University of Science, VNU-HCM
Faculty of Information Technology

Data Structure and Algorithm

Binary Search Tree
Balanced Tree

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HCM City

Outline

- Red-Black Tree
- AA Tree

Red-Black Tree

- Red-Black Tree is a binary search tree which complies with the following rules :

  [1] Every node has a colour either red or black

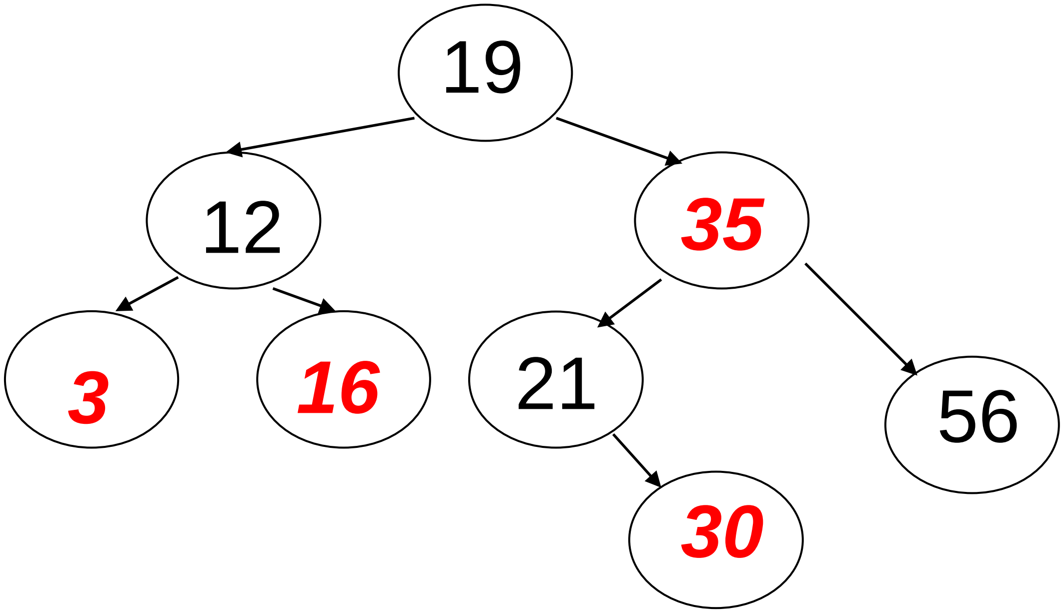
  [2] The root of tree is always black

  [3] If a node is red, its children must be black

  [4] Every path from a node (including root) to any of its descendant NULL node has the same number of black nodes.

  [5] The NULL node is black.

Example



Example

- Which following trees are Red-Black Tree?
  - If not, why?

```
      30
     / \
   20   NIL
   / \
 10   NIL
```

```
       30
      /   \
    20     NIL
    / \
  10   NIL
```

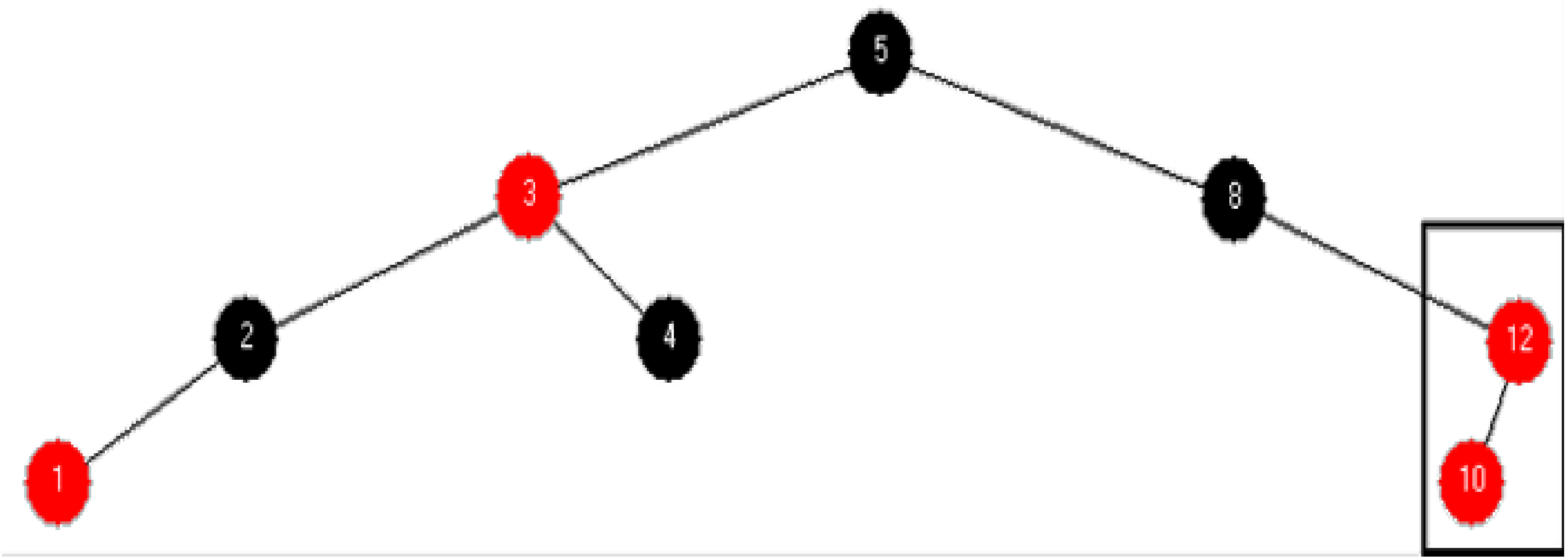
```
       30
      /   \
    20     NIL
   /   \
 10     NIL
```

```
        20
       /   \
     10      30
    /  \    /  \
 NIL   NIL NIL NIL
```

```
        20
       /   \
     10      30
    /  \    /  \
 NIL   NIL NIL NIL
```

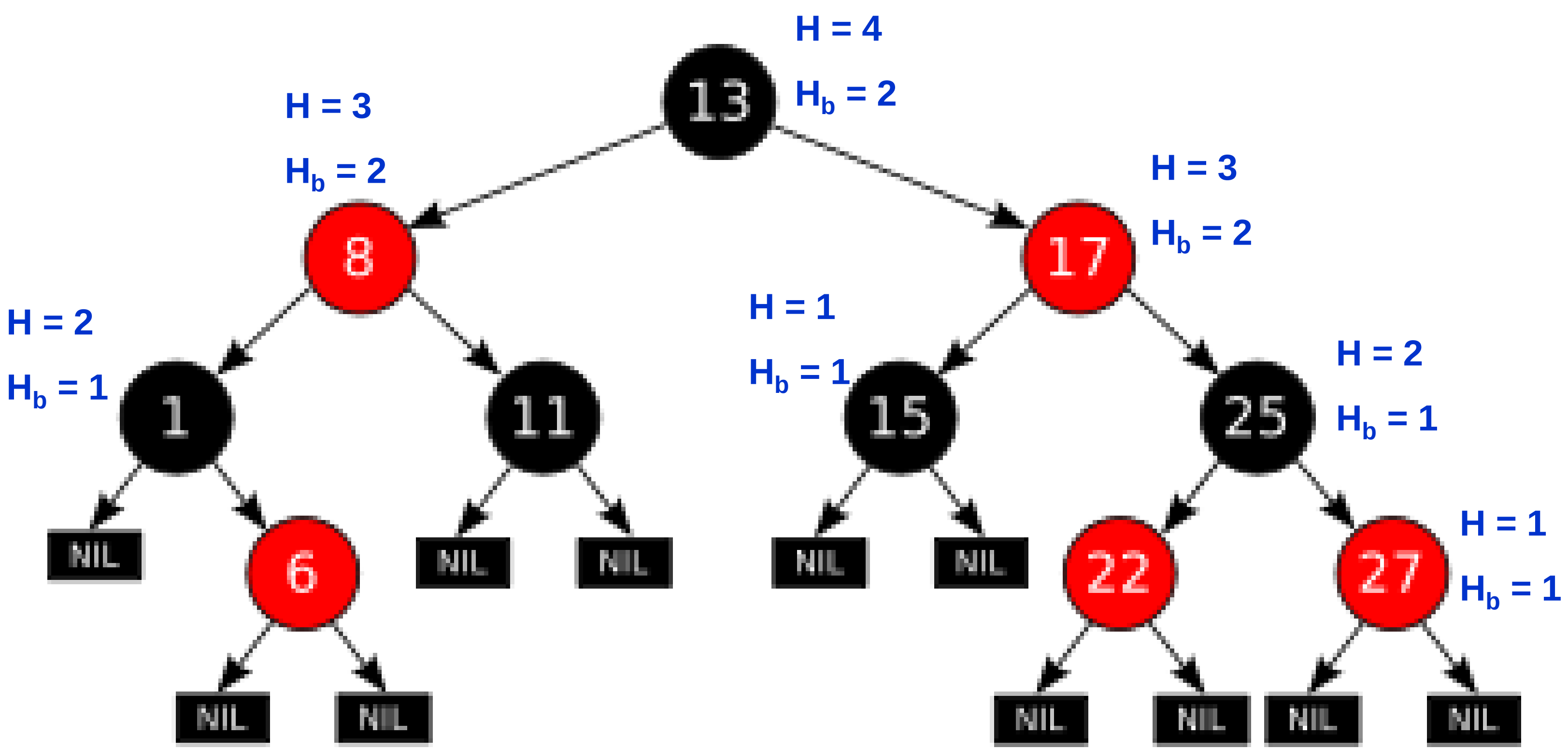
Consequences

- If a Red node has children, it must have 2 children and they must be Black. (Why?)
- If a Black node has only one child, the child must be a Red leaf. (Why?)



Red-Black True

- Black height – $h_b(x)$: is the number of black nodes on the path from node x to its descendant NULL node (x is not included).



Data Structure

```
typedef enum {BLACK, RED} NodeColor;

typedef int DataType;

typedef struct NodeTag {
      DataType          key;       // Data
      NodeColor         color;
      struct NodeTag   *pLeft;
      struct NodeTag   *pRight;
      struct NodeTag   *pParent;   //for easy traverse
} RBNode;

typedef struct RBNode* RBTREE;
```

Intert new element

- Insert node:
  - Execute like binary tree search.
  - The new node is inserted at leaf node and assigned a Red color (Why?)
    - Rule 4
  - Check rules:
    - If the parent node is Black → it doesn't matter.
    - If the parent node is Red → violates parent-child are Red → adjust to balance tree

Insertion function

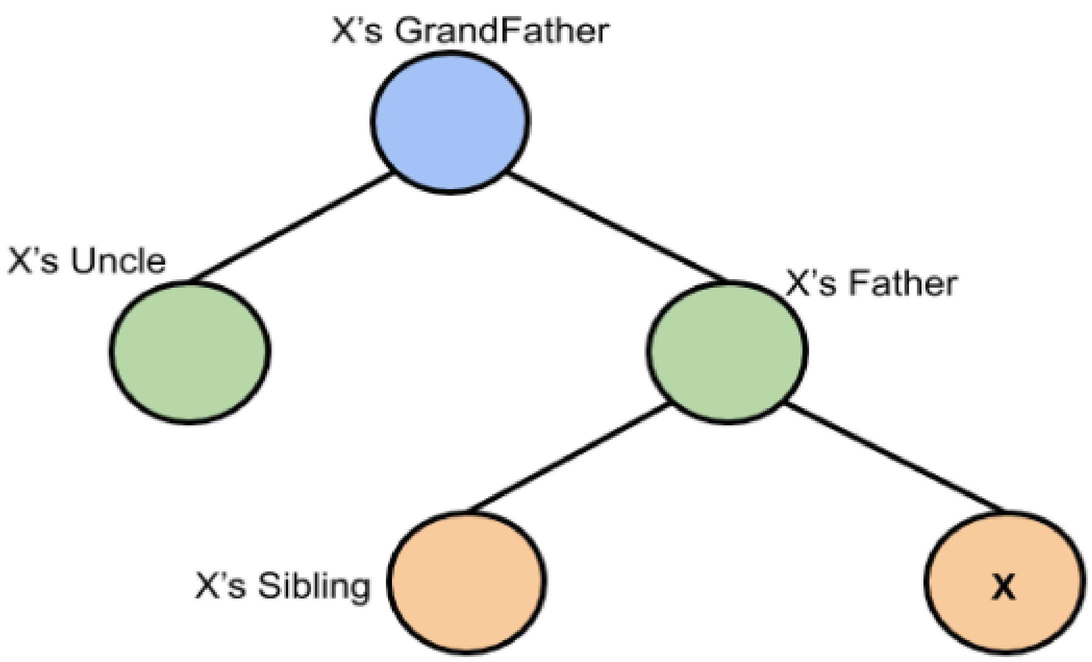
```
RB_Insert_Node(T, z)                // T: tree; z: new node
    y ← NULL; x ← root[T];          // Travese tree to
    while x ≠ NULL {                // go to leaf node
       y ← x                        // y: is parent node of x
       if (key[z] < key[x]) x ← left[x];
       else x ← right[x];
    }
    parent[z] ← y;                  // assign parent of z
    if (y == NULL) root[T] ← z;     // if parent is null -> z: root
    else if (key[z] < key[y]) left[y] ← z; //choose branch for z
         else right[y] ← z;
    left[z] ← NULL
    right[z] ← NULL
    color[z] ← RED                  // new node (z) is Red
    RB_Insert_FixUp(T, z)           // check and balance tree
```

Balance tree

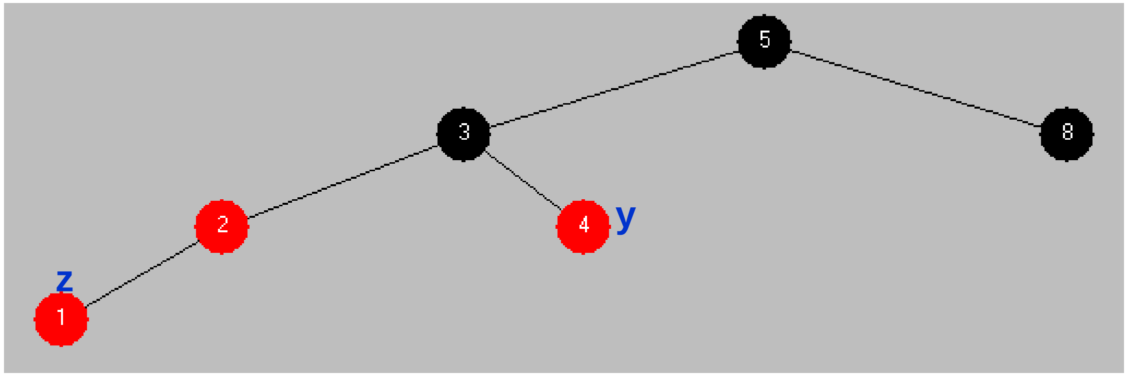
- There are 3 basic steps to do the balancing
  - Recoloring
  - Rotation
    - Left-Rotation
    - Right-Rotation

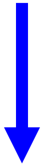
Recoloring

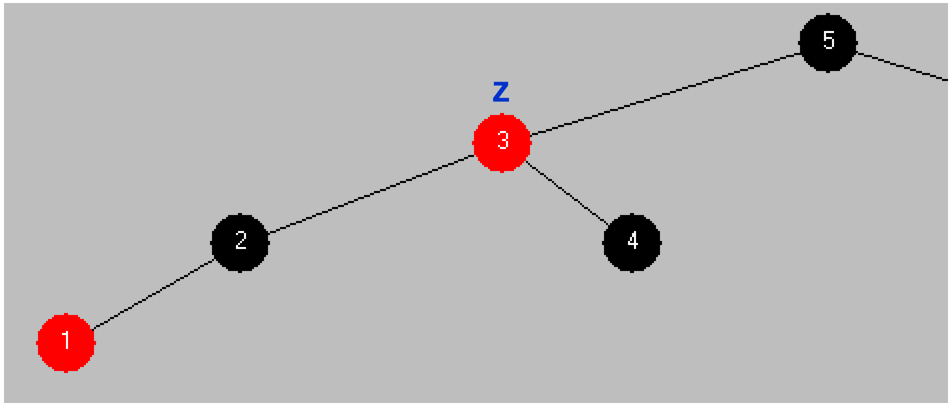
- Recoloring is the change in colour of the node.
  – If it is red then change it to black and vice versa.



Recoloring



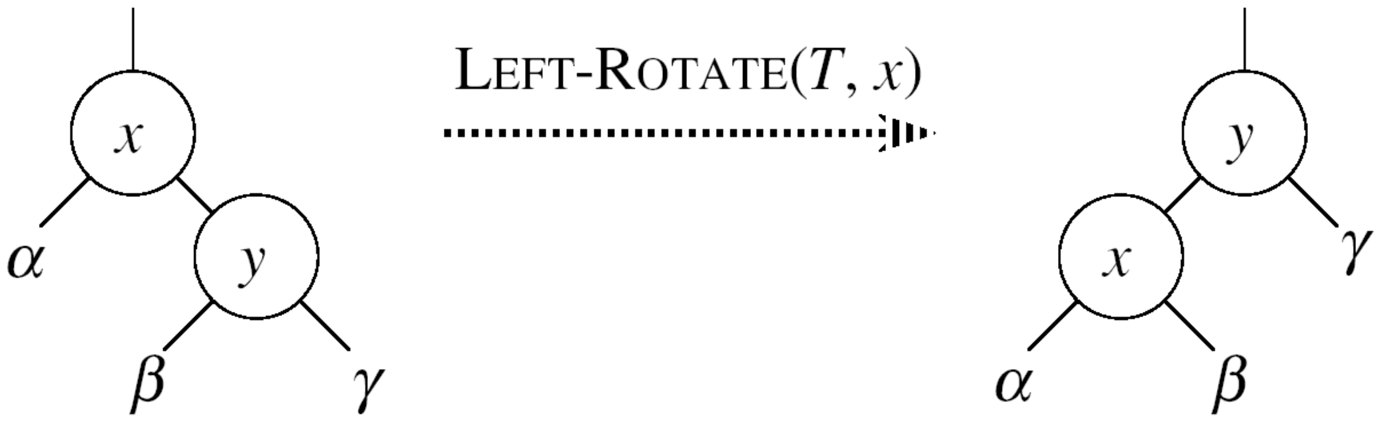




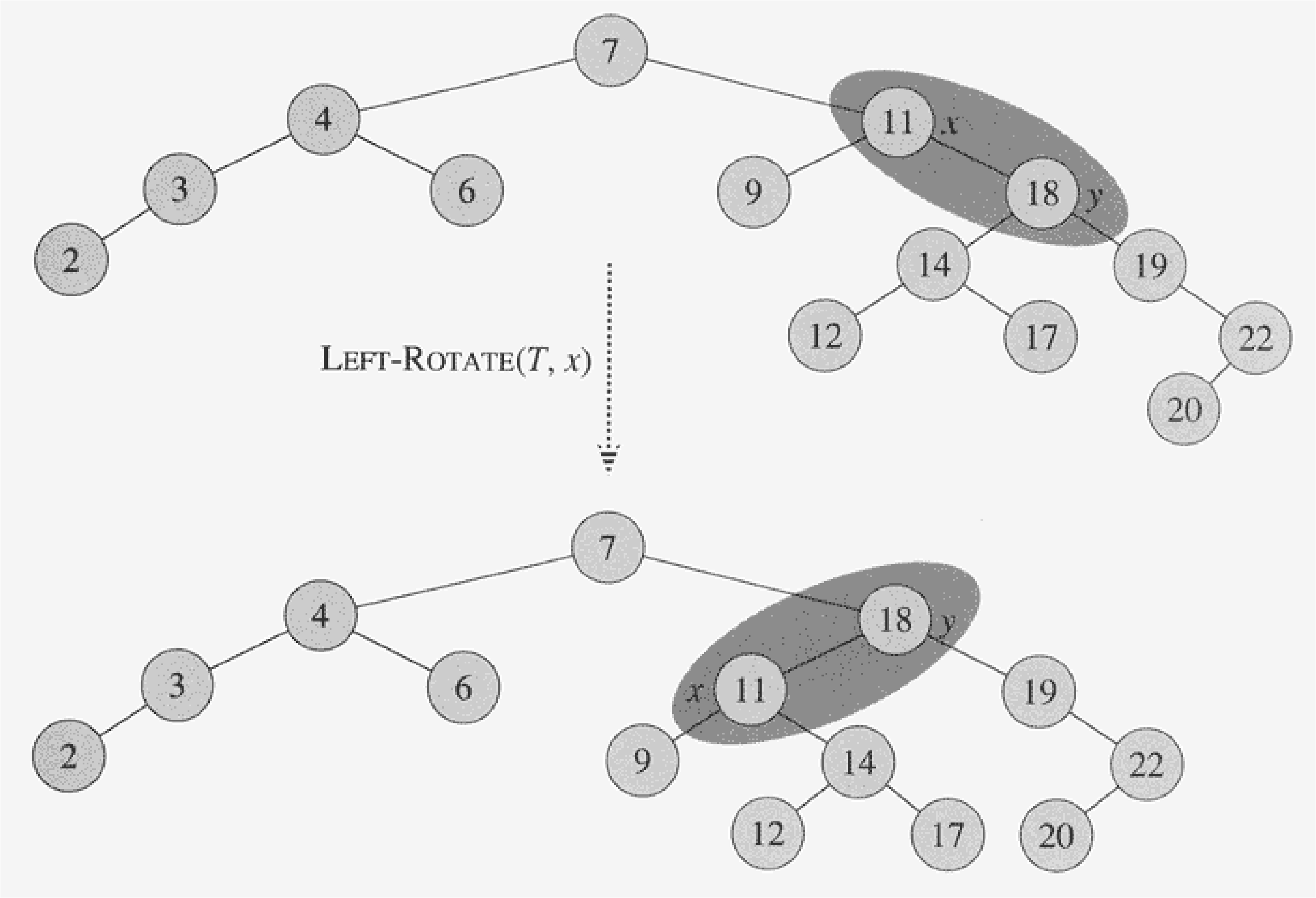

```
color[parent[z]] ← black
color[y] ← black
color[parent[parent[z]]] ← red
z = parent[parent[z]]
```

