Assignment 7

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Please note that

- **Released date**: 21st Apr.
- **Due date**: 2nd May. by 11:59pm.
- Late submission is **NOT** accepted.
- Please submit your answers as a PDF file with a name like `"118010XXX_HW7.pdf"` (Your student ID + HW No.). You may either typeset you answers directly using computers, or scan your handwritten answers. (We recommend you use the printers on campus to scan. If you use your smartphone to scan, please limit the file size $\leq$ 10MB).
- Please make sure that your submitted file is clear and readable. Submitted file that can not be opened or not readable will get 0 point.

Question 1. Find the angle between the vectors $\mathbf{v}$ and $\mathbf{w}$:

$$\mathbf{v} = (2,1,3)^T, \mathbf{w} = (6,3,9)^T$$

Solution

The angle θ between two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ is given by

$$\begin{aligned}\cos\theta &= \frac{\mathbf{v}^T\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}, \quad 0 \leq \theta \leq \pi \\ &= 1\end{aligned}$$

So $\theta = 0^\circ$

Question 2. For the following pair of vectors $\mathbf{x}$ and $\mathbf{y}$, find the vector projection $\mathbf{p}$ of $\mathbf{x}$ onto $\mathbf{y}$ and verify that $\mathbf{p}$ and $\mathbf{x} - \mathbf{p}$ are orthogonal:

$$\mathbf{x} = (3,5)^T, \mathbf{y} = (1,1)^T$$

Solution

Vector projection of $\mathbf{x}$ onto $\mathbf{y}$:

$$\mathbf{p} = \frac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\mathbf{y} = (4, 4)^T$$

$$\mathbf{x} - \mathbf{p} = (3, 5)^T - (4, 4)^T = (-1, 1)^T$$

$$\mathbf{p}^T(\mathbf{x} - \mathbf{p}) = 0$$

Question 3. Let $\mathbf{x}$ and $\mathbf{y}$ be linearly independent vectors in $\mathbb{R}^2$. If $\|\mathbf{x}\| = 2$ and $\|\mathbf{y}\| = 3$, what, if anything, can we conclude about the possible values of $\left|\mathbf{x}^T\mathbf{y}\right|$?

Solution

If x and y are linearly independent and θ is the angle between the vectors, then $|\cos\theta| < 1$ and hence

$$\left|\mathrm{x}^T\mathrm{y}\right| = \|\mathrm{x}\|\|\mathrm{y}\||\cos\theta| \le 6$$

Question 4. If $\mathbf{x} = (x_1, x_2)^T, \mathbf{y} = (y_1, y_2)^T$, and $\mathbf{z} = (z_1, z_2)^T$ are arbitrary vectors in $\mathbb{R}^2$, prove that:

(a) $\mathbf{x}^T\mathbf{x} \ge 0$

(b) $\mathbf{x}^T\mathbf{y} = \mathbf{y}^T\mathbf{x}$

(c) $\mathbf{x}^T(\mathbf{y} + \mathbf{z}) = \mathbf{x}^T\mathbf{y} + \mathbf{x}^T\mathbf{z}$

Solution

(a) $\mathbf{x}^T\mathbf{x} = \mathbf{x}_1^2 + \mathbf{x}_2^2 \ge 0$

(b) $\mathbf{x}^T\mathbf{y} = \mathbf{x}_1\mathbf{y}_1 + \mathbf{x}_2\mathbf{y}_2 = \mathbf{y}^T\mathbf{x}$

(c) $\mathbf{x}^T(\mathbf{y} + \mathbf{z}) = \mathbf{x}_1\mathbf{y}_1 + \mathbf{x}_1\mathbf{z}_1 + \mathbf{x}_2\mathbf{y}_2 + \mathbf{x}_2\mathbf{z}_2 = \mathbf{x}^T\mathbf{y} + \mathbf{x}^T\mathbf{z}$

Question 5. For the following matrix, determine a basis for each of the subspaces $Row(A^T)$, $Null(A)$, $Row(A)$, and $Null(A^T)$:

$$A = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}$$

Solution

We can find bases for $Row(A)$ and $Null(A)$ by transforming A into reduced row echelon form:

$$\begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix} \to \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix}$$

