Assignment 3

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Please note that

- **Released date**: 1st Mar.
- **Due date**: 9th Mar. by 11:59pm.
- Late submission is **NOT** accepted.
- Please submit your answers as a PDF file with a name like `"118010XXX_HW7.pdf"` (Your student ID + HW No.). You may either typeset you answers directly using computers, or scan your handwritten answers. (We recommend you use the printers on campus to scan. If you use your smartphone to scan, please limit the file size $\leq$ 10MB).
- Please make sure that your submitted file is clear and readable. Submitted file that can not be opened or not readable will get 0 point.

Question 1. Consider the vectors $\boldsymbol{x}_1 = (8,6)^T$ and $\boldsymbol{x}_2 = (4,-1)^T$ in $\mathbb{R}^2$.

(a) Determine the length of each vector.

(b) Let $\boldsymbol{x}_3 = \boldsymbol{x}_1 + \boldsymbol{x}_2$. Determine the length of $\boldsymbol{x}_3$. How does its length compare with the sum of the lengths of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$?

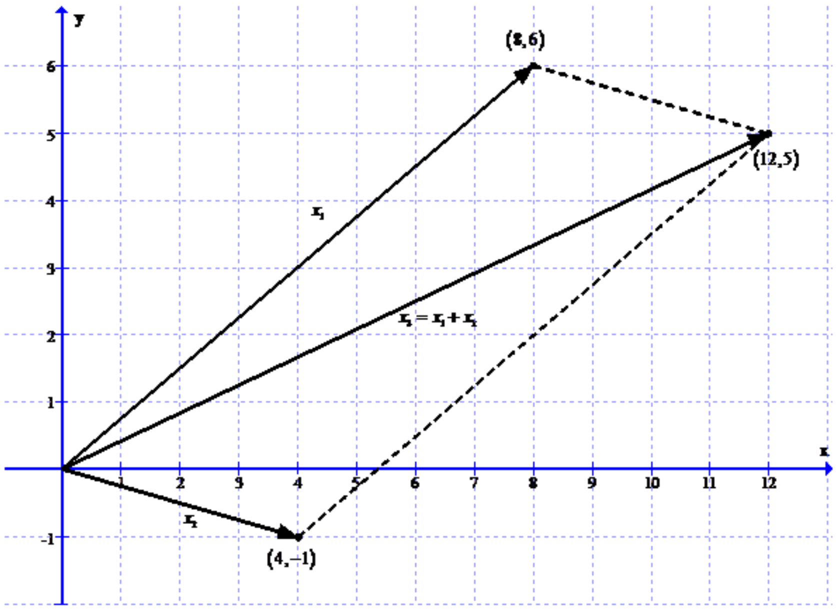
(c) Draw a graph illustrating how $\boldsymbol{x}_3$ can be constructed geometrically using $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Use this graph to give a geometrical interpretation of your answer to the question in part (b).

Solution

(a) $\|\boldsymbol{x}_1\| = \sqrt{8^2+6^2} = 10$, $\|\boldsymbol{x}_2\| = \sqrt{4^2+(-1)^2} = \sqrt{17}$

(b) $\boldsymbol{x}_3 = (12,5)^T$, $\|\boldsymbol{x}_3\| = \sqrt{12^2+5^2} = 13$, $\|\boldsymbol{x}_3 < \|\boldsymbol{x}_1\| + \|\boldsymbol{x}_2\|$

(c) Here both $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are placed at the origin and a parallelogram is formed. The line segment from $(0,0)$ to $(8,6)$ represents the length of $\boldsymbol{x}_1$ and the line segment from $(0,0)$ to $(4,-1)$ represents the length of



$\boldsymbol{x}_2$. Hence the diagonal of the parallelogram is $\boldsymbol{x}_3 = \boldsymbol{x}_1 + \boldsymbol{x}_2$.

Question 2. Let C be the set of complex numbers. Define addition on C by

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

and define scalar multiplication by

$$\alpha(a + bi) = \alpha a + \alpha bi$$

for all real numbers α. Show that C is a vector space with these operations.

