**NYU CSCI-UA 480-072: Introduction to Robot Intelligence** **Fall 2023**

# Problem Set 3

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### Question 1

$$\vec{a} = (a_1, a_2, a_3)$$
$$\vec{b} = (b_1, b_2, b_3)$$
$$\vec{c} = (c_1, c_2, c_3)$$

(a)

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

We can switch row 1 with row 2 of the first matrix, and then switch row 3 with row 1. After the two consecutive row interchange operations, the first matrix is same as the second. Also, we know that row interchange multiplies the determinant by -1, and two consecutive row interchanges preserve the determinant value. Therefore, we can get

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

(b)

$$\vec{a} \times (\vec{b} \times \vec{c}) = (a_1, a_2, a_3) \times (b_1c_2 - b_2c_1, b_3c_1 - b_1c_3, b_1c_2, b_2c_1)$$
$$= ((a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3), (a_3(b_1c_2 - b_2c_1) - a_1(b_1c_2 - b_1c_3), (a_1(b_3c_1 - b_1c_3) - a_2(b_1c_2 - b_2c_1))$$
$$= (a_2c_2b_1 + a_3c_3b_1 - a_2b_2c_1 - a_3b_3c_1, a_1c_1b_2 + a_3c_3b_2 - a_1b_1c_2 - a_3b_3c_2, a_1c_1b_3 + a_2c_2b_3 - a_1b_1c_3 - a_2b_2c_3)$$

$$(\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} = (a_1c_1 + a_2c_2 + a_3c_3)(b_1, b_2, b_3) - (a_1b_1 + a_2b_2 + a_3b_3)(c_1, c_2, c_3)$$
$$= (a_2c_2b_1 + a_3c_3b_1 - a_2b_2c_1 - a_3b_3c_1, a_1c_1b_2 + a_3c_3b_2 - a_1b_1c_2 - a_3b_3c_2, a_1c_1b_3 + a_2c_2b_3 - a_1b_1c_3 - a_2b_2c_3)$$

Therefore, we can get

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

### Question 2

(a) (1). Cabinet:
Links: The main body of the cabinet itself is a single rigid link. It may have multiple doors, each of which can be considered as a link.
Joints: Revolute joints are used for the doors of the cabinet, allowing the doors to rotate around the axis between the main body and the door.

(2). Bicycle:
Links: The bicycle frame is the main rigid link, consisting of the top tube, down tube, seat tube. Also, the two wheels, each with spokes and tire are also links. Pedals are also links.

Joints: The handlebars are attached to the frame using a fork and headset, allowing them to rotate. Also, The wheels are connected to the frame using revolute joints. Besides, the saddle can either rise or drop, which is enabled by using a prismatic joint.

(3). Skateboard:
Links: The skateboard deck is the main rigid link, and the trucks of skateboard are seperate linked. Joints: Wheels are attached to the trucks using axles, forming revolute joints. The trucks are connected to the deck using revolute joints, allowing the skateboard to turn.

(b)

$$||T(\vec{u}) - T(\vec{w})||^2 = ||R\vec{u} + \vec{t} - R\vec{w} - \vec{t}||^2 = ||R(\vec{u} - \vec{w})||^2$$

Since R is only a rotation matrix and is also an orthogonal matrix, it preserves the norm. Therefore, we can get

$$||R(\vec{u} - \vec{w})||^2 = ||\vec{u} - \vec{w}||^2$$

Then we can get

$$||T(\vec{u}) - T(\vec{w})||^2 = ||\vec{u} - \vec{w}||^2$$

(c) We know that for an n-dimensional space, we have n degrees of freedom for translation, as we can move the rigid body along n independent axes, so translation generates n degrees of freedom.

Since the rotation happens on a plane, it takes 2 dimensions. Therefore, we can get the following result.
For the first rotation, we have n-1 choices.

For the second rotation, we have n-2 choices.

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For the (n-1)th rotation, we only have 1 choice, since only two independent dimensions are left.

Therefore, translation generates

$$(n-1) + (n-2) + \cdots + 1 = \frac{n(n-1)}{2}$$

degrees of freedom.

$$\text{Total degrees of freedom} = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$$

**Question 3**

1. (a) Since $R_1$ and $R_2$ are orthogonal matrix, then

$$(R_1)^T R_1 = Id_3 \text{ and } (R_2)^T R_2 = Id_3$$

We can deduce that

$$(R')^T R' = (R_1 R_2)^T R_1 R_2 = (R_2)^T (R_1)^T R_1 R_2 = (R_2)^T Id_3 R_2 = (R_2)^T R_2 = Id_3$$

Therefore, $R'$ is an orthogonal matrix.
Also, since $det(R_1) = 1$ and $det(R_2) = 1$,

$$det(R') = det(R_1 R_2) = det(R_1) det(R_2) = 1$$

Therefore, $det(R') = 1$.
R' is a valid rotation matrix that ensures the closure property.