So $\{3,4\}^T$ form a basis for the row space of A. if $x \in Null(A)$, it follows that:

$$3\mathbf{x}_1 + 4\mathbf{x}_2 = 0$$

Thus $(-4,3)^T$ form a basis for $Null(A)$
We can also find bases for $Row(A^T)$ and $Null(A^T)$ by transforming A^T into reduced row echelon form:

$$\begin{pmatrix} 3 & 6 \\ 4 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

Thus, $(1,2)^T$ form a basis for the row space of A^T. if $x \in Null(A^T)$, it follows that:

$$\mathbf{x}_1 + 2\mathbf{x}_2 = 0$$

Thus $(-2,1)^T$ form a basis for $Null(A^T)$
Conclude:
$\left\{(3,4)^T\right\}$ basis for $Row(A)$,
$\left\{(-4,3)^T\right\}$ basis for $Null(A)$,
$\left\{(1,2)^T\right\}$ basis for $Row(A^T)$,
$\left\{(-2,1)^T\right\}$ basis for $Null(A^T)$;

Question 6. Let S be the subspace of $\mathbb{R}^3$ spanned by $\mathbf{x} = (1,-1,1)^T$.

(a) Find a basis for $S^\perp$.

(b) Give a geometrical description of S and $S^\perp$.

Solution

(a) Since $S = span\{(1,-1,1)^T\}$, $S^\perp$ is the solution set of $S^Tx = 0$ for any $x \in \mathbb{R}^3$, which is also the $Null(S^T)$

$$Null(S^T) = \{x_1, x_2, x_3 | x_1 - x_2 + x_3 = 0\}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Thus, the vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ form a basis for $S^{\perp}$.

(b) all vectors in S is perpendicular to every vector in $S^{\perp}$

Question 7. (a) Let S be the subspace of $\mathbb{R}^3$ spanned by the vectors $\mathbf{x} = (x_1, x_2, x_3)^T$ and $\mathbf{y} = (y_1, y_2, y_3)^T$.Let

$$A = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$$

Show that $S^{\perp} = Null(A)$.

(b) Find the orthogonal complement of the subspace of $\mathbb{R}^3$ spanned by $(1, 2, 1)^T$ and $(1, -1, 2)^T$.

Solution

(a) Let $(x, y, z) \in S^T$

By the definition of orthogonal complement,

$$\mathbf{x} \cdot (x, y, z) = 0$$
$$\mathbf{y} \cdot (x, y, z) = 0$$
$$(x_1, x_2, x_3)^T (x, y, z) = 0$$
$$(y_1, y_2, y_3)^T (x, y, z) = 0$$
$$x_1 x + x_2 y + x_3 z = 0$$
$$y_1 x + y_2 y + y_3 z = 0$$

This system can be written in matrix notation as,

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

This implies, $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ (or) $(x, y, z) \in N(A)$

Thus, $(x, y, z) \in S^T \Rightarrow (x, y, z) \in N(A)$.
In the same way, observe that $(x, y, z) \in N(A) \Rightarrow (x, y, z) \in S^T$.
Therefore, $S^\perp = N(A)$.
(b) From part (a), for the given vectors,

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

From part (a), the orthogonal component of the subspace S is $S^\perp = N(A)$.
To find the null space of A, reduce this matrix into echelon form as follows.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 1 \end{bmatrix} \quad (\text{ By } R_2 \to R_2 - R_1)$$

From this, the reduced system of equations is,

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ -3x_2 + x_3 &= 0 \end{aligned}$$

This implies,

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ x_3 &= 3x_2 \end{aligned}$$

Suppose that $x_2 = t, t \in \mathbb{R}$. Then, $x_3 = 3t$.
Put $x_2 = t, x_3 = 3t$ in the equation $x_1 + 2x_2 + x_3 = 0$.

$$\begin{aligned} x_1 + 2t + 3t &= 0 \\ x_1 &= -5t \end{aligned}$$

Therefore, the null space of A is,

$$\begin{aligned} N(A) &= \left\{(x_1, x_2, x_3)^T\right\} \\ &= \left\{(-5t, t, 3t)^T : t \in \text{i}\right\} \\ &= \left\{(-5, 1, 3)^T t : t \in \text{i}\right\} \end{aligned}$$

The orthogonal complement of the subspace S of $\mathbb{R}^3$ is,

$$\begin{aligned} S^\perp &= N(A) \\ &= \left\{(-5, 1, 3)^T t : t \in \mathbb{R}\right\} \end{aligned}$$

Question 8. Let S be the subspace of $\mathbb{R}^4$ spanned by $\mathbf{x}_1 = (1, 0, -2, 1)^T$ and $\mathbf{x}_2 = (0, 1, 3, -2)^T$. Find a basis for $S^\perp$.