Solution

The vector space must follow the following axioms (if they are real vector spaces in C):
For all x, y, z in C, and A, B in R
1) $x + y = y + x$
2) $(x + y) + z = x + (y + z)$
3) There exists a 0 in C such that $0 + x = x$
4) For each x in C, there exists $a - x$ in C such that $-x + x = 0$
5) $\alpha(x + y) = \alpha x + \alpha y$
6) $(\alpha + \beta)x = \alpha x + \beta x$
7) $(\alpha\beta)x = \alpha(\beta x)$
8) $1.x = x$
To show that C is a vector space we must show that all eight axioms are

satisfied. Here I just show the first two proofs for example.

1)

We must show that addition, as defined by (1), is commutative.

Let $x = a + bi$ and $y = c + di$ be an arbitrary pair of complex numbers. Here a, b, c, d are arbitrary real numbers. We need to show $x + y = y + x$. We have

$$\begin{aligned} x + y &= (a + bi) + (c + di) \\ &= (a + c) + (b + d)i \\ &= (c + a) + (d + b)i \\ &= (c + di) + (a + bi) \\ &= y + x \end{aligned}$$

Since this equality holds for all $x, y \in C$, we have established that addition, as defined by (1) is commutative.

2)

To show that addition, as defined by (1) is associative. Let, $x = a + bi, y = c + di$ and $z = p + qi$ be arbitrary complex numbers. We need to show $x + (y + z) = (x + y) + z$. We have

$$\begin{aligned} x + (y + z) &= x + ((c + di) + (p + qi)) \\ &= (a + bi) + ((c + p) + (d + q)i) \\ &= (a + (c + p)) + (b + (d + q))i \\ &= ((a + c) + p) + ((b + d) + q)i \end{aligned}$$

On the other hand,

$$\begin{aligned} (x + y) + z &= ((a + bi) + (c + di)) + z \\ &= ((a + c) + (b + d)i) + (p + qi) \\ &= ((a + c) + p) + ((b + d) + q)i \end{aligned}$$

Thus, to shown that $x + (y + z) = (x + y) + z$ for all $x, y, z \in C$. Thus addition, as defined by (1), is associative.

Question 3. Let S be the set of all ordered pairs of real numbers. Define scalar multiplication and addition on S by

$$\alpha\,(x_1, x_2) = (\alpha x_1, \alpha x_2)$$
$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, 0)$$

We use the symbol $\oplus$ to denote the addition operation for this system in order to avoid confusion with the usual addition $\mathbf{x}+\mathbf{y}$ of row vectors. Show that S, together with the ordinary scalar multiplication and the addition operation $\oplus$, is not a vector space. Which of the eight axioms fail to hold?

Solution

To show that S is not a vector space it is enough to show that at least one of the axioms of vector space is fail.
Check identity property. Let $(x_1, x_2) \in S$. Consider $(x_1, x_2) \oplus (0, 0)$

$$\begin{aligned}(x_1, x_2) \oplus (0, 0) &= (x_1 + 0, 0)\\ &= (x_1, 0)\\ &\neq (x_1, x_2)\end{aligned}$$

So, the set S does not satisfy the identity property. Hence, S is not a vector space.

Question 4. Let S denote the set of all infinite sequences of real numbers with scalar multiplication and addition defined by

$$\begin{aligned}\alpha\{a_n\} &= \{\alpha a_n\}\\ \{a_n\} + \{b_n\} &= \{a_n + b_n\}\end{aligned}$$

Show that S is a vector space.