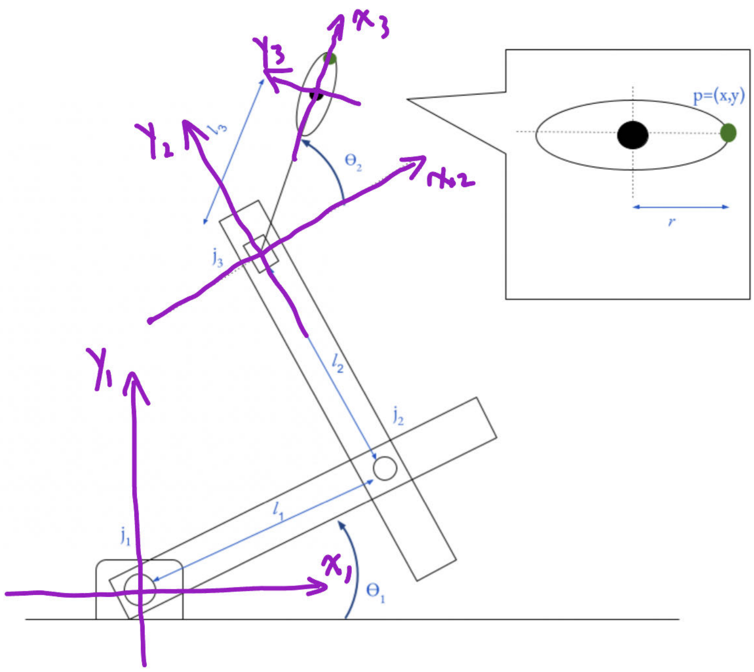


Figure 3.1: The way I construct axes

(b) We can take $e = Id_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We know that

$$R_1 Id_3 = Id_3 R_1 = R_1 \text{ and } R_2 Id_3 = Id_3 R_2 = R_2$$

Obviously, $Id_3$ is an orthogonal matrix. Also, we know that $det(Id_3) = 1$.
Therefore, $e = Id_3$ is a valid rotation matrix that ensures the identity property.

(c) We can take $R_1^{-1} = R_1^T$ and $R_2^{-1} = R_2^T$.
Since $R_1$ and $R_2$ are orthogonal matrix, then

$$(R_1)^T R_1 = Id_3 \text{ and } (R_2)^T R_2 = Id_3$$

Also, we know that $R_1^T$ and $R_2^T$ are orthogonal matrix, since $R_1$ and $R_2$ are orthogonal matrix.
For the determinant,

$$det(R_1^T) = det(R_1) = 1 \text{ and } det(R_2^T) = det(R_2) = 1$$

Therefore, there exists $R_1^{-1} = R_1^T$ and $R_2^{-1} = R_2^T$ ensure the inverse property.

2. Fig. 3.1 shows how I construct axes in order to compute p(x, y).

We know that in $x_3, y_3$, $P_3 = (r, 0, 1)$.
In order to transform from $P_3$ to $P_2$, the transformation matrix should be

$$H_2 = \begin{bmatrix} cos(\theta_2) & -sin(\theta_2) & l_3 cos(\theta_2) \\ sin(\theta_2) & cos(\theta_2) & l_3 sin(\theta_2) \\ 0 & 0 & 1 \end{bmatrix}$$

In order to transform from $P_2$ to $P_1$, the transformation matrix should be

$$H_1 = \begin{bmatrix} cos(\theta_1) & -sin(\theta_1) & l_1 cos(\theta_1) - l_2 sin(\theta_1) \\ sin(\theta_1) & cos(\theta_1) & l_1 sin(\theta_1) + l_2 cos(\theta_1) \\ 0 & 0 & 1 \end{bmatrix}$$

Then we can get

$$P_1 = H_1 H_2 \begin{bmatrix} r \\ 0 \\ 1 \end{bmatrix} = H_1 \begin{bmatrix} (r+l_3)cos(\theta_2) \\ (r+l_3)sin(\theta_2) \\ 1 \end{bmatrix} = \begin{bmatrix} (r+l_3)cos(\theta_1+\theta_2) + l_1 cos(\theta_1) - l_2 sin(\theta_1) \\ (r+l_3)sin(\theta_1+\theta_2) + l_1 sin(\theta_1) + l_2 cos(\theta_1) \\ 1 \end{bmatrix}$$

Then we omit 1 in the last entry, and we get the following result.

$$p(x,y) = \begin{bmatrix} (r+l_3)cos(\theta_1+\theta_2) + l_1 cos(\theta_1) - l_2 sin(\theta_1) \\ (r+l_3)sin(\theta_1+\theta_2) + l_1 sin(\theta_1) + l_2 cos(\theta_1) \end{bmatrix}$$