Left-Rotation

- Left-Rotation:



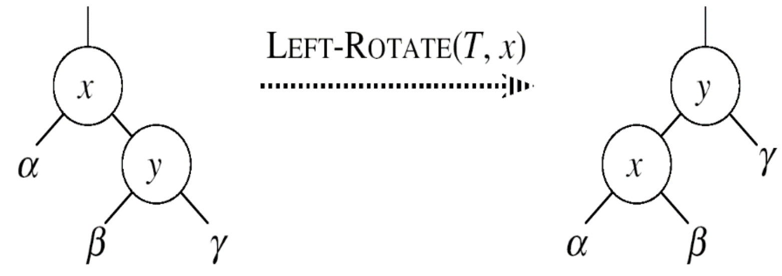
Left-Rotation



Example

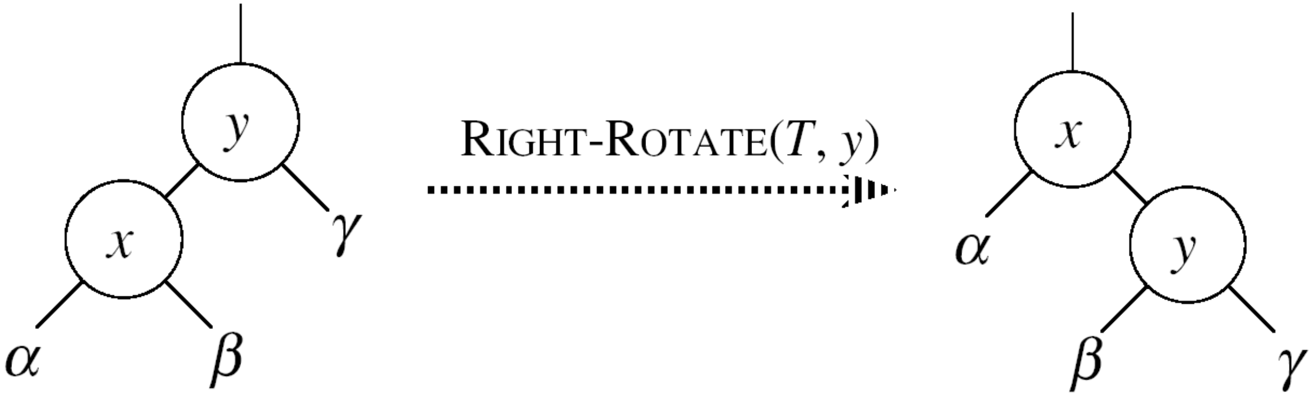
Left-Rotation

```
RB_Left_Rotate(T, x)
    y ← right[x];
    right[x] ← left[y];
    if (left[y] ≠ NULL) parent[left[y]] ← x;
    parent[y] ← parent[x];
    if (parent[x] == NULL) root[T] ← y;
    else if (x == left[parent[x]])
                left[parent[x]] ← y;
         else right[parent[x]] ← y;
    left[y] ← x;
    parent[x] ← y;
```



Right-Rotation

- Right-Rotation:



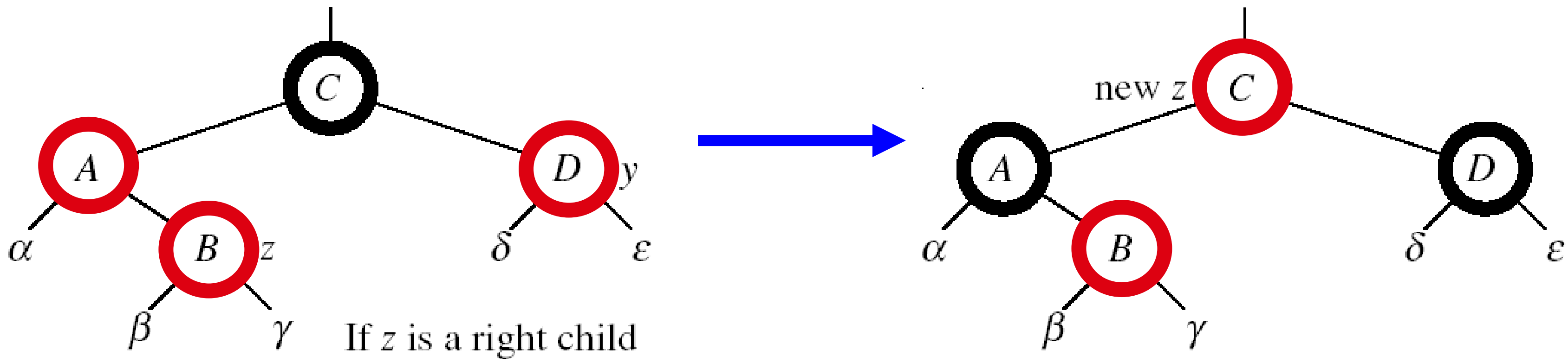
- **`RB_Right_Rotate(T, x):`** similar to the left rotation function

Balance tree

- The algorithms have mainly two cases depending upon the colour of the uncle.
  - If the uncle is red, we do recolour.
  - If the uncle is black, we do rotations and/or recolouring.
- Base on two main cases, we list 6 smaller cases for easy control.

Balance tree

- Case 1: parent is red, uncle is red, it is red → Use recoloring method

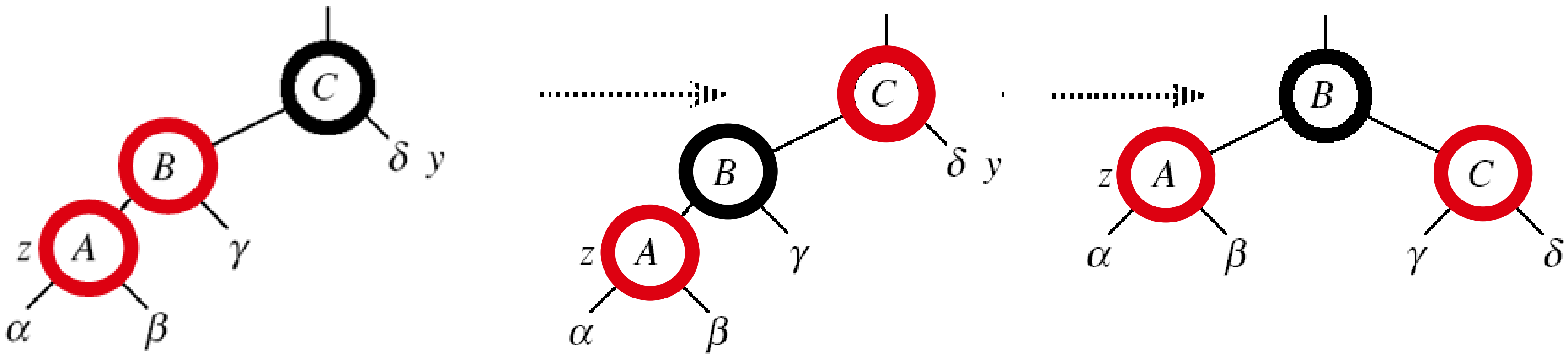


If z is a right child

```
color[parent[z]] ← black
color[y] ← black
color[parent[parent[z]]] ← red
z = parent[parent[z]]
```

Balance tree

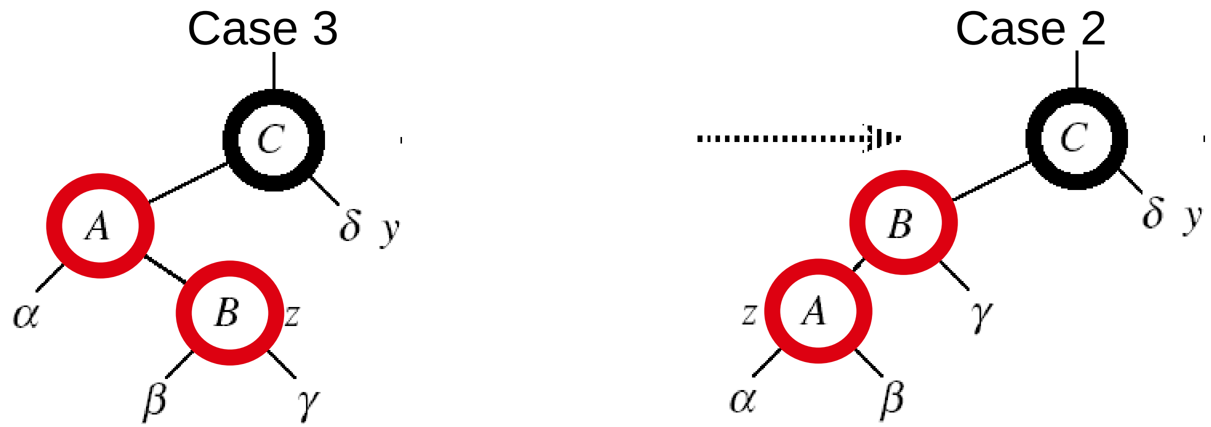
- Case 2: parent is red, uncle is black, it is a left child while parent is left child of grandfather (or otherwise) → Use recoloring and rotation method




```
color[parent[z]] ← black
color[parent[parent[z]]] ← red
RIGHT-ROTATE(T, parent[parent[z]])
```

Balance tree

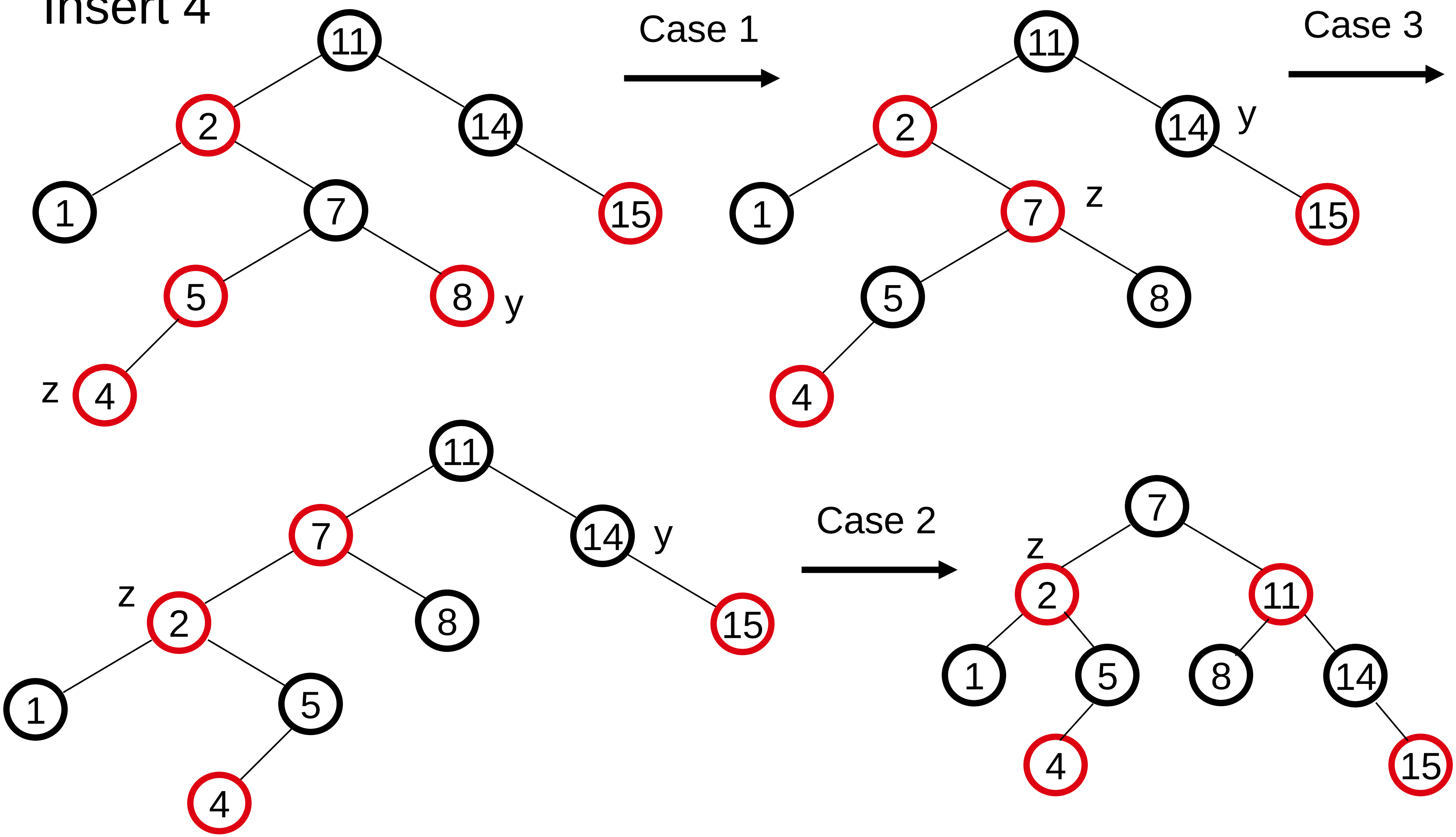
- Case 3: parent is red, uncle is black, it is a right child while parent is left child of grandfather (or otherwise) -> rotate to convert to case 2




```
z ← parent[z]
LEFT-ROTATE(T, z)
"Treat as the case 2"
```

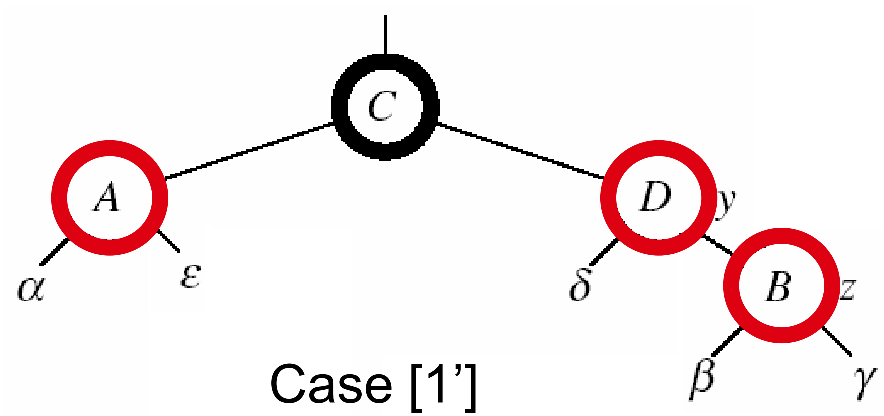
Example

Insert 4

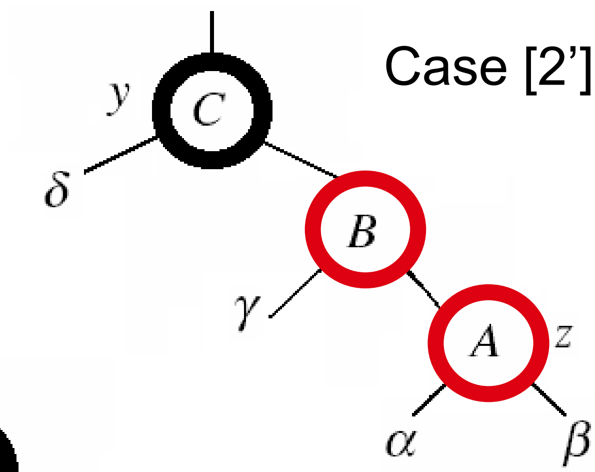


Balance tree

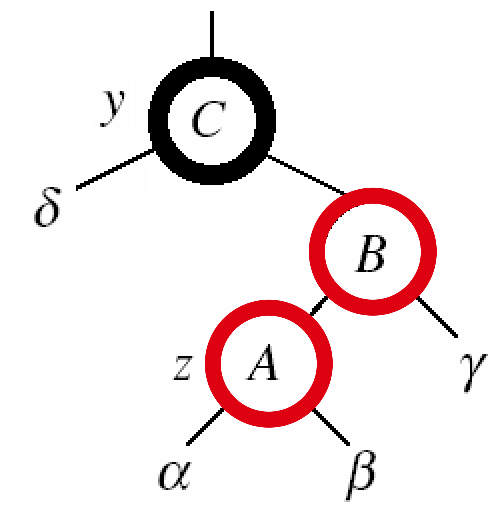
- Remain cases:
  – 3 similar cases [1'], [2'], [3'] are symmetrical with [1], [2], [3] through the y axis



Case [1’]



Case [2’]



Case [3’]

Balance tree

```
RB_Insert_FixUp(T, z)
    while (parent[z] != NULL && color[parent[z]] == RED)
    {
       // Case [1], [2], [3]
       if (parent[z] == left[parent[parent[z]]]) {
              y ← right[parent[parent[z]]];
              if (color[y] == RED) “Case 1”;
              else {
                 if (z == right[parent[z]]) “Case 3”;
                 else “Case 2”;
              }
       }
       else …// case 1′], [2′], [3′]
    }
    color[root[T]] ← BLACK
```

Comments

- Evaluate the Insert node operation :
  - The cost of adding a new element (z): $\boldsymbol{O}(log_2 N)$
  - Cost of RB_Insert_FixUp: $\boldsymbol{O}(log_2 N)$
  - Total cost: $\boldsymbol{O}(log_2 N)$

Delete a node

- Delete a node:
  - Execute like binary tree search.
  - However:
    - For the case of 2 children, only the value is replaced, not the color.
  - After deleting, there are 2 cases to consider :
    - The actually deleted node is the red node
    - The actually deleted node is the black node

Delete a node

- Delete a node:
  - If the actually deleted node is red: no violation
    - Every node has a colour either red or black → OK
    - The root of tree is always black → OK
    - The NULL node is black → OK
    - If a node is red, its children must be black → OK because do not create 2 red consecutive nodes
    - Every path from a node (including root) to any of its descendant NULL node has the same number of black nodes → OK because it doesn't change the number of black nodes

Delete a node

- Delete a node:
  - If the actually deleted node is **black**: a violation may occur
    - Every node has a colour either red or black → OK
    - The root of tree is always black → not OK ! Because it can delete root and replace it with a red node
    - The NULL node is black → OK
    - If a node is red, its children must be black → not OK ! Because it is possible to create 2 consecutive red nodes
    - Every path from a node (including root) to any of its descendant NULL node has the same number of black nodes → not OK ! Because it reduces the number of black nodes

Delete a node

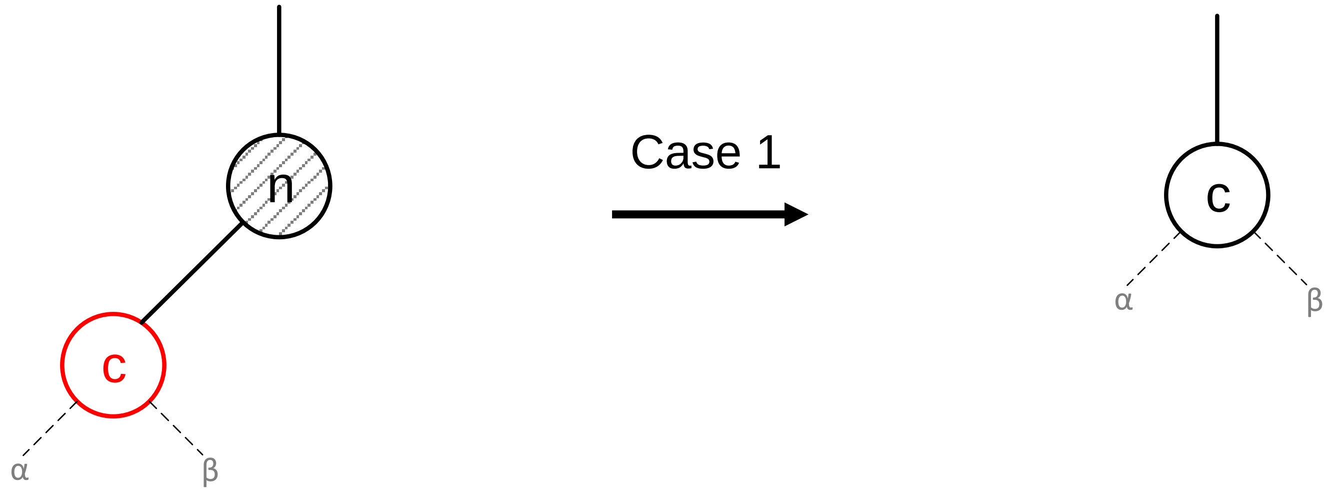
- So, before removing the actual deleted node:
  - If the actually deleted node is red, no adjustment is required
  - If the actually deleted node is **black**, it will need to be adjusted to rebalance the tree

Balance

- Delete the black node:
  - Comment: *if the actual black node is deleted, there can be at most 1 child.*
  - We divide it into 2 cases:
    - **Case 1**: the child of the delete black node is in red
    - **Case 2**: the child of the deleted black node is in **black**
      - Note that the null node is black, so in the case of the deleted node there are no children, we still consider that it still has black children.

