\,\sigma^2 < \infty$

$\therefore$, we have the ***autocorrelation function (ACF) of MA (2)*** as:

$\rho_j = \frac{\gamma_j}{\gamma_0}$

For $j = 0, \rho_0 = \frac{\gamma_0}{\gamma_0} = \frac{(1+\theta_1^2+\theta_2^2)\,\sigma^2}{(1+\theta_1^2+\theta_2^2)\,\sigma^2} = 1$

For $j = 1, \rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{(\theta_1+\theta_1\theta_2)\sigma^2}{(1+\theta_1^2+\theta_2^2)\,\sigma^2} = \frac{(\theta_1+\theta_1\theta_2)}{(1+\theta_1^2+\theta_2^2)}$

For $j = 2, \rho_1 = \frac{\gamma_2}{\gamma_0} = \frac{\theta_2\sigma^2}{(1+\theta_1^2+\theta_2^2)\,\sigma^2} = \frac{\theta_2}{(1+\theta_1^2+\theta_2^2)}$

For $j \geq 3, \rho_j = 0$

Problem 3-b:

For the ***AR (2) model*** where y_t is a stationary process

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t , \qquad \text{for } |\lambda| < 1$$

Where the mean, $\boldsymbol{E}(y_t) = \mu \; for \; all \; t, \phi$ is the AR (1) parameter, and ϵ_t is a white noise sequence.

For y_t to be stable, the roots of the characteristic equation lie within the unit circle in the complex plane. Using λ, we can describe the characteristic equation for y_t as follow:

$\lambda^2 - \phi_1 \lambda - \phi_2 = 0$

$\lambda = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2}$, where $|\lambda| < 1$ for stationary process.

The properties of ϵ_t:

Mean:

$\boldsymbol{E}(\epsilon_t) = 0,$

$\boldsymbol{Cov}(\epsilon_i, \epsilon_t) = 0$, for all $\boldsymbol{i} \neq \boldsymbol{t}$

Variance: $\boldsymbol{Var}(\epsilon_t) = \sigma^2$

Looking at the second-order properties of AR (1), we have:

Mean: $\boldsymbol{E}(y_t) = \mu$

To find μ, we can use the y_t equation:

$\mu = \boldsymbol{E}(y_t) = c + \phi_1 \boldsymbol{E}(y_{t-1}) + \phi_2 \boldsymbol{E}(y_{t-2}) + \boldsymbol{E}(\epsilon_t) = c + \phi_1 \mu + \phi_2 \mu + 0$

$\therefore, \; \mu = \frac{c}{1 - \phi_1 - \phi_2}$

Variance:

Substituting c from where we find μ,

$y_t = (\mu - \phi_1 \mu - \phi_2 \mu) + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t$

$y_t - \mu = \phi_1 (y_{t-1} - \mu) + \phi_2 (y_{t-2} - \mu) + \epsilon_t$

$$\begin{aligned}
\boldsymbol{\gamma_0} = \boldsymbol{Var}(y_t) &= \boldsymbol{E}\,[y_t - \boldsymbol{E}(y_t)]^2 \\
&= \boldsymbol{E}\,[y_t - \mu]^2 \\
&= \boldsymbol{E}\,[(\phi_1 (y_{t-1} - \mu) + \phi_2 (y_{t-2} - \mu) + \epsilon_t)(y_t - \mu)] \\
&= \boldsymbol{E}\,[(\phi_1 (y_t - \mu)(y_{t-1} - \mu) + \phi_2 (y_t - \mu)(y_{t-2} - \mu) + (y_t - \mu)\epsilon_t)] \\
&= \boldsymbol{E}\,[(\phi_1 (y_t - \mu)(y_{t-1} - \mu)] + \boldsymbol{E}\,[\phi_2 (y_t - \mu)(y_{t-2} - \mu)] + \boldsymbol{E}\,[(y_t - \mu)\epsilon_t)] \\
&= \phi_1 \gamma_1 + \phi_2 \gamma_2 + \boldsymbol{E}\,[(\phi_1 (y_{t-1} - \mu) + \phi_2 (y_{t-2} - \mu) + \epsilon_t)\,\epsilon_t)]
\end{aligned}$$

