Übung 5

Dienstag, 26. April 2022 13:26

Aufgabe 1

$$A = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$$

a)

1)

$$B_j = I - D^{-1}A$$

$$\Leftrightarrow B_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 4 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 4 \end{pmatrix}$$

$$\Leftrightarrow B_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \cdot \begin{pmatrix} 4 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 4 \end{pmatrix}$$

$$\Leftrightarrow B_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & 1 \end{pmatrix}$$

$$\Leftrightarrow B_j = \begin{pmatrix} 0 & -\frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} & 0 \end{pmatrix}$$

$$\chi_{B_j}(\lambda) = \det(B_j - \lambda I) \overset{!}{=} 0$$

$$\Leftrightarrow \det\left(\begin{pmatrix} 0 & -\frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = 0$$

$$\Leftrightarrow -\frac{1}{4} \det \begin{pmatrix} 4\lambda & 2 & 1 \\ 2 & 4\lambda & 2 \\ 1 & 2 & 4\lambda \end{pmatrix} = 0$$

$$\Leftrightarrow -\frac{1}{4}\left((64\lambda)^3 + 4 + 4) - (4\lambda + 16\lambda + 16\lambda\right) = 0$$

$$\Leftrightarrow -\frac{1}{4}\left(64\lambda^3 - 36\lambda + 8\right) = 0$$

$$\Leftrightarrow -16\lambda^3 + 9\lambda - 2 = 0$$

$$\Rightarrow \lambda_1 = \frac{1}{4}, \quad \lambda_2 = \frac{-1-\sqrt{33}}{8}, \quad \lambda_3 = \frac{-1+\sqrt{33}}{8}$$

$$\Rightarrow \rho(B_j) = \max(|\lambda_1|, |\lambda_2|, |\lambda_3|) = \left| \frac{-1-\sqrt{33}}{8} \right| \approx 0{,}8431$$

2)

$$M = (D - E)^{-1}$$

$$M = \left(\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ -1 & -2 & 0 \end{pmatrix} \right)$$

$$\Rightarrow M^{-1} = \begin{pmatrix} 4 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ -\frac{1}{8} & \frac{1}{4} & 0 \\ 0 & -\frac{1}{8} & \frac{1}{4} \end{pmatrix}$$

$$B_{GS} = I - M^{-1} \cdot A$$

$$\Leftrightarrow B_{GS} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ -\frac{1}{8} & \frac{1}{4} & 0 \\ 0 & -\frac{1}{8} & \frac{1}{4} \end{pmatrix} \cdot \begin{pmatrix} 4 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 4 \end{pmatrix}$$

$$B_{GS} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{3}{8} \\ 0 & 0 & \frac{3}{4} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2} & -\frac{1}{4} \\ 0 & \frac{1}{4} & -\frac{3}{8} \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

$$\chi_{B_{GS}}(\lambda) = \det(B_{GS} - \lambda I) \overset{!}{=} 0$$

$$\Leftrightarrow \det\left(\begin{pmatrix} 0 & -\frac{1}{2} & -\frac{1}{4} \\ 0 & \frac{1}{4} & -\frac{3}{8} \\ 0 & 0 & \frac{1}{4} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = 0$$

$$\Leftrightarrow \frac{1}{8} \det \begin{pmatrix} -\lambda & -4 & -2 \\ 0 & 2-\lambda & -3 \\ 0 & 0 & 2-\lambda \end{pmatrix}$$

$$\Leftrightarrow -\lambda \cdot \left(\frac{1}{4}\lambda\right)^2 = 0$$

$$\Rightarrow \lambda_1 = 0, \quad \lambda_2 = \frac{1}{4}$$

$$\Rightarrow \rho(B_{GS}) = \max(|\lambda_1|, |\lambda_2|) = \frac{1}{4}$$

b)

$$A = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}, \quad x_0 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

1) $$\boxed{x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum a_{ij} x_j^{(k)} \right)}$$

$$x_1^{(1)} = \frac{1}{4} \left(3 - (2 \cdot 0 + 1 \cdot 0) \right) = \frac{3}{4}$$

$$x_2^{(1)} = \frac{1}{4} \left(0 - (2 \cdot 0 + 2 \cdot 0) \right) = 0$$

$$x_3^{(1)} = \frac{1}{4} \left(3 - (1 \cdot 0 + 2 \cdot 0) \right) = \frac{3}{4}$$

$$\Rightarrow x_1 = \left(\frac{3}{4}, 0, \frac{3}{4} \right)^T$$

- - - - - - - - -

$$x_1^{(2)} = \frac{1}{4} \left(3 - (2 \cdot 0 + 1 \cdot \tfrac{3}{4}) \right) = \frac{9}{16}$$

$$x_2^{(2)} = \frac{1}{4} \left(0 - (2 \cdot \tfrac{3}{4} + 2 \cdot \tfrac{3}{4}) \right) = -\frac{3}{4}$$

$$x_3^{(2)} = \frac{1}{4} \left(3 - (1 \cdot \tfrac{3}{4} + 2 \cdot 0) \right) = \frac{9}{16}$$