Case 1: red child

- Case 1: child c of the deleted black node n is red
  - After deleting node n, we just need to change the color of child node c to black.



Case 2: black child

- Case 2: child c of the deleted black node n is black.
  - Let s be a sibling node of n, p be the parent node of n.
  - We will in turn perform the following steps:
    - Step 2.1: If n is the root then end. If n has a father then go to the next step.
    - Step 2.2: If s is red, invert parent and sibling colors. Then left rotate tree at p. Go to the next step.
    - Step 2.3: If s is black, the children of s are black and parent of p is black, we change the color s to red. Go back to step 2.1 with new node n is the parent of the current n (correction propagation).
    - Step 2.4: If s is black, the children of s are black but p is red, we reverse the parent and sibling color. Go to **end**.
    - Step 2.5: If s is black, the left child of s is red, the right child of is black, and n is its father's left child, we change color of s with its left child and right rotate at s. At this point, node n will have new siblings. Go to the next step.
    - Step 2.6: If s is black, the right child of s is red and n is its father's left child, we reverse the s and p color, coloring black for the right child of s. Left rotate at p. Go to **end**.
  - Note:
    - *In steps 2.2, 2.5 and 2.6, we will do the same but vice versa if n is the right child of its father.*
    - *Deletion of n, only performed when the steps are completed and n is the actual n need to be deleted, not n of the correction propagation processes.*

Pseudo Code

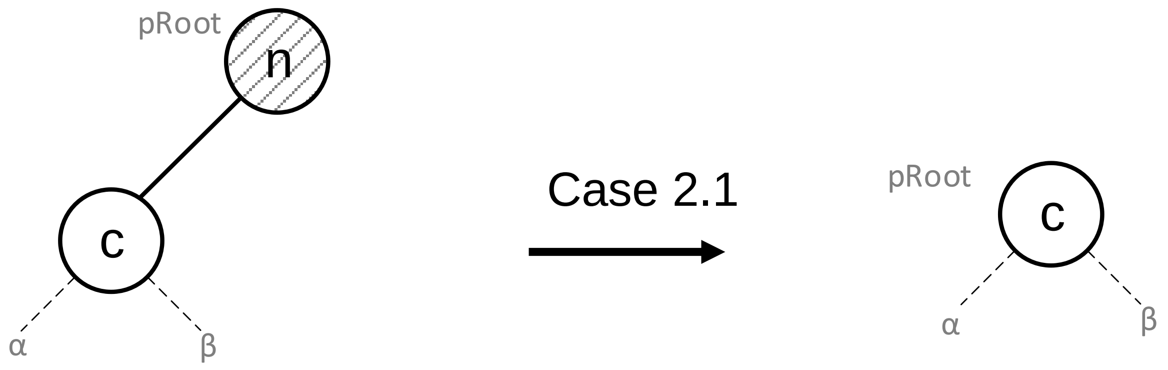
```
void replace_node(node* n, node* child) {
    child->parent = n->parent;
    if (n == n->parent->left)
        n->parent->left = child;
    else
        n->parent->right = child;
}

void delete_one_child(node* n)
{
    if (n->color == BLACK) {
        if (child->color == RED)
                child->color = BLACK;
        else
                processCase2(child);
    }

    node* child = is_leaf(n->right) ? n->left : n->right;
    replace_node(n, child);
    delete n;
}
```

Pseudo Code

```
void processCase2(node* n)
{
    doStep2_1(n);
}
```

Case 2 (black child) in detail

- Case 2.1:
  - If n has no parent (n is the root): end
    - If n is the root and there is 1 black, the deletion occurs as usual without losing the tree balance.
    - This step also stops the correction propagation.
    - Note: do not delete n if it is an correction propagation node, not the actual node being deleted.



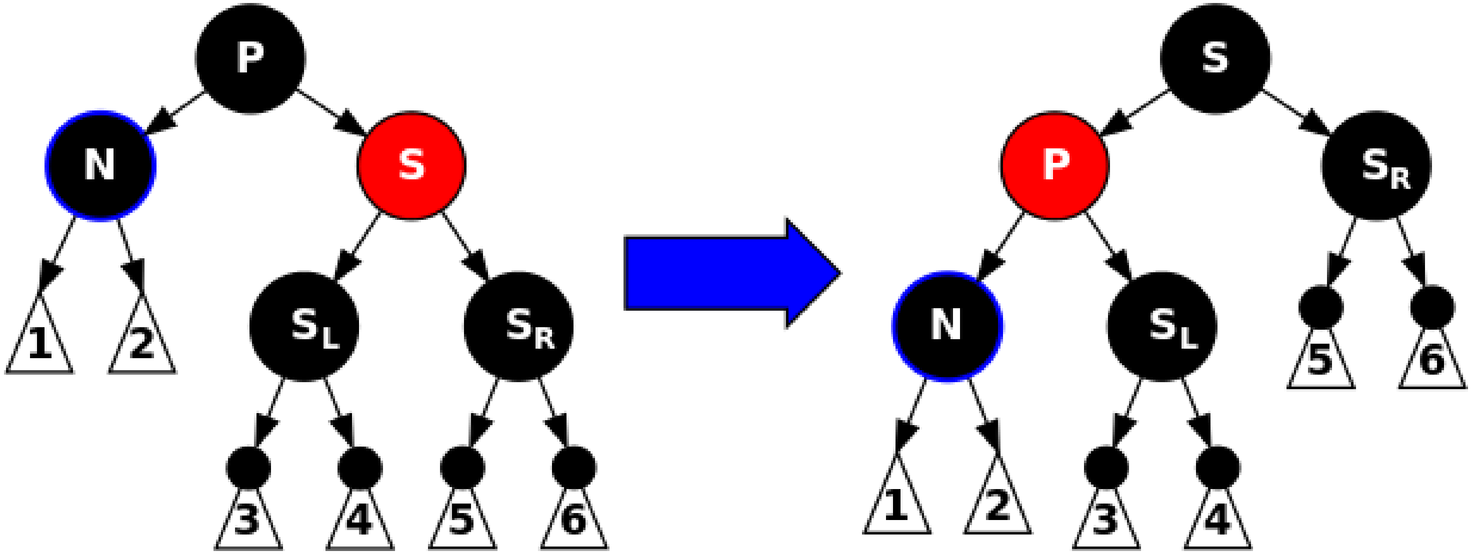
  - If n has a father, go to step 2.2

Pseudo Code

```
void doStep2_1(struct node* n)
{
    if (n->parent != NULL)
        doStep2_2(n);
}
```

Case 2 (black child) in detail

- Step 2.2 (n is black, s is red):
  - Reversing father and sibling color.
  - Left rotate at p.
  - Go to step 2.3

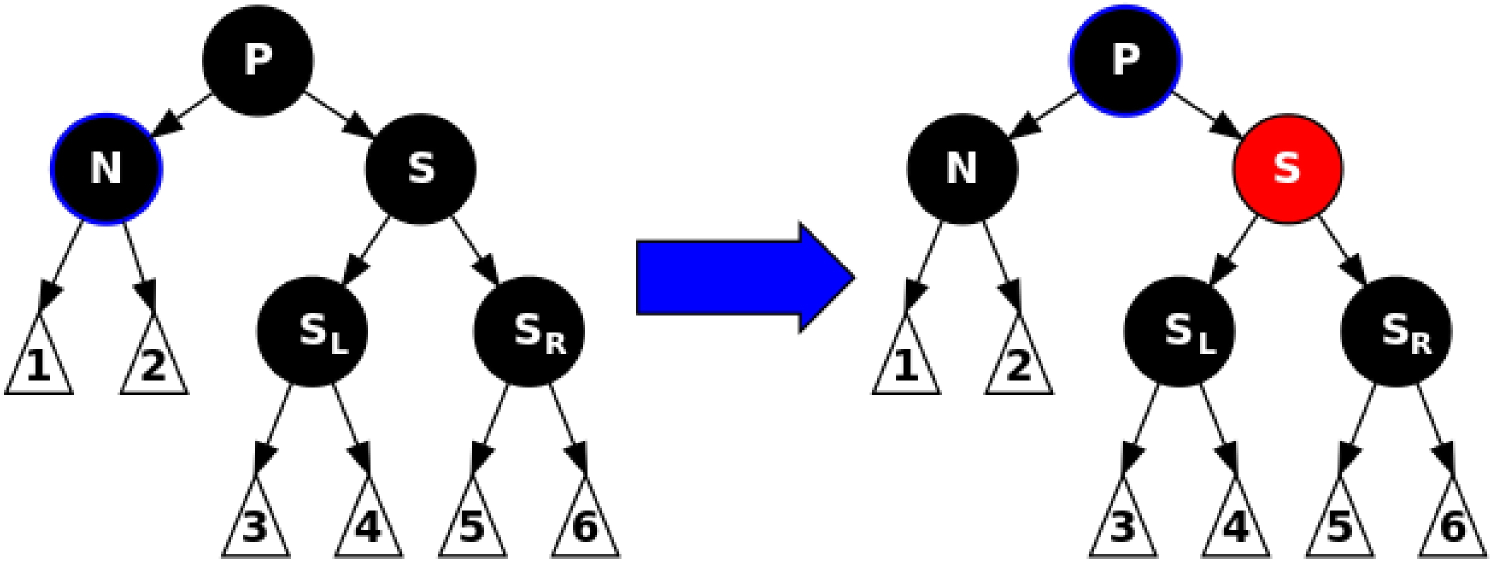


Pseudo Code

```
void doStep2_2(node* n)
{
    node* s = sibling(n);

    if (s->color == RED) {
        n->parent->color = RED;
        s->color = BLACK;
        if (n == n->parent->left)
            rotate_left(n->parent);
        else
            rotate_right(n->parent);
    }
    doStep2_3(n);
}
```

Case 2 (black child) in detail

- Bước 2.3 (n is black, s is black, children of s are black, p is black):
  - Change color s to red.
  - Go back to step 2.1 with n is current p (correction propagation)

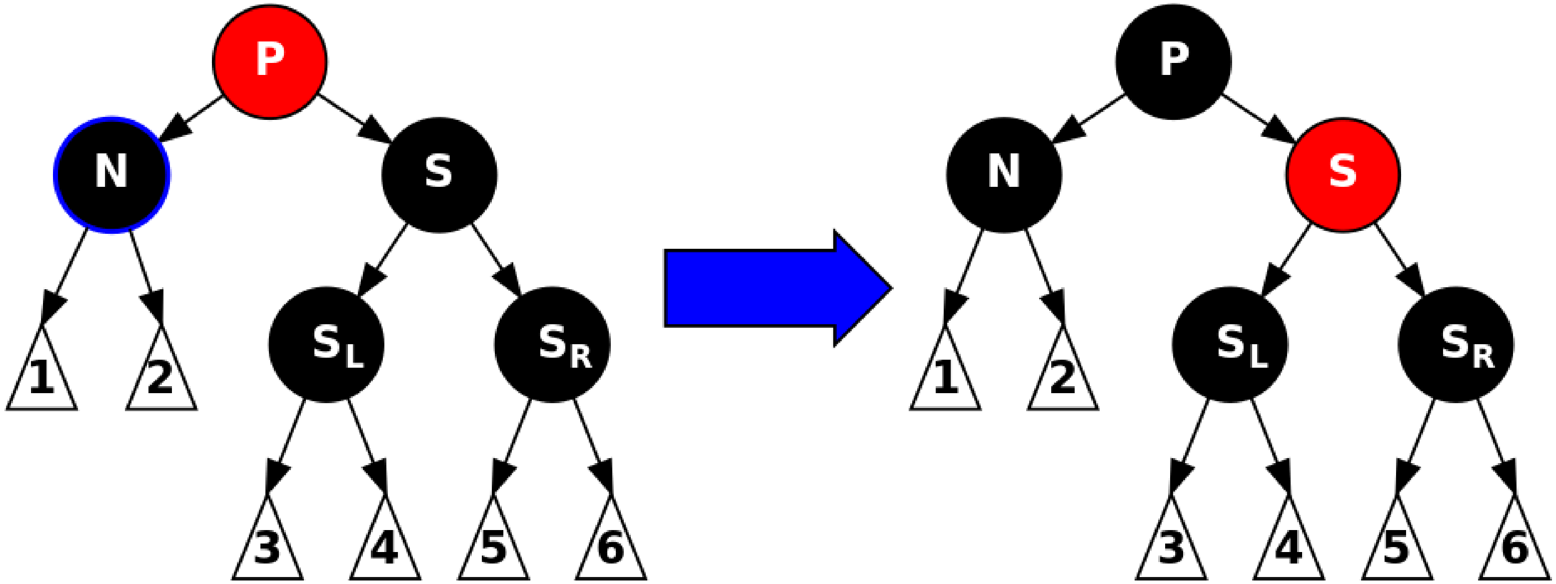


Pseudo Code

```
void doStep2_3(node* n)
{
    node* s = sibling(n);

    if ((n->parent->color == BLACK) &&
        (s->color == BLACK) &&
        (s->left->color == BLACK) &&
        (s->right->color == BLACK)) {
            s->color = RED;
            doStep2_1(n->parent);
    }
    else
        doStep2_4(n);
}
```

Case 2 (black child) in detail

- Step 2.4 (n is black, s is black, children of s is black, p is red):
  - Reversing parent and sibling color.
  - End

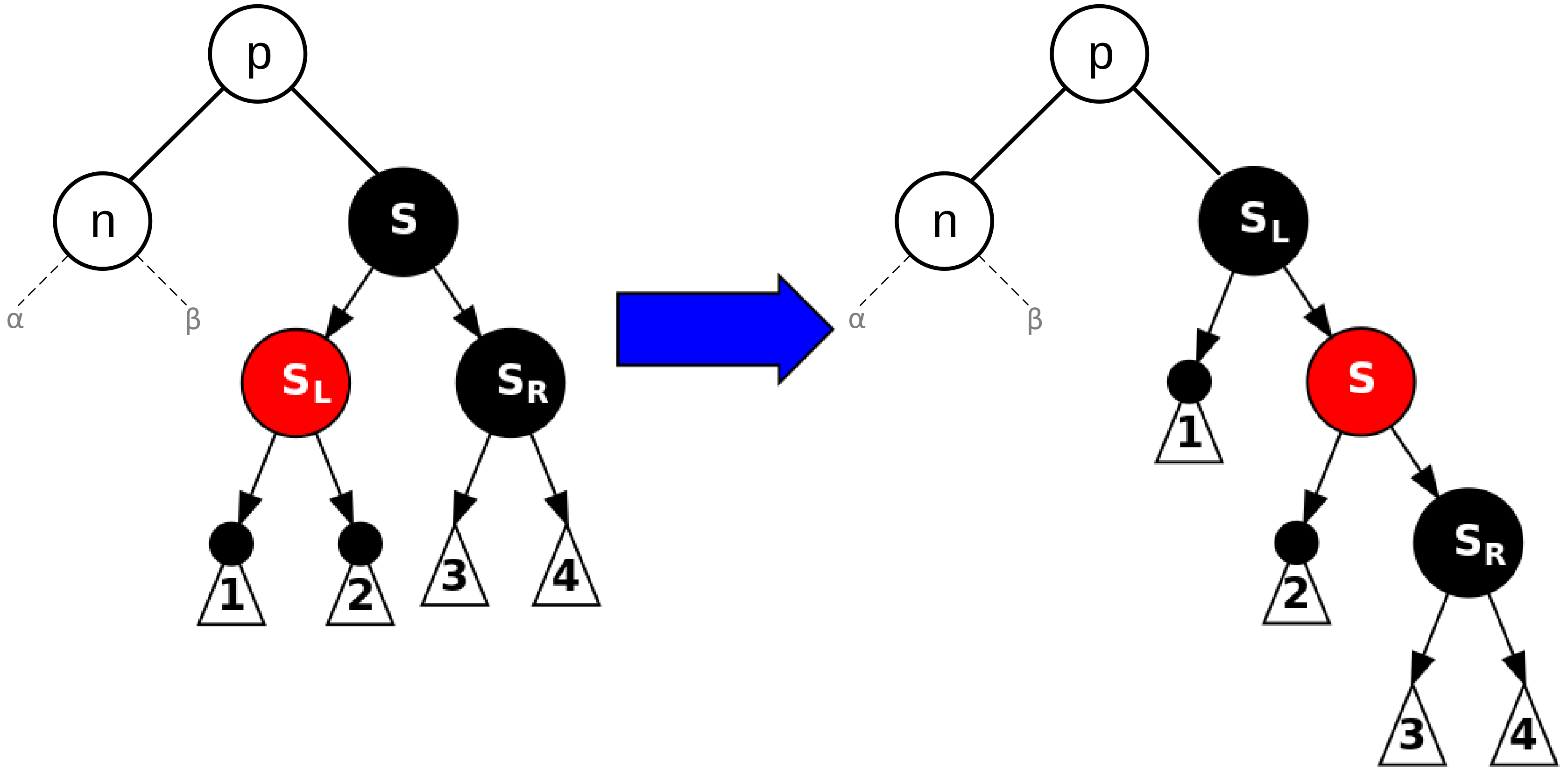


Pseudo Code

```
void doStep2_4(node* n)
{
    node* s = sibling(n);

    if ((n->parent->color == RED) &&
        (s->color == BLACK) &&
        (s->left->color == BLACK) &&
        (s->right->color == BLACK)) {
            s->color = RED;
            n->parent->color = BLACK;
    }
    else
        doStep2_5(n);
}
```

Case 2 (black child) in detail

- Step 2.5 (n is black, s is black, left child of s is red, right child of s is black, n is left child):
  - Reversing s and its left child color.
  - Right rotate at s.
  - Go to step 2.6

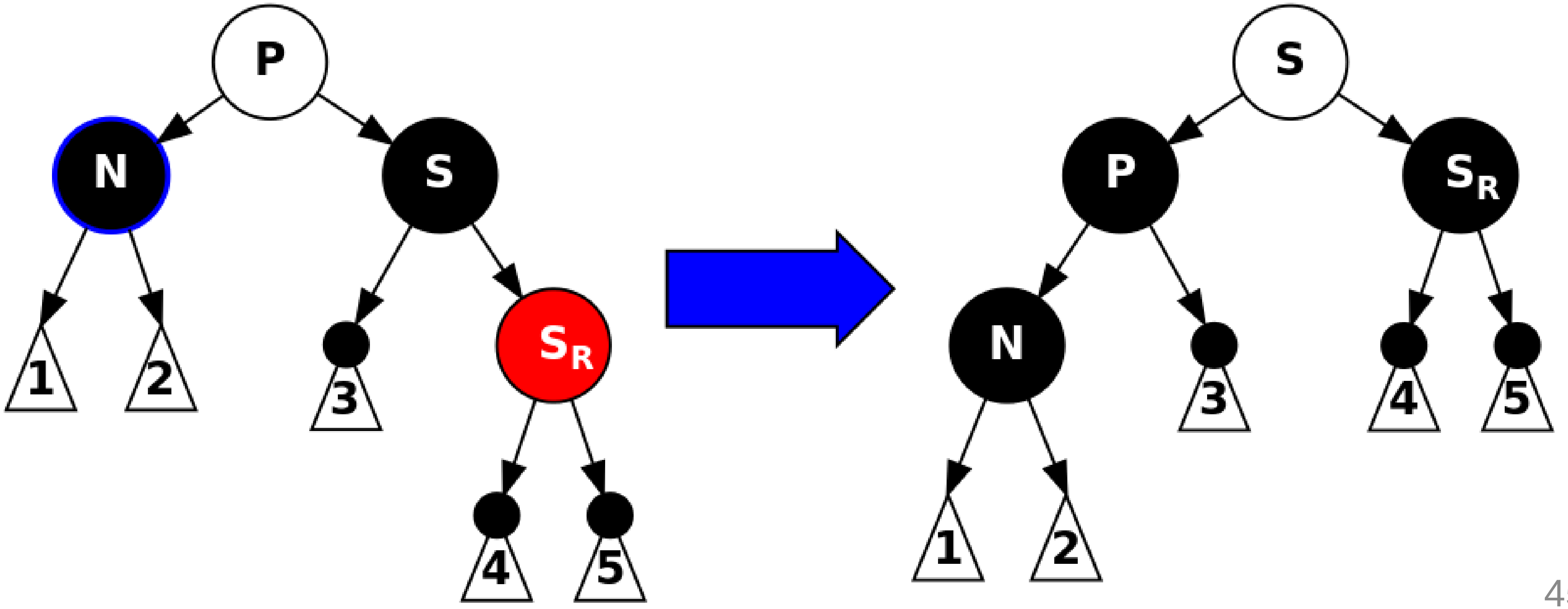


Pseudo Code

```
void doStep2_5(node* n)
{
    node* s = sibling(n);

    if (s->color == BLACK) {
        if ((n == n->parent->left) &&
            (s->right->color == BLACK) &&
            (s->left->color == RED)) {
            s->color = RED;
            s->left->color = BLACK;
                rotate_right(s);
        }
        else if ((n == n->parent->right) &&
            (s->left->color == BLACK) &&
            (s->right->color == RED)) {
            s->color = RED;
            s->right->color = BLACK;
                rotate_left(s);
        }
    }
    doStep2_6(n);
}
```

Case 2 (black child) in detail

- Step 2.6 (n is black, s is black, right child of s is red, n is left child):
  - Reversing s and p color
  - Color black for the right child of s
  - Left rotate at p
  - End.