$$= \phi_1\gamma_1 + \phi_2\gamma_2 + \boldsymbol{E}\,[(\phi_1\,(y_{t-1} - \mu)\epsilon_t + \phi_2\,(y_{t-2} - \mu)\epsilon_t + \epsilon_t\epsilon_t)]$$

$$= \phi_1\gamma_1 + \phi_2\gamma_2 + \mathbf{0} + \mathbf{0} + \sigma^2$$

$$\boldsymbol{\gamma_0} = \phi_1\gamma_1 + \phi_2\gamma_2 + \sigma^2$$

Autocovariance:

$$\gamma_j = \boldsymbol{E}\,[\phi_1(y_t - \mu)(y_{t-j} - \mu)] = \boldsymbol{E}[(\phi_1\,(y_{t-1} - \mu) + \phi_2\,(y_{t-2} - \mu) + \epsilon_t)(y_{t-j} - \mu)]$$

$$= \phi_1\boldsymbol{E}\,[(y_{t-1} - \mu)(y_{t-j} - \mu) + \phi_2\boldsymbol{E}\,[(y_{t-2} - \mu)(y_{t-j} - \mu)] + \boldsymbol{E}\,[\epsilon_t(y_{t-j} - \mu)]$$

$\boldsymbol{\gamma_j} = \phi_1\,\gamma_{j-1} + \phi_2\,\gamma_{j-2}$, for $j \geq 1$

∴, we have the ***autocorrelation function (ACF) of AR (2)*** as:

$$\rho_j = \frac{\gamma_j}{\gamma_0} = \frac{\phi_1\,\gamma_{j-1} + \phi_2\,\gamma_{j-2}}{\gamma_0}$$

$$= \phi_1\frac{\gamma_{j-1}}{\gamma_0} + \phi_2\frac{\gamma_{j-2}}{\gamma_0}$$

$\rho_j = \phi_1\,\rho_{j-1} + \phi_2\,\rho_{j-2}$, for $j \geq 1$

For $j = 0, \rho_0 = \frac{\gamma_0}{\gamma_0} = \frac{\phi_1\gamma_1 + \phi_2\gamma_2 + \sigma^2}{\phi_1\gamma_1 + \phi_2\gamma_2 + \sigma^2} = 1$

Using $\rho_k = \rho_{-k}$

For $j = 1$,

$$\rho_1 = \phi_1\,\rho_{1-1} + \phi_2\,\rho_{1-2} = \phi_1\,\rho_0 + \phi_2\,\rho_1$$

$$\rho_1 = \frac{\phi_1}{1-\phi_2}$$

For $j = 2, \rho_2 = \phi_1\,\rho_{2-1} + \phi_2\,\rho_{2-2} = \phi_1\,\rho_1 + \phi_2$

For $j \geq 3, \rho_j = \phi_1\,\rho_{j-1} + \phi_2\,\rho_{j-2}$

Note: for $\boldsymbol{\gamma_0} = \boldsymbol{Var}(y_t) = \phi_1\gamma_1 + \phi_2\gamma_2 + \sigma^2$, since we now know $\rho_1 and\ \rho_2$, we can find $\gamma_1\ and\ \gamma_2$.

$$\boldsymbol{\gamma_0} = \phi_1\left(\frac{\phi_1}{1-\phi_2}\right)\gamma_0 + \phi_2(\phi_1\rho_1 + \phi_2)\gamma_0 + \sigma^2$$

$$\boldsymbol{\gamma_0}\left(1 - \phi_1\left(\frac{\phi_1}{1-\phi_2}\right) - \phi_2\left(\phi_1\left(\frac{\phi_1}{1-\phi_2}\right) + \phi_2\right)\right) = \sigma^2$$

$$\boldsymbol{\gamma_0}\left(\frac{(1-\phi_2) - {\phi_1}^2 - \phi_2\,{\phi_1}^2 - {\phi_2}^2(1-\phi_2)}{(1-\phi_2)}\right) = \sigma^2$$

$$\boldsymbol{\gamma_0} = \frac{(1-\phi_2)\,\sigma^2}{(1+\phi_2)[(1-\phi_2)^2 - \phi_1^2]}$$

Now, we apply what we get for ACF for AR(2) into this function:

$$y_t = 0.6y_{t-1} + 0.3y_{t-2} + \epsilon_t$$

We know that $\phi_1 = 0.6$ and $\phi_2 = 0.3$, then we get:

$\boldsymbol{\rho_0} = \mathbf{1.0000}$

$\boldsymbol{\rho_1} = \frac{\phi_1}{1-\phi_2} = \frac{0.6}{1-0.3} = \mathbf{0.8571}$

$\boldsymbol{\rho_2} = \phi_1\,\rho_1 + \phi_2 = 0.6(0.8571) + 0.3 = \mathbf{0.8143}$

For $j \geq 3, \rho_j = \phi_1\,\rho_{j-1} + \phi_2\,\rho_{j-2}$

$\boldsymbol{\rho_3} = \phi_1\,\rho_2 + \phi_2\,\rho_1 = 0.6(0.8143) + 0.3(0.8571) = \mathbf{0.7457}$

$\boldsymbol{\rho_4} = \phi_1\,\rho_3 + \phi_2\,\rho_2 = 0.6(0.7457) + 0.3(0.8143) = \mathbf{0.6917}$

$\boldsymbol{\rho_5} = \phi_1\,\rho_4 + \phi_2\,\rho_3 = 0.6(0.6917) + 0.3(0.7457) = \mathbf{0.6387}$

$\boldsymbol{\rho_6} = \phi_1\,\rho_5 + \phi_2\,\rho_4 = 0.6(0.6387) + 0.3(0.6917) = \mathbf{0.5908}$

$\boldsymbol{\rho_7} = \phi_1\,\rho_6 + \phi_2\,\rho_5 = 0.6(0.5908) + 0.3(0.6387) = \mathbf{0.5461}$

$\boldsymbol{\rho_8} = \phi_1\,\rho_7 + \phi_2\,\rho_6 = 0.6(0.5461) + 0.3(0.5908) = \mathbf{0.5049}$

$\boldsymbol{\rho_9} = \phi_1\,\rho_8 + \phi_2\,\rho_7 = 0.6(5049) + 0.3(0.5461) = \mathbf{0.4667}$

$\boldsymbol{\rho_{10}} = \phi_1\,\rho_9 + \phi_2\,\rho_8 = 0.6(0.4667) + 0.3(0.5049) = \mathbf{0.4315}$

Etc.

Problem #3-c:

For the **ARMA (1,2) model** where y_t is,

$$y_t = c + \phi_1 y_{t-1} + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}$$

Where the mean, $\boldsymbol{E}(y_t) = \mu, \phi$ is the AR (1) parameter, θ is the MA (2) parameter, and ϵ_t is a white noise sequence.

The properties of ϵ_t:

Mean:

$\boldsymbol{E}(\epsilon_t) = 0,$

$\boldsymbol{E}(\epsilon_t\epsilon_{t-k}) = 0$, for all $\boldsymbol{k} \geq \boldsymbol{1}$

$\boldsymbol{Cov}(\epsilon_i, \epsilon_t) = 0$, for all $\boldsymbol{i} \neq \boldsymbol{t}$

Variance: $\boldsymbol{Var}(\epsilon_t) = \sigma^2$

Looking at the second-order properties of AR (1), we have:

To find μ, we can use the y_t equation:

$\mu = \boldsymbol{E}(y_t) = c + \phi_1\boldsymbol{E}(y_{t-1}) + \boldsymbol{E}(\epsilon_t) + \boldsymbol{E}(\theta_1\epsilon_{t-1}) + \boldsymbol{E}(\theta_2\epsilon_{t-2}) = c + \phi_1\mu + 0$

$\therefore,\ \mu = \frac{c}{1-\phi_1}$

Variance:

Substituting c from where we find μ,

$y_t = (\mu - \phi_1\mu) + \phi_1 y_{t-1} + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}$

$y_t - \mu = \phi_1\,(y_{t-1} - \mu) + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}$

Variance:

$$\begin{aligned}
\boldsymbol{\gamma_0} = \boldsymbol{Var}(y_t) &= \boldsymbol{E}\,[y_t - \boldsymbol{E}(y_t)]^2 \\
&= \boldsymbol{E}\,[y_t - \mu]^2 \\
&= \boldsymbol{E}\,[\phi_1\,(y_{t-1} - \mu) + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}]^2 \\
&= {\phi_1}^2\boldsymbol{E}\,[\,(y_{t-1} - \mu)]^2 + \boldsymbol{E}\,[{\epsilon_t}^2] + \theta_1^2\boldsymbol{E}[\epsilon_{t-1}^2] + \theta_2^2\boldsymbol{E}[\epsilon_{t-2}^2] + \\
&\quad \boldsymbol{E}\,[2\phi_1\,(y_{t-1} - \mu)\epsilon_t] + \boldsymbol{E}\,[2\phi_1\,(y_{t-1} - \mu)\theta_1\epsilon_{t-1}] + \boldsymbol{E}\,[2\phi_1\,(y_{t-1} - \mu)\theta_2\epsilon_{t-2}] + \\
&\quad \boldsymbol{E}\,[2\epsilon_t\theta_1\epsilon_{t-1}] + \boldsymbol{E}\,[2\epsilon_t\theta_2\epsilon_{t-2}] + \boldsymbol{E}\,[2\theta_1\epsilon_{t-1}\theta_2\epsilon_{t-2}]
\end{aligned}$$

Using the properties of ϵ_t, we get:

$$\begin{aligned}
&= {\phi_1}^2\gamma_0 + \sigma^2 + \theta_1^2\sigma^2 + \theta_2^2\sigma^2 + 0 + 2\phi_1\theta_1\sigma^2 + 0 + 0 + 0 + 0 \\
&= {\phi_1}^2\gamma_0 + \sigma^2 + \theta_1^2\sigma^2 + \theta_2^2\sigma^2 + 2\phi_1\theta_1\sigma^2
\end{aligned}$$

$$\gamma_0(1-{\phi_1}^2) \quad = \sigma^2[1+\theta_1^2+\theta_2^2+2\phi_1\theta_1]$$

$$\boldsymbol{\gamma_0} = \frac{\sigma^2[1+\theta_1^2+\theta_2^2+2\phi_1\theta_1]}{1-{\phi_1}^2}$$

When $\theta_1 = 0$ and $\theta_2 = 0 \Rightarrow \mathrm{AR}(1) \Rightarrow \gamma_0 = \frac{\sigma^2}{1-{\phi_1}^2}$

When $\phi_1 = 0 \Rightarrow \mathrm{MA}(2) \Rightarrow \gamma_0 = \sigma^2[1+\theta_1^2+\theta_2^2]$

$\gamma_1 = \boldsymbol{E}\,[(y_t - \mu)(y_{t-1} - \mu)] = \boldsymbol{E}\{\,[\phi_1\,(y_{t-1}-\mu) + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}][(y_{t-1} - \mu)]\}$

$\gamma_1 = \boldsymbol{E}[\phi_1\,(y_{t-1}-\mu)(y_{t-1} - \mu) + \epsilon_t(y_{t-1} - \mu) + \theta_1\epsilon_{t-1}(y_{t-1} - \mu) + \theta_2\epsilon_{t-2}(y_{t-1} - \mu)]$

$\gamma_1 = \phi_1\boldsymbol{E}[\,(y_{t-1}-\mu)^2] + \boldsymbol{E}[\epsilon_t(y_{t-1} - \mu)] + \theta_1\boldsymbol{E}[\epsilon_{t-1}(y_{t-1} - \mu)] + \theta_2\boldsymbol{E}[\epsilon_{t-2}(y_{t-1} - \mu)]$

$\gamma_1 = \phi_1\gamma_0 + \theta_1\sigma^2$

$\gamma_1 = \phi_1\left[\frac{\sigma^2[1+\theta_1^2+\theta_2^2+2\phi_1\theta_1]}{1-{\phi_1}^2}\right] + \theta_1\sigma^2$

$\gamma_1 = \left[\frac{\phi_1[1+\theta_1^2+\theta_2^2+2\phi_1\theta_1]}{1-{\phi_1}^2} + \theta_1\right]\sigma^2$

When $\theta_1 = 0$ and $\theta_2 = 0 \Rightarrow \mathrm{AR}(1) \Rightarrow \gamma_1 = \frac{\phi_1\sigma^2}{1-{\phi_1}^2}$

When $\phi_1 = 0 \Rightarrow \mathrm{MA}(2) \Rightarrow \gamma_1 = \theta_1\sigma^2$

$\gamma_2 = \boldsymbol{E}\,[(y_t - \mu)(y_{t-2} - \mu)] = \boldsymbol{E}\{\,[\phi_1\,(y_{t-1}-\mu) + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}][(y_{t-2} - \mu)]\}$

