Lince Rumainum
DSA 5203
Time Series Analysis
Final Exam

Problem #1-1

For the ***AR (1) model*** where x_t is a stationary, ergodic process

$$x_t = 0.5 + 0.1x_{t-1} + \epsilon_t$$

Where the mean, $\boldsymbol{E}(y_t) = \mu \; for \; all \; t, \phi = 0.1$ is the AR (1) parameter, and ϵ_t is a white noise sequence.

The properties of ϵ_t:

Mean:

$\boldsymbol{E}(\epsilon_t) = 0,$

$\boldsymbol{Cov}(\epsilon_i, \epsilon_t) = 0$, for all $\boldsymbol{i} \neq \boldsymbol{t}$

Variance: $\boldsymbol{Var}(\epsilon_t) = \sigma^2$

Looking at the second-order properties of AR (1), we have:

Mean: $\boldsymbol{E}(\mathrm{x}_t) = \mu$

To find μ, we can use the x_t equation:

$\mu = \boldsymbol{E}(x_t) = c + \phi \boldsymbol{E}(x_{t-1}) + \boldsymbol{E}(\epsilon_t) = c + \phi\mu + 0$

$\therefore, \; \mu = \frac{c}{1-\phi} = \frac{0.5}{1-0.1} = 0.556$

Variance:

Substituting c from where we find μ,

$y_t = (\mu - \phi\mu) + \phi y_{t-1} + \epsilon_t$

$y_t - \mu = \phi\,(y_{t-1} - \mu) + \epsilon_t$

$\boldsymbol{\gamma_0} = \boldsymbol{Var}(y_t) = \boldsymbol{E}\,[y_t - \boldsymbol{E}(y_t)]^2$

$= \boldsymbol{E}\,[y_t - \mu]^2$

$= \boldsymbol{E}\,[\phi\,(y_{t-1} - \mu) + \epsilon_t]^2 = \boldsymbol{E}\,[\,\phi^2\,(y_{t-1} - \mu) + {\epsilon_t}^2 + 2\phi\,(y_{t-1} - \mu)]^2$

$= \boldsymbol{E}\,[\,\phi^2\,(y_{t-1} - \mu)^2 + 2\phi\,(y_{t-1} - \mu)\epsilon_t + {\epsilon_t}^2]$

$= \phi^2 \boldsymbol{E}\,[\,(y_{t-1} - \mu)^2] + 0 + \boldsymbol{E}\,[{\epsilon_t}^2]$

$\gamma_0 = \phi^2\gamma_0 + \sigma^2$

$\gamma_0 = \frac{\sigma^2}{1-\phi^2}$

Autocovariance:

$$\gamma_j = \boldsymbol{E}\,[(y_t - \mu)(y_{t-j} - \mu)] = \boldsymbol{E}[(\phi\,(y_{t-1} - \mu) + \epsilon_t)(y_{t-j} - \mu)]$$

$$= \boldsymbol{E}\,[\phi\,(y_{t-1} - \mu)(y_{t-j} - \mu) + \epsilon_t(y_{t-j} - \mu)]$$

$$= \phi\,\boldsymbol{E}\,[\,(y_{t-1} - \mu)(y_{t-j} - \mu) + 0]$$

$$= \phi\,\gamma_{j-1}$$

Iterating γ_j,

$$\gamma_j = \phi^j \gamma_0 = \phi^j \left(\frac{\sigma^2}{1-\phi^2}\right)$$

∴, we have the ***autocorrelation function (ACF) of AR (1)*** as:

$$\rho_j = \frac{\gamma_j}{\gamma_0} = \frac{\phi^j \gamma_0}{\gamma_0}$$

$$\rho_j = \phi^j \text{, for j } \geq 1$$

$$\boldsymbol{\rho_1} = \phi^1 = 0.1^1 = \mathbf{0.1}$$

$$\boldsymbol{\rho_2} = \phi^2 = 0.1^2 = \mathbf{0.01}$$

Problem #1-2

For the ***AR (4) model*** where x_t is a stationary, ergodic process

$$x_t = 0.5 + 0.1x_{t-4} + \epsilon_t$$

Where the mean, $\boldsymbol{E}(\mathrm{x}_t) = \mu\ for\ all\ t, \phi_4 = 0.1$ is the AR (4) parameter, and ϵ_t is a white noise sequence.

