I. Proposición 8.

Sea A y B subconjuntos cerrados de $\mathbb{R}^n$, entonces.

b) $A^{\mathrm{o}} \cap B^{\mathrm{o}} = (A \cap B)^{\mathrm{o}}$

1.- $A^{\mathrm{o}} \cap B^{\mathrm{o}} \subseteq (A \cap B)^{\mathrm{o}}$

$$\begin{aligned} x \in (A^{\mathrm{o}} \cap B^{\mathrm{o}}) &\Rightarrow (x \in A^{\mathrm{o}}) \wedge (x \in B^{\mathrm{o}}) \Rightarrow (\exists B_r(x) \subset A) \wedge (\exists B_r(x) \subset B) \Rightarrow \exists B_r(x) \subset (A \cap B) \\ &\Rightarrow x \in (A \cap B)^{\mathrm{o}} \\ &\therefore A^{\mathrm{o}} \cap B^{\mathrm{o}} \subseteq (A \cap B)^{\mathrm{o}} \end{aligned} \tag{1}$$

2.- $(A \cap B)^{\mathrm{o}} \subseteq A^{\mathrm{o}} \cap B^{\mathrm{o}}$

$$\begin{aligned} x \in (A \cap B)^{\mathrm{o}} &\Rightarrow \exists B_(x) \subset (A \cap B) \Rightarrow (\exists B_r(x) \subset A) \wedge (\exists B_r(x) \subset B) \Rightarrow (x \in A^{\mathrm{o}}) \wedge (x \in B^{\mathrm{o}}) \\ &\Rightarrow x \in (A \cap B)^{\mathrm{o}} \\ &\therefore (A \cap B)^{\mathrm{o}} \subseteq A^{\mathrm{o}} \cap B^{\mathrm{o}} \end{aligned} \tag{2}$$

$\therefore A^{\mathrm{o}} \cap B^{\mathrm{o}} = (A \cap B)^{\mathrm{o}}$

c) $\overline{A} \cup \overline{B} = \overline{A \cup B}$

1.- $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$

$$\begin{aligned} x \in (A^{\mathrm{o}} \cap B^{\mathrm{o}}) &\Rightarrow (x \in \overline{A}) \vee (x \in \overline{B}) \Rightarrow (\forall B_r(x), B_r(x) \cap A \neq \emptyset) \vee (\forall B_r(x), B_r(x) \cap B \neq \emptyset) \\ &\Rightarrow (\forall B_r(x), B_r(x) \cap (A \cup B) \neq \emptyset) \Rightarrow x \in (\overline{A \cup B}) \\ &\therefore \overline{A} \cup \overline{B} \subseteq \overline{A \cup B} \end{aligned} \tag{3}$$

2.- $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$

$$\begin{aligned} x \in (\overline{A \cup B}) &\Rightarrow \forall B_r(x), B_r(x) \cap (A \cup B) \neq \emptyset \Rightarrow (\forall B_r(x), B_r(x) \cap A \neq \emptyset) \cup (\forall B_r(x), B_r(x) \cap B \neq \emptyset) \\ &\Rightarrow (\forall B_r(x), B_r(x) \cap A \neq \emptyset) \vee (\forall B_r(x), B_r(x) \cap B \neq \emptyset) \\ &\Rightarrow (x \in \overline{A}) \vee (x \in \overline{B}) \Rightarrow x \in (\overline{A} \cup \overline{B}) \\ &\therefore \overline{A \cup B} \subseteq \overline{A} \cup \overline{B} \end{aligned} \tag{4}$$

$\therefore \overline{A} \cup \overline{B} = \overline{A \cup B}$

d) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$

Por la proposición 3 inciso b) sabemos que $A \subset \overline{A}$ y $B \subset \overline{B}$ por lo que $A \cap B \subset \overline{A} \cap \overline{B}$ y dado que $\overline{A} \cap \overline{B}$ es cerrado $\Rightarrow \overline{A \cap B} \subset \overline{A} \cap \overline{B}$. Si consideramos $A = (0, 1)$ y $B = (1, 2) \Rightarrow \overline{A \cap B} = \emptyset$ pero $\overline{A} = [0, 1]$ y $\overline{B} = [1, 2] \Rightarrow \overline{A} \cap \overline{B} = 1$.