The Answers for Assignment 8

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Question 1. Find the eigenvalues and the corresponding eigenspaces for each of the following matrices:

(a) $\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 6 & -4 \\ 3 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{pmatrix}$

Solution

For an $n \times n$ matrix A, its eigenvalues are the solutions to the characteristic equation $\det(A - \lambda I) = 0$, and the corresponding eigenspace for eigenvalue λ is the null space of $A - \lambda I$.

(a) Eigenvalues are $\lambda_1 = 5$, $\lambda_2 = -1$.

Corresponding eigenspaces are $\text{Span}\{(1,1)^T\}$ and $\text{Span}\{(1,-2)^T\}$.

(b) Eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 3$.

Corresponding eigenspaces are $\text{Span}\{(1,1)^T\}$ and $\text{Span}\{(4,3)^T\}$.

(c) Eigenvalue is $\lambda = 0$.

Corresponding eigenspace is $\text{Span}\{(1,0,0)^T\}$.

(d) Eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 1$.

Corresponding eigenspaces are:

$\text{Span}\{(1,1,0)^T\}$ and $\text{Span}\{(1,0,0)^T, (0,1,-1)^T\}$.

(e) Eigenvalues are $\lambda_1 = -2, \lambda_2 = 1$ and $\lambda_3 = 4$

Corresponding eigenspaces are $\text{Span}\{(1,1,-5)^T\}, \text{Span}\{(1,0,0)^T\}$ and $\text{Span}\{(1,1,1)^T\}$.

Question 2. Let A be an $n \times n$ matrix. Prove that A is singular if and only if $\lambda = 0$ is an eigenvalue of A.

Solution

Proof. A is singular if and only if the homogenous equation $A\mathbf{x} = \mathbf{0}$ has a non-zero solution. If such a solution exist, and let it be $\mathbf{x}_0$, then we have $A\mathbf{x}_0 = 0\mathbf{x}_0$ and $\lambda = 0$ is an eigenvalue of A. Conversely, an eigenvalue 0 implies the equation $A\mathbf{x} = 0\mathbf{x}$ has a non-zero solution, which indicates A is singular. □

Question 3. Let A be a nonsingular matrix and let λ be an eigenvalue of A. Show that $1/\lambda$ is an eigenvalue of A^{-1}.

Solution

Proof. λ is an eigenvalue of A means the equation $A\mathbf{x} = \lambda\mathbf{x}$ has non-zero solution for $\mathbf{x}$. Let $\mathbf{x}_0$ be such a solution (an eigenvector of A), let $\mathbf{y}_0 = A\mathbf{x}_0 = \lambda\mathbf{x}_0$. Because A is invertible, we have $\mathbf{y}_0 \neq \mathbf{0}$ and $A^{-1}\mathbf{y}_0 = A^{-1}A\mathbf{x}_0 = \mathbf{x}_0 = \frac{1}{\lambda}\mathbf{y}_0$, therefore, $1/\lambda$ is an eigenvalue of A^{-1}. □

Question 4. An $n \times n$ matrix A is said to be *idempotent* if $A^2 = A$. Show that if λ is an eigenvalue of an idempotent matrix, then λ must be either 0 or 1.

Solution

Eigenvalues are solutions to the equation $|A - \lambda I| = 0$. For idempotent matrix A, we can construct the following multiplication and have:

$$(A - \lambda I)(A + (\lambda - 1)I) = A^2 - A - \lambda(\lambda - 1)I = -\lambda(\lambda - 1)I.$$

Then, we have the determinant $|(A - \lambda I)(A + (\lambda - 1)I)| = -\lambda(\lambda - 1)$. Now, consider if λ is an eigenvalue of A, we must have $|A - \lambda I| = 0$ and because

$$-\lambda(\lambda - 1) = |(A - \lambda I)(A + (\lambda - 1)I)| = |(A - \lambda I)||(A + (\lambda - 1)I)| = 0,$$

λ must be either 0 or 1.

Question 5. Let A be an $n \times n$ matrix and let $B = A + I$. Is it possible for A and B to be similar? Explain.

Solution

Because $B = A + I$, the characteristic equation for A is $\det(A - \lambda I) = 0$ and for B is $\det(B - \lambda I) = \det(A - (\lambda - 1)I) = 0$. We see that if λ_0 is an eigenvalue of A, then $\lambda_0 + 1$ is an eigenvalue of B. Since A and B have different sets of eigenvalues, they cannot be similar.

