Module 11.2
Examples

S. Lakshmivarahan

School of Computer Science
University of Oklahoma
Norman, OK, 73071
USA

1) Consider the so called <u>Air-Line model</u>: $(0, 1, 1)\times(0, 1, 1)_{12}$ given by $(1\text{-}L)(1\text{-}L^{12})y_t =$ $(1+\theta_1 L)(1+\bar{\theta}_1 L^{12})\varepsilon_t$ with $\theta_1 = 0.740$ and $\bar{\theta}_1 = -0.671$, $\varepsilon_t \sim \text{WGN}(0, 1)$.
   (a) Compute ACF and PACF for this model.

Let $w_t = (1-L)(1-L^{12})y_t = (1+\theta_1 L)(1+\bar{\theta}_1 L^{12})\varepsilon_t$.

$$E(w_t) = E[(1+\theta_1 L)(1+\bar{\theta}_1 L^{12})\varepsilon_t] = E[\varepsilon_t + \theta_1\varepsilon_{t-1} + \bar{\theta}_1\varepsilon_{t-12} + \theta_1\bar{\theta}_1\varepsilon_{t-13}] = 0$$

$$\begin{aligned} E(w_t^2) &= E[(\varepsilon_t + \theta_1\varepsilon_{t-1} + \bar{\theta}_1\varepsilon_{t-12} + \theta_1\bar{\theta}_1\varepsilon_{t-13})^2] \\ &= E(\varepsilon_t^2) + \theta_1^2 E(\varepsilon_{t-1}^2) + \bar{\theta}_1^2 E(\varepsilon_{t-12}^2) + \theta_1^2\bar{\theta}_1^2 E(\varepsilon_{t-13}^2) = 1 + \theta_1^2 + \bar{\theta}_1^2 + \theta_1^2\bar{\theta}_1^2 \\ &= (1+\theta_1^2)(1+\bar{\theta}_1^2) < \infty \end{aligned}$$

Hence, time series w_t is weakly stationary. Let $\gamma(k)$ be the auto-covariance and $\rho(k)$ be the auto-correlation of w_t at lag $k \geq 0$.

$$\begin{aligned} \gamma(k) = \gamma(t, t+k) &= E(w_t w_{t+k}) \\ &= E[(\varepsilon_t + \theta_1\varepsilon_{t-1} + \bar{\theta}_1\varepsilon_{t-12} + \theta_1\bar{\theta}_1\varepsilon_{t-13})(\varepsilon_{t+k} + \theta_1\varepsilon_{t+k-1} + \bar{\theta}_1\varepsilon_{t+k-12} \\ &\quad + \theta_1\bar{\theta}_1\varepsilon_{t+k-13})] \end{aligned}$$

When $k = 0$, $\gamma(0) = E(w_t^2) = (1+\theta_1^2)(1+\bar{\theta}_1^2)$ and $\rho(0) = 1$.

When $k = 1$, $\gamma(1) = E(w_t w_{t+1}) = \theta_1 E(\varepsilon_t^2) + \theta_1\bar{\theta}_1^2 E(\varepsilon_{t-12}^2) = \theta_1(1+\bar{\theta}_1^2)$ and $\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{\theta_1}{1+\theta_1^2} \approx 0.478$.

When $k = 11$, $\gamma(11) = E(w_t w_{t+11}) = \theta_1\bar{\theta}_1 E(\varepsilon_{t-1}^2) = \theta_1\bar{\theta}_1$ and $\rho(11) = \frac{\gamma(11)}{\gamma(0)} = \frac{\theta_1\bar{\theta}_1}{(1+\theta_1^2)(1+\bar{\theta}_1^2)} \approx -0.221$.

When $k = 12$, $\gamma(12) = E(w_t w_{t+12}) = \bar{\theta}_1 E(\varepsilon_t^2) + \theta_1^2\bar{\theta}_1 E(\varepsilon_{t-1}^2) = (1+\theta_1^2)\bar{\theta}_1$ and $\rho(12) = \frac{\gamma(12)}{\gamma(0)} = \frac{\bar{\theta}_1}{1+\bar{\theta}_1^2} \approx -0.463$.

When $k = 13$, $\gamma(13) = E(w_t w_{t+13}) = \theta_1\bar{\theta}_1 E(\varepsilon_t^2) = \theta_1\bar{\theta}_1$ and $\rho(13) = \frac{\gamma(13)}{\gamma(0)} = \frac{\theta_1\bar{\theta}_1}{(1+\theta_1^2)(1+\bar{\theta}_1^2)} \approx -0.221$.

Otherwise, $\gamma(k) = 0$ and $\rho(k) = 0$.

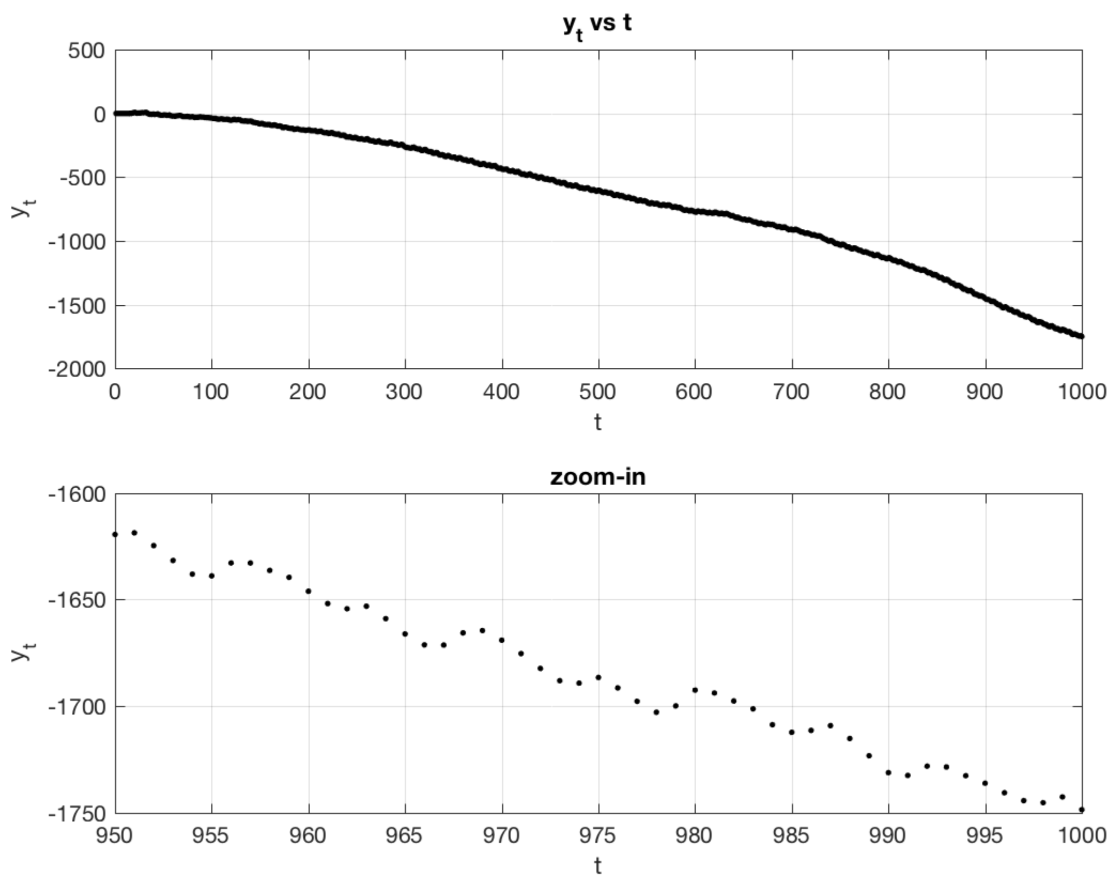
$\varphi_{11} = \rho(1) \approx 0.478$.

$$\varphi_{22} = \frac{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & \rho(2) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{vmatrix}} = \frac{\rho(2) - \rho^2(1)}{1-\rho^2(1)} = -\frac{\rho^2(1)}{1-\rho^2(1)} \approx -0.296.$$

$$\varphi_{33} = \frac{\begin{vmatrix} 1 & \rho(1) & \rho(1) \\ \rho(1) & 1 & \rho(2) \\ \rho(2) & \rho(1) & \rho(3) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & \rho(2) \\ \rho(1) & 1 & \rho(1) \\ \rho(2) & \rho(1) & 1 \end{vmatrix}} = \frac{\begin{vmatrix} 1 & \rho(1) & \rho(1) \\ \rho(1) & 1 & 0 \\ 0 & \rho(1) & 0 \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & 0 \\ \rho(1) & 1 & \rho(1) \\ 0 & \rho(1) & 1 \end{vmatrix}} = \frac{\rho^3(1)}{1-2\rho^2(1)} \approx 0.201.$$

$\varphi_{kk} = \frac{|\Gamma^*(k)|}{|\Gamma(k)|} = \frac{|rho^*(k)|}{|rho(k)|}$, where $|A|$ is the determinant of A and $\Gamma^*(k)$ is the matrix obtained by replacing its last column by $\rho(1:k)$.