The properties of ϵ_t:

Mean:

$\boldsymbol{E}(\epsilon_t) = 0$,

$\boldsymbol{Cov}(\epsilon_i, \epsilon_t) = 0$, for all $\boldsymbol{i} \neq \boldsymbol{t}$

Variance: $\boldsymbol{Var}(\epsilon_t) = \sigma^2$

Looking at the second-order properties of AR (4), we have:

Mean: $\boldsymbol{E}(\mathrm{x}_t) = \mu$

To find μ, we can use the x_t equation:

$$\mu = \boldsymbol{E}(x_t) = c + \phi_4 \boldsymbol{E}(x_{t-4}) + \boldsymbol{E}(\epsilon_t) = c + \phi_4 \mu + 0$$

$$\therefore,\ \mu = \frac{c}{1-\phi_4} = \frac{0.5}{1-0.1} = 0.556$$

To find general equation for μ, we can use the $x_t = c + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \phi_3 x_{t-3} + \phi_4 x_{t-4} + \epsilon_t$ equation:

$$\mu = \boldsymbol{E}(\mathrm{x}_t) = c + \phi_1 \boldsymbol{E}(x_{t-1}) + \phi_2 \boldsymbol{E}(x_{t-2}) + \phi_3 \boldsymbol{E}(x_{t-3}) + \phi_4 \boldsymbol{E}(x_{t-4}) + \boldsymbol{E}(\epsilon_t)$$

$$= c + \phi_1 \mu + \phi_2 \mu + \phi_3 \mu + \phi_4 \mu + 0$$

$$\therefore, \ \mu = \frac{c}{1-\phi_1-\phi_2-\phi_3-\phi_4}$$

Variance:

Substituting c from where we find μ,

$$\mathrm{x}_t = (\mu - \phi_1\mu - \phi_2\mu - \phi_3\mu - \phi_4\mu) + \phi_1 \mathrm{x}_{t-1} + \phi_2 \mathrm{x}_{t-2} + \phi_3 \mathrm{x}_{t-3} + \phi_4 \mathrm{x}_{t-4} + \epsilon_t$$

$$\mathrm{x}_t - \mu = \phi_1 (y_{t-1} - \mu) + \phi_2 (y_{t-2} - \mu) + \phi_4 (y_{t-4} - \mu) + \phi_2 (y_{t-4} - \mu) + \epsilon_t$$

$$\boldsymbol{\gamma_0} = \boldsymbol{Var}(\mathrm{x}_t) = \boldsymbol{E}\,[\mathrm{x}_t - \boldsymbol{E}(\mathrm{x}_t)]^2$$

$$= \boldsymbol{E}\,[y_t - \mu]^2$$

$$= \boldsymbol{E}\,[(\phi_1 (x_{t-1} - \mu) + \phi_2 (x_{t-2} - \mu) + (\phi_3 x_{t-3} - \mu) + (\phi_4 x_{t-4} - \mu) + \epsilon_t)(\mathrm{x}_t - \mu)]$$

$$= \phi_1\gamma_1 + \phi_2\gamma_2 + \phi_3\gamma_3 + \phi_4\gamma_4$$

$$+\boldsymbol{E}\,[(\phi_1 (y_{t-1} - \mu) + \phi_2 (y_{t-2} - \mu) + (\phi_3 x_{t-3} - \mu) + (\phi_4 x_{t-4} - \mu) + \epsilon_t)\,\epsilon_t)]$$