Question 6. Let Q be an orthogonal matrix.

(a) Show that if λ is an eigenvalue of Q, then $|\lambda| = 1$.

(b) Show that $|\det(Q)| = 1$.

Solution

(a) *Proof.* If λ is an eigenvalue of Q, and let $\mathbf{x}_0$ be a corresponding eigenvector, then we have $Q\mathbf{x}_0 = \lambda\mathbf{x}_0$ and $\mathbf{x}_0 \neq \mathbf{0}$. Because Q is an orthogonal matrix, we have:

$$\lambda^2 \mathbf{x}_0^T \mathbf{x}_0 = (\lambda\mathbf{x}_0)^T(\lambda\mathbf{x}_0) = (Q\mathbf{x}_0)^T(Q\mathbf{x}_0) = \mathbf{x}_0^T(Q^TQ)\mathbf{x}_0 = \mathbf{x}_0^T\mathbf{x}_0$$

Because $\mathbf{x}_0^T\mathbf{x}_0 \neq 0$, the above equation implies $\lambda^2 = 1$, or $|\lambda| = 1$. □

(b) *Proof.* For orthogonal matrix Q, we have $Q^T = Q^{-1}$, and,

$$(\det(Q))^2 = \det(Q)\det(Q^T) = \det(Q)\det(Q^{-1}) = \det(QQ^{-1}) = \det(I) = 1.$$

Hence, $|\det(Q)| = 1$. □

Question 7. In each of the following, factor the matrix A into a product XDX^{-1}, where D is diagonal:

(a)

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

Solution

(a) The eigenvalues of this matrix is $\lambda_1 = 1$ and $\lambda_2 = -1$. And $x_1 = (1,1)^T$ is an eigenvector belonging to eigenvalue λ_1, $x_2 = (-1,1)^T$ is an eigenvector belonging to eigenvalue λ_2, so A can be written as

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(b) The calculation process is the same with (a), A can be written as

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & \frac{5}{3} \\ 0 & -1 & -1 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Question 8. For each of the matrices in last Exercise, use the XDX^{-1} factorization to compute A^6.

Solution

(a) $A^6 = XD^6X^{-1}$, so

$$A^6 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^6 & 0 \\ 0 & (-1)^6 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b)

$$A^6 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2^6 & 0 & 0 \\ 0 & 1^6 & 0 \\ 0 & 0 & (-1)^6 \end{bmatrix} \begin{bmatrix} 1 & 2 & \frac{5}{3} \\ 0 & -1 & -1 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 64 & 125 & 105 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Question 9. For each of the following, find a matrix B such that $B^2 = A$.

(a)

$$A = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 9 & -5 & 3 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

(a) The eigenvalues of A are 1 and 0. Let X be the matrix whose columns are eigenvectors belonging to eigenvalues of matrix A and $D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Since $D^2 = D$, It follows that $A^2 = (XDX^{-1})^2 = XD^2X^{-1} = A$. So $B = A$.

(b) Since

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

So

$$B = XD^{\frac{1}{2}}X^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

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Question 10. Let A be a diagonalizable matrix whose eigenvalues are all either 1 or -1. Show that $A^{-1} = A$.

Solution

Assuming $A = XDX^{-1}$ where $D = diag(d_1, d_2, ..., d_n)$ and $d_i = 1/-1$. We get $D^{-1} = D$, So $A^{-1} = (XDX^{-1})^{-1} = XD^{-1}X^{-1} = XDX^{-1} = A$.

Question 11. Show that any 3×3 matrix of the form $\begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & b \end{bmatrix}$ is defective.

Solution

Because the eigenvalues are $\lambda_1 = \lambda_2 = a, \lambda_3 = b$. When the eigenvalue is a, $(A - aI)X = 0$, we can get the eigenvector $X_1 = X_2 = (1, 0, 0)^T$. According to the definition of defective matrix, we know that matrix A is defective.

Question 12. For each of the following, find all possible values of the scalar α that make the matrix defective or show that no such values exist.

(a)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 3\alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}$$

Solution

(a) We firstly calculate the eigenvalues of A are $\lambda_1 = 0, \lambda_2 = 2, \lambda_3 = \alpha$.

- When $\alpha = 0$, we can get the eigenvectors are $X_1 = (1, -1, 0)^T, X_2 = (1, 1, 0)^T, X_3 = (0, 0, 1)^T$.
- When $\alpha = 2$, the eigenvectors are $X_1 = (1, -1, 0)^T, X_2 = (1, 1, 0)^T, X_3 = (0, 1, 1)^T$.

So no such α exist.

(b) The eigenvalues of A are $\lambda_1 = \alpha, \lambda_2 = \alpha, \lambda_3 = 3\alpha$.

- When $\alpha = 0$, the eigenvectors are $X_1 = (1, 0, 0)^T$ and $X_2 = (0, 0, 1)^T$. There are 2 linearly independent eigenvectors corresponding to eigenvalues $\lambda_{1,2,3} = 0$, so matrix A is defective for $\alpha = 0$.
- When $\alpha \neq= 0$, There are 2 linearly independent eigenvectors $(1, -2\alpha, 0)^T$ and $(0, 0, 1)^T$ corresponding to eigenvalues $\lambda_{1,2} = 0$. There are 3 linearly independent eigenvectors for matrix A. This shows that matrix A is not defective for $\alpha \neq= 0$.

So $\alpha = 0$ makes the matrix defective.

Question 13. Let A be a 4×4 matrix and let λ be an eigenvalue of multiplicity 3. If $A - \lambda I$ has rank 1, is A defective? Explain.

Solution

By Rank-Nullity Theorem we have that $rank(A-\lambda I)+nullity(A-\lambda I)=4$, which implies that $dim(A-\lambda I)=nullity(A-\lambda I)=4-\text{rank}(A-\lambda I)=3$. Therefore, there are 3 linearly independent eigenvectors belongs to eigenvalue λ. Since eigenvectors belonging to distinct eigenvalues are linearly independent, A is not defective.

Question 14. Let A be an $n\times n$ matrix with an eigenvalue λ of multiplicity n. Show that A is diagonalizable if and only if $A=\lambda I$.

Solution

A is diagonalizable if $dim(N(A-\lambda I))=n$, that is $rank(A-\lambda I)=0$. So $A=\lambda I$.

Question 15. Let A be a diagonalizable $n\times n$ matrix. Prove that if B is any matrix that is similar to A, then B is diagonalizable.

Solution

A is diagonalizable, then $X^{-1}AX=D$. If B is similar to A then $A=S^{-1}BS$. Substitute A then we get $D=X^{-1}(S^{-1}BS)X=(SX)^{-1}B(SX)$. So, B is diagonalizable by SX.

Question 16. Show that if A and B are two $n\times n$ matrices with the same diagonalizing matrix X, then $AB=BA$.

Solution

Assuming $A=XDX^{-1}$ and $B=XEX^{-1}$, where D,E are diagonal matrix and $DE=ED$. $AB=(XDX^{-1})(XEX^{-1})=XDEX^{-1}=XEDX^{-1}=(XEX^{-1})(XDX^{-1})=BA$.

Question 17. For each of the following pairs of vectors $\mathbf{z}$ and $\mathbf{w}$, compute (i) $||\mathbf{z}||$, (ii) $||\mathbf{w}||$, (iii) $\langle \mathbf{z},\mathbf{w}\rangle$, and (iv) $\langle \mathbf{w},\mathbf{z}\rangle$:

(a)

$$\mathbf{z}=\begin{bmatrix}4+2i\\4i\end{bmatrix},\mathbf{w}=\begin{bmatrix}-2\\2+i\end{bmatrix}$$

(b)

$$\mathbf{z} = \begin{bmatrix} 1+i \\ 2i \\ 3-i \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 2-4i \\ 5 \\ 2i \end{bmatrix}$$

Solution

(a) $||\mathbf{z}|| = \sqrt{\mathbf{z}^H\mathbf{z}} = 6$, $||\mathbf{w}|| = 3$, $\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^H\mathbf{z} = -4+4i$, $\langle \mathbf{w}, \mathbf{z} \rangle = -4-4i$.

(b) $||\mathbf{z}|| = 4$, $||\mathbf{w}|| = 7$, $\langle \mathbf{z}, \mathbf{w} \rangle = -4+10i$, $\langle \mathbf{w}, \mathbf{z} \rangle = -4-10i$.

Question 18. Let

$$\mathbf{z}_1 = \begin{bmatrix} \frac{1+i}{2} \\ \frac{1-i}{2} \end{bmatrix}, \mathbf{z}_2 = \begin{bmatrix} \frac{i}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

(a) Show that $\{\mathbf{z}_1, \mathbf{z}_2\}$ is an orthonormal set in $\mathbb{C}^2$.

(b) Write the vector $\mathbf{z} = \begin{bmatrix} 2+4i \\ -2i \end{bmatrix}$ as a linear combination of $\mathbf{z}_1$ and $\mathbf{z}_2$.

Solution

(a) $||\mathbf{z}_1|| = \sqrt{\mathbf{z}_1^H\mathbf{z}_1} = \sqrt{[\frac{1-i}{2}, \frac{1+i}{2}] \begin{bmatrix} \frac{1+i}{2} \\ \frac{1-i}{2} \end{bmatrix}} = \sqrt{\frac{1}{2}+\frac{1}{2}} = 1$, similarly, $||\mathbf{z}_2|| = 1$.

$\langle \mathbf{z}_1, \mathbf{z}_2 \rangle = \mathbf{z}_2^H\mathbf{z}_1 = [\frac{i}{\sqrt{2}}, \frac{-1}{\sqrt{2}}] \cdot \begin{bmatrix} \frac{1+i}{2} \\ \frac{1-i}{2} \end{bmatrix} = 0$. Hence, $\{\mathbf{z}_1, \mathbf{z}_2\}$ is an orthonormal set.

(b) Assume $\mathbf{z} = c_1\mathbf{z}_1 + c_2\mathbf{z}_2$.

$\langle \mathbf{z}, \mathbf{z}_1 \rangle = c_1\langle \mathbf{z}_1, \mathbf{z}_1 \rangle + c_2\langle \mathbf{z}_2, \mathbf{z}_1 \rangle = c_1$, similarly, $\langle \mathbf{z}, \mathbf{z}_2 \rangle = c_2$.

Then we can calculate that $c_1 = 4, c_2 = 2\sqrt{2}$, hence $\mathbf{z} = 4\mathbf{z}_1 + 2\sqrt{2}\mathbf{z}_2$.

Question 19. Let $\{\mathbf{u}_1, \mathbf{u}_2\}$ be an orthonormal basis for $\mathbb{C}^2$, and let $\mathbf{z} = (4+2i)\mathbf{u}_1 + (6-5i)\mathbf{u}_2$.

(a) What are the values of $\mathbf{u}_1^H\mathbf{z}, \mathbf{z}^H\mathbf{u}_1, \mathbf{u}_2^H\mathbf{z}$ and $\mathbf{z}^H\mathbf{u}_2$?

(b) Determine the value of $||\mathbf{z}||$.

Solution

(a) $\mathbf{u}_1^H\mathbf{z} = \langle \mathbf{z}, \mathbf{u}_1 \rangle = 4+2i$

$\mathbf{z}^H\mathbf{u}_1 = \overline{\langle \mathbf{z}, \mathbf{u}_1 \rangle} = 4-2i$

$\mathbf{u}_2^H\mathbf{z} = 6-5i$

$\mathbf{z}^H\mathbf{u}_2 = 6+5i$

(b) $||\mathbf{z}|| = \sqrt{\langle \mathbf{z}, \mathbf{z} \rangle} = 9$

Question 20. Which of the matrices that follow are Hermitian? Normal?

(a)
$$\begin{bmatrix} 1-i & 2 \\ 2 & 3 \end{bmatrix}$$

(b)
$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

(c)
$$\begin{bmatrix} \frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}i \end{bmatrix}$$

Solution

(a) $A^H = \begin{bmatrix} 1+i & 2 \\ 2 & 3 \end{bmatrix}$. Here $A \neq A^H$, hence A is not Hermitian.

$AA^H = \begin{bmatrix} 6 & 8-2i \\ 8+2i & 13 \end{bmatrix}, A^H A = \begin{bmatrix} 6 & 8+2i \\ 8-2i & 13 \end{bmatrix}$. Here $AA^H \neq A^H A$, hence A is not normal.

(b) Not Hermitian but normal.

(c) Not Hermitian and also not normal.

Question 21. Find an orthogonal or unitary diagonalizing matrix for each of the following:

(a)
$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 1 & 3+i \\ 3-i & 4 \end{bmatrix}$$

(c)
$$\begin{bmatrix} 2 & i & 0 \\ -i & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Solution

(a) $\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (\lambda-1)(\lambda-3), \lambda_1 = 1, \lambda_2 = 3, D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

The corresponding eigenvector is $\mathbf{x}_1 = (1,-1)^T, \mathbf{x}_2 = (1,1)^T$.

$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \times (1,-1)^T, \mathbf{u}_2 = \frac{1}{\sqrt{2}} \times (1,1)^T, U = [\mathbf{u}_1, \mathbf{u}_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

(b) $D = \begin{bmatrix} -1 & 0 \\ 0 & 6 \end{bmatrix}, U = \begin{bmatrix} \frac{3+i}{\sqrt{14}} & \frac{3+i}{\sqrt{35}} \\ \frac{-2}{\sqrt{14}} & \frac{5}{\sqrt{35}} \end{bmatrix}$.

(c) $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, U = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}$.

Question 22. Let U be a unitary matrix. Prove that

(a) U is normal.

(b) $||U\mathbf{x}|| = ||\mathbf{x}||$ for all $\mathbf{x} \in \mathbb{C}^n$.

(c) if λ is an eigenvalue of U, then $|\lambda| = 1$.

Solution

(a) Since U is a unitary matrix, $U^H U = I$, then $U^{-1} = U^H, U^H U = UU^H = I$, U is normal.

(b) By $U^H U = I$, we have $||U\mathbf{x}||^2 = \langle U\mathbf{x}, U\mathbf{x}\rangle = (U\mathbf{x})^H U\mathbf{x} = \mathbf{x}^H U^H U\mathbf{x} = \mathbf{x}^H I\mathbf{x} = \mathbf{x}^H\mathbf{x} = \langle \mathbf{x}, \mathbf{x}\rangle = ||\mathbf{x}||^2$.Therefore, $||U\mathbf{x}|| = ||\mathbf{x}||$.

(c) Let λ be an eigenvalue of U and let $\mathbf{x}$ be an eigenvector belonging to λ, then $||U\mathbf{x}|| = ||\lambda\mathbf{x}|| = |\lambda|||\mathbf{x}||$. By (b), $||\mathbf{x}|| = |\lambda|||\mathbf{x}||$. Since $||\mathbf{x}|| \neq 0$, $|\lambda| = 1$.

Question 23. Let $\mathbf{u}$ be a unit vector in $\mathbb{C}^n$ and define $U = I - 2\mathbf{u}\mathbf{u}^H$. Show that U is both unitary and Hermitian and, consequently, is its own inverse.

Solution

Hermitian: $U^H = (I - 2\mathbf{u}\mathbf{u}^H)^H = I - 2(\mathbf{u}^H)^H\mathbf{u}^H = I - 2\mathbf{u}\mathbf{u}^H = U$.

Unitary: $U^H U = UU = (I - 2\mathbf{u}\mathbf{u}^H)(I - 2\mathbf{u}\mathbf{u}^H) = I - 4\mathbf{u}\mathbf{u}^H + 4\mathbf{u}\mathbf{u}^H = I$, hence $U = U^{-1}$.

Question 24. find the singular value decomposition of each of the following matrices:

(a)

$$\begin{bmatrix} 2 & -2 \\ 1 & 2 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solution

(a) First, find the transpose of u, v. Then, find the matrix of singular decomposition basis.

$$A = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Question 25. Find the matrix associated with each of the following quadratic forms:

(a) $3x^2 - 5xy + y^2$

(b) $2x^2 + 3y^2 + z^2 + xy - 2xz + 3yz$

Solution

(a) $\begin{bmatrix} 3 & -\frac{5}{2} \\ -\frac{5}{2} & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & \frac{1}{2} & -1 \\ \frac{1}{2} & 3 & \frac{3}{2} \\ -1 & \frac{3}{2} & 1 \end{bmatrix}$

Question 26. Show that if A is a symmetric positive definite matrix, then A is nonsingular and A^{-1} is also positive definite.

Solution

Let A be $n \times n$ symmetric positive definite matrix, we have that, $det(A) > 0$, therefore A is nonsingular. Since $AA^{-1} = I_n$, we have that $(AA^{-1})^T = (A^{-1})^T A^T = (I_n)^T = I_n$. Therefore $(A^{-1})^T = (A^T)^{-1}$. Since A is symmetric we have that $A = A^T$ and $(A^{-1})^T = A^{-1}$ and therefore A^{-1} is symmetric. Since A is a positive definite matrix, all eigenvalues of A are positive and the eigenvalues of A^{-1} are the inverses of the eigenvalues of A. Hence since A^{-1} is symmetric we have that A is nonsingular and positive definite.