Pseudo Code

```
void doStep2_6(node* n)
{
    node* s = sibling(n);

    s->color = n->parent->color;
    n->parent->color = BLACK;

    if (n == n->parent->left) {
        s->right->color = BLACK;
        rotate_left(n->parent);
    }
    else {
        s->left->color = BLACK;
        rotate_right(n->parent);
    }
}
```

Red Black Tree

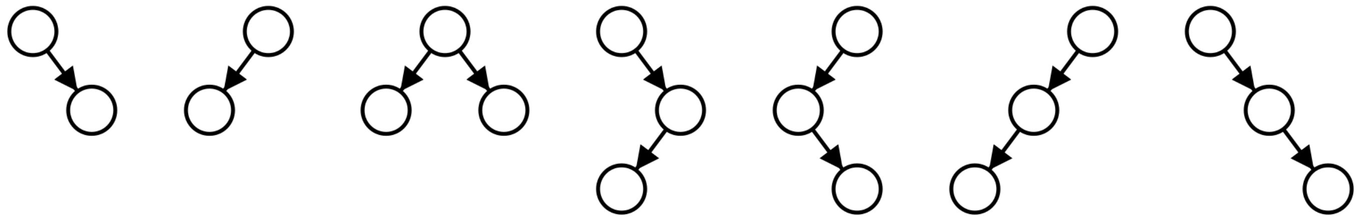
- Comments:
  - Advantages:
    - Search $\boldsymbol{O}(log_2N)$
    - Insert $\boldsymbol{O}(log_2N)$
    - Delete $\boldsymbol{O}(log_2N)$
    - Minimum $\boldsymbol{O}(log_2N)$
    - Maximum $\boldsymbol{O}(log_2N)$
  - Disadvantages:
    - Must store the color attribute and pointer to the parent node
    - More complicated than AVL and AA trees

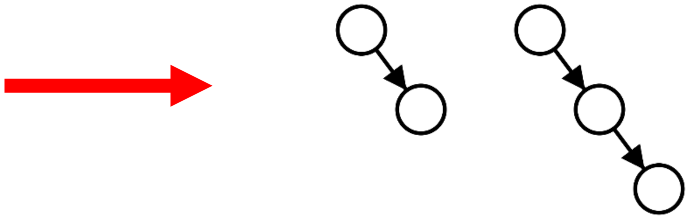
Outline

- Red-Black Tree
- AA Tree

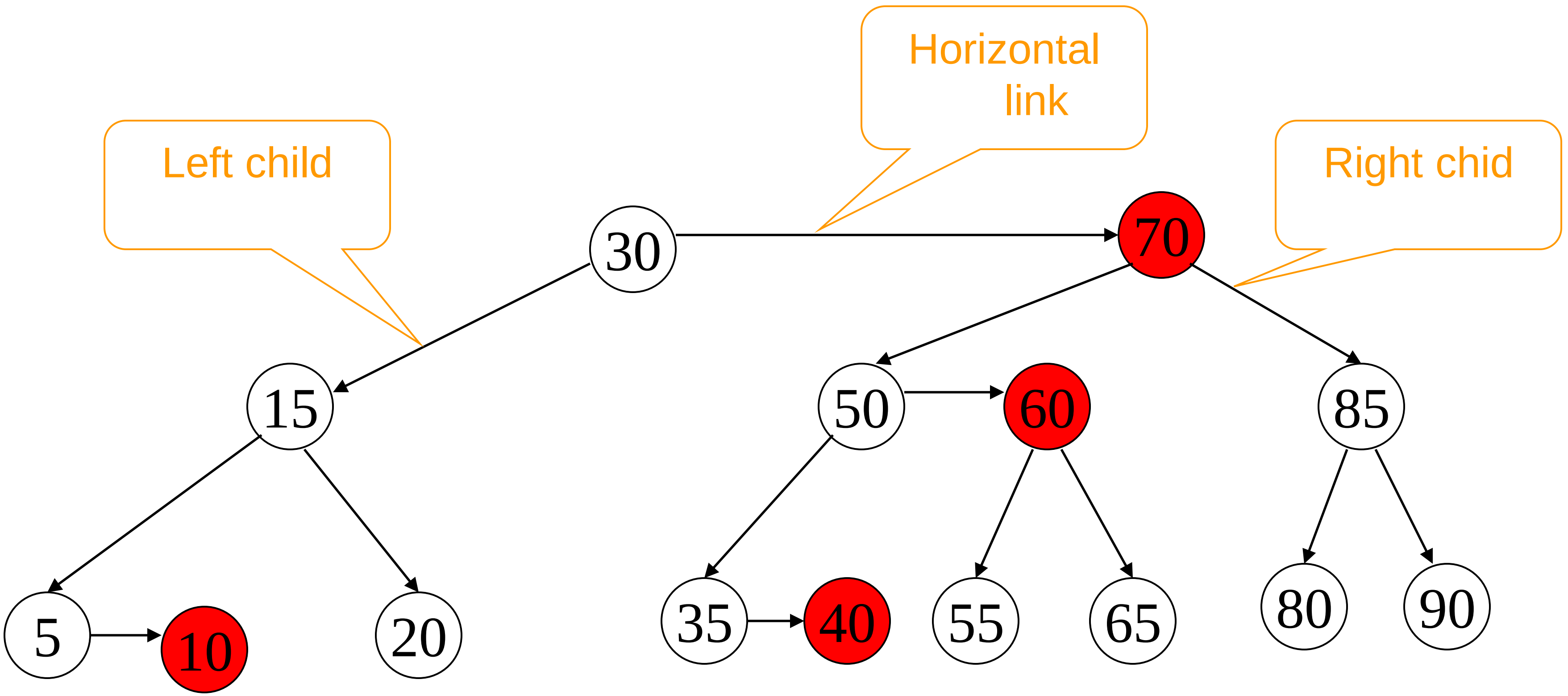
AA (Arne Andersson) Tree

- AA trees are a variation of the red-black tree
  - Unlike red-black trees, red nodes on an AA tree can only be added as a right child.
  - A red-black tree need to consider seven different shapes to properly balance the tree. An AA tree on the other hand only needs to consider two shapes.



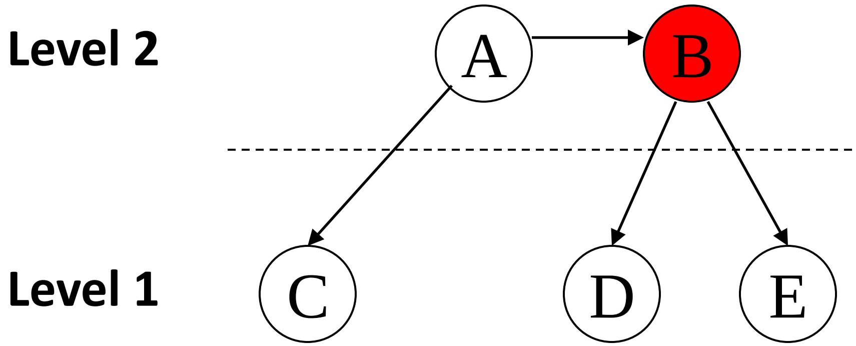


Example


Horizontal link
Left child
Right chid
30
70
15
50
60
85
5
10
20
35
40
55
65
80
90

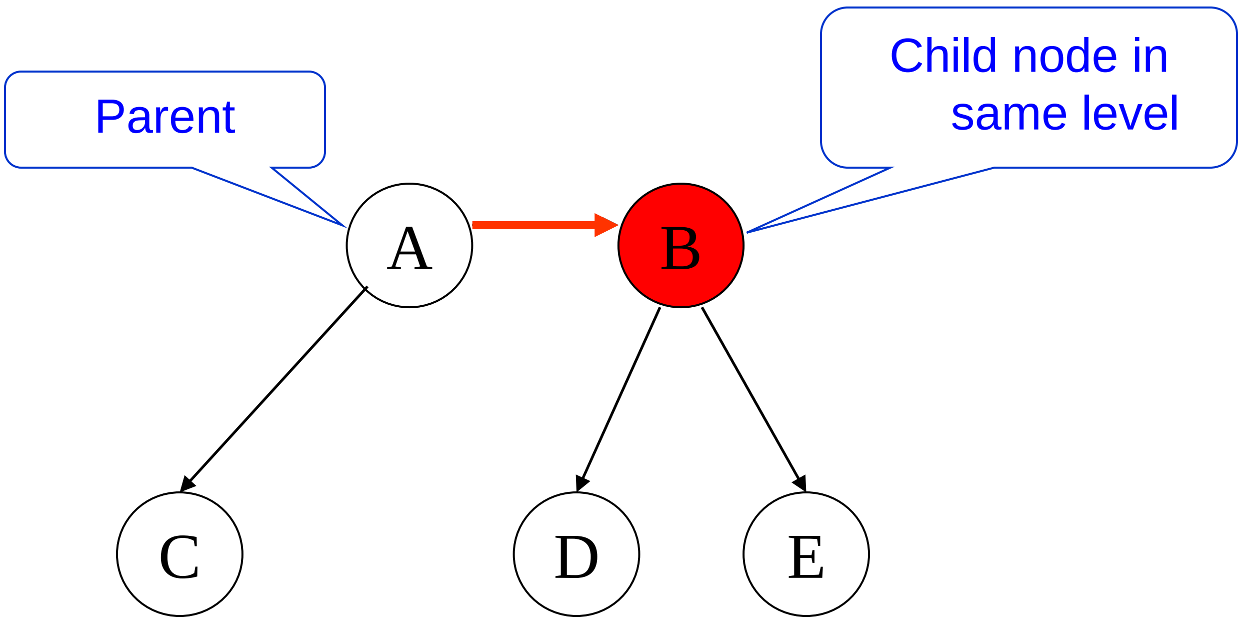
Level of a node

- Level of a node: is the number of left links from that node to NULL
  - The level of the NULL node is 0
  - The level of the leaf node is 1



Horizontal link

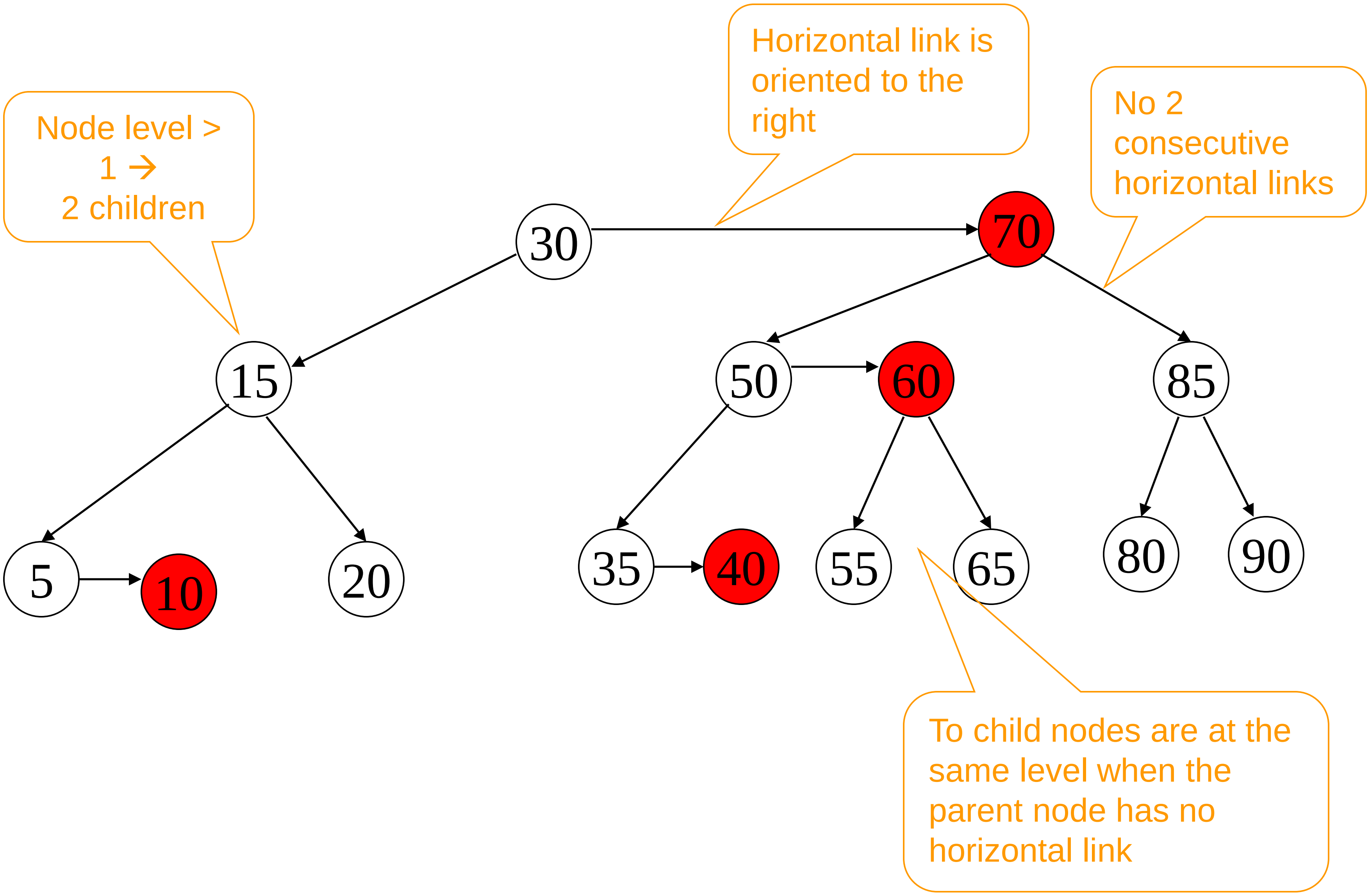
- Horizontal link: an link between a node and its children at the same level



AA Tree

- AA tree is a binary search tree that conforms to the following rules:
  - Horizontal link is always oriented to the right
  - There are no 2 consecutive horizontal links
  - Every node with level greater than 1 will have 2 children
  - If a node has no right horizontal link then its children are at the same level

Example



Node level > 1 → 2 children
Horizontal link is oriented to the right
No 2 consecutive horizontal links
30
70
15
50
60
85
5
10
20
35
40
55
65
80
90
To child nodes are at the same level when the parent node has no horizontal link

Data Structure

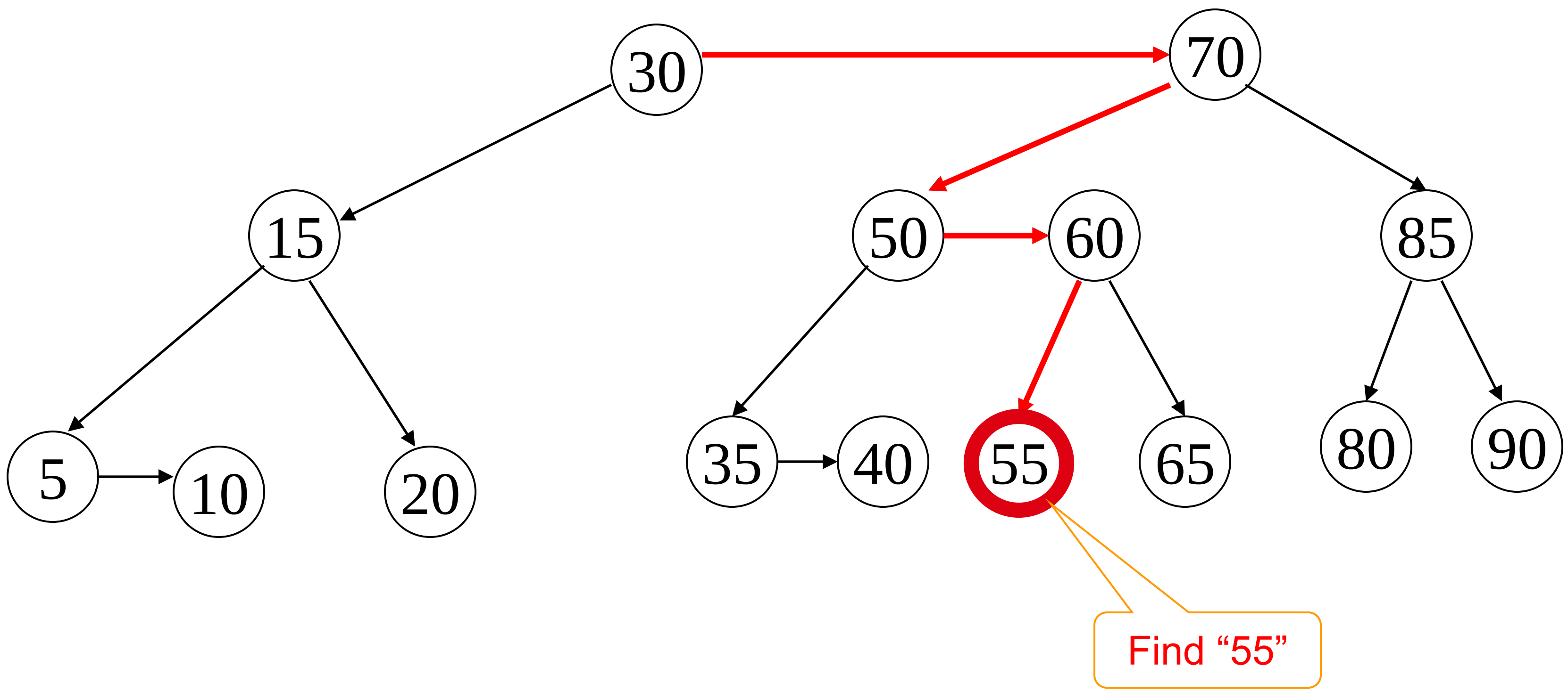
```
typedef int DataType;

typedef struct NodeTag {
      DataType           key;        // Data
      struct NodeTag    *pLeft;
      struct NodeTag    *pRight;
      int                level;      // Left of node
} AANode;

typedef struct AANode* AATREE;
```

Some properties of AA Tree

- Some properties:
  - The level of every leaf node is one.
  - The level of every left child is exactly one less than that of its parent.
  - The level of every right child is equal to or one less than that of its parent.
  - The level of every right grandchild is strictly less than that of its grandparent.
  - Every node of level greater than one has two children.