Solution

The orthogonal complement of S,

$$S^{\perp} = \left\{\mathbf{x} \in \mathbb{R}^4 : \mathbf{x}^T\mathbf{y} = 0, \forall \mathbf{y} \in S\right\}$$

Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

As the subspace S is spanned by $\mathbf{x}_1 = (1, 0, -2, 1)^T, \mathbf{x}_2 = (0, 1, 3, -2)^T$ only, consider the equation,

$$\begin{aligned} \mathbf{x}^T\mathbf{x}_1 &= 0 \\ (x_1, x_2, x_3, x_4)\,(1, 0, -2, 1)^T &= 0 \\ x_1 - 2x_3 + x_4 &= 0 \ldots\ldots\ (1) \end{aligned}$$

And, consider the equation,

$$\begin{aligned} \mathbf{x}^T\mathbf{x}_2 &= 0 \\ (x_1, x_2, x_3, x_4)\,(0, 1, 3, -2)^T &= 0 \\ x_2 + 3x_3 - 2x_4 &= 0 \ldots\ldots (2) \end{aligned}$$

The number of variables is 4, and the number of equations is 2 to solve. So, choose two variables arbitrarily to solve the equations (1) and (2).
Suppose that $x_3 = t, x_4 = r$, where $t, r \in \mathbb{R}$.
Then, by the equations (1) and (2),

$$\begin{aligned} x_1 - 2t + r &= 0 \\ x_2 + 3t - 2r &= 0 \end{aligned}$$

This implies,

$$\begin{aligned} x_1 &= 2t - r \\ x_2 &= -3t + 2r \end{aligned}$$

Therefore,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= \begin{bmatrix} 2t - r \\ -3t + 2r \\ t \\ r \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} r$$

The orthogonal complement of S,

$$S^{\perp} = \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} r \in \mathbb{R}^4 : t, r \in \mathbb{R} \right\}$$

So, the orthogonal complement of S is spanned by the linearly independent vectors $(2, -3, 1, 0)^T$ and $(-1, 2, 0, 1)^T$.
Therefore, a basis for $S^{\perp}$ is $\left\{(2, -3, 1, 0)^T, (-1, 2, 0, 1)^T\right\}$

Question 9. For each of the following systems $A\mathbf{x} = \mathbf{b}$, find all least squares solutions:

(a) $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 1 & 3 \\ -1 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -2 \\ 0 \\ 8 \end{pmatrix}$

Solution

Definition: If A is an $m \times n$ matrix of rank n, the normal equations

$$A^T A\mathbf{x} = A^T\mathbf{b}$$

have a unique solution

$$\mathbf{x} = \left(A^T A\right)^{-1} A^T\mathbf{b}$$

and $\mathbf{x}$ is the unique least squares solution of the system $A\mathbf{x} = \mathbf{b}$.

(a) Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Substitute the values of $A, \mathbf{x}$, and $\mathbf{b}$ in $A^T A\mathbf{x} = A^T\mathbf{b}$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{bmatrix}^T \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

When solving this linear system, we can obtain

$$x_1 + 2x_2 = 1$$

Note that there are two variables (namely, x_1, x_2) and one equation. So, solution of it must contain 1 variable as free variable.
Suppose x_2 be the free variable.
That is, $x_2 = \alpha, \alpha \in \mathbb{R}$
Then from the equation $x_1 + 2x_2 = 1$ get $x_1 = 1 - 2\alpha$
Hence the least squares solution is, $(x_1, x_2)^T = \left\{(1 - 2\alpha, \alpha)^T \mid \alpha \in \mathbb{R}\right\}$.
(b) Similar with (a), we can solve the linear system to obtain:

$$x_1 + 2x_3 = 2$$
$$x_2 + x_3 = 1$$

Note that there are three variables (namely, x_1, x_2, x_3) and two equations. So, solution of it must contains 1 variable as free variable.
Suppose x_3 be the free variable.
That is, $x_3 = \alpha, \alpha \in \mathbb{R}$
Then from the equation $x_2 + x_3 = 1$ get $x_2 = 1 - \alpha$ from the equation $x_