$$= \phi_1\gamma_1 + \phi_2\gamma_2 + \phi_3\gamma_3 + \phi_4\gamma_4$$

$$+\boldsymbol{E}\,[(\phi_1 (\mathrm{x}_{t-1} - \mu)\epsilon_t + \phi_2 (\mathrm{x}_{t-2} - \mu)\epsilon_t + (\phi_3 x_{t-3} - \mu)\epsilon_t + (\phi_4 x_{t-4} - \mu)\epsilon_t + \epsilon_t\epsilon_t)]$$

$$= \phi_1\gamma_1 + \phi_2\gamma_2 + \phi_3\gamma_3 + \phi_4\gamma_4 + \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} + \sigma^2$$

$$\boldsymbol{\gamma_0} = \phi_1\gamma_1 + \phi_2\gamma_2 + \phi_3\gamma_3 + \phi_4\gamma_4 + \sigma^2$$

Autocovariance:

$$\gamma_j = \boldsymbol{E}\big[\phi_1(\mathrm{x}_t - \mu)(\mathrm{x}_{t-j} - \mu)\big]$$

$$= \boldsymbol{E}[(\phi_1 (x_{t-1} - \mu) + \phi_2 (x_{t-2} - \mu) + (\phi_3 x_{t-3} - \mu) + (\phi_4 x_{t-4} - \mu) + \epsilon_t)(\mathrm{x}_{t-\mathrm{j}} - \mu)]$$

$$= \phi_1 \boldsymbol{E}\,[(\mathrm{x}_{t-1} - \mu)(\mathrm{x}_{t-j} - \mu) + \phi_2 \boldsymbol{E}\,[(\mathrm{x}_{t-2} - \mu)(\mathrm{x}_{t-j} - \mu)] + \phi_3 \boldsymbol{E}\,[(\mathrm{x}_{t-3} - \mu)(\mathrm{x}_{t-j} - \mu)]$$

$$+\phi_4 \boldsymbol{E}\,[(\mathrm{x}_{t-4} - \mu)(\mathrm{x}_{t-j} - \mu)] + \boldsymbol{E}\,[\epsilon_t(\mathrm{x}_{t-j} - \mu)]$$

$$\boldsymbol{\gamma_j} = \phi_1\,\gamma_{j-1} + \phi_2\,\gamma_{j-2} + \phi_3\,\gamma_{\mathrm{j}-3} + \phi_4\,\gamma_{j-4}\ \text{, for } j \geq 1$$

$\therefore$, we have the ***autocorrelation function (ACF) of AR (4)*** as:

$$\rho_j = \frac{\gamma_j}{\gamma_0} = \frac{\phi_1\,\gamma_{j-1} + \phi_2\,\gamma_{j-2} + \phi_3\,\gamma_{\mathrm{j}-3} + \phi_4\,\gamma_{j-4}}{\gamma_0}$$

$$= \phi_1 \frac{\gamma_{j-1}}{\gamma_0} + \phi_2 \frac{\gamma_{j-2}}{\gamma_0} + \phi_3 \frac{\gamma_{j-3}}{\gamma_0} + \phi_4 \frac{\gamma_{j-4}}{\gamma_0}$$

$$\rho_j = \phi_1\,\rho_{j-1} + \phi_2\,\rho_{j-2} + \phi_3\,\rho_{j-3} + \phi_4\,\rho_{j-4}\ \text{, for } j \geq 1$$

For $j = 0, \rho_0 = \frac{\gamma_0}{\gamma_0} = \frac{\phi_1\gamma_1+\phi_2\gamma_2+\phi_3\gamma_3+\phi_4\gamma_4+\sigma^2}{\phi_1\gamma_1+\phi_2\gamma_2+\phi_3\gamma_3+\phi_4\gamma_4+\sigma^2} = 1$

Using $\rho_k = \rho_{-k}$

For $j = 1$,

$$\rho_1 = \frac{\phi_1+\phi_3\,\rho_2+\phi_4\,\rho_3}{1-\phi_2}$$

For $j = 2, \rho_2 = \phi_1\,\rho_{2-1} + \phi_2\,\rho_{2-2} + \phi_3\,\rho_{2-3} + \phi_4\,\rho_{2-4} = \phi_1\,\rho_1 + \phi_2 + \phi_3\,\rho_2 + \phi_4\,\rho_3$

For $j = 3, \rho_3 = \phi_1\,\rho_{3-1} + \phi_2\,\rho_{3-2} + \phi_3\,\rho_{3-3} + \phi_4\,\rho_{3-4} = \phi_1\,\rho_2 + \phi_2\rho_1 + \phi_3 + \phi_4\,\rho_1$

For $j = 4, \rho_4 = \phi_1\,\rho_{4-1} + \phi_2\,\rho_{4-2} + \phi_3\,\rho_{4-3} + \phi_4\,\rho_{4-4} = \phi_1\,\rho_3 + \phi_2\rho_2 + \phi_3\,\rho_1 + \phi_4$

For $j \geq 5, \rho_j = \phi_1\,\rho_{j-1} + \phi_2\,\rho_{j-2} + +\phi_3\,\rho_{j-3} + \phi_4\,\rho_{j-4}$

For problem 1-2, we know

$$\phi_1 = 0,\ \phi_2 = 0,\ \phi_3 = 0,\ \phi_4 = 0.1$$

For $\boldsymbol{j = 4}, \rho_4 = \phi_1\,\rho_3 + \phi_2\rho_2 + \phi_3\,\rho_1 + \phi_4 = \mathbf{0.1}$

Problem #1-3

For the ***AR (4) model*** where x_t is a stationary, ergodic process

$$x_t = 0.5 + 0.1x_{t-1} + 0.1x_{t-4} + \epsilon_t$$

Where the mean, $\boldsymbol{E}(x_t) = \mu\ for\ all\ t, \phi_1 = 0.1, \phi_4 = 0.1$ is the AR (4) parameter, and ϵ_t is a white noise sequence.

(*Using what we found in Problem 1-2*)

For problem 1-3, we know

$$\phi_1 = 0.1, \phi_2 = 0, \phi_3 = 0, \phi_4 = 0.1$$

For $j = 0, \boldsymbol{\rho_0} = \frac{\gamma_0}{\gamma_0} = \frac{\phi_1\gamma_1+\phi_2\gamma_2+\phi_3\gamma_3+\phi_4\gamma_4+\sigma^2}{\phi_1\gamma_1+\phi_2\gamma_2+\phi_3\gamma_3+\phi_4\gamma_4+\sigma^2} = \mathbf{1}$

For $\boldsymbol{j = 1}, \boldsymbol{\rho_1} = \frac{\phi_1+\phi_3\,\rho_2+\phi_4\,\rho_3}{1-\phi_2} = \phi_1 + \phi_4\,\rho_3 = 0.1 + 0.1\rho_3 = 0.\mathbf{1011236}$

For $j = 2, \rho_2 = \phi_1\,\rho_1 + \phi_2 + \phi_3\,\rho_2 + \phi_4\,\rho_3 = \phi_1\,\rho_1 + \phi_4\,\rho_3$

$$= 0.1\,\rho_1 + 0.1(0.1\,\rho_2 + 0.1\,\rho_1) = 0.011$$

For $j = 3, \rho_3 = \phi_1\,\rho_2 + \phi_2\rho_1 + \phi_3 + \phi_4\,\rho_1 = \phi_1\,\rho_2 + \phi_4\,\rho_1 = 0.1\,\rho_2 + 0.1\rho_1 = 0.011236$

For $\boldsymbol{j = 4}, \boldsymbol{\rho_4} = \phi_1\,\rho_3 + \phi_2\rho_2 + \phi_3\,\rho_1 + \phi_4 = \phi_1\,\rho_3 + \phi_4 = 0.1\,\rho_3 + 0.1 = \mathbf{0.1011236}$