Solution

Still need to prove 8 axioms. Here I show two examples.
5) Let $\alpha \in S$ and $a_n, b_n \in S$. Then

$$\begin{aligned}\alpha\{a_n + b_n\} &= \{\alpha(a_n + b_n)\}\\ &= \{\alpha a_n + \alpha b_n\}\\ &= \{\alpha a_n\} + \{\alpha b_n\}\\ &= \alpha\{a_n\} + \alpha\{b_n\}\end{aligned}$$

So, $\alpha\{a_n + b_n\} = \alpha\{a_n\} + \alpha\{b_n\}$ Hence axiom 5 is satisfied.
6) Let $\alpha, \beta, a_n \in S$, then

$$\begin{aligned}(\alpha + \beta)\{a_n\} &= \{(\alpha + \beta)a_n\}\\ &= \{\alpha a_n + \beta a_n\}\\ &= \{\alpha a_n\} + \{\beta a_n\}\\ &= \alpha\{a_n\} + \beta\{a_n\}\end{aligned}$$

Hence axiom 6 is satisfied.

Question 5. We can define a one-to-one correspondence between the elements of $V = \{\text{all polynomials with degree} < n\}$ and $\mathbb{R}^n$ by

$$\begin{aligned} p(x) &= a_1 + a_2 x + \cdots + a_n x^{n-1} \\ &\leftrightarrow (a_1, \ldots, a_n)^T = \mathbf{a} \end{aligned}$$

Show that if $p \leftrightarrow \mathbf{a}$ and $q \leftrightarrow \mathbf{b}$, then

(a) $\alpha p \leftrightarrow \alpha \mathbf{a}$ for any scalar α.

(b) $p + q \leftrightarrow \mathbf{a} + \mathbf{b}$.

[In general, two vector spaces are said to be i somorphic if their elements can be put into a one-toone correspondence that is preserved under scalar multiplication and addition as in (a) and (b).]

Solution

(a) Let α be any scalar and $p(x) = a_1 + a_2 x + \cdots + a_n x^{n-1} \in V$.

Then, $(\alpha p)(x)$ can be written as follows:

$$\begin{aligned} (\alpha p)(x) &= \alpha p(x) \\ &= \alpha \left(a_1 + a_2 x + \cdots + a_n x^{n-1}\right) \\ &= \alpha a_1 + \alpha a_2 x + \cdots + \alpha a_n x^{n-1} \\ &\leftrightarrow (\alpha a_1, \alpha a_2 x, \ldots, \alpha a_n) \\ &= \alpha (a_1, a_2, \ldots, a_n) \\ &= \alpha \mathbf{a} \end{aligned}$$

Therefore, $\alpha p \leftrightarrow \alpha \mathbf{a}$, for any scalar α.

(b) Let $p(x) = a_1 + a_2 x + \cdots + a_n x^{n-1}, q(x) = b_1 + b_2 x + \cdots + b_n x^{n-1} \in V$.

Then, $(p + q)(x)$ can be written as follows:

$$\begin{aligned} (p+q)(x) &= p(x) + q(x) \\ &= \left(a_1 + a_2 x + \cdots + a_n x^{n-1}\right) + \left(b_1 + b_2 x + \cdots + b_n x^{n-1}\right) \\ &= a_1 + a_2 x + \cdots + a_n x^{n-1} + b_1 + b_2 x + \cdots + b_n x^{n-1} \\ &= (a_1 + b_1) + (a_2 + b_2) x + \cdots + (a_n + b_n) x^{n-1} \\ &\leftrightarrow ((a_1 + b_1), (a_2 + b_2), \ldots, (a_n + b_n)) \\ &= (a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) \\ &= \mathbf{a} + \mathbf{b} \end{aligned}$$

Therefore, $p + q \leftrightarrow \mathbf{a} + \mathbf{b}$, for all $p, q \in V$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$.

Question 6. Determine whether the following are subspaces of $\mathbb{R}^{2\times 2}$:

(a) The set of all 2×2 diagonal matrices

(b) The set of all 2×2 triangular matrices

(c) The set of all 2×2 lower triangular matrices

(d) The set of all 2×2 matrices A such that $a_{12} = 1$

(e) The set of all 2×2 matrices B such that $b_{11} = 0$

(f) The set of all symmetric 2×2 matrices

(g) The set of all singular 2×2 matrices

Solution

(a), (c), (e), and (f) are subspaces; (b), (d), and (g) are not.