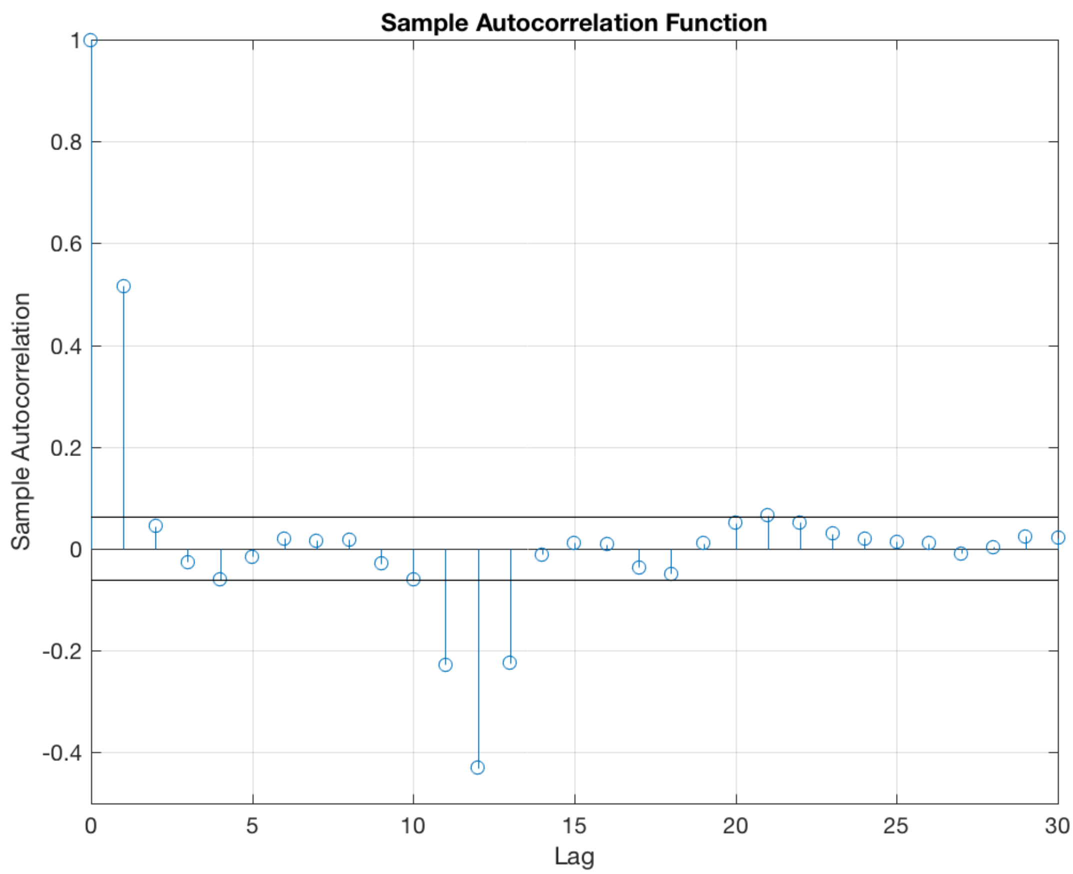
(b) Generate 500 samples from this model and plot y_t vs t.




```
% Generate at least 500 samples from this model and plot y(t) vs t
rng('default')
sampleSize = 1000;
et = randn(1, sampleSize);
yt = zeros(1, sampleSize);
wt = zeros(1, sampleSize);
theta1 = 0.740;
theta1_bar = -0.671;
for t = 14:sampleSize
yt(t) = yt(t-1)+yt(t-12)-yt(t-13)+et(t)+theta1_bar*et(t-12)+theta1*et(t-1)+theta1*theta1_bar*et(t-13);
wt(t) = et(t)+theta1_bar*et(t-12)+theta1*et(t-1)+theta1*theta1_bar*et(t-13);
```

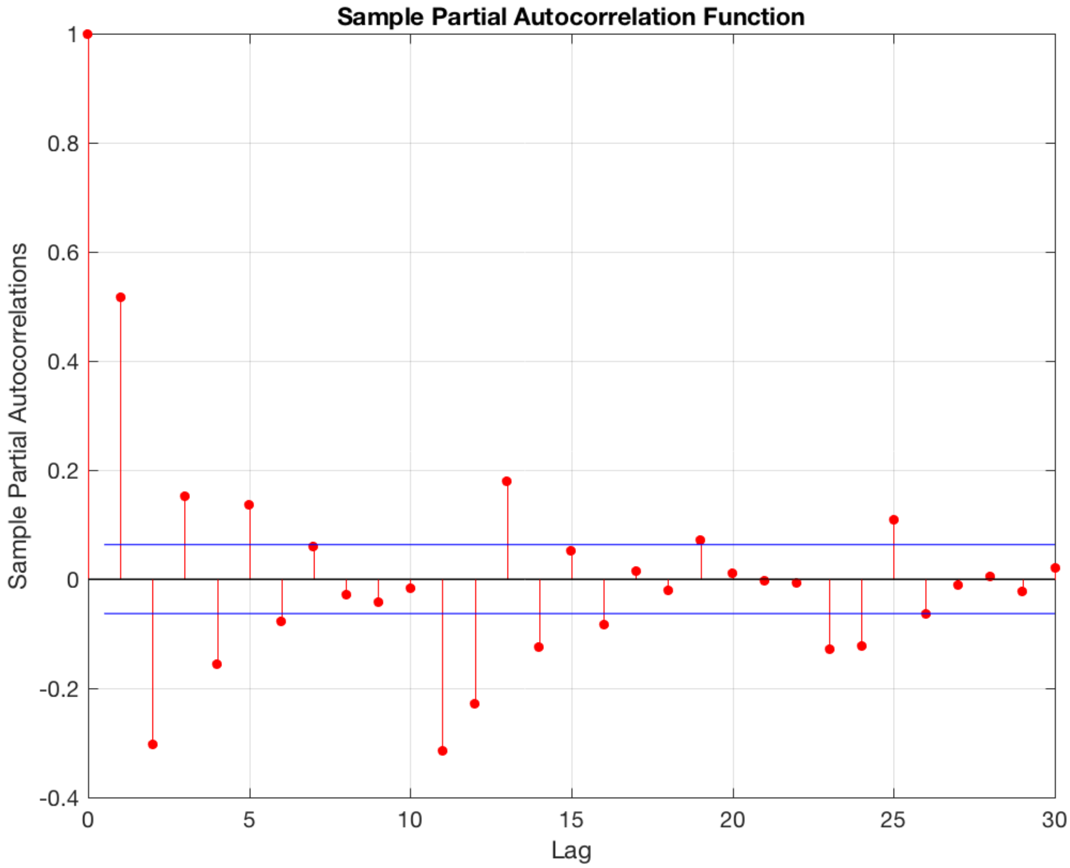
```
end
figure
subplot(2,1,1)
plot(1:sampleSize, yt, 'k.');
grid on, box on
xlabel('t')
ylabel('y_t')
title('y_t vs t')
subplot(2,1,2)
plot(sampleSize-50:sampleSize, yt(sampleSize-50:sampleSize), 'k.');
grid on, box on
xlabel('t')
ylabel('y_t')
title('zoom-in')
saveas(gcf, 'figure_1_a.png')
```

(c) Compute and plot numerical ACF and PACF and verify your computation in (a).



Numerical ACF

To compare with the theoretical ones, the values of sample ACF at lags 0, 1, 11, 12, and 13 are $\rho(0) = 1, \rho(1) = 0.516, \rho(11) = -0.2274, \rho(12) = -0.4305, \rho(13) = -0.2249$.



Numerical PACF

To compare with the theoretical ones, the values of sample PACF at lags 1, 2, and 3 are $\varphi_{11} = 0.5158, \varphi_{22} = -0.303, \varphi_{33} = 0.1509.$

```
% Compute and plot numerical ACF and PACF and verify your computation in (a)

numOfLags = 30;

[acor, lags] = xcorr(wt,numOfLags,'coeff');

ci = 1.96/sqrt(length(wt));

figure

hold on

stem(lags(numOfLags + 1:end),acor(numOfLags + 1:end));

plot(lags(numOfLags + 1:end), ci.*ones(size(lags(numOfLags + 1:end))), 'k');

plot(lags(numOfLags + 1:end),-ci.*ones(size(lags(numOfLags + 1:end))), 'k');

hold off

grid on, box on

ax = gca;
```

```
ax.YTick = [-1.0 -0.8 -0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8 1.0];
xlabel('Lag');
ylabel('Sample Autocorrelation')
title('Sample Autocorrelation Function')
saveas(gcf, 'figure_1_c_1.png')
figure
parcorr(wt,numOfLags);
saveas(gcf, 'figure_1_c_2.png')
```

2) (a) Fit three classes of models to the data generated in (1b):
   - Autoregressive: $(1-L^{12})(1-L)(1-aL^{12})y_t = (1+bL)\varepsilon_t$
   - Multiplicative: $(1-L^{12})(1-L)y_t = (1 + b_1 L)(1 + \bar{b}_1 L^{12})\varepsilon_t$
   - Additive: $(1-L^{12})(1-L)y_t = (1+b_1L+b_2L^{12})\varepsilon_t$

For Autoregressive model: $(1-L^{12})(1-L)(1-aL^{12})y_t = (1+bL)\varepsilon_t$

```
ARMdl = arima('Constant',0,'D',1,'Seasonality',12,'SARLags',12,'MALags',1);
ESTARMdl = estimate(ARMdl, yt');
```

ARIMA(0,1,1) Model Seasonally Integrated with Seasonal AR(12):

Conditional Probability Distribution: Gaussian

Parameter	Value	Standard Error	t Statistic
Constant	0	Fixed	Fixed
SAR{12}	-0.454284	0.0281491	-16.1385
MA{1}	0.732133	0.0206501	35.4543
Variance	1.06666	0.0472192	22.5895

Hence, the estimated model is $(1-L^{12})(1-L)(1+0.454284L^{12})y(t) = (1+0.732133L)\varepsilon(t)$

For Multiplicative model: $(1-L^{12})(1-L)y_t = (1 + b_1 L)(1 + \bar{b}_1 L^{12})\varepsilon_t$

```
MulMdl = arima('Constant',0,'D',1,'Seasonality',12,'SMALags',12,'MALags',1);
ESTMulMdl = estimate(MulMdl, yt');
```

ARIMA(0,1,1) Model Seasonally Integrated with Seasonal MA(12):

Conditional Probability Distribution: Gaussian

Parameter	Value	Standard Error	t Statistic
Constant	0	Fixed	Fixed
MA{1}	0.732037	0.0208733	35.0706
SMA{12}	-0.628737	0.0258515	-24.3211
Variance	0.978481	0.0418935	23.356

Hence, the estimated model is $(1-L^{12})(1-L)y(t) = (1+0.732037L)(1-0.628737L^{12})\varepsilon(t)$

For Additive model: $(1-L^{12})(1-L)y_t = (1+b_1L+b_2L^{12})\varepsilon_t$

AddMdl = arima('Constant',0,'D',1,'Seasonality',12,'MALags',[1,12]);

ESTAddMdl = estimate(AddMdl, yt');

ARIMA(0,1,12) Model Seasonally Integrated:

Conditional Probability Distribution: Gaussian

Parameter	Value	Standard Error	t Statistic
Constant	0	Fixed	Fixed
MA{1}	0.605409	0.0206318	29.3435
MA{12}	-0.341099	0.021155	-16.1238
Variance	1.17874	0.0518605	22.7291

Hence, the estimated model is $(1-L^{12})(1-L)y(t) = (1+0.605409L-0.341099L^{12})\varepsilon(t)$

(b) Compute the residuals from each model and compute the ACF of the residuals and plot.

