Lince Rumainum
DSA 5203
Time Series Analysis
Homework 6

For the ***MA (2) model*** where y_t is a stationary, ergodic process

$y_t = \mu + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}$ ---- (1)

Where the mean, $\boldsymbol{E}(y_t) = \mu, \theta$ is the MA (2) parameter, and ϵ_t is a white noise sequence.

$y_t = \mu + (1 + \theta_1\mathrm{L} + \theta_2\mathrm{L}^2)\epsilon_t = \mu + \psi(\mathrm{L})\epsilon_t$ ----(2)

Where $\psi(\mathrm{L}) = 1 + \theta_1\mathrm{L} + \theta_2\mathrm{L}^2$

Recall, we know that the autocovariances of MA(2) model are (eq. (3)):

$\gamma_0 = (1 + \theta_1^2 + \theta_2^2)\,\sigma^2$

$\gamma_1 = (\theta_1 + \theta_1\theta_2)\sigma^2$

$\gamma_2 = \theta_2\sigma^2$

$\gamma_\mathrm{j} = 0 \text{ for } |j| > 2$

The generating function for ACF is:

$$\mathrm{g_y(z)} = \sum_{-\infty}^{\infty} \gamma_\mathrm{j}\mathrm{z}^\mathrm{j}, (\gamma_{-\mathrm{j}} = \gamma_\mathrm{j})$$

So, for MA(2) model we have:

$\mathrm{g_y(z)} = \gamma_{-2}\mathrm{z}^{-2} + \gamma_{-1}\mathrm{z}^{-1} + \gamma_0\mathrm{z}^0 + \gamma_1\mathrm{z}^1 + \gamma_2\mathrm{z}^2 = \gamma_{-2}\mathrm{z}^{-2} + \gamma_{-1}\mathrm{z}^{-1} + \gamma_0 + \gamma_1\mathrm{z} + \gamma_2\mathrm{z}^2$ ---- (4)

Because of symmetricity $\gamma_{-2} = \gamma_2$ and $\gamma_{-1} = \gamma_1$

Substituting what we know from the autocovariances of MA(2) model on eq. (3) to eq. (4) we get:

$\mathrm{g_y(z)} = \theta_2\sigma^2\mathrm{z}^{-2} + (\theta_1 + \theta_1\theta_2)\sigma^2\mathrm{z}^{-1} + (1 + \theta_1^2 + \theta_2^2)\,\sigma^2 + (\theta_1 + \theta_1\theta_2)\sigma^2\mathrm{z} + \theta_2\sigma^2\mathrm{z}^2$

$\mathrm{g_y(z)} = \sigma^2[\theta_2\mathrm{z}^{-2} + (\theta_1 + \theta_1\theta_2)\mathrm{z}^{-1} + (1 + \theta_1^2 + \theta_2^2) + (\theta_1 + \theta_1\theta_2)\mathrm{z} + \theta_2\mathrm{z}^2]$ ---- (5)

$= \sigma^2[\theta_2\mathrm{z}^{-2} + \theta_1\mathrm{z}^{-1} + \theta_1\theta_2\mathrm{z}^{-1} + (1 + \theta_1^2 + \theta_2^2) + \theta_1\mathrm{z} + \theta_1\theta_2\mathrm{z} + \theta_2\mathrm{z}^2]$

$= \sigma^2(1 + \theta_1\mathrm{z} + \theta_2\mathrm{z}^2)(1 + \theta_1\mathrm{z}^{-1} + \theta_2\mathrm{z}^{-2})$

$= \sigma^2\psi(\mathrm{z})\psi(\mathrm{z}^{-1})$

Factoring *z* from eq. (5), we have:

$\mathrm{g_y(z)} = \sigma^2[(1 + \theta_1^2 + \theta_2^2) + (\theta_1 + \theta_1\theta_2)(\mathrm{z}^{-1} + \mathrm{z}) + \theta_2(\mathrm{z}^{-2} + \mathrm{z}^2)]$---- (6)

Setting $z = e^{-i\omega} = cos\ \omega - i\ sin\ \omega$, and the symmetric properties of $\gamma_{-\mathrm{j}} = \gamma_\mathrm{j}$, we have the power spectrum of MA(2) as:

$S_y(\omega) = \frac{\sigma^2}{2\pi}[(1 + \theta_1^2 + \theta_2^2) + 2(\theta_1 + \theta_1\theta_2)\ cos\ \omega + 2\theta_2\ cos\ 2\omega]$ ---- (7)

$\therefore$, the power spectrum for MA(2):

$S_y(\omega) = \frac{1}{2\pi}[\gamma_0 + 2\gamma_1\ cos\ \omega + 2\gamma_2\ cos\ 2\omega]$ ---- (8)

Plot for $S_y(\omega)$ for MA(2):

$\frac{2\pi}{\sigma^2} S_y(\omega) = (1 + \theta_1^2 + \theta_2^2) + 2(\theta_1 + \theta_1\theta_2)\ cos\ \omega + 2\theta_2\ cos\ 2\omega$ ---- (9)

Since $cos\ 0 = \cos 2\pi = 1, \cos\frac{\pi}{2} = 0, \cos\pi = -1,$

At $\omega = 0$, eq. (9) becomes:

$$\frac{2\pi}{\sigma^2} S_y(\omega) = (1 + \theta_1^2 + \theta_2^2) + 2(\theta_1 + \theta_1\theta_2)(1) + 2\theta_2(1)$$

$$= (1 + \theta_1^2 + \theta_2^2) + 2\theta_1 + 2\theta_1\theta_2 + 2\theta_2$$

$$= (1 + \theta_1 + \theta_2)^2$$

At $\omega = \frac{\pi}{2}$, eq. (9) becomes:

$$\frac{2\pi}{\sigma^2} S_y(\omega) = (1 + \theta_1^2 + \theta_2^2) + 2(\theta_1 + \theta_1\theta_2)(0) + 2\theta_2(-1)$$

$$= 1 - 2\theta_2 + \theta_1^2 + \theta_2^2$$

At $\omega = \pi$, eq. (9) becomes:

$$\frac{2\pi}{\sigma^2} S_y(\omega) = (1 + \theta_1^2 + \theta_2^2) + 2(\theta_1 + \theta_1\theta_2)(-1) + 2\theta_2(1)$$

$$= (1 + \theta_1^2 + \theta_2^2) - 2\theta_1 - 2\theta_1\theta_2 + 2\theta_2$$

$$= (1 - \theta_1 + \theta_2)^2$$

the function decreases from $(1 + \theta_1 + \theta_2)^2$ to $(1 - \theta_1 + \theta_2)^2$