$\gamma_2 = \boldsymbol{E}[\phi_1\,(y_{t-1}-\mu)(y_{t-2} - \mu) + \epsilon_t(y_{t-1} - \mu) + \theta_1\epsilon_{t-1}(y_{t-2} - \mu) + \theta_2\epsilon_{t-2}(y_{t-2} - \mu)]$

$\gamma_2 = \phi_1\boldsymbol{E}[\,(y_{t-1}-\mu)(y_{t-2} - \mu)] + \boldsymbol{E}[\epsilon_t(y_{t-2} - \mu)] + \theta_1\boldsymbol{E}[\epsilon_{t-1}(y_{t-2} - \mu)] + \theta_2\boldsymbol{E}[\epsilon_{t-2}(y_{t-2} - \mu)]$

$\gamma_2 = \phi_1\gamma_1 + \theta_2\sigma^2$

$\gamma_2 = \phi_1\left[\left[\frac{\phi_1[1+\theta_1^2+\theta_2^2+2\phi_1\theta_1]}{1-{\phi_1}^2} + \theta_1\right]\sigma^2\right] + \theta_2\sigma^2$

$\gamma_2 = \left[\frac{{\phi_1}^2[1+\theta_1^2+\theta_2^2+2\phi_1\theta_1]}{1-{\phi_1}^2} + \phi_1\theta_1 + \theta_2\right]\sigma^2$

When $\theta_1 = 0$ and $\theta_2 = 0 \Rightarrow \mathrm{AR}(1) \Rightarrow \gamma_2 = \frac{\phi_1\sigma^2}{1-{\phi_1}^2}$

When $\phi_1 = 0 \Rightarrow \mathrm{MA}(2) \Rightarrow \gamma_2 = \theta_2\sigma^2$

$\gamma_3 = \boldsymbol{E}\,[(y_t - \mu)(y_{t-3} - \mu)] = \boldsymbol{E}\{\,[\phi_1\,(y_{t-1}-\mu) + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}][(y_{t-3} - \mu)]\}$

$\gamma_3 = \boldsymbol{E}[\phi_1\,(y_{t-1}-\mu)(y_{t-3} - \mu) + \epsilon_t(y_{t-1} - \mu) + \theta_1\epsilon_{t-1}(y_{t-3} - \mu) + \theta_2\epsilon_{t-2}(y_{t-3} - \mu)]$

$\gamma_3 = \phi_1\boldsymbol{E}[\,(y_{t-1}-\mu)(y_{t-3} - \mu)] + \boldsymbol{E}[\epsilon_t(y_{t-3} - \mu)] + \theta_1\boldsymbol{E}[\epsilon_{t-1}(y_{t-3} - \mu)] + \theta_2\boldsymbol{E}[\epsilon_{t-2}(y_{t-3} - \mu)]$

$\gamma_3 = \phi_1\gamma_2$

$\gamma_j = \phi_1\gamma_{j-1}\, for\, j \geq 3$ as in AR(1)

∴, we have the ***autocorrelation function (ACF) of ARMA (1,2)*** as:

$\rho_0 = \frac{\gamma_0}{\gamma_0} = 1$

$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\phi_1\gamma_0+\theta_1\sigma^2}{\gamma_0}$

$\rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{\phi_1\gamma_1+\theta_2\sigma^2}{\gamma_0}$

$\rho_{\mathrm{j}} = \frac{\phi_1\gamma_{j-1}}{\gamma_0}\, for\, j \geq 3$

Where

$$\boldsymbol{\gamma_0} = \frac{\sigma^2[1+\theta_1^2+\theta_2^2+2\phi_1\theta_1]}{1-{\phi_1}^2}$$

$$\gamma_1 = \phi_1\gamma_0 + \theta_1\sigma^2 = \left[\frac{\phi_1[1+\theta_1^2+\theta_2^2+2\phi_1\theta_1]}{1-{\phi_1}^2} + \theta_1\right]\sigma^2$$

$$\gamma_2 = \phi_1\gamma_1 + \theta_2\sigma^2 = \left[\frac{{\phi_1}^2[1+\theta_1^2+\theta_2^2+2\phi_1\theta_1]}{1-{\phi_1}^2} + \phi_1\theta_1 + \theta_2\right]\sigma^2$$

$$\gamma_j = \phi_1\gamma_{j-1}\, for\, j \geq 3$$