For Autoregressive model: $(1-L^{12})(1-L)(1-aL^{12})y_t = (1+bL)\varepsilon_t$

$(1-L^{12})(1-L)(1-aL^{12})y(t) = (1-L^{12})(1-L)(y(t)-ay(t-12))$

$= (1-L^{12})(y(t)-y(t-1)-ay(t-12)+ay(t-13)) = y(t)-y(t-1)-ay(t-12)+ay(t-13)-y(t-12)+y(t-13)+ay(t-24)-ay(t-25)$

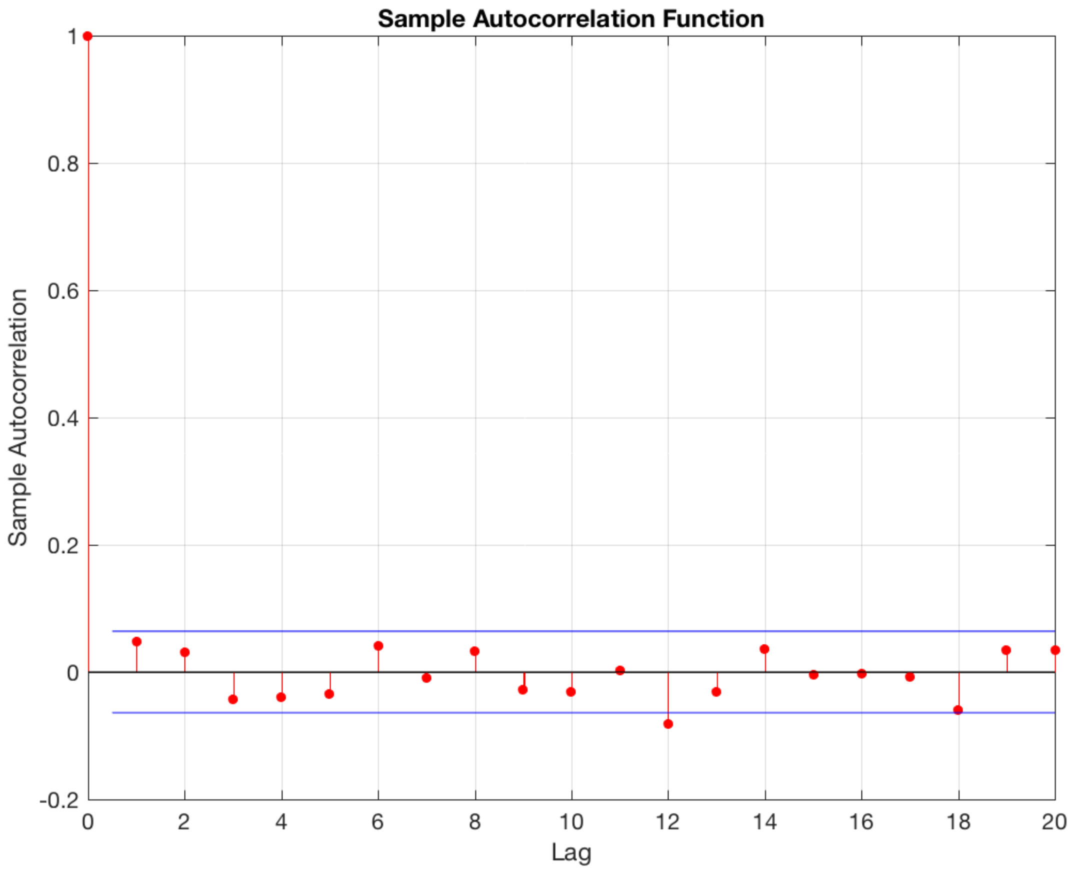
$= y(t)-y(t-1)-(a+1)y(t-12)+(a+1)y(t-13)+ay(t-24)-ay(t-25)$

$= \varepsilon(t) + b\varepsilon(t-1)$

Hence, $y(t) = y(t-1) + (a+1)y(t-12) - (a+1)y(t-13) - ay(t-24) + ay(t-25) + \varepsilon(t) + b\varepsilon(t-1)$

Matlab code is given below.

```
a = -0.454284;
b =  0.732133;
et_AR = zeros(1, sampleSize);
for t = 1:sampleSize
   if t < 26
      yt_AR = yt(t);
   else
      yt_AR = yt(t-1)+(a+1)*yt(t-12)-(a+1)*yt(t-13)-a*yt(t-24)+a*yt(t-25)+b*et_AR(t-1);
   end
   et_AR(t) = yt(t) - yt_AR;
end
figure
autocorr(et_AR(26:end));
saveas(gcf, 'figure_2_b_1.png')
```



ACF of the residual from the **Autoregressive** model

For Multiplicative: $(1-L^{12})(1-L)y_t = (1+b_1L)(1+\bar{b}_1L^{12})\varepsilon_t$

$(1-L^{12})(1-L)y(t) = (1-L^{12})(y(t)-y(t-1)) = y(t)-y(t-1)-y(t-12)+y(t-13)$

$= (1+b_1L)(\varepsilon(t)+\bar{b}_1\varepsilon(t-12))$

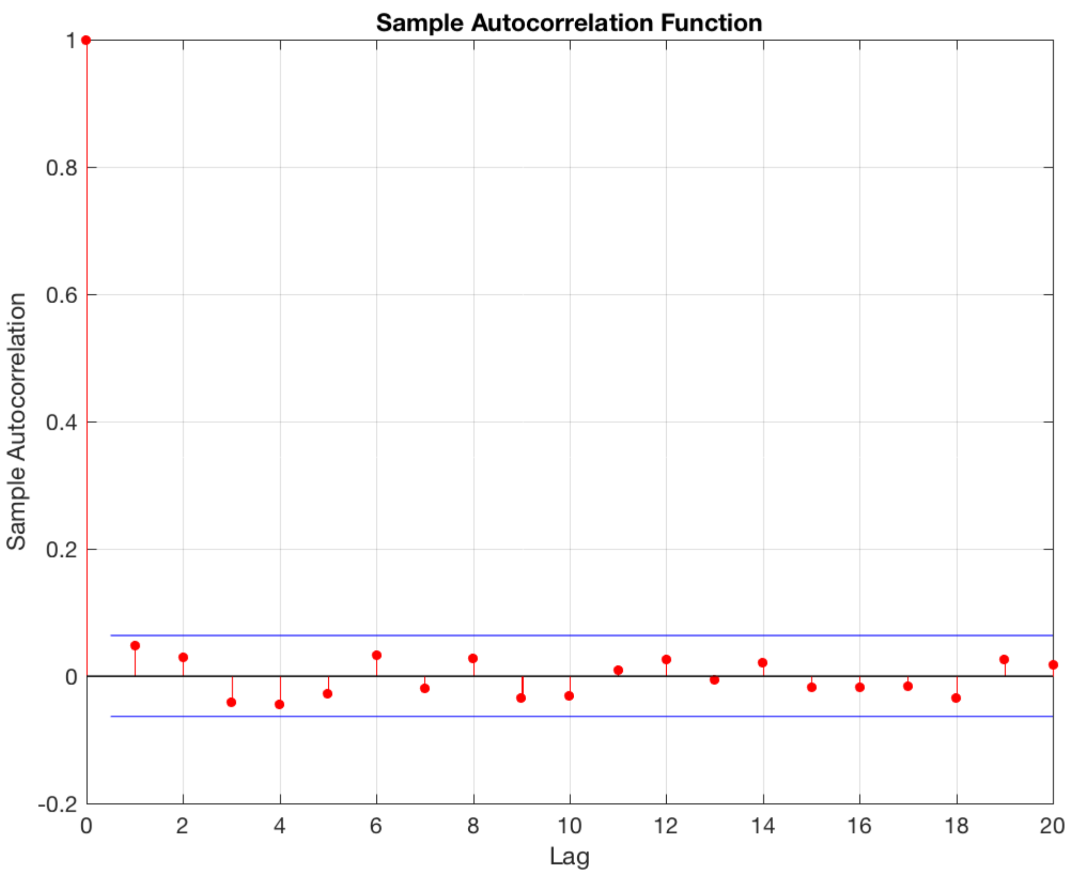
$= \varepsilon(t)+b_1\varepsilon(t-1)+\bar{b}_1\varepsilon(t-12)+b_1\bar{b}_1\varepsilon(t-13)$

Hence, $y(t) = y(t-1)+y(t-12)-y(t-13)+\varepsilon(t)+b_1\varepsilon(t-1)+\bar{b}_1\varepsilon(t-12)+b_1\bar{b}_1\varepsilon(t-13)$

Matlab code is shown below.

```
b1 = 0.732037;

b1_bar = -0.628737;

et_Mul = zeros(1, sampleSize);

for t = 1:sampleSize
```

```
    if t < 14
        yt_Mul = yt(t);
    else
        yt_Mul = yt(t-1)+yt(t-12)-yt(t-13)+b1*et_Mul(t-1)+b1_bar*et_Mul(t-12)+b1*b1_bar*et_Mul(t-13);
    end
    et_Mul(t) = yt(t) - yt_Mul;
end
figure
autocorr(et_Mul(14:end));
saveas(gcf, 'figure_2_b_2.png')
```