Question 7. Determine whether the following are subspaces of vector space

$V = \{\text{all polynomials with degree} < 4\}$ (be careful!):

(a) The set of polynomials in V of even degree

(b) The set of all polynomials of degree 3

(c) The set of all polynomials $p(x)$ in V such that $p(0) = 0$

(d) The set of all polynomials in V having at least one real root

Solution

Only the set in part (c) is a subspace of V.

Question 8. Determine whether the following are subspaces of $C[-1, 1]$:

(a) The set of functions f in $C[-1, 1]$ such that $f(-1) = f(1)$

(b) The set of odd functions in $C[-1, 1]$

(c) The set of continuous nondecreasing functions on $[-1, 1]$

(d) The set of functions f in $C[-1, 1]$ such that $f(-1) = 0$ and $f(1) = 0$

(e) The set of functions f in $C[-1, 1]$ such that $f(-1) = 0$ or $f(1) = 0$

Solution

Firstly, we should know some basic definitions about the subspaces:

If Si s a nonempty subset of a vector space V, and S satisfies the conditions:

(i) $\alpha x \in S$ whenever $x \in S$ for any scalar α

(ii) $x + y \in S$ whenever $x \in S$ and $y \in S$;
Then S is said to be a subspace of V.
(a) Suppose that A is a set of functions f in $C[-1, 1]$ such that $f(-1) = f(1)$. Now that if we have two functions f and g in A, then we have:

$$(f + g)(-1) = f(-1) + g(-1) = f(1) + g(1) = (f + g)(1)$$

therefore, the function $(f + g)$ is in the set A.
And we also have:

$$\alpha f(-1) = \alpha f(1)$$

hence, αf is also in A.
So from the above definition, this set of functions is a subspace of $C[-1, 1]$.
(b) For a odd functions f, we have

$$f(-x) = -f(x)$$

similar with (a), we assume another function in this odd functions set that $g(-x) = -g(x)$
then we have:

$$(f + g)(-x) = f(-x) + g(-x) = -f(x) - g(x) = -(f + g)(x)$$

and:

$$\alpha f(-x) = -\alpha f(x)$$

so $(f + g)$ and αf are also in this set.
Hence, the set of odd functions is a subspace of $C[-1, 1]$.
(c) For a set of non-decreasing functions, we can take two examples of non-decreasing functions f and g such that:

$$f(x) = x \quad g(x) = x$$

so we have:

$$(f + g)(x) = f(x) + g(x) = x + x = 2x$$

and:

$$\alpha f(x) = \alpha x$$

since α can be any scalar, if we choose α is -1, then

$$\alpha f(x) = -x$$

it's not a non-decreasing functions.
So αf are not in this set.
Hnce the set of non-decreasing functions are not the subspace of $C[-1, 1]$.
(d) If we have two functions f and g in A such that:

$$f(-1) = 0 \quad and \quad f(1) = 0$$

$$g(-1) = 0 \quad and \quad g(1) = 0$$

Now,

$$(f + g)(-1) = f(-1) + g(-1) = 0 + 0 = 0$$

$$(f + g)(1) = f(1) + g(1) = 0 + 0 = 0$$

and

$$\alpha f(-1) = \alpha 0 = 0$$

$$\alpha f(1) = \alpha 0 = 0$$

so $(f + g)$ and αf are also in this set.
Hence, the set of functions is a subspace of $C[-1, 1]$.
(e) Take two functions f and g in this set such that: $f(-1)$=0 and $f(1) \neq 0$, and $g(-1) \neq 0$ and $g(1) = 0$,
then we have:

$$(f + g)(-1) = f(-1) + g(-1) = g(-1) \neq 0$$

and

$$(f + g)(1) = f(1) + g(1) = f(1) \neq 0$$

Therefore, the $(f + g)$ is not in this set, so this set of functions are not a subspace of $C[-1, 1]$.

