NYU CSCI-UA 480-072: Introduction to Robot Intelligence	**Fall 2023**

Problem Set 1

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Exercise 1

The probability for Player 1 winning in the first round is $\frac{1}{5}$.
The probability for Player 1 winning in the second round is $\frac{1}{5} \cdot \frac{3}{4} \cdot \frac{4}{5}$.
The probability for Player 1 winning in the third round is $\frac{1}{5} \cdot \left(\frac{3}{4}\right)^2 \cdot \left(\frac{4}{5}\right)^2$.
So on and so forth ...

Then we can get the probability that Player 1 wins

$$
\begin{aligned}
&= \frac{1}{5} + \frac{1}{5} \cdot \frac{3}{5} + \frac{1}{5} \cdot \left(\frac{3}{5}\right)^2 + \frac{1}{5} \cdot \left(\frac{3}{5}\right)^3 + \ldots \\
&= \frac{1}{5} \cdot \sum_{n=0}^{\infty} \frac{3}{5} \\
&= \frac{1}{5} \cdot \frac{1}{1 - \frac{3}{5}} \\
&= \frac{1}{2}
\end{aligned}
$$

Exercise 2

Based on the information given, we can get the proportion of each category as following:
True Positive: 0.009
False Positive: 0.099
False Negative: 0.001
True Negative: 0.891

Then the probability that you have COVID given that you tested positive

$$= \frac{TP}{TP + FP} = \frac{0.009}{0.009 + 0.099} = \frac{1}{12}$$

Exercise 3

To show that f(x) is a PDF, we have to prove its nonnegativity and the integral of f(x) over the entire range of X equals 1. The first condition is obviously True. The proof of the second condition is shown below:

$$\int f(x)\, dx = \int_0^{\infty} \frac{1}{1+x}\, dx = \lim_{k \to \infty} ln\, |1+k| - ln\, (1) = \infty$$

Since the integral of f(x) over the entire range of X is not 1, f(x) is not a PDF.

Exercise 4

In order to find $P(X + Y \leq 1)$, we should do the following calculation

$$\int \int f(x) f(y) dx dy, \text{ where } 0 \leq x \leq 1, 0 \leq y \leq 1, \text{and } x + y \leq 1$$

We use the polar coordinate to solve the integration problem.

$$x = rcos(\theta)$$
$$y = rsin(\theta)$$
$$0 \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2}$$

Then we can get

$$P(X + Y \leq 1) = 2\int_0^1 r^2 dr \int_0^{\frac{\pi}{2}} cos(\theta)sin(\theta)d(\theta) = \frac{1}{3}$$

Exercise 5

From the given information, we know the PDF and CDF are

$$f(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ 0, & otherwise \end{cases}$$

$$F_X(x) = \begin{cases} 1, & \text{if } x \geq 1 \\ x, & \text{if } 0 < x < 1 \\ 0, & \text{if } x \leq 0 \end{cases}$$

Based on the condition that $Y = e^X$, we know that

$$F_Y(y) = P(e^X \leq y) = P(X \leq ln(y))$$

$$F_Y(y) = \begin{cases} 0, & \text{if } 0 < y \leq 1 \\ ln(y), & \text{if } 1 < y < e \\ 1, & \text{if } y \geq e \end{cases}$$

Take the derivative, we can get the PDF $F_Y(y)$ as following

$$f_Y(y) = \begin{cases} \frac{1}{y}, & \text{if } 1 < y < e \\ 0, & \text{otherwise} \end{cases}$$

Then we calculate the expectation of Y as follows

$$\begin{aligned} E[Y] &= \int_{-\infty}^{\infty} y f_Y(y) dy \\ &= \int_1^e y \cdot \frac{1}{y} dy \\ &= \int_1^e dy \\ &= e - 1 \end{aligned}$$

Exercise 6

From the given condition, we know that $\mu = 5$ is the mean of X, and $\sigma = \sqrt{5}$ is the standard deviation of X.
Applying Central Limit Theorem, we can get

$$\begin{aligned} P\left(\bar{X} - 5.5 \leq 0\right) &= P\left(\bar{X} - 5 < 0.5\right) \\ &= P\left(\frac{\sqrt{n}(\bar{X} - 5)}{\sqrt{5}} < \frac{\sqrt{n} \cdot 0.5}{\sqrt{5}}\right) \\ &= P\left(\frac{\sqrt{n}(\bar{X} - 5)}{\sqrt{5}} < 2.5\right) \\ &\approx P(Z < 2.5) \\ &\approx 0.9938 \end{aligned}$$

Exercise 7

According to the chain rule,

$$\frac{\partial E}{\partial X_0} = \frac{\partial E}{X_p} \cdot \frac{\partial X_p}{\partial X_{p-1}} \cdot \frac{\partial X_{p-1}}{\partial X_{p-2}} \cdots \frac{\partial X_1}{\partial X_0}$$

According to the formula, we know that

$$\frac{\partial E}{\partial X_p} = 2||c - X_p||$$
$$\frac{\partial X_n}{\partial X_{n-1}} = \frac{\partial f(W_n, X_{n-1})}{\partial X_{n-1}}$$

Then we plug in, and we can get

$$\begin{aligned}\frac{\partial E}{\partial X_0} &= \frac{\partial E}{X_p} \cdot \frac{\partial X_p}{\partial X_{p-1}} \cdot \frac{\partial X_{p-1}}{\partial X_{p-2}} \cdots \frac{\partial X_1}{\partial X_0} \\ &= 2||c - X_p|| \cdot \prod_{0}^{p-1} \frac{\partial f(W_i, X_{i-1})}{\partial X_{i-1}}\end{aligned}$$

Exercise 8

We know that

$$A = \begin{bmatrix} 2 & 6 & 7 \\ 3 & 1 & 2 \\ 5 & 3 & 4 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & 3 & 5 \\ 6 & 1 & 3 \\ 7 & 2 & 4 \end{bmatrix}$$

Then we can get

$$Ax = \begin{bmatrix} 2 & 6 & 7 \\ 3 & 1 & 2 \\ 5 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 50 \\ 17 \\ 35 \end{bmatrix}$$
$$A^T x = \begin{bmatrix} 2 & 3 & 5 \\ 6 & 1 & 3 \\ 7 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 33 \\ 27 \\ 36 \end{bmatrix}$$
$$x^T A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 6 & 7 \\ 3 & 1 & 2 \\ 5 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 33 & 27 & 36 \end{bmatrix}$$

Exercise 9

(a) Let the matrix be A.

$$Det(A) = -1 \neq 0$$

Therefore, A is invertible. Then we calculate the A^{-1}.

$$\begin{bmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 10 & 3 & 4 \end{bmatrix} \Bigg| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 1 & 1/3 & 1/2 \\ 0 & 0 & 1 \\ 0 & 1/3 & 1 \end{bmatrix} \Bigg| \begin{bmatrix} 1/6 & 0 & 0 \\ 1 & -2 & 0 \\ 5/3 & 0 & -1 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 1 & 1/3 & 1/2 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Bigg| \begin{bmatrix} 1/6 & 0 & 0 \\ 2/3 & 2 & -1 \\ 1 & -2 & 0 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Bigg| \begin{bmatrix} -1/2 & -2 & 1 \\ 2 & 6 & -3 \\ 1 & -2 & 0 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Bigg| \begin{bmatrix} -1 & -1 & 1 \\ 2 & 6 & -3 \\ 1 & -2 & 0 \end{bmatrix}$$
$$A^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 6 & -3 \\ 1 & -2 & 0 \end{bmatrix}$$

(b) Let the matrix be B.

$$Det(B) = 0$$

Therefore, B is not invertible.

Exercise 10

Let the matrix be A. Then in order to get the eigenvalues and eigenvectors of the matrix A, we focus on the equation $(A - \lambda I)\, x = 0$. We should let $Det(A - \lambda I) = 0$.

$$\begin{aligned} & Det(A - \lambda I) = 0 \\ \Rightarrow & -\lambda\left(\lambda^2 - 2\lambda + 1\right) - 2 + 2\lambda = 0 \\ \Rightarrow & (\lambda^2 - 1)(\lambda - 2) = 0 \\ \Rightarrow & \lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 2 \end{aligned}$$

When $\lambda = -1$,

$$\begin{aligned} A - \lambda I = & \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \\ -2 & 2 & 2 \end{bmatrix} \\ \to & \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 2 & 1 \end{bmatrix} \\ \to & \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

The eigenvector is $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

When $\lambda = 1$,

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} 0 & 0 & -1 \\ 1 & -1 & 0 \\ -2 & 2 & 0 \end{bmatrix} \\ &\to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

The eigenvector is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

When $\lambda = 2$,

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} -1 & 0 & -1 \\ 1 & -2 & 0 \\ -2 & 2 & -1 \end{bmatrix} \\ &\to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

The eigenvector is $\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$.