ACF of the residual from the **Multiplicative** model

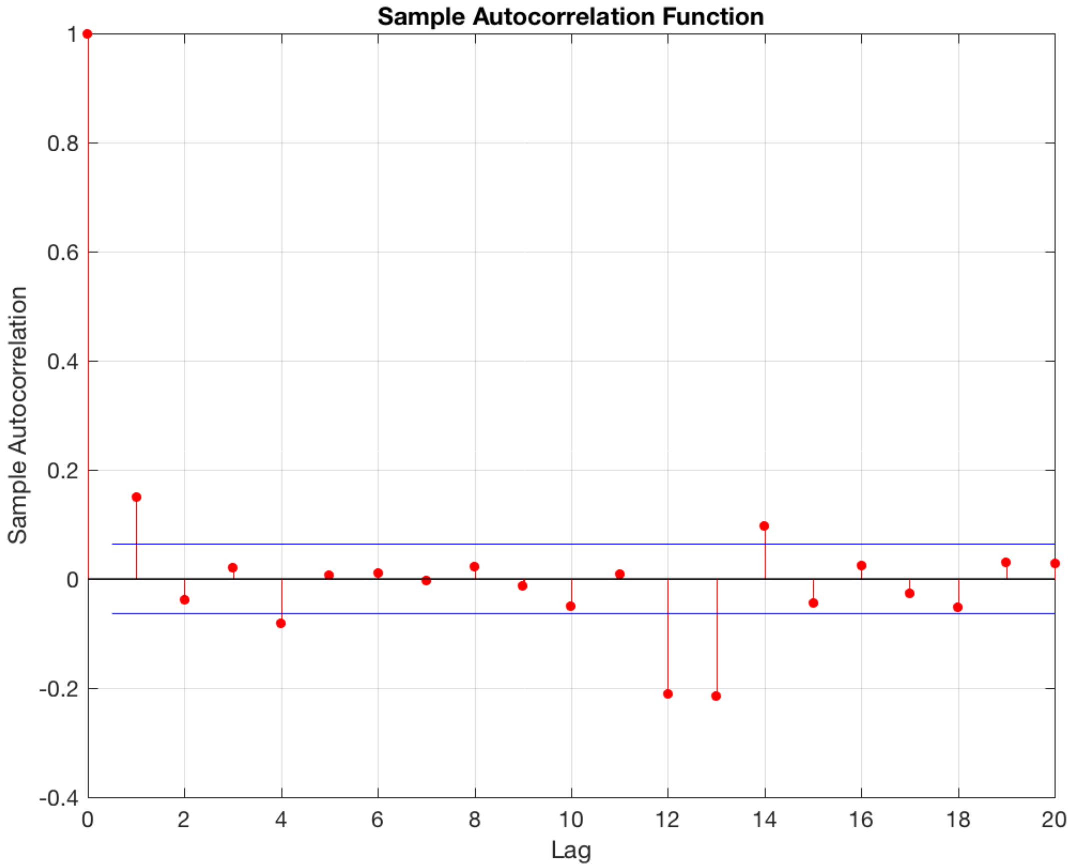
For Additive model: $(1-L^{12})(1-L)y_t = (1+b_1L+b_2L^{12})\varepsilon(t)$

$(1-L^{12})(1-L)y(t) = (1-L^{12})(y(t)-y(t-1)) = y(t)-y(t-1)-y(t-12)+y(t-13) = \varepsilon(t)+b_1\varepsilon(t-1)+b_2\varepsilon(t-12)$

Hence, $y(t) = y(t-1)+y(t-12)-y(t-13)+\varepsilon(t)+b_1\varepsilon(t-1)+b_2\varepsilon(t-12)$.

Matlab code is given below.

```
b1 = 0.605409;

b2 = -0.341099;

et_Add = zeros(1, sampleSize);

for t = 1:sampleSize

   if t < 14

      yt_Add = yt(t);

   else

      yt_Add = yt(t-1)+yt(t-12)-yt(t-13)+b1*et_Add(t-1)+b2*et_Add(t-12);

   end

   et_Add(t) = yt(t) - yt_Add;

end

figure

autocorr(et_Add(14:end));

saveas(gcf, 'figure_2_b_3.png')
```



ACF of the residual from the **Additive** model

(c) Compute AIC and find the best model.

Matlab code is given below.

```
logL = zeros(3,1);

[~,~,logL(1)] = estimate( ARMdl,yt','print',false);

[~,~,logL(2)] = estimate(MulMdl,yt','print',false);

[~,~,logL(3)] = estimate(AddMdl,yt','print',false);

aic = aicbic(logL, [3; 3; 3], sampleSize*ones(3,1));
```

AIC values for the three models are 2908.4, 2822.1, and 3008.3. Since it has the smallest value for AIC, the **Multiplicative** model is the best. The Air-line model is given by

$$(1-L)(1-L^{12})y_t = (1+0.740L)(1-0.671L^{12})\varepsilon_t$$

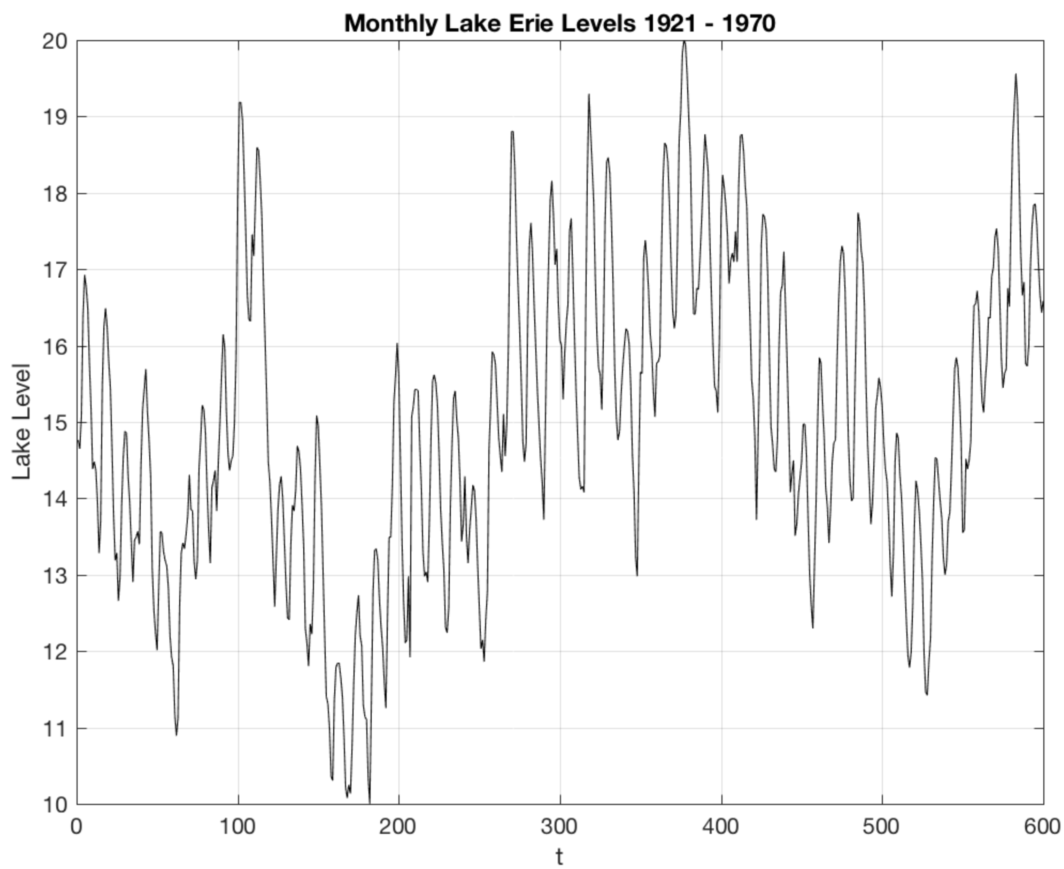
The chosen multiplicative model that fits the data is

$$(1-L^{12})(1-L)y_t = (1+0.732L)(1-0.629L^{12})\varepsilon_t$$

Hence, we know that the method introduced in the class works.

3) (a) Pick two time series with trend, seasonality and randomness from the data set discussed in the class.
(b) Plot ACF, PACF, choose a small subset of models, estimate the parameters, residual plot and use AIC to pick the best model in each case.

The first time series is Monthly Lake Erie Levels 1921-1970. The time series y(t) is shown below.



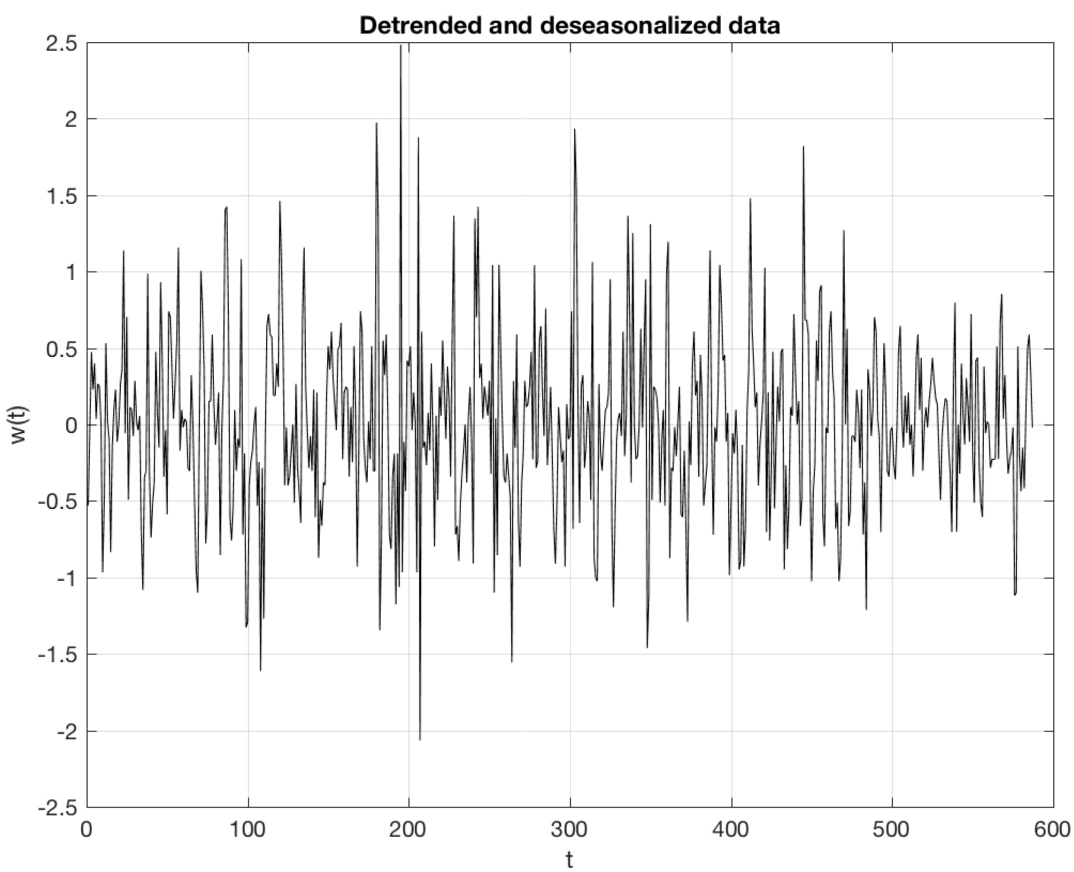
Remove the trend and seasonal components from the time series y(t).