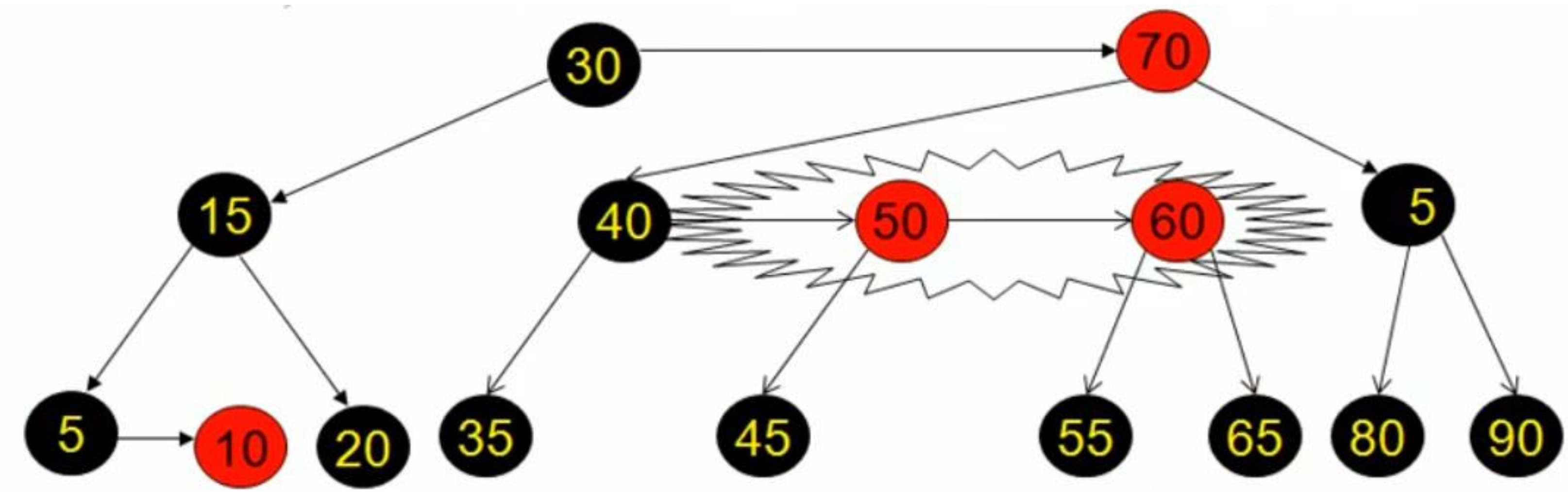
Traverse in AA Tree

- Traverse in Tree: similar with BST



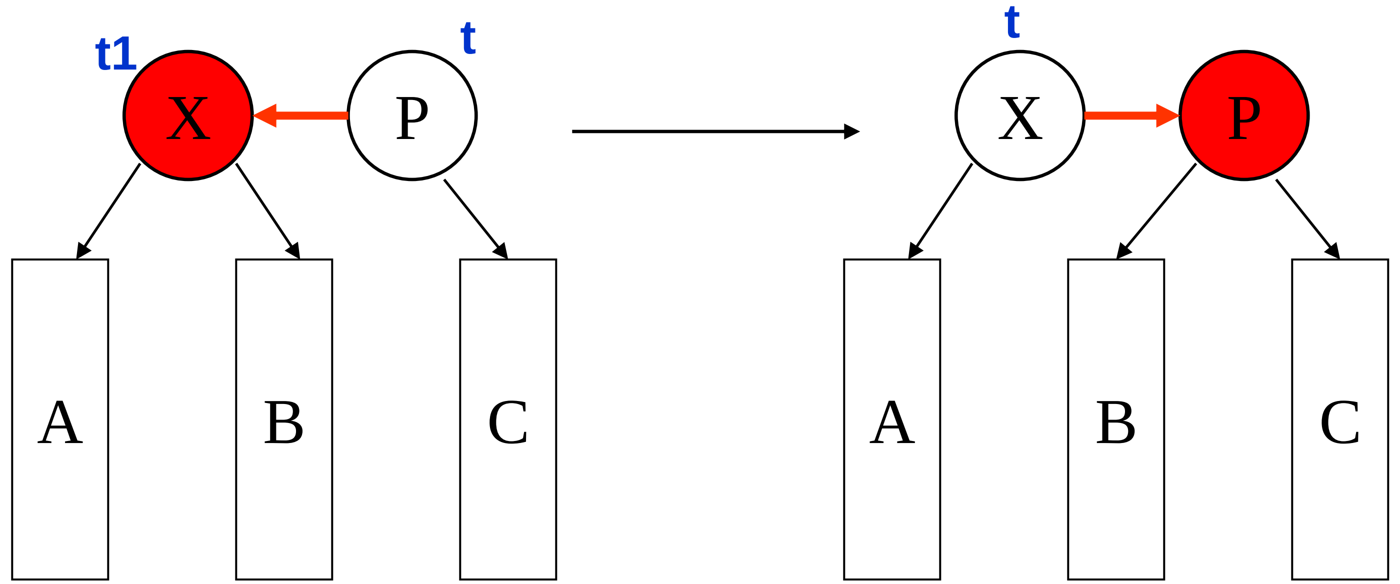
Balance

- Insertions and deletions may transiently cause an AA tree to become unbalanced
  - Only two distinct operations are needed for restoring balance: "skew" and "split".



Skew

- Skew is a right rotation to replace a subtree containing a left horizontal link with one containing a right horizontal link instead

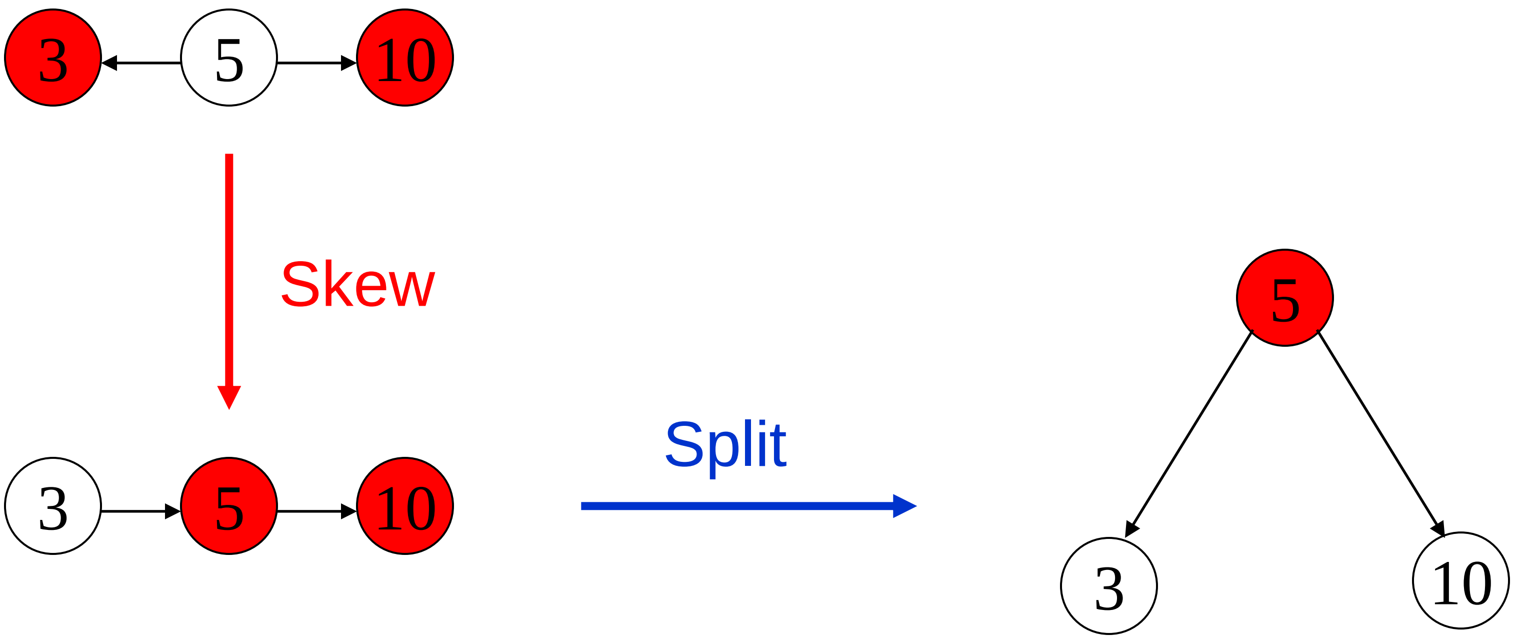


AA – Tree (tt)

```
AATREE right_rotate(AATREE &t)
{
    AATREE t1;
    t1 = t->pLeft;
    t->pLeft = t1->pRight;
    t1->pRight = t;

    return t1;
}

AATREE Skew(AATREE &t)
{
    if (t->pLeft != NULL)
        if (t->pLeft->level == t->level)
            t = right_rotate(t);
    return t;
}
```

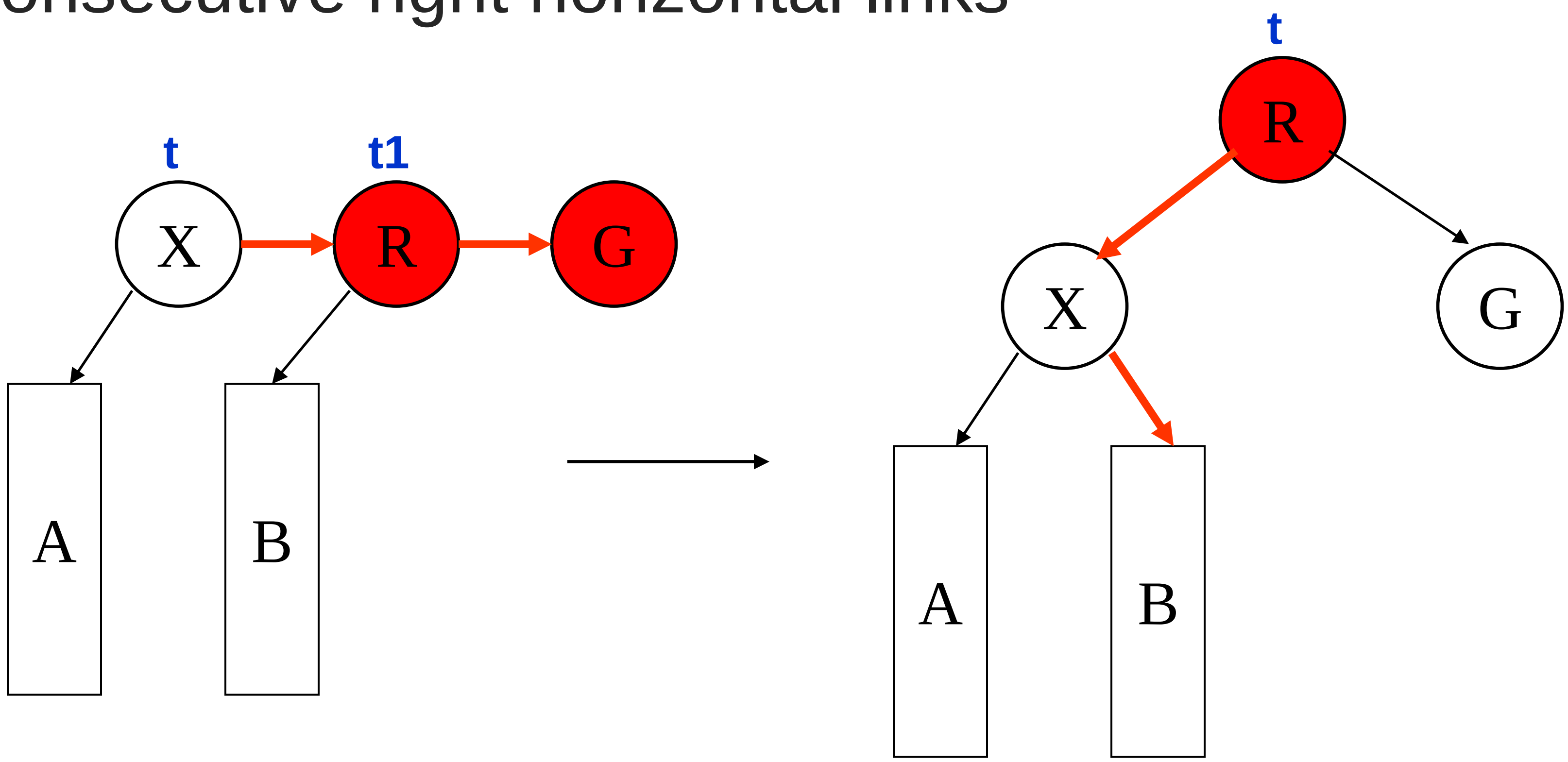
Multiple right-row horizontal links

- Skew can cause multiple right-row horizontal links → using Split to adjust



Split

- Split is a left rotation and level increase to replace a subtree containing two or more consecutive right horizontal links with one containing two fewer consecutive right horizontal links



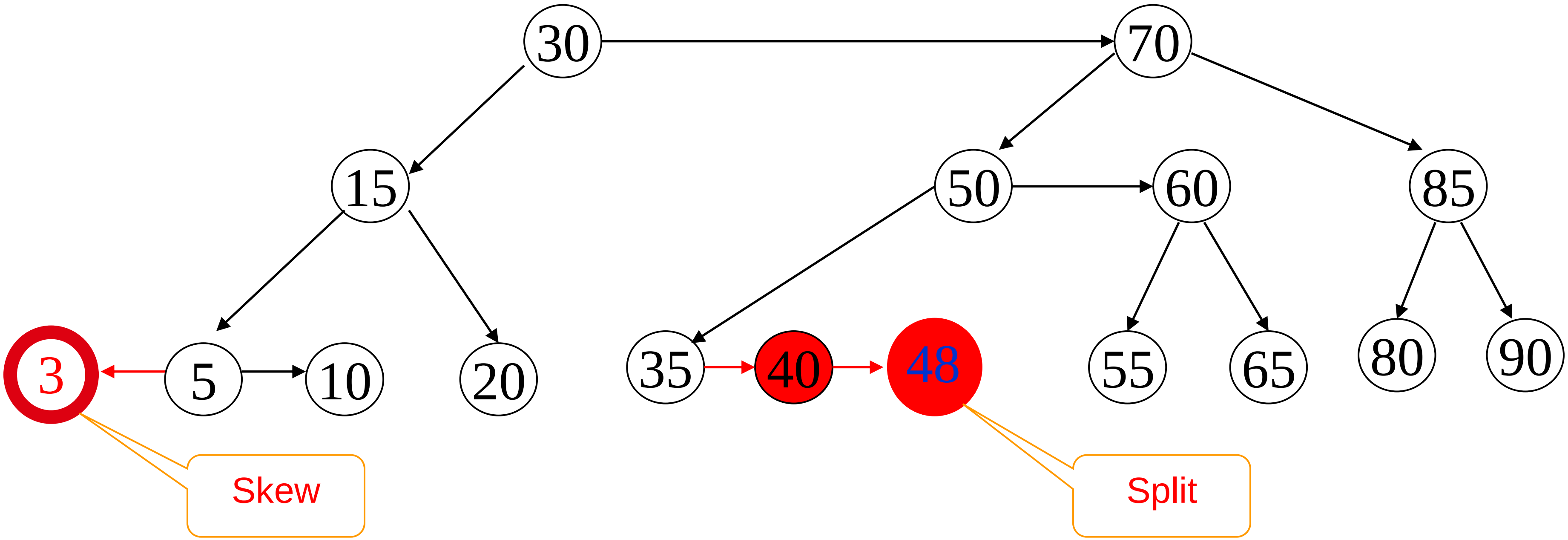
AA – Tree (tt)

```
AATREE left_rotate(AATREE &t)
{
   AATREE t1;
   t1 = t->pRight;
   t->pRight = t1->pLeft;
   t1->pLeft = t;
   t1->level++;
   return t1;
}
AATREE Split(AATREE &t)
{
     if (t->pRight !=NULL)
         if (t->pRight->pRight != NULL)
           if (t->pRight->pRight->level == t->level)
                 t = left_rotate(t);
     return t;
}
```

Insert a new node

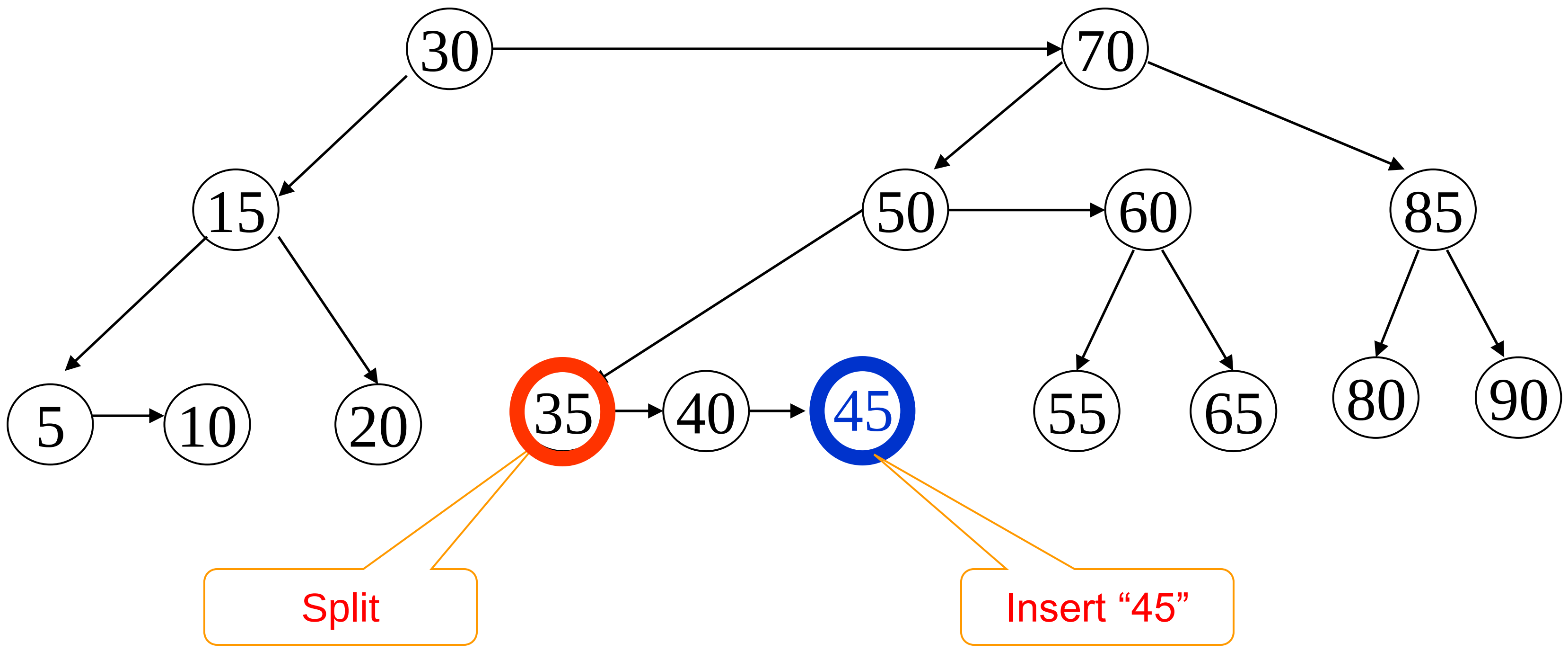
- Insert a new node
  - Execute like binary tree search.
    - Insert a node is always done at the node with level = 1.
  - When node added on the left side
    - It will create a left horizontal link → balance with Skew
  - When node added on the right side
    - It will create a right horizontal link
    - If it causes two right horizontal links → balance with Split

Insert a new node



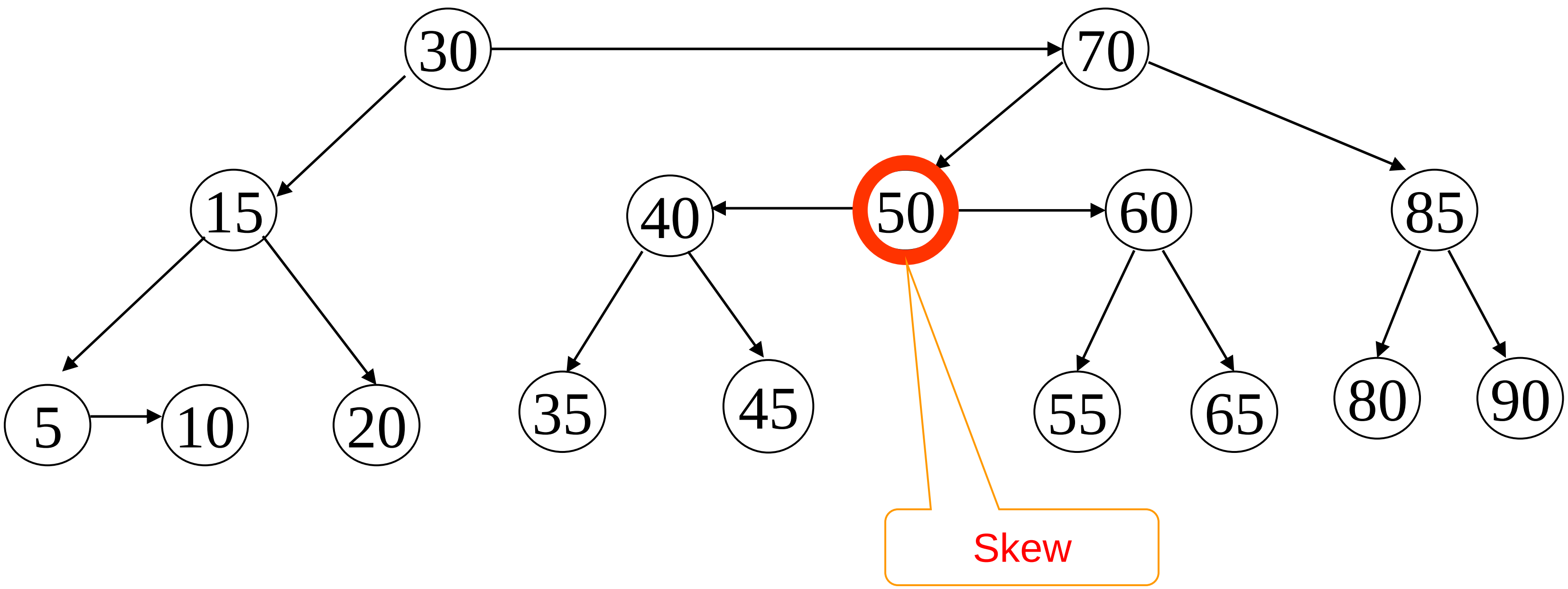
Example

- Insert node 45:



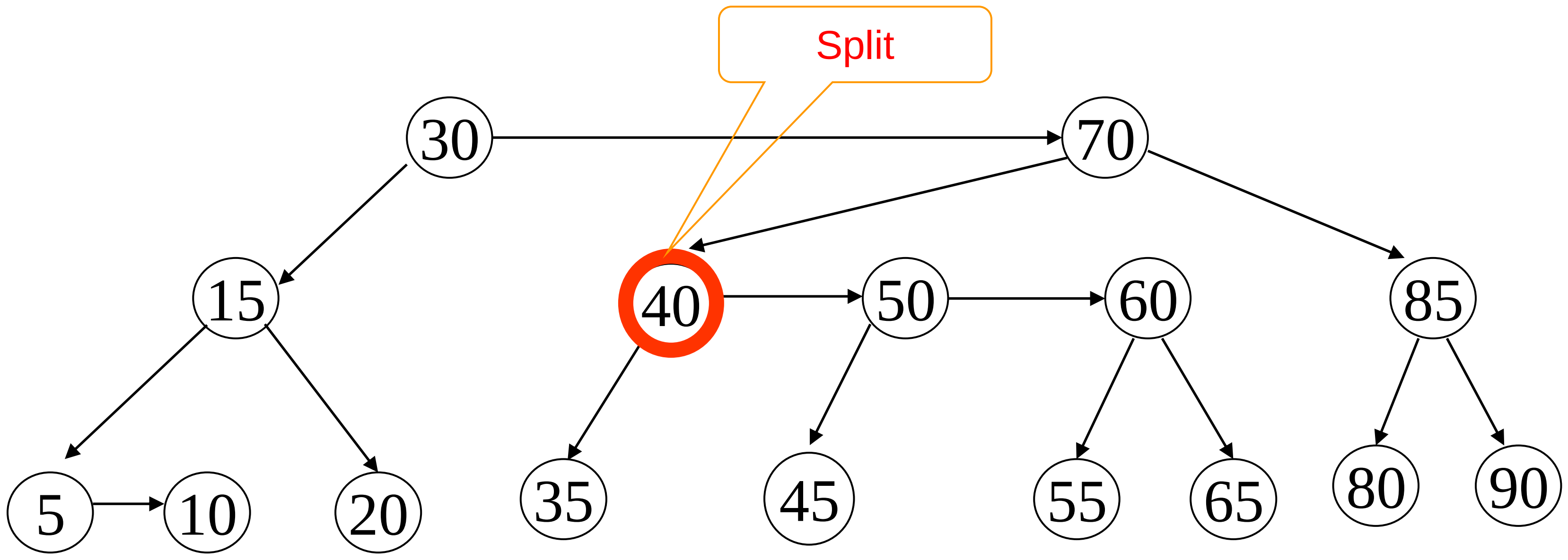
Example

- After split at 35:



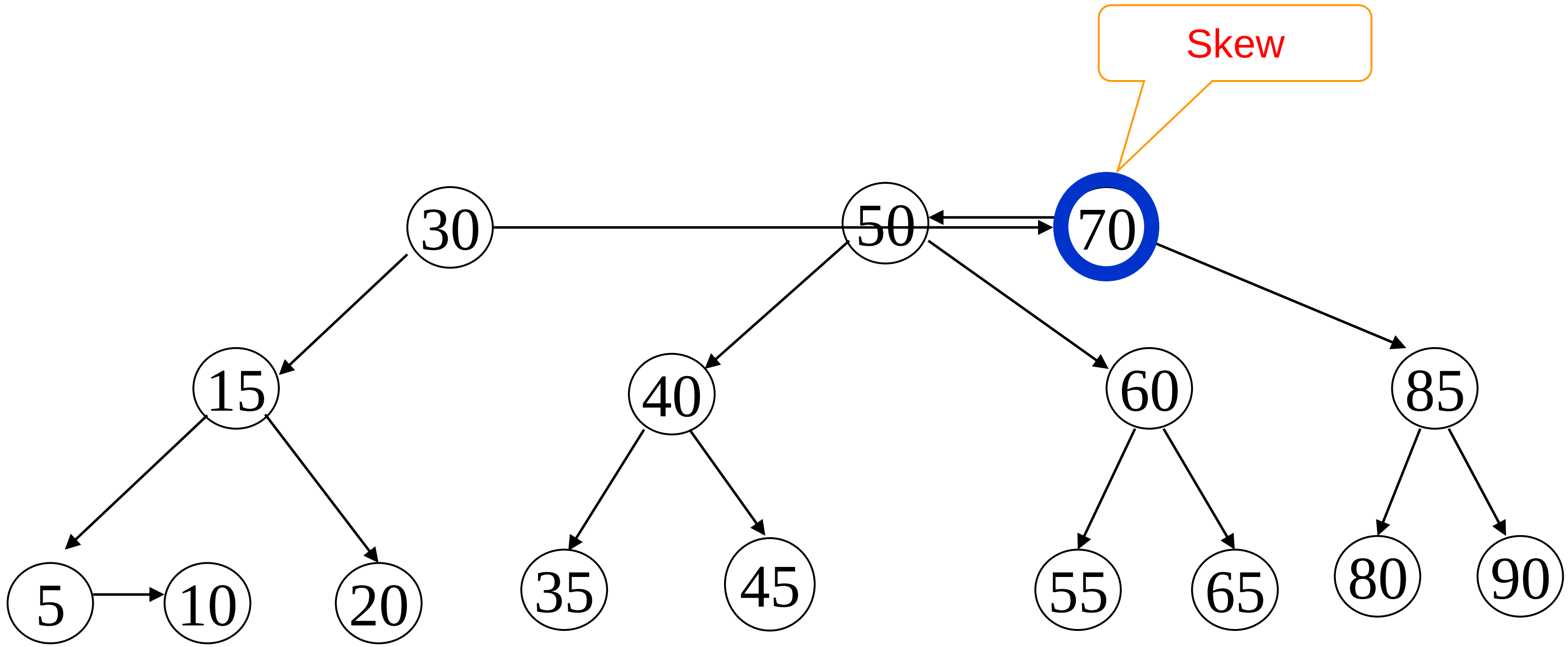
Example

- After skew at 50:



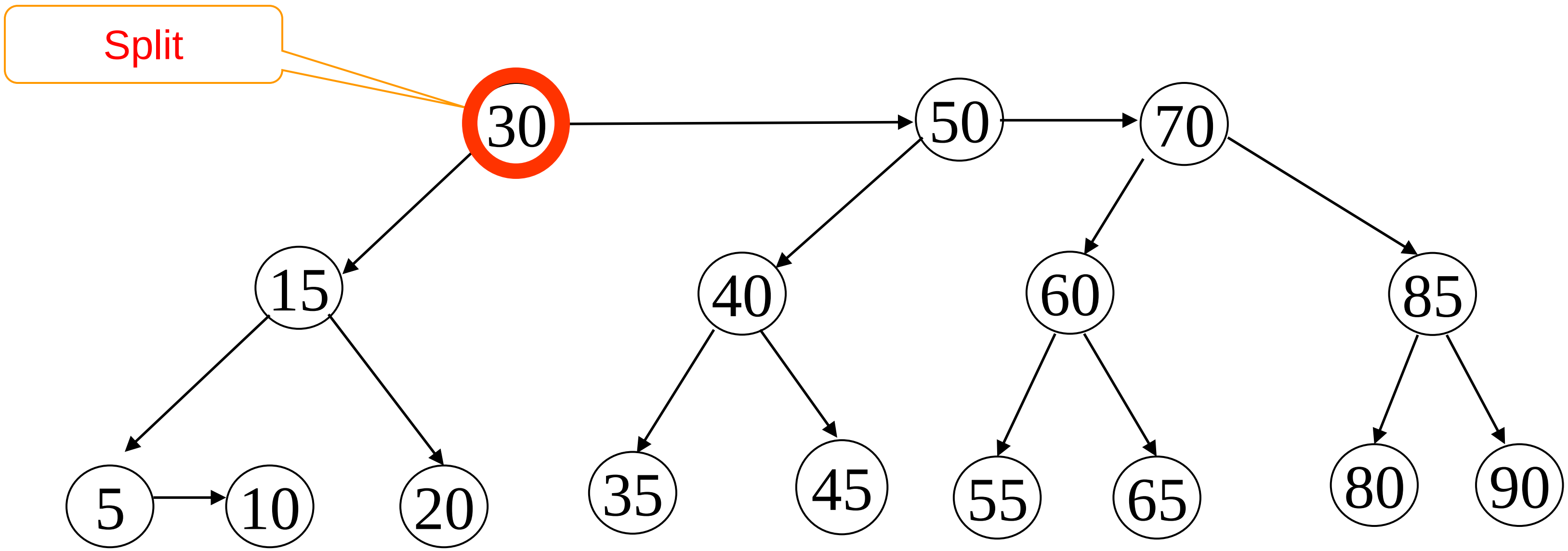
Example

- After split at 40:



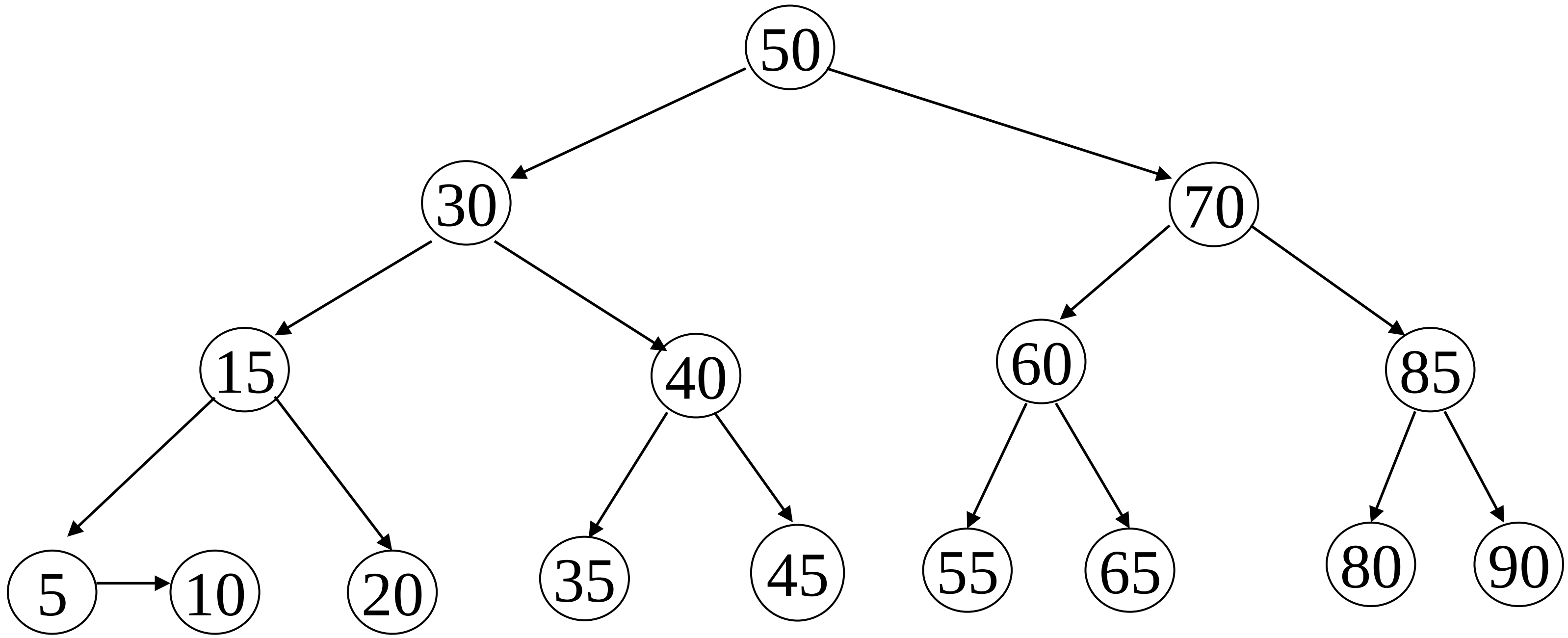
Example

- After skew at 70:



Example

- After split at 30:



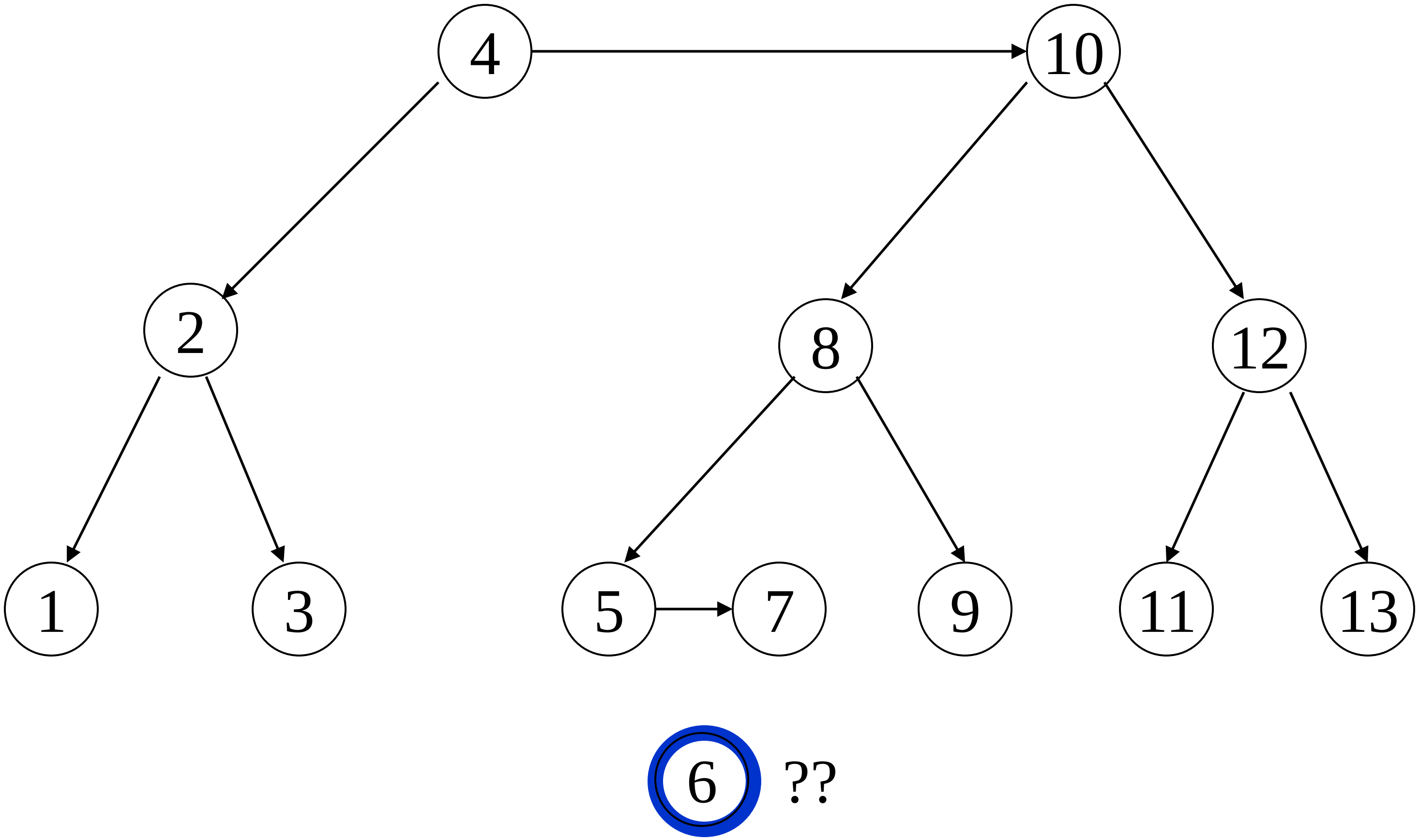
→ STOP !

AA – Tree (tt)

```
AATREE AA_Insert_Node(DataType x, AATREE t)
{
   if(t == NULL)      {
        t = new AANode;
        t->key = x;
        t->pLeft = t->pRight = NULL;
        t->level = 1;
   }
   else if(x < t->key)
               t->pLeft = AA_Insert_Node(x, t->pLeft);
        else if(x > t->key)
                 t->pRight = AA_Insert_Node(x, t->pRight);
        else return t;  // duplicate

   /* Perform skew and then split. The conditionals that
   determine whether or not a rotation will occur or not are
   inside of the procedures, as given above.*/

   t = Skew(t);
   t = Split(t);

   return t;
}
```

Exercise

- Insert node with 6:



Delete a node

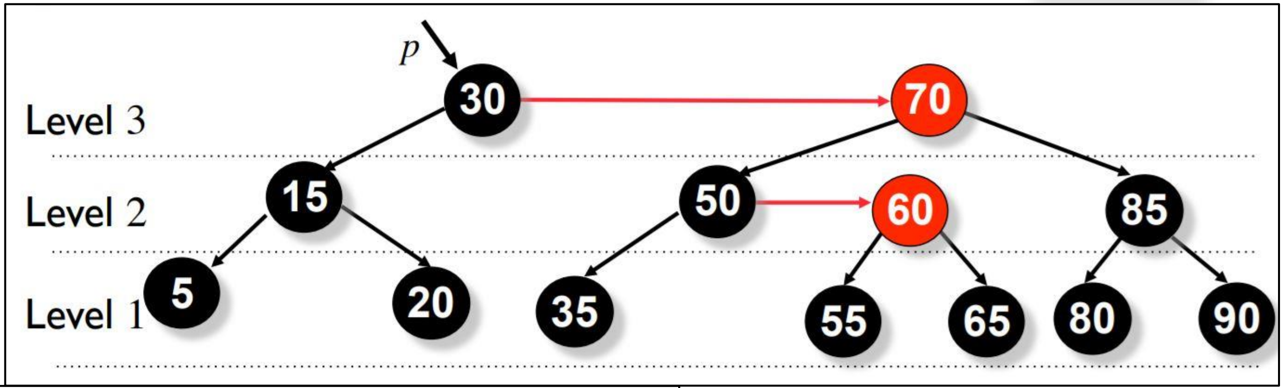
- Delete a node
  - Execute like binary tree search.
    - The actually deleted node is always in level 1.
  - After a removal (balance):
    - The first step is to lower the level of any nodes whose children are two levels below them (who are missing children).
    - Then, the entire level must be skewed and split.