Question 9. Given

$$\mathbf{x}_1 = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} 2 \\ 6 \\ 6 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -9 \\ -2 \\ 5 \end{pmatrix}$$

(a) Is $\mathbf{x} \in \operatorname{Span}(\mathbf{x}_1, \mathbf{x}_2)$?

(b) Is $\mathbf{y} \in \text{Span}\,(\mathbf{x}_1, \mathbf{x}_2)$?

Prove your answers.

Solution

For problems which we want to determine whether a vector belongs to a span, we can use **non-all-zero** linear combination of this span to represent this vector.
(a) if we solve the linear system:

$$ax_1 + bx_2 = x$$

we cann't get the solution, so vector x is not belongs to $Span\{x_1, x_2\}$.
(b) Similarly, we can find $y \in Span\{x_1, x_2\}$. and $y = 3x_1 - 2x_3$

Question 10. Let A be a 4×3 matrix and let

$$\mathbf{c} = 2\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3$$

(a) If $N(A) = \{\mathbf{0}\}$, what can you conclude about the solutions to the linear system $A\mathbf{x} = \mathbf{c}$?

(b) If $N(A) \neq \{\mathbf{0}\}$, how many solutions will the system $A\mathbf{x} = \mathbf{c}$ have?

Explain.

Solution

(a) If $N(A) = \{0\}$, we can know that 0 is the only solution of the system $Ax = 0$. Therefore, there aren't any other solution of the system $Ax = c$, except $(2, 1, 1)^T$
(b) If $N(A) \neq \{0\}$, we can know that the system $Ax = 0$ has a solution $y \neq 0$. Thus, any vector of the form $(2, 1, 1)^T + \alpha y$, where $\alpha \in R$ is a solution of the system $Ax = 0$.
Therefore, there are infinitely many solutions of the system $Ax = c$.

Question 11. Which of the following sets span the polynomial space P_3=\{all polynomials with degree ≤ 3\} ? Justify your answers.

(a) $\{1, x^2, x^2 - 2\}$

(b) $\{2, x^2, x, 2x + 3\}$

(c) $\{x + 2, x + 1, x^2 - 1\}$

(d) $\{x + 2, x^2 - 1\}$

Solution

(a) Determine whether it's possible to find constants α_1, α_2 and α_3 such that:

$$\alpha_1 \cdot 1 + \alpha_2 \cdot x^2 + \alpha_3 \cdot (x^2 - 1) = c_1 + c_2 x + c_3 x^2$$

we can get that:

$$\begin{aligned} \alpha_1 - 2\alpha_3 &= c_1 \\ \alpha_2 + \alpha_3 &= c_3 \\ 0 &= c_2 \end{aligned}$$

If $c_2 \neq 0$, then the system of equations has no solution. Therefore, all polynomials in P_3 are not in $Span\{1, x^2, x^2 - 2\}$.
Therefore, this sets are not a spanning set for P_3.
(b) Similarly, let $B = \{2, x^2, x, 2x + 3\}$, we determine the linear combination of

$$\alpha_1 \cdot 2 + \alpha_2 x^2 + \alpha_3 x + \alpha_4 (2x + 3) = c_1 + c_2 x + c_3 x^2$$

we can get that:

$$\begin{aligned} \alpha_1 &= \frac{1}{2} c_1 - \frac{3}{2} \alpha_4 \\ \alpha_2 &= c_3 \\ \alpha_3 &= c_2 - 2\alpha_4 \end{aligned}$$

and α_4 is free.
Therefore, B is a spanning set for P_3.
(c) Similarly, we can get that:

$$\begin{aligned} \alpha_1 &= c_1 - c_2 + c_3 \\ \alpha_2 &= -c_1 + 2c_2 - c_3 \\ \alpha_3 &= c_3 \end{aligned}$$

Therefore, this set is a spanning set for P_3.
(d) Similarly, we can get that:

$$\begin{aligned} 2\alpha_1 - \alpha_2 &= c_1 \\ \alpha_1 &= c_2 \\ \alpha_2 &= c_3 \end{aligned}$$

If $c_1 \neq 2c_2 - c_3$, then this system has no solution.
Therefore, this set is not a spanning set for P_3.