$w(t) = (1-L^{12})(1-L)y(t) = (1-L^{12})(y(t)-y(t-1))=y(t)-y(t-1)-y(t-12)+y(t-13)$

Matlab code is given below.

```
wt = zeros(sampleSize-13,1);

for t = 14:sampleSize
```

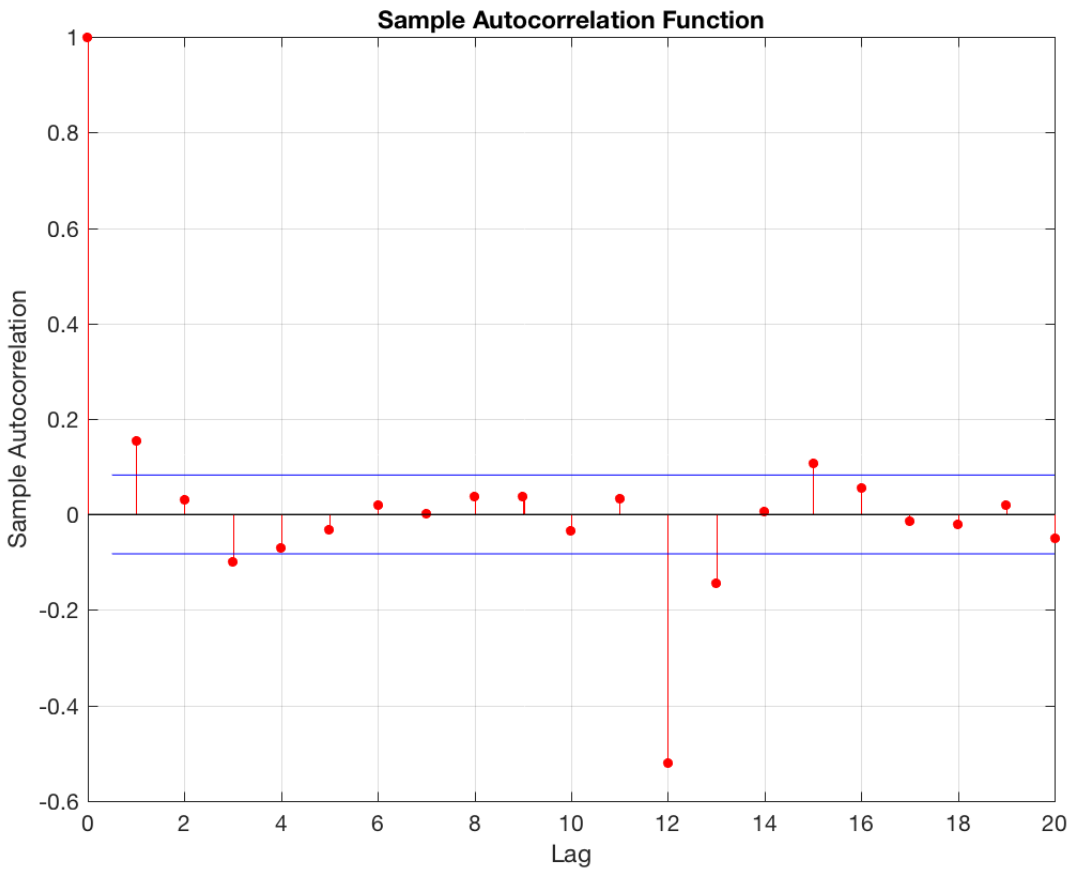
```
    wt(t-13) = yt(t)-yt(t-1)-yt(t-12)+yt(t-13);
end
figure
plot(wt,'k-')
grid on, box on
xlabel('t')
ylabel('w(t)')
title('Detrended and deseasonalized data')
saveas(gcf, 'figure_3_1_2.png')
```



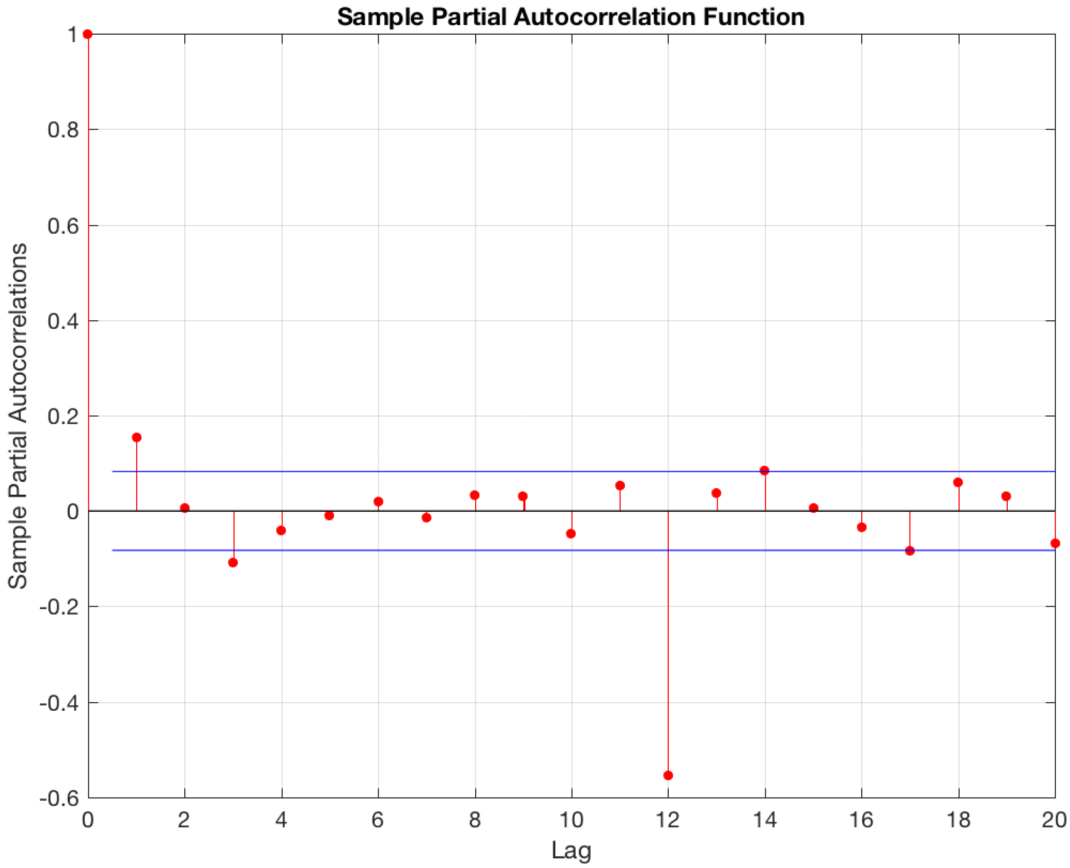
Plot ACF and PACF.

```
figure
```

```
autocorr(wt);

saveas(gcf, ‘figure_3_1_3.png’)

figure

parcorr(wt);

saveas(gcf, ‘figure_3_1_4.png’)
```



Numerical ACF



Numerical PACF

Candidate One: $(1-L)(1-L^{12})(1-aL)y(t) = (1+bL^{12})\varepsilon(t)$

Estimate the parameters first.

MdlOne = arima('Constant',0,'D',1,'Seasonality',12,'SMALags',12,'ARLags',1);

ESTMdlOne = estimate(MdlOne, yt);

ARIMA(1,1,0) Model Seasonally Integrated with Seasonal MA(12):

Conditional Probability Distribution: Gaussian

Parameter	Value	Standard Error	t Statistic
Constant	0	Fixed	Fixed
AR{1}	0.196521	0.0304167	6.46095
SMA{12}	-0.898694	0.0193983	-46.3284
Variance	0.173445	0.0070879	24.4706

Hence, the model is $(1-L)(1-L^{12})(1-0.196521L)y(t) = (1-0.898694L^{12})\varepsilon(t)$.

Second, compute the residual.

$(1-L)(1-L^{12})(1-aL)y(t) = (1-L-L^{12}+L^{13})(1-aL)y(t) = (1-aL-L+aL^2-L^{12}+aL^{13}+L^{13}-aL^{14})y(t) = \varepsilon(t)+b\varepsilon(t-12)$

$= (1-(a+1)L+aL^2-L^{12}+(a+1)L^{13}-aL^{14})y(t) = \varepsilon(t)+b\varepsilon(t-12)$

$= y(t)-(a+1)y(t-1)+ay(t-2)-y(t-12)+(a+1)y(t-13)-ay(t-14) = \varepsilon(t)+b\varepsilon(t-12)$

Hence,

$$y(t) = (a+1)y(t-1)-ay(t-2)+y(t-12)-(a+1)y(t-13)+ay(t-14)+\varepsilon(t)+b\varepsilon(t-12)$$

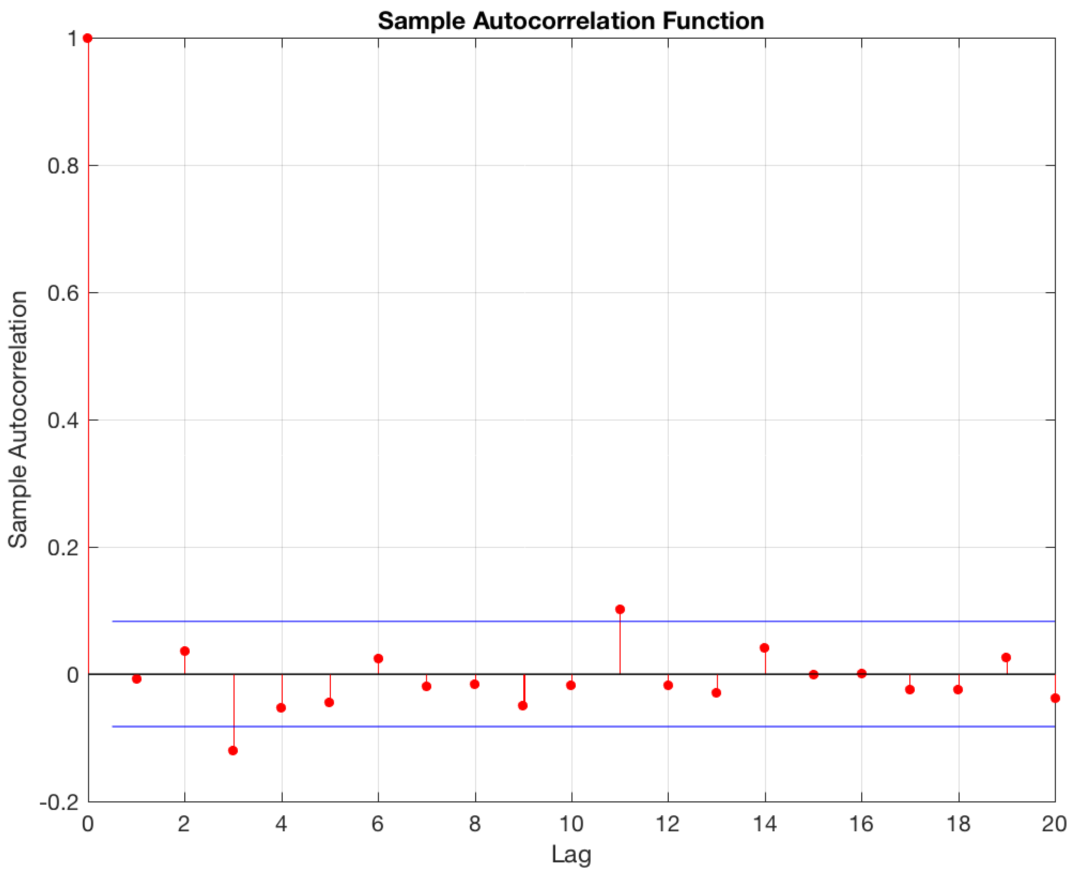
Matlab code is given below.

```
a = 0.196521;
b = -0.898694;
et = zeros(1, sampleSize);
for t = 1:sampleSize
  if t < 15
    yt_one = yt(t);
  else
    yt_one = (a+1)*yt(t-1)-a*yt(t-2)+yt(t-12)-(a+1)*yt(t-13)+a*yt(t-14)+b*et(t-12);
  end
  et(t) = yt(t) - yt_one;
end
```

figure

autocorr(et(15:end));

saveas(gcf, 'figure_3_1_5.png')



ACF of the residual from **Candidate One**

Third, compute the AIC.

logL = zeros(1,1);

[~,~,logL(1)] = estimate(MdlOne,yt,'print',false);

aic = aicbic(logL, 3, sampleSize);

For Candidate One, the value of AIC is 657.5902.

Candidate Two: $(1-L)(1-L^{12})(1-aL^{12})y(t) = (1+bL)\varepsilon(t)$

Estimate the parameters first.

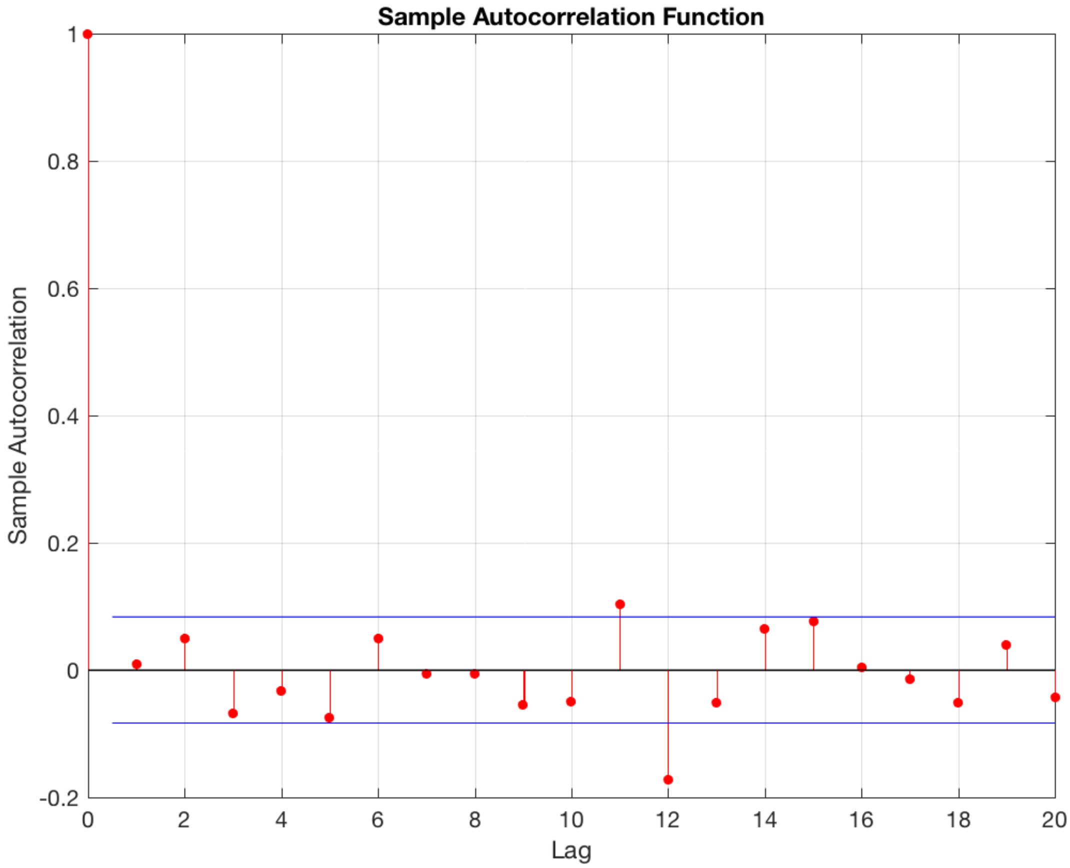
```
MdlTwo = arima('Constant',0,'D',1,'Seasonality',12,'SARLags',12,'MALags',1);

ESTMdlTwo = estimate(MdlTwo, yt);

   ARIMA(0,1,1) Model Seasonally Integrated with Seasonal AR(12):
   ---------------------------------------------------------------
   Conditional Probability Distribution: Gaussian

                                  Standard          t
   Parameter       Value           Error        Statistic
   ---------------  ------------  --------------  -----------
      Constant              0         Fixed          Fixed
       SAR{12}     -0.547764    0.0278092      -19.6973
         MA{1}      0.180805     0.0290469        6.2246
      Variance      0.227529     0.0104549       21.7629
```

Second, compute the residual.

y(t) = y(t-1)+(a+1)y(t-12)-(a+1)y(t-13)-ay(t-24)+ay(t-25)+ε(t)+bε(t-1)

Matlab code is given below.

```
a = -0.547764;
b = 0.180805;
et = zeros(1, sampleSize);
for t = 1:sampleSize
   if t < 26
      yt_two = yt(t);
   else
      yt_two = yt(t-1)+(a+1)*yt(t-12)-(a+1)*yt(t-13)-a*yt(t-24)+a*yt(t-25)+b*et(t-1);
   end
   et(t) = yt(t) - yt_two;
end
figure
autocorr(et(26:end));
saveas(gcf, 'figure_3_1_6.png')
```



ACF of the residual from **Candidate Two**

Third, compute the AIC.

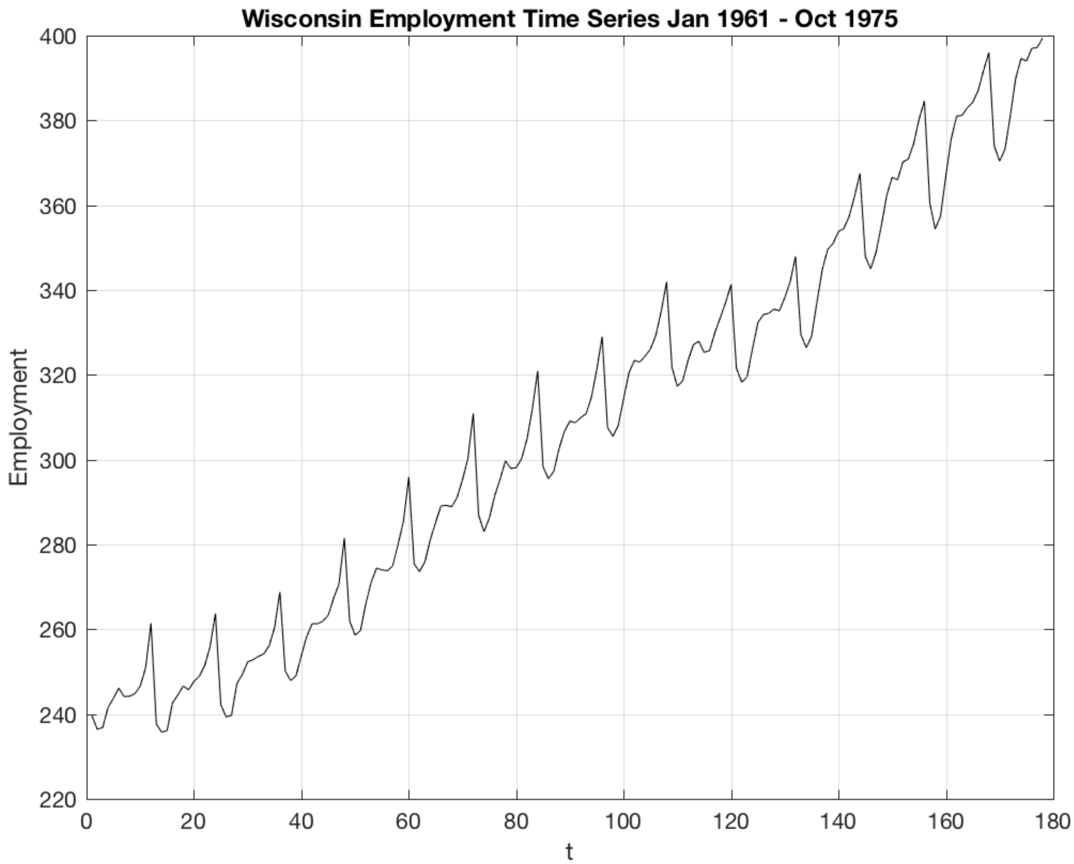
logL = zeros(1,1);

[~,~,logL(1)] = estimate(MdlTwo,yt,'print',false);

aic = aicbic(logL, 3, sampleSize);

For Candidate Two, the value of AIC is 820.4404. From the above discussion, we conclude that Candidate One is the better model for Monthly Lake Erie Levels.