$$\Rightarrow x_2 = \left(\frac{9}{16}, -\frac{3}{4}, \frac{9}{16} \right)^T$$

- - - - - - - - -

$$x_1^{(3)} = \frac{1}{4} \left(3 - (2 \cdot -\tfrac{3}{4} + 1 \cdot \tfrac{9}{16}) \right) = \frac{63}{64}$$

$$x_2^{(3)} = \frac{1}{4} \left(0 - (2 \cdot \tfrac{9}{16} + 2 \cdot \tfrac{9}{16}) \right) = -\frac{9}{16}$$

$$x_3^{(3)} = \frac{1}{4} \left(3 - (1 \cdot \tfrac{9}{16} + 2 \cdot -\tfrac{3}{4}) \right) = \frac{63}{64}$$

$$\Rightarrow x_3 = \left(\frac{63}{64}, -\frac{9}{16}, \frac{63}{64} \right)^T$$

- - - - - - - - -

$$x_1^{(4)} = \frac{1}{4} \left(3 - (2 \cdot -\tfrac{9}{16} + 1 \cdot \tfrac{63}{64}) \right) = \frac{201}{256}$$

$$x_2^{(4)} = \frac{1}{4} \left(0 - (2 \cdot \tfrac{63}{64} + 2 \cdot \tfrac{63}{64}) \right) = -\frac{63}{64}$$

$$x_3^{(4)} = \frac{1}{4} \left(3 - (1 \cdot \tfrac{63}{64} + 2 \cdot -\tfrac{9}{16}) \right) = \frac{201}{256}$$

$$\Rightarrow x_4 = \left(\frac{201}{256}, -\frac{63}{64}, \frac{201}{256} \right)^T$$

- - - - - - - - -

2) $$\boxed{x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j<i} a_{ij} x_j^{(k+1)} - \sum_{j>i} a_{ij} x_j^{(k)} \right)}$$

$$x_1^{(1)} = \frac{1}{4} \left(3 - (2 \cdot 0 + 1 \cdot 0) - (0) \right) = \frac{3}{4}$$

$$x_2^{(1)} = \frac{1}{4} \left(0 - (2 \cdot \tfrac{3}{4}) - (2 \cdot 0) \right) = -\frac{6}{16} = -\frac{3}{8}$$

$$x_3^{(1)} = \frac{1}{4} \left(3 - (0) - (1 \cdot \tfrac{3}{4} + 2 \cdot -\tfrac{3}{8}) \right) = \frac{3}{4}$$

$$\Rightarrow x_1 = \left(\frac{3}{4}, -\frac{3}{8}, \frac{3}{4} \right)$$

- - - - - - - - - - - - - - - - - - - -

$$x_1^{(2)} = \frac{1}{4} \left(3 - (2 \cdot -\tfrac{3}{8} + 1 \cdot \tfrac{3}{4}) - (0) \right) = \frac{3}{4}$$

$$x_2^{(2)} = \frac{1}{4} \left(0 - (2 \cdot \tfrac{3}{4}) - (2 \cdot \tfrac{3}{4}) \right) = -\frac{3}{4}$$

$$x_3^{(2)} = \frac{1}{4} \left(3 - (0) - (1 \cdot -\tfrac{3}{4} + 2 \cdot -\tfrac{3}{4}) \right) = \frac{21}{16}$$

$$\Rightarrow x_2 = \left(\frac{3}{4}, -\frac{3}{4}, \frac{21}{16} \right)$$

- - - - - - - - - - - - - - - - - - - -

$$x_1^{(3)} = \frac{1}{4} \left(3 - (2 \cdot -\tfrac{3}{8} + 1 \cdot \tfrac{3}{4}) - (0) \right) = \frac{3}{4}$$

$$x_2^{(3)} = \frac{1}{4} \left(0 - (2 \cdot \tfrac{3}{4}) - (2 \cdot \tfrac{3}{4}) \right) = -\frac{3}{4}$$

$$x_3^{(3)} = \frac{1}{4} \left(3 - (0) - (1 \cdot -\tfrac{3}{4} + 2 \cdot -\tfrac{3}{4}) \right) = \frac{21}{16}$$

$$\Rightarrow x_3 = \left(\frac{3}{4}, -\frac{3}{4}, \frac{21}{16} \right) = x_2$$

↳ würde sich bis x_5 so wiederholen

$$x_1^{(5)} = \frac{1}{4}\left(3 - \left(2 \cdot -\frac{63}{64} + 1 \cdot \frac{201}{256}\right)\right) = \frac{1071}{1024}$$

$$x_2^{(5)} = \frac{1}{4}\left(0 - \left(2 \cdot \frac{201}{256} + 2 \cdot \frac{201}{256}\right)\right) = -\frac{201}{256}$$

$$x_3^{(5)} = \frac{1}{4}\left(3 - \left(1 \cdot \frac{201}{256} + 2 \cdot -\frac{63}{64}\right)\right) = \frac{1071}{1024}$$

$$\Rightarrow x_5 = \left(\frac{1071}{1024}, -\frac{201}{256}, \frac{1071}{1024}\right)^T$$