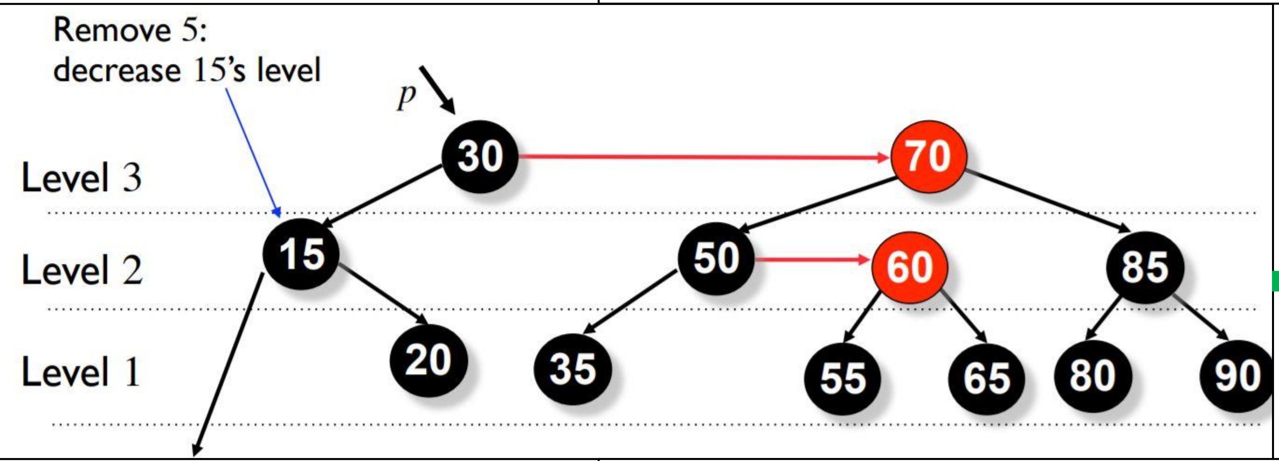
Decrease level

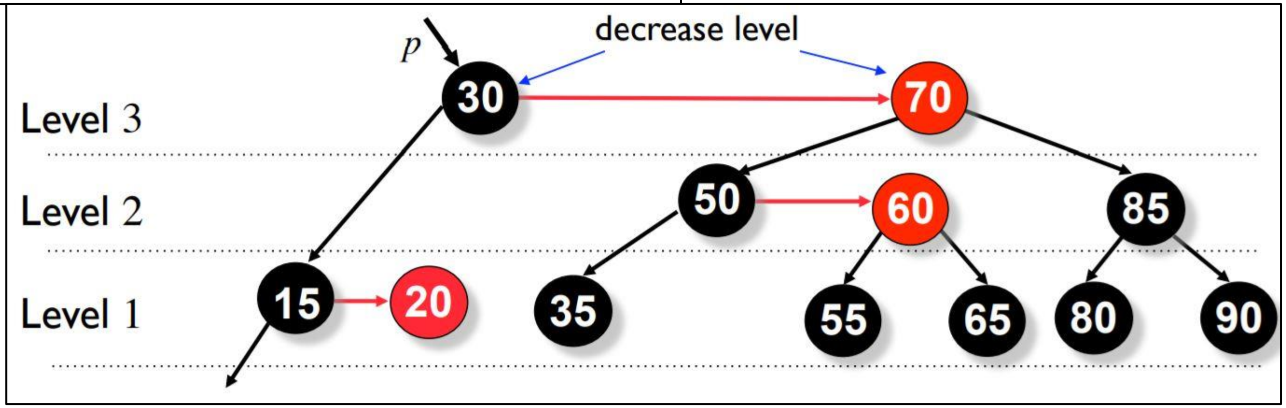
- In the process of deletion, a node can lose one of its children. As a result, we may need to decrease this node's level in the tree.
  - The ideal level of any node is one more than the minimum level of its two children.
  - If we discover that p's current level is higher than this ideal value, we set it to its proper value.
  - If p's right child is a red node (that is, p.right.level == p.level prior to the adjustment), then the level of p.right needs to be decreased as well.

Decrease level

- Delete node 5







Decrease level

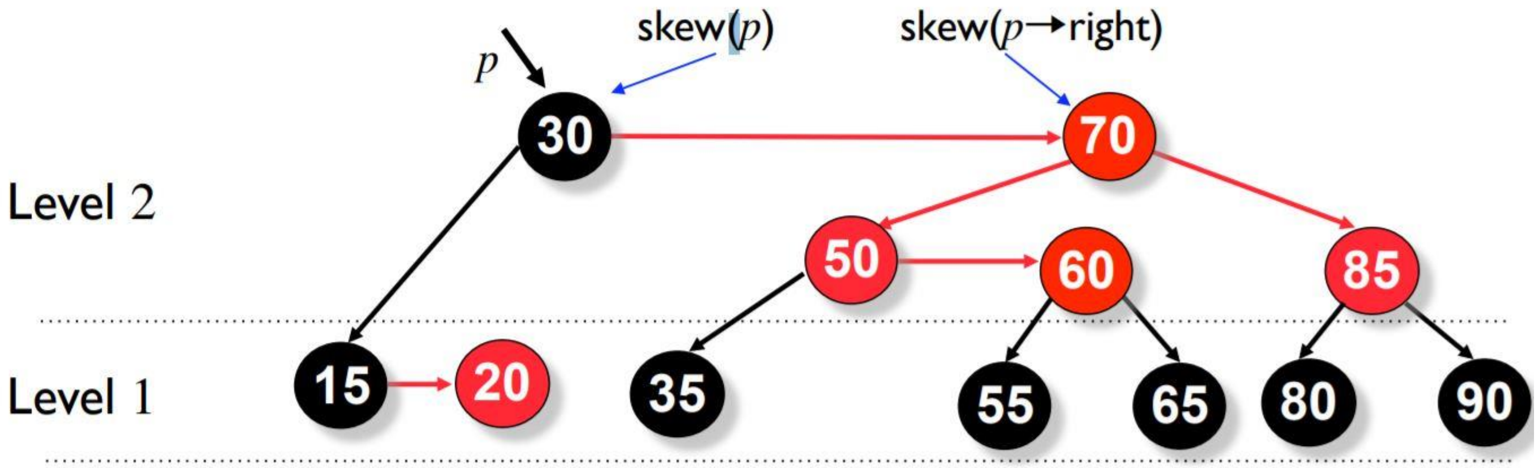
```
AANode AA_Update_Level(AANode p)
{
   int idealLevel = 1 + min(p.left.level, p.right.level);
   if (p.level > idealLevel) { // p's level is too high?
        p.level = idealLevel; // decrease its level
        if (p.right.level > idealLevel) // p's right child red?
              p.right.level = idealLevel; //fix its level as well
   }
   return p;
}
```

Skew and Split

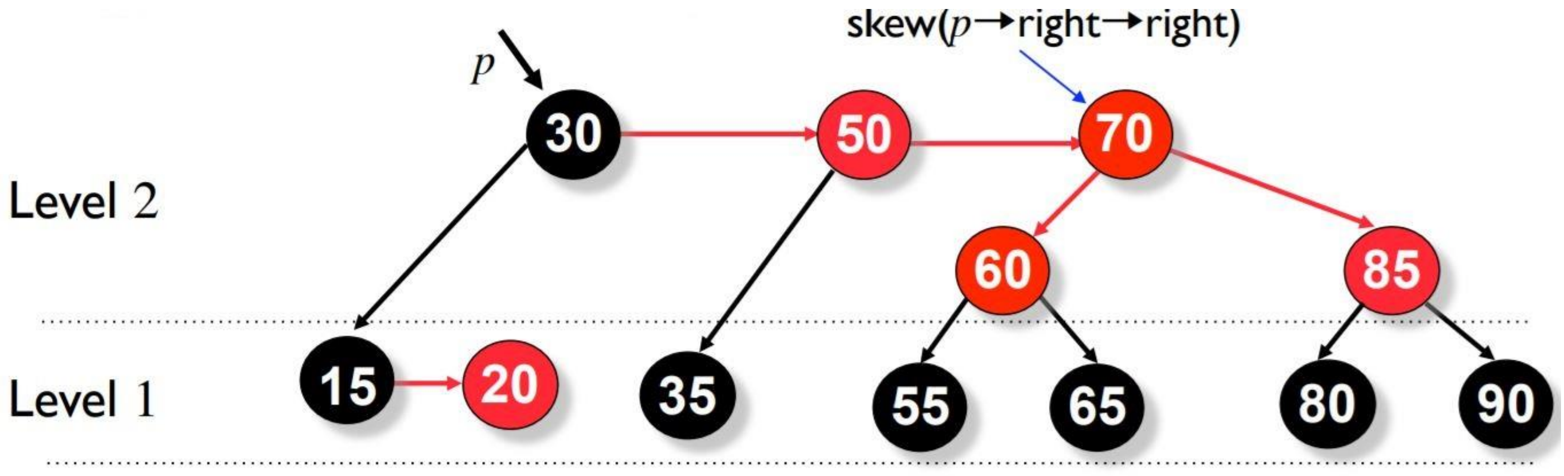
- After update level of any node:
  - Perform skew operations for p, p.right, and p.right.right.
  - Next, perform two splits, one at p, and the other to its right-right grandchild, which becomes its right child after the first split.

Skew and Split

- Do the skew operation on p and p->right

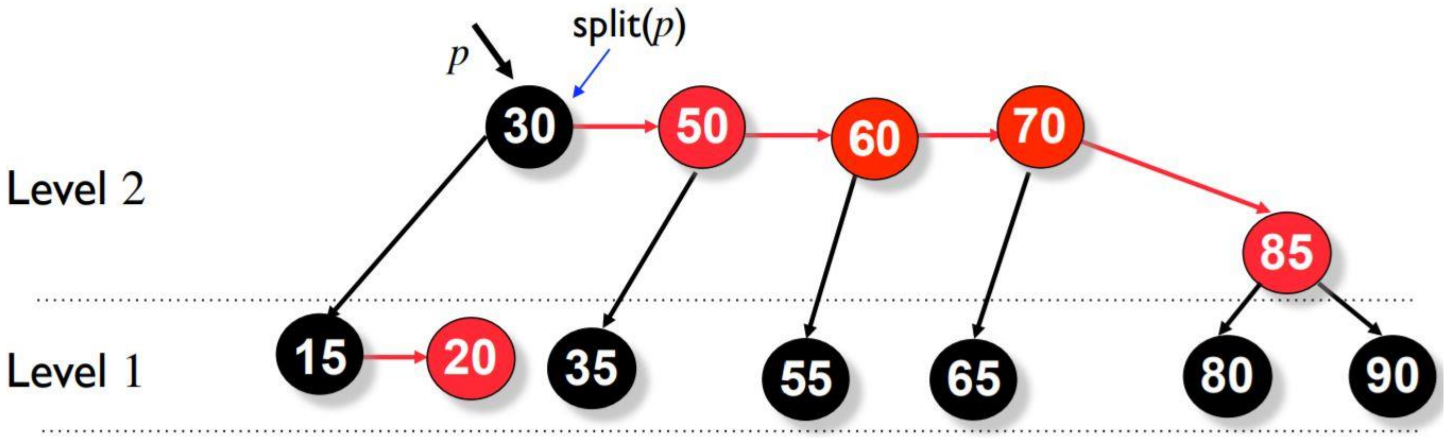


- Do the skew operation on p->right->right

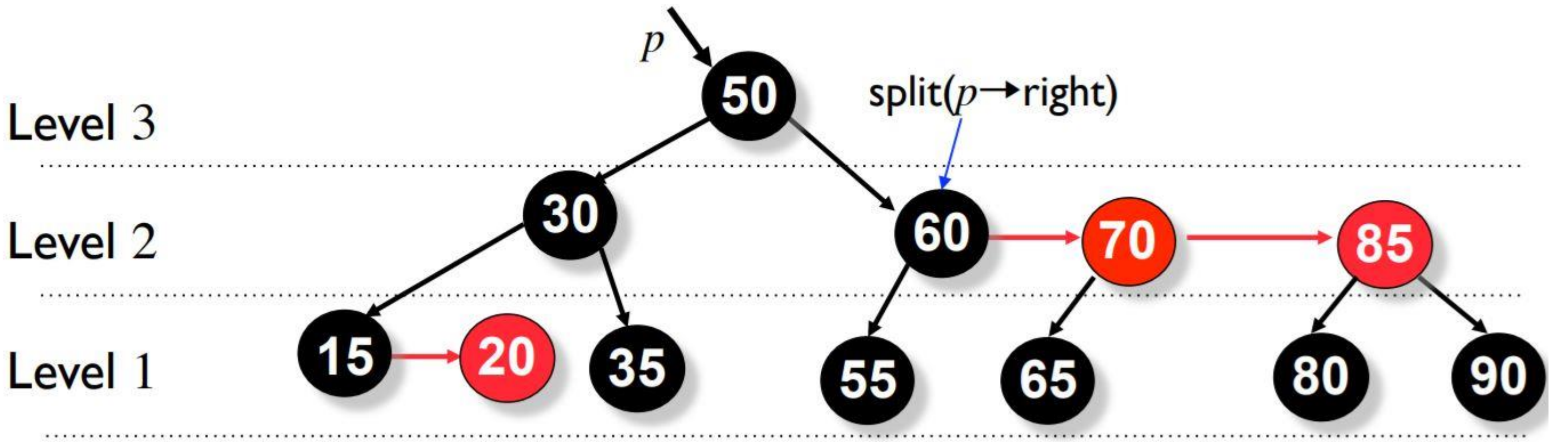


Skew and Split

- Do split operation on p

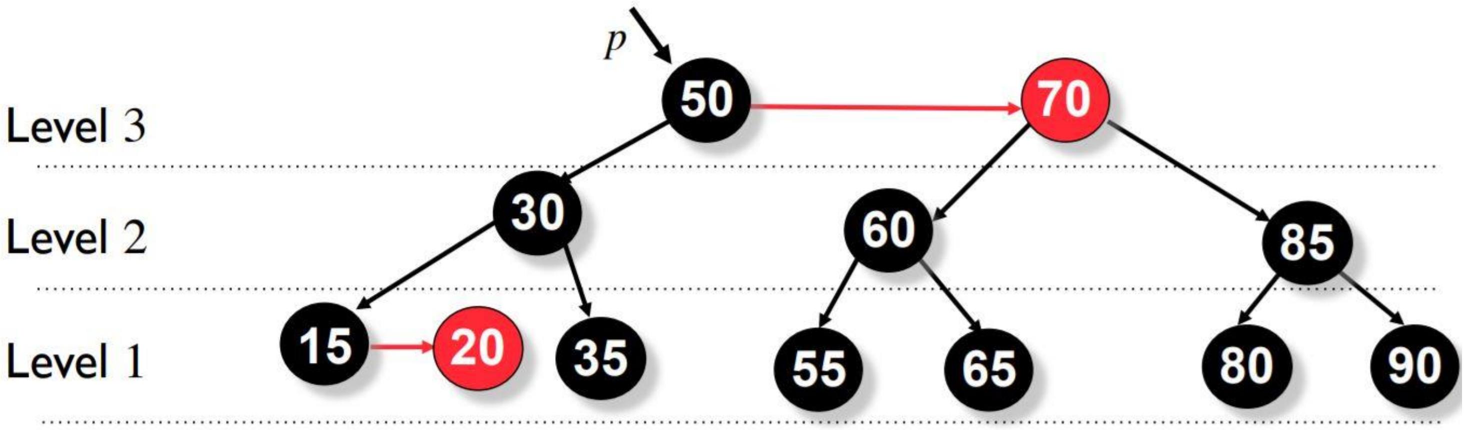


- Do split operation on p->right



Skew and Split

- After final split



Balance Tree

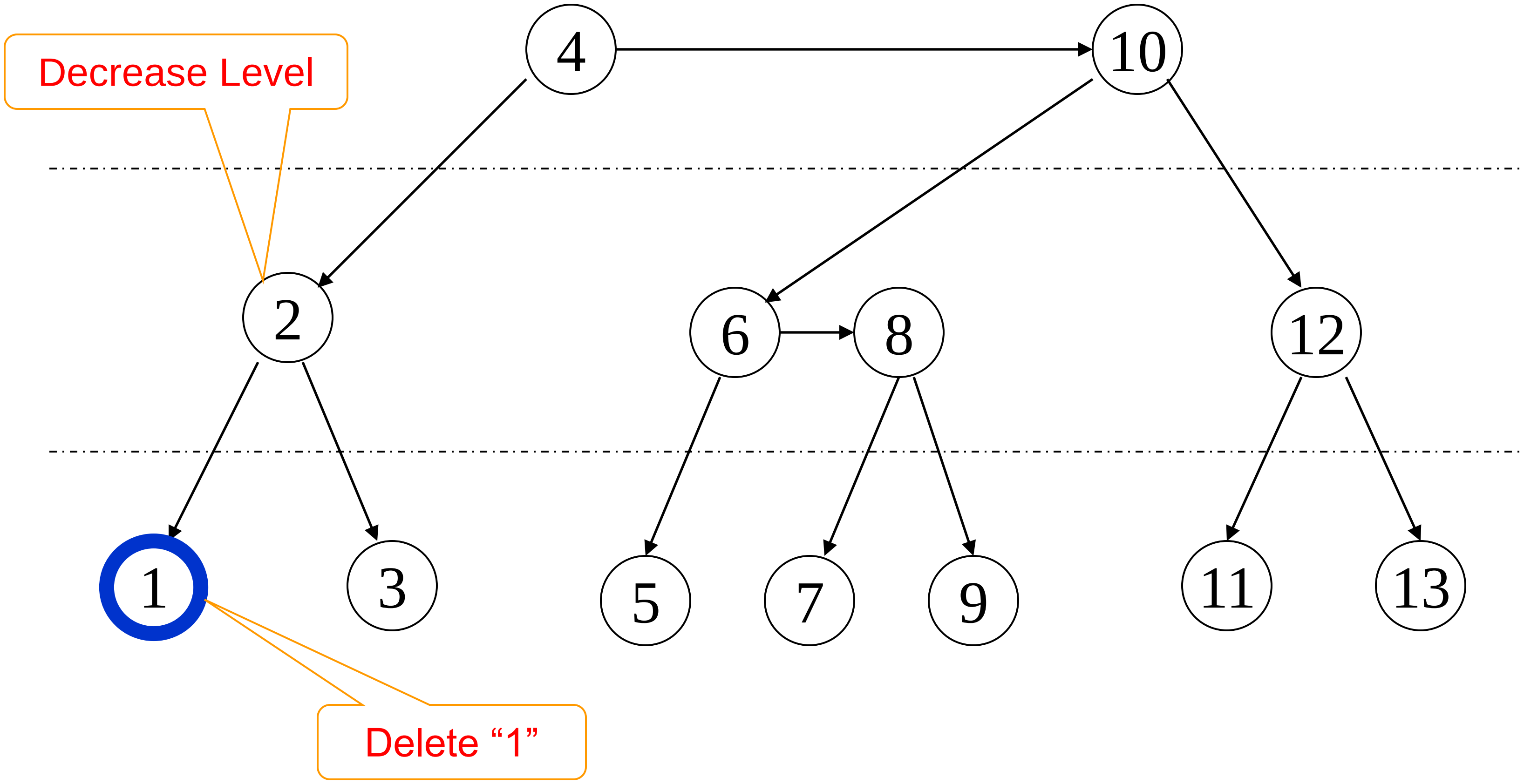
```
AANode AA_Balance_Tree(AANode p)
{
   p = AA_Update_Level(p); // update p's level

   p = skew(p);                  // skew p
   p.right = skew(p.right); // and p's right child
   p.right.right = skew(p.right.right); // and p's right-right grandchild

   p = split(p);                      // split p
   p.right = split(p.right);          // and p's (new) right child

   return p;
}
```

AA Tree Deletion Algorithm

```
AANode delete(Key x, AANode p) {
   if (p == nil) // fell out of tree?
        throw KeyNotFoundException; // ...error - no such key
   else {
        if (x < p.key) // look in left subtree
                p.left = delete(x, p.left);
        else if (x > p.key) // look in right subtree
                p.right = delete(x, p.right);
        else { // found it!
                if (p.left == nil && p.right == nil) // leaf node?
                        return nil; // just unlink the node
                else if (p.left == nil) { // no left child?
                        // get replacement from right
                        AANode r = inorderSuccessor(p);
                        // copy replacement contents here
                        p.copyContentsFrom(r);
                        // delete replacement
                        p.right = delete(r.key, p.right);
                }
                else { // no right child?
                        // get replacement from left
                        AANode r = inorderPredecessor(p);
                        p.copyContentsFrom(r); // copy replacement contents
                        p.left = delete(r.key, p.left); // delete replacement
                }
        }
        return AA_Balance_Tree(p); // fix structure after deletion
   }
}
```

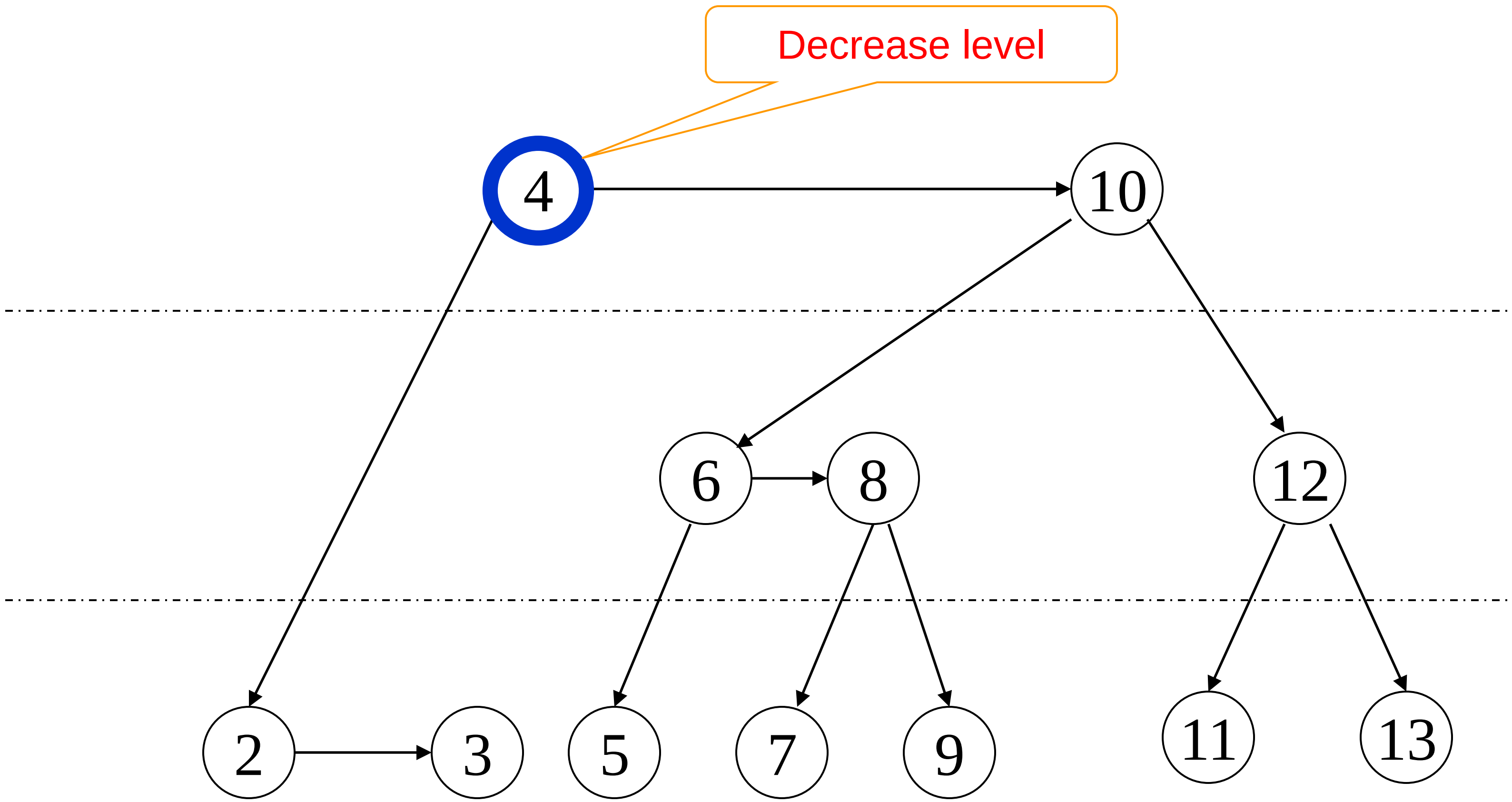
Example

- Delete node 1:



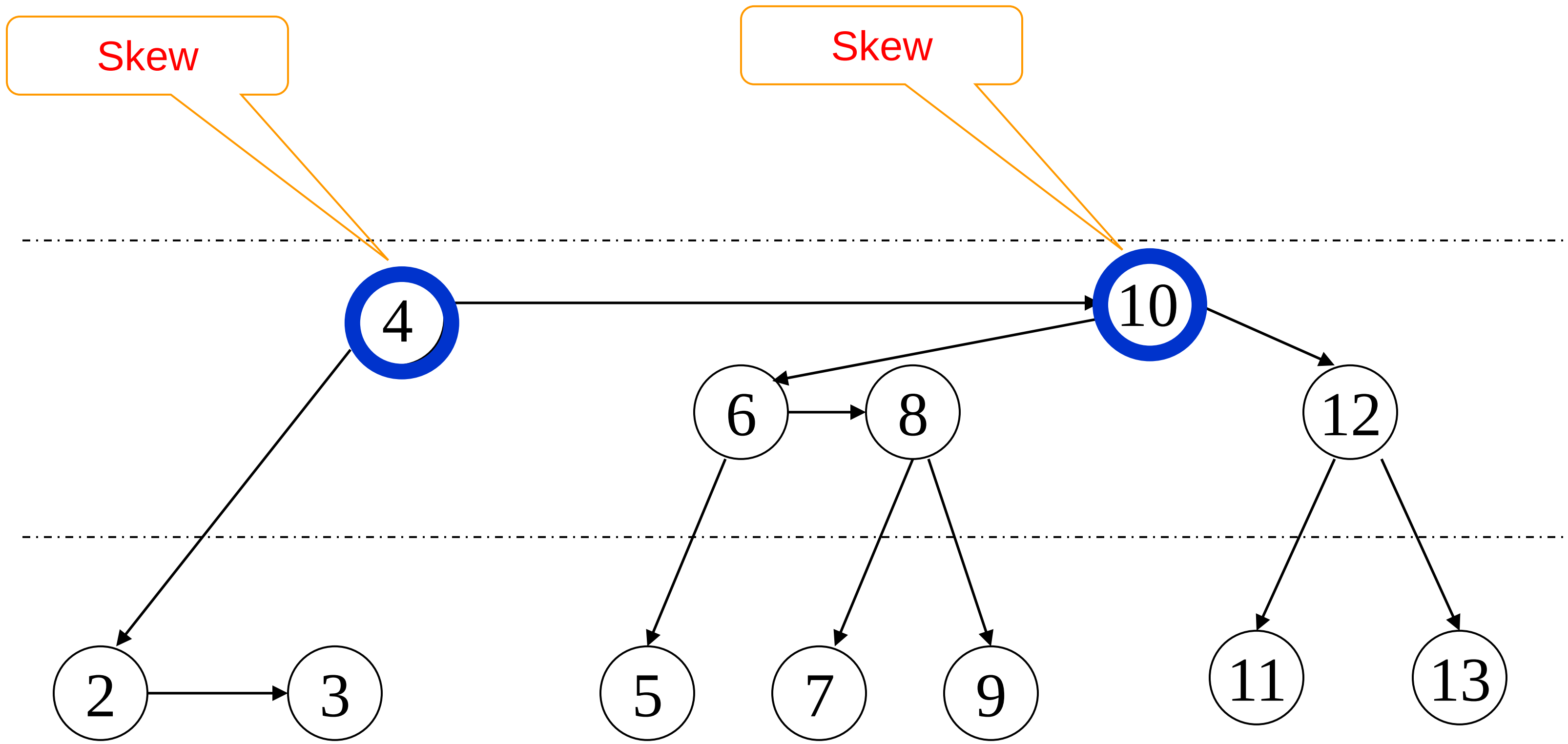
Example

- After decrease level of 2:



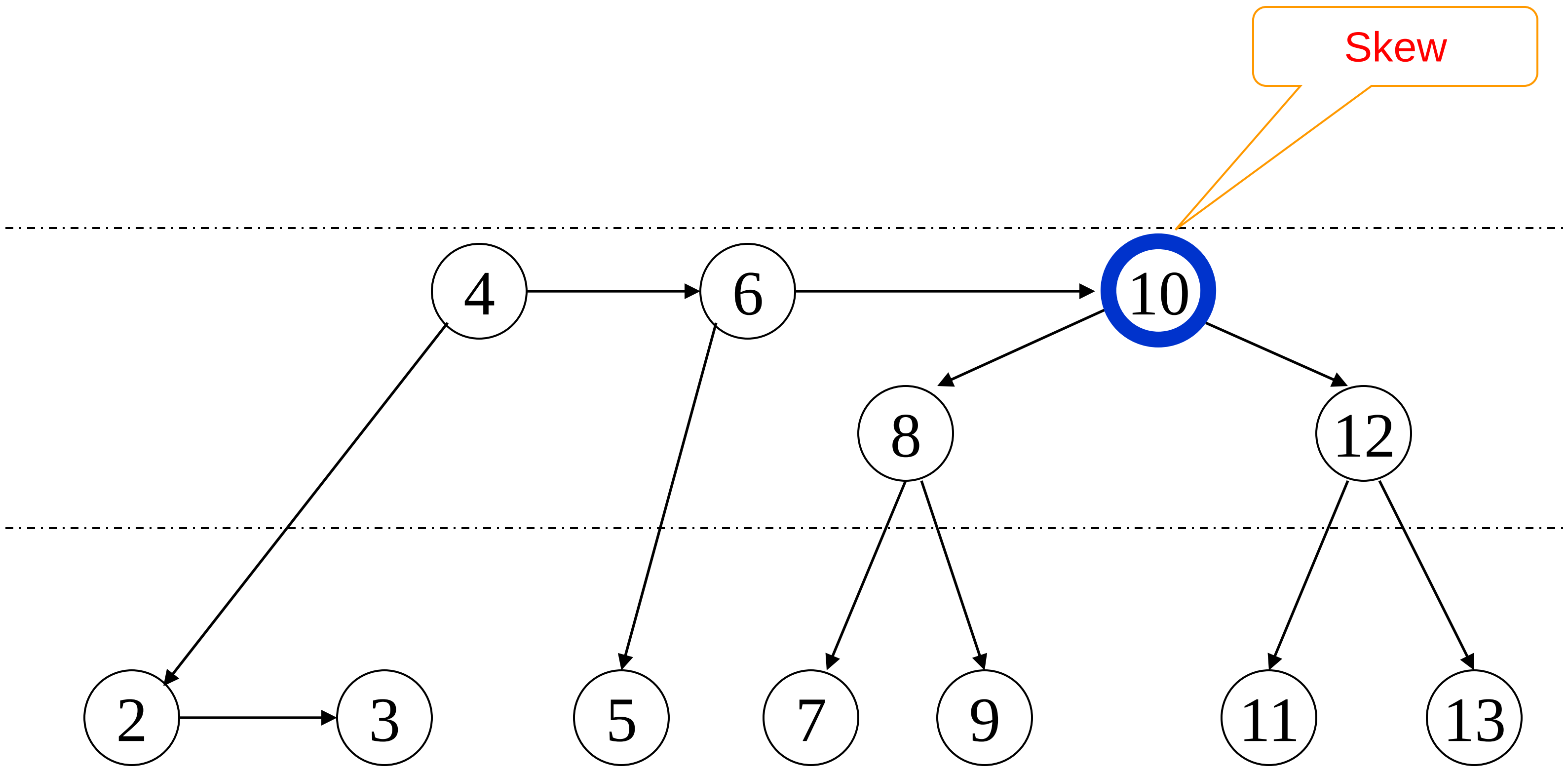
Example

- After decrease level of 4 and 10, skew at node 4 and 10:



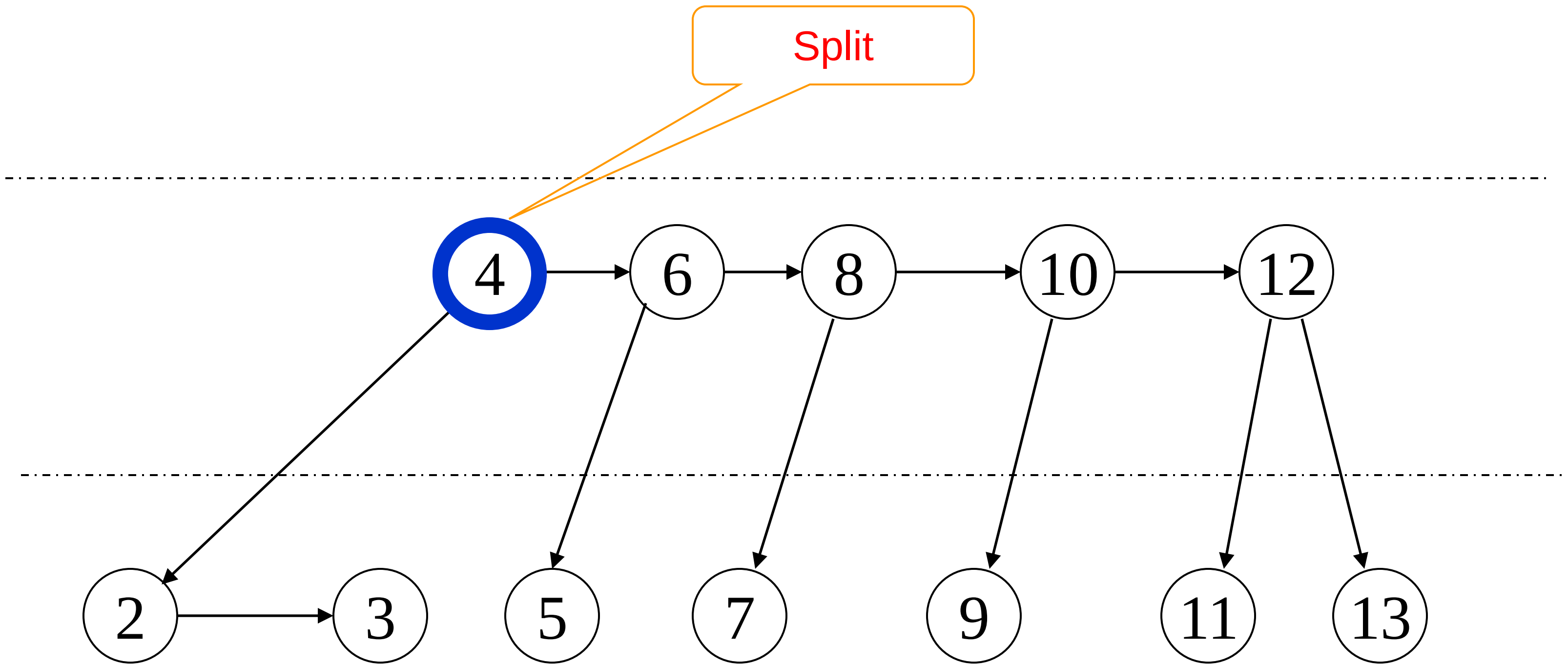
Example

- After skew at node 4 and 10, skew at node p→right→right (node 10):



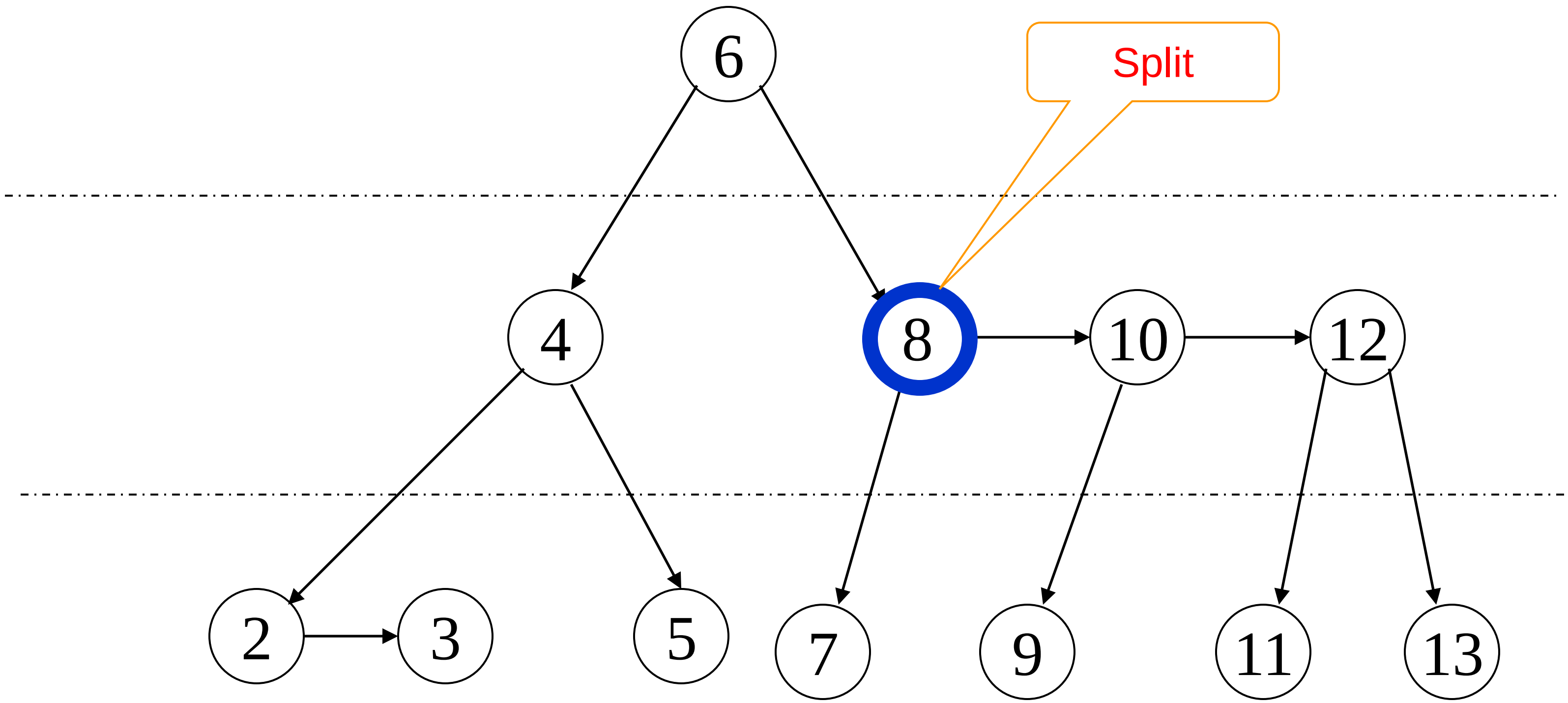
Example

- After skew at node p→right→right (node 10):



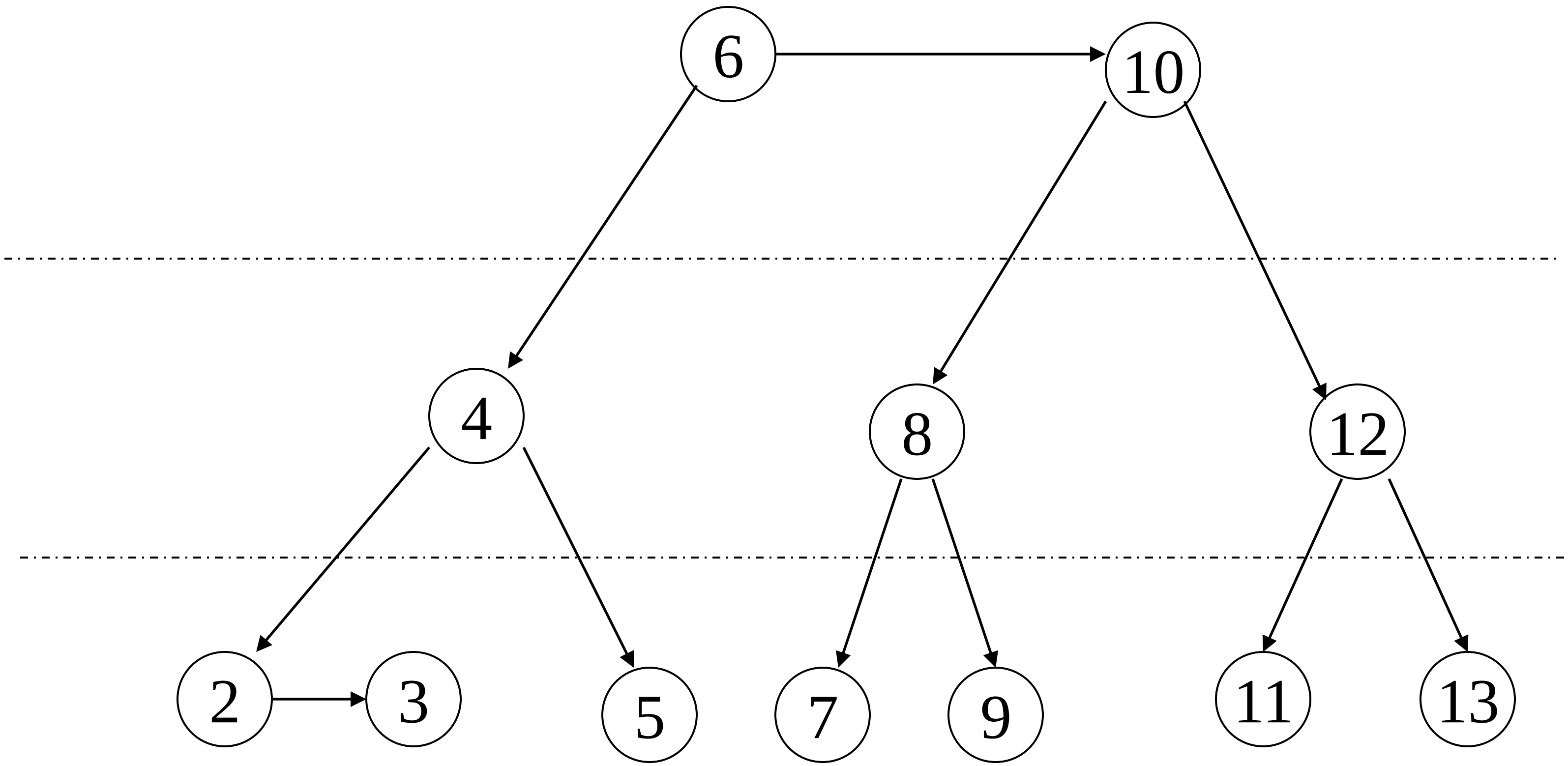
Example

- After split at node 4, split at node 8 (p→right):



Example

- After split at node 8 (p→right):



→ STOP !

AA Tree

- Comments:
  - Complexity $\boldsymbol{O}(log_2 N)$
  - No need to save pointer to parent node (pParent)
  - Simpler than the Red-Black tree

Exercise

- Create AA-Tree by inserting the following values in order:

10, 85, 15, 70,20, 60, 30, 50, 65, 80,90, 40, 5, 55, 35, 2, 45, 17, 99

- After building AA Tree from previous data, delete values in tree in order:

30, 90, 5, 2, 85, 17, 55, 20, 65

The End.