The second time series chosen is Wisconsin Employment Time Series. The time series y(t) is shown below.



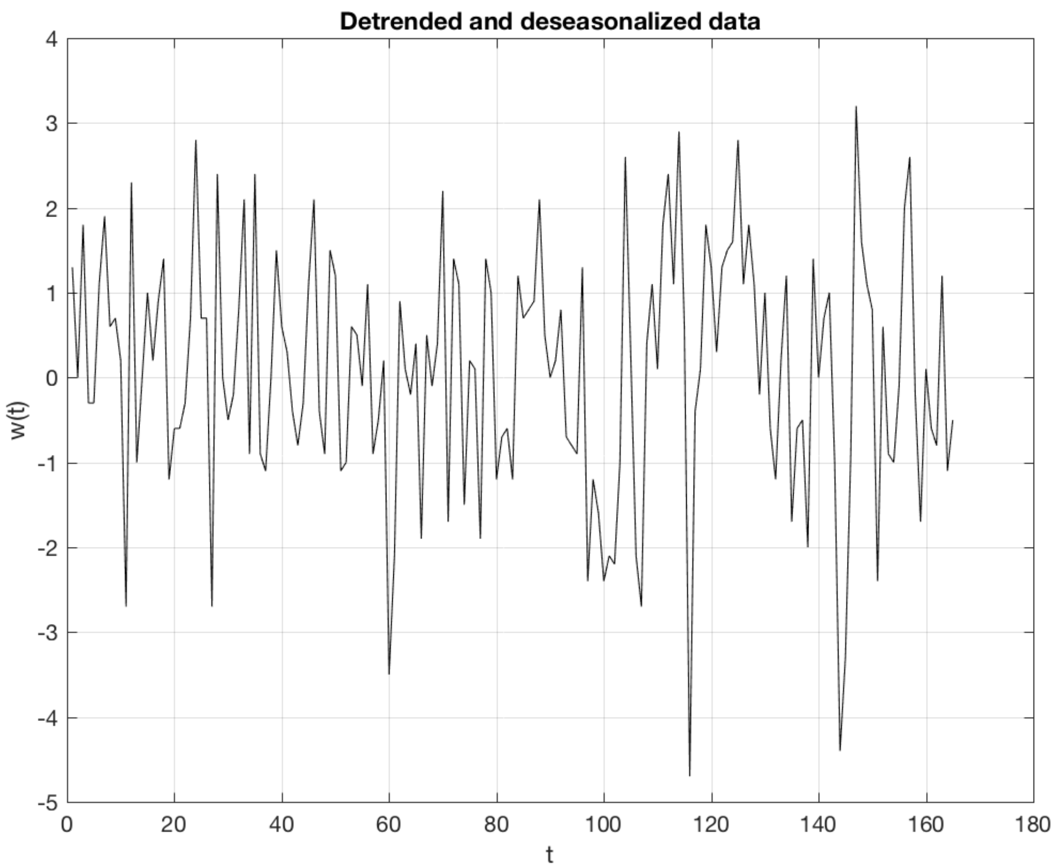
Remove the trend and seasonal components from the time series y(t).

$w(t) = (1-L^{12})(1-L)y(t) = (1-L^{12})(y(t)-y(t-1))=y(t)-y(t-1)-y(t-12)+y(t-13)$

Matlab code is given below.

```
wt = zeros(sampleSize-13,1);

for t = 14:sampleSize

   wt(t-13) = yt(t)-yt(t-1)-yt(t-12)+yt(t-13);

end

figure

plot(wt,'k-')

grid on, box on

xlabel('t')

ylabel('w(t)')

title('Detrended and deseasonalized data')
```

saveas(gcf, 'figure_3_2_2.png')



Plot ACF and PACF.

figure

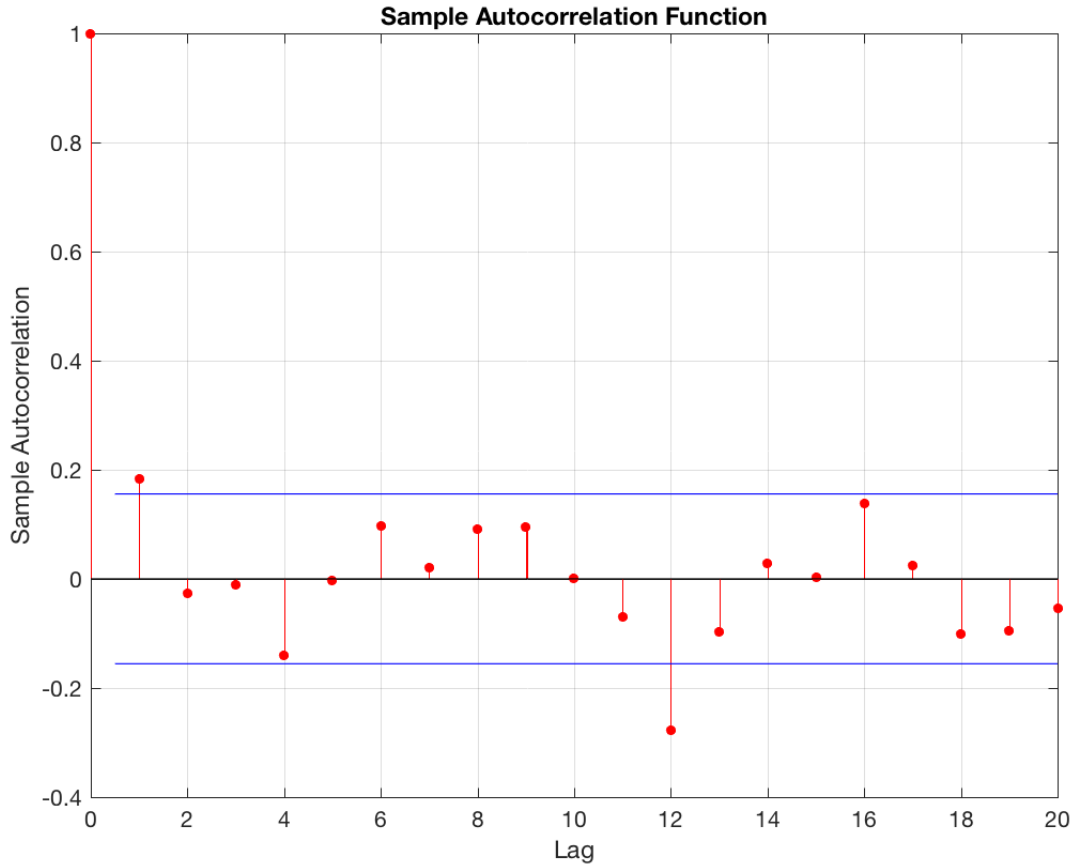
autocorr(wt);

saveas(gcf, 'figure_3_2_3.png')

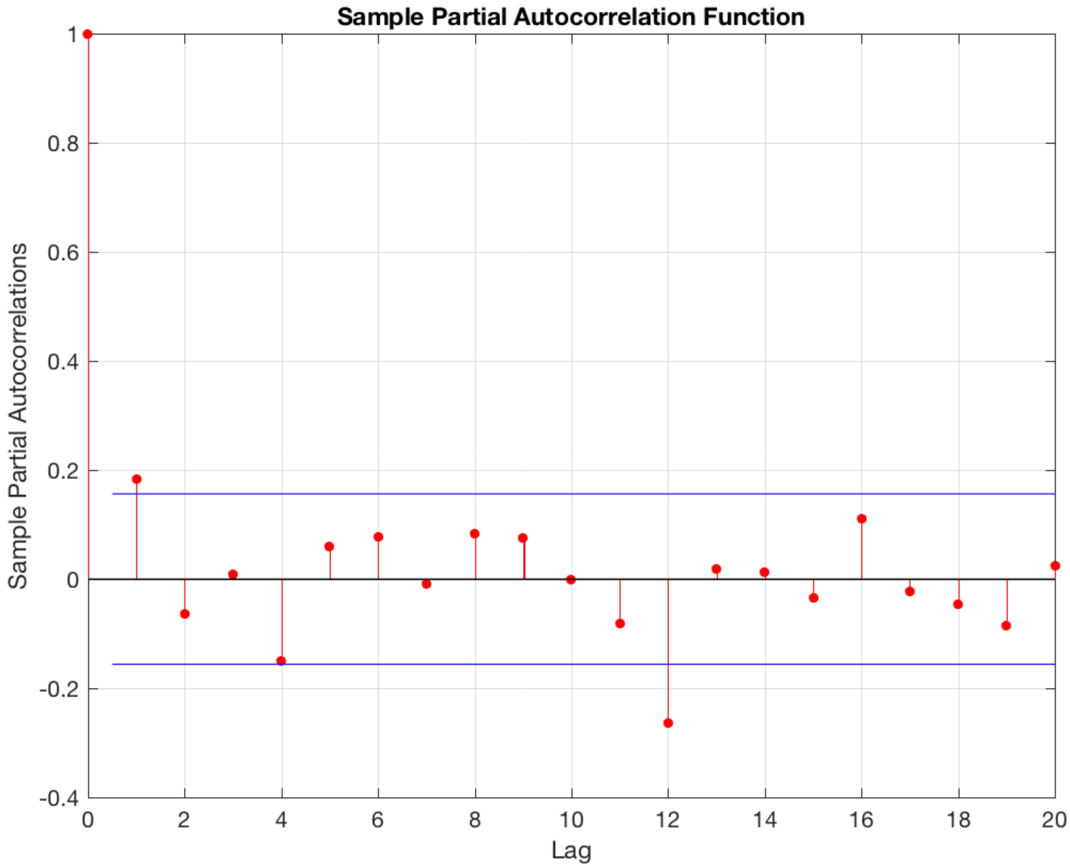
figure

parcorr(wt);

saveas(gcf, 'figure_3_2_4.png')



Numerical ACF



Numerical PACF

Candidate One: (1-L)(1-L^{12})(1-aL)y(t) = (1+bL^{12})ε(t)

Estimate the parameters first.

MdlOne = arima('Constant',0,'D',1,'Seasonality',12,'SMALags',12,'ARLags',1);

ESTMdlOne = estimate(MdlOne, yt);

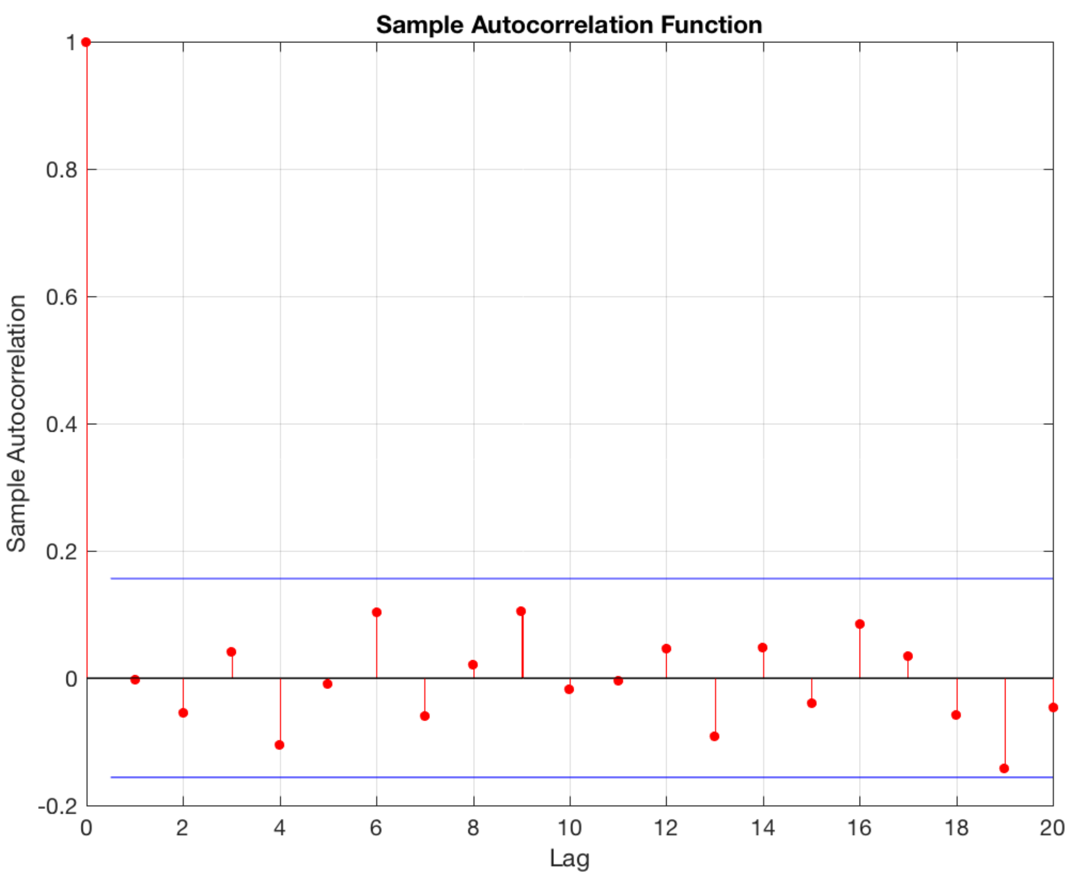
```
    ARIMA(1,1,0) Model Seasonally Integrated with Seasonal MA(12):
    --------------------------------------------------------------
    Conditional Probability Distribution: Gaussian

                                  Standard          t
     Parameter        Value         Error        Statistic
    -----------   -----------   ------------    -----------
     Constant              0         Fixed         Fixed
        AR{1}       0.162678     0.0660975       2.46118
      SMA{12}      -0.386864     0.0677269      -5.71213
     Variance        1.67756       0.14791        11.341
```

Second, compute the residual.

$$y(t) = (a+1)y(t-1)-ay(t-2)+y(t-12)-(a+1)y(t-13)+ay(t-14)+\varepsilon(t)+b\varepsilon(t-12)$$

Matlab code is given below.

```
a = 0.162678;
b = -0.386864;
et = zeros(1, sampleSize);
for t = 1:sampleSize
   if t < 15
      yt_one = yt(t);
   else
      yt_one = (a+1)*yt(t-1)-a*yt(t-2)+yt(t-12)-(a+1)*yt(t-13)+a*yt(t-14)+b*et(t-12);
   end
   et(t) = yt(t) - yt_one;
end
figure
autocorr(et(15:end));
saveas(gcf, 'figure_3_2_5.png')
```



ACF of the residual from **Candidate One**

Third, compute the AIC.

```
logL = zeros(1,1);
[~,~,logL(1)] = estimate( MdlOne,yt,'print',false);
aic = aicbic(logL, 3, sampleSize);
```

For Candidate One, the value of AIC is 603.2285.

Candidate Two: $(1-L)(1-L^{12})(1-aL^{12})y(t) = (1+bL)\varepsilon(t)$

Estimate the parameters first.

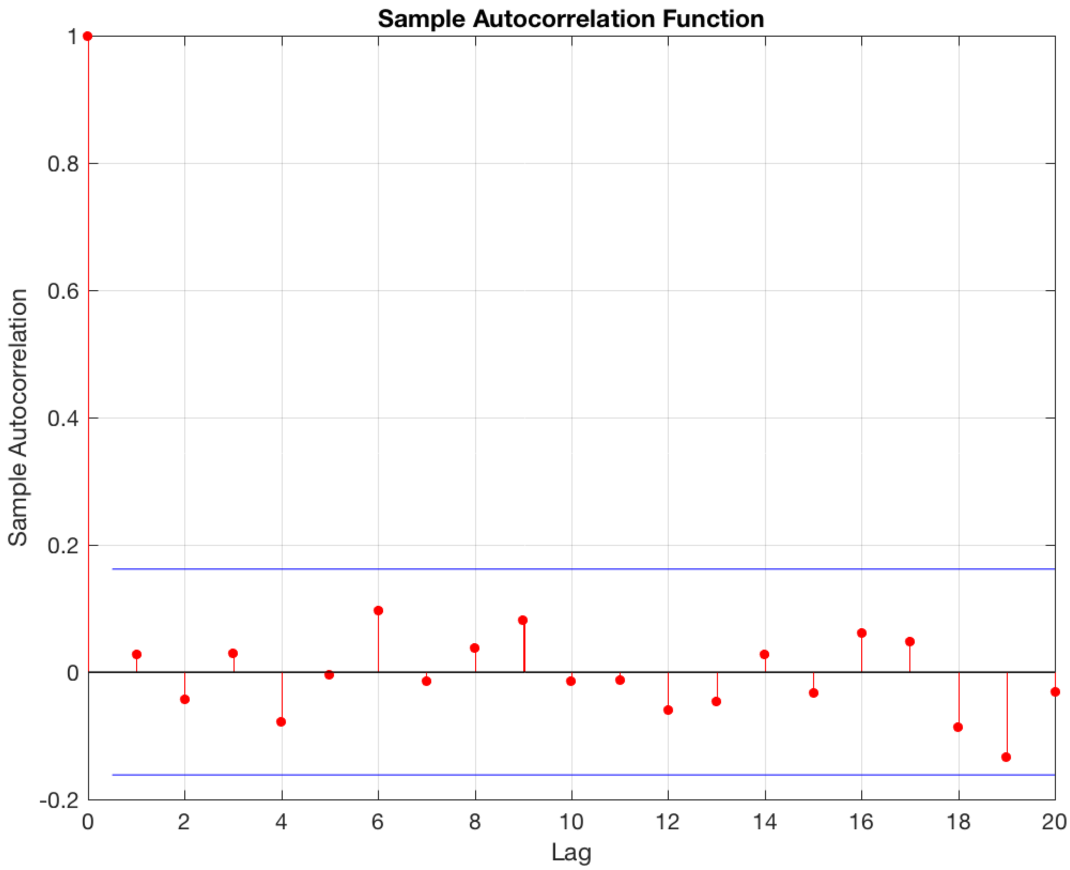
```
MdlTwo = arima('Constant',0,'D',1,'Seasonality',12,'SARLags',12,'MALags',1);
ESTMdlTwo = estimate(MdlTwo, yt);
   ARIMA(0,1,1) Model Seasonally Integrated with Seasonal AR(12):
   --------------------------------------------------------------
   Conditional Probability Distribution: Gaussian
                                   Standard          t
   Parameter       Value            Error         Statistic
   --------------  -------------  ------------   -----------
    Constant                0         Fixed          Fixed
     SAR{12}       -0.302958      0.0681809       -4.44345
       MA{1}        0.181748      0.0707099        2.57034
    Variance         1.70825       0.154665        11.0448
```

Second, compute the residual.

$y(t) = y(t-1)+(a+1)y(t-12)-(a+1)y(t-13)-ay(t-24)+ay(t-25)+\varepsilon(t)+b\varepsilon(t-1)$

Matlab code is given below.

```
a = -0.302958;
b = 0.181748;
et = zeros(1, sampleSize);
for t = 1:sampleSize
   if t < 26
      yt_two = yt(t);
   else
      yt_two = yt(t-1)+(a+1)*yt(t-12)-(a+1)*yt(t-13)-a*yt(t-24)+a*yt(t-25)+b*et(t-1);
   end
   et(t) = yt(t) - yt_two;
end
figure
autocorr(et(26:end));
saveas(gcf, 'figure_3_2_6.png')
```



Third, compute the AIC.

```
logL = zeros(1,1);
[~,~,logL(1)] = estimate( MdlTwo,yt,'print',false);
aic = aicbic(logL, 3, sampleSize);
```

For Candidate Two, the value of AIC is 606.4553.

From the above discussion, we conclude that Candidate One is the better model for Wisconsin employment time series.