Question 12. Prove that if S is a subspace of $\mathbb{R}^1$, then either $S = \{\mathbf{0}\}$ or $S = \mathbb{R}^1$.

Solution

Assume that a non-zero vector be $\mathrm{x} = \{x_1\}$ in $\mathbb{R}^1$.

Assume some cases.

Case 1:

When $S = \{0\}$

(i)

For any scalar value $\alpha, \alpha \cdot 0 = 0 \in S$ So, $\alpha\mathrm{x} \in S$ Whenever $\mathrm{x} \in S$ for any scalar α.

(ii)

$$0 + 0 = 0 \in S$$

So, $\mathrm{x} + \mathrm{y} \in S$, whenever $\mathrm{x} \in S$ and $y \in S$. This proves S is a subspace of $\mathbb{R}^1$.

Case 2:

When $\mathrm{S} = \{x_1, \mathrm{x}_2\}$

(i) For any scalar value $\alpha, \alpha \cdot x_1 = \alpha x_1 \notin S$.

(ii)

$$x_1 + x_2 \notin S$$

This gives that S is not a subspace of $\mathbb{R}^1$.

Case 3:

When $S = \mathbb{R}^1$

(i) For any scalar value $\alpha, \alpha \cdot x = \alpha x \in S$ So, $\alpha\mathrm{x} \in S$ whenever $\mathrm{x} \in S$ for any scalar α.

(ii)

$$x + y \in S$$

So, $\mathrm{x} + \mathrm{y} \in S$, whenever $\mathrm{x} \in S$ and $y \in S$. This proves S is a subspace of $\mathbb{R}^1$.

So, in the above cases we see, that S is a subspace of $\mathbb{R}^1$ only when either $\mathbf{S} = \{0\}$ or $\mathbf{S} = \mathbb{R}^1$.

In other cases S is not a subspace of $\mathbb{R}^1$.

Question 13. Let U and V be subspaces of a vector space W. Prove that their intersection $U \cap V$ is also a subspace of W

Solution

Given U and V are subspaces of a vector space W.

We need to prove that $U \cap V$ is a subspace of W.

If S is a nonempty subset of a vector space V, and S satisfies the conditions
(i) $\alpha x \in S$ whenever $x \in S$ for any scalar α
(ii) $x + y \in S$ whenever $x \in S$ and $y \in S$
Then S is said to be a subspace of V.
Let α be a scalar and the vectors x and y are elements of both $U \cap V$.
Then $x, y \in U$ and $x, y \in V$.
Since U and V are subspaces it follows that

$$\alpha x \in U \text{ and } x + y \in U$$

$\alpha x \in V$ and $x + y \in V$
Therefore $\alpha x \in U \cap V$ and $x + y \in U \cap V$
Thus $U \cap V$ is a subspace of W.

Question 14. Determine whether the following vectors are linearly independent in $\mathbb{R}^3$:

$$(a) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad (b) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$(c) = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \qquad (d) = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -4 \end{bmatrix}$$

$$(e) = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Solution

(a) Suppose there are scalars c_1, c_2, c_3 such that

$$c_1(1,0,0)^T + c_2(0,1,1)^T + c_3(1,0,1)^T = 0$$

rewritten the equation as a linear system:

$$\begin{aligned} c_1 + c_3 &= 0 \\ c_2 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

Apply Gaussian elimination to reduce to row echelon form, we obtain the unique solution

$$c_1 = c_2 = c_3 = 0$$

Thus these vector are linear independent.

(b) Suppose there are scalars c_1, c_2, c_3, c_4 such that

$$c_1(1,0,0)^T + c_2(0,1,1)^T + c_3(1,0,1)^T + c_4(1,2,3)^T = 0$$

then

$$\begin{aligned} c_1 + c_3 + c_4 &= 0 \\ c_2 + 2c_4 &= 0 \\ c_2 + c_3 + 3c_4 &= 0 \end{aligned}$$

Apply Gaussian elimination to reduce to row echelon form, since c_4 is free variable, thus these vectors are linear dependent.

(c) Suppose there are scalars c_1, c_2, c_3 such that

$$c_1(2,1,-2)^T + c_2(3,2,-2)^T + c_3(2,2,0)^T = 0$$

Then:

$$\begin{aligned} 2c_1 + 3c_2 + 2c_3 &= 0 \\ c_1 + 2c_2 + 2c_2 &= 0 \\ -2c_1 - 2c_2 &= 0 \end{aligned}$$

Apply Gaussian elimination to reduce to row echelon form, since c_3 is free vaiable, thus these vectors are linear dependent.

(d) Suppose there are scalars c_1, c_2, c_3 such that

$$c_1(2,1,-2)^T + c_2(-2,-1,2)^T + c_3(4,2,-4)^T = 0$$

rewritten the equation as a linear system:

$$\begin{aligned} 2c_1 - 2c_2 + 4c_3 &= 0 \\ c_1 - c_2 + 2c_2 &= 0 \\ -2c_1 + 2c_2 - 4c_3 &= 0 \end{aligned}$$

Apply Gaussian elimination to reduce to row echelon form, since $c-2, c_3$ are free variables, thus these vectors are linear dependent.

(e) Suppose there are scalars c_1, c_2 such that

$$c_1(1,1,3)^T + c_2(0,2,1)^T = 0$$

rewritten the equation as a linear system:

$$\begin{aligned} c_1 + c_3 &= 0 \\ c_2 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

Apply Gaussian elimination to reduce to row echelon form, we obtain the unique solution

$$c_1 = c_2 = 0$$

Thus these vectors are linear independent.

Question 15. For each of the sets of vectors in Question 14, describe geometrically the span of the given vectors.

Solution

(a) and (b) are all of 3-space;

(c) is a plane through (0, 0, 0);

(d) is a line through (0, 0, 0);

(e) is a plane through (0, 0, 0)

Question 16. Determine whether the following vectors are linearly independent in $\mathbb{R}^{2\times 2}$:

(a) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$

Solution

(a) Suppose there are scalars c_1, c_2 such that

$$c_1 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} c_1 & c_2 \\ c_1 & c_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Equating entries leads to the system

$$c_1 = c_2 = 0$$

Thus these matrices are linear independent.

(b) Suppose there are scalars c_1, c_2, c_3 such that

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} c_1 & c_2 \\ c_3 & c_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Equating entries leads to the system

$$c_1 = c_2 = c_3 = 0$$

Thus these matrices are linear independent.

(c) Suppose there are scalars c_1, c_2, c_3 such that

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = 0$$

$$\begin{bmatrix} c_1 + 2c_3 & c_2 + 3c_3 \\ 0 & c_1 + 2c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Equating entries leads to the system

$$c_1 + 2c_3 = 0$$

$$c_2 + 3c_3 = 0$$

c_3 is free variable. Thus these matrices are linear dependent.

Question 17. Determine whether the following vectors are linearly independent in $P3$:

(a) $1, x^2, x^2 - 2$

(b) $2, x^2, x, 2x + 3$

(c) $x + 2, x + 1, x^2 - 1$

(d) $x + 2, x^2 - 1$

Solution

(a) Suppose there are scalars c_1, c_2, c_3 such that

$$c_1 * 1 + c_2 * x^2 + c_3 * (x^2 - 2) = 0$$
$$(c_3 + c_2) * x^2 + c_1 - 2c_3 = 0$$

Equating coefficients leads to the system

$$c_2 + c_3 = 0$$
$$c_1 - 2c_3 = 0$$

Thus c_3 are free variable, these vectors are **linear dependent**.

(b) Suppose there are scalars c_1, c_2, c_3, c_4 such that

$$c_1 * 2 + c_2 * x^2 + c_3 * x + c_4 * (2x + 3) = 0$$
$$c_2 x^2 + (c_3 + 2c_4)x + 2c_1 + 3c_4 = 0$$

Equating coefficients leads to the system

$$c_2 = 0$$
$$c_3 + 2c_4 = 0$$
$$2c_1 + 3c_4 = 0$$

Thus c_4 are free variable, these vectors are **linear dependent**.

(c) Suppose there are scalars c_1, c_2, c_3 such that

$$c_1 * (x + 2) + c_2 * (x + 1) + c_3 * (x^2 - 1) = 0$$
$$c_3 * x^2 + (c_1 + c_2) * x + 2c_3 + c_2 - c_3 = 0$$

Equating coefficients leads to the system

$$c_3 = 0$$
$$c_1 + c_2 = 0$$
$$2c_1 + c_2 = 0$$

Thus $c_1 = c_2 = c_3 = 0$ are free variable, these vectors are **linear independent**.

(b) Suppose there are scalars $c_1, c_2, ..., c_k$ such that

$$c_1 * (x + 2) + c_2 * (x^2 - 1) = 0$$
$$c - 2 * x^2 + c_1 * x - 2c_1 - c_2 = 0$$

Equating coefficients leads to the system

$$c_2 = 0$$
$$c_1 = 0$$

Thus $c_1 = c_2 = 0$ are free variable, these vectors are **linear independent**.

Question 18. Let $x_1, ..., x_k$ be linearly independent vectors in $\mathbb{R}^n$, and let A be a nonsingular $n \times n$ matrix. Define $y_i = Ax_i$ for $i = 1, ..., k$. Show that $y_1, ..., y_k$ are linearly independent.

Solution

Proof. Suppose there are scalars $c_1, c_2, ..., c_k$ such that

$$c_1 y_1 + c_2 y_2 + ... + c_k y_k = 0$$

then

$$c_1 y_1 + c_2 y_2 + ... + c_k y_k = 0$$
$$c_1 A x_1 + c_2 A x_2 + ... + c_k A x_k = 0$$
$$A(c_1 x_1 + c_2 x_2 + ... + c_k x_k) = 0$$

Since A is nonsingular, the only solution for $Ax = 0$ is the trivial solution $x = 0$, Hence,

$$c_1 x_1 + c_2 x_2 + ... + c_k x_k = 0$$

Since $x_1, x_2, ...x_k$ be linear independent, this implies

$$c_1 = c_2 = ... = c_k = 0$$

Thus, $y_1, y_2, ..., y_k$ are linear independent.

Question 19. Let $\{v_1, ..., v_n\}$ be spanning set for the vector space V, and let v be any other vector in V. Show that $v, v_1, ..., v_n$ are linearly dependent.

Solution

Proof. Since $\{v_1, ..., v_n\}$ is a spanning set for V, every vector in V can be written as a linear combination of $v_1, ..., v_n$. Thus $v = c_1v_1 + ..., c_nv_n$ for some scalars $c_1, ..., c_n$. But then

$$(-1)v + c_1v_1 + ...c_nv_n = 0$$

Therefore $v, v_1, ..., v_n$ are linear dependent.

Question 20. Let $v_1, v_2, ..., v_n$ be linearly independent vectors in vector space V. Show that $v_2, ..., v_n$ cannot span V.

Solution

Proof. Since $v_1, v_2, ..., v_n$ are linear independent vectors in V, if

$$c_1v_1 + c_2v_2 + ...c_nv_n = 0$$

then all the scalars $c_1, ..., c_n$ must equal 0. Thus there is no way to write

$$v_1 = c_2v_2 + ...c_nv_n$$

In other words, v_1 cannot be written as a linear combination of $v_2, ..., v_n$. Therefore, $v_2, ..., v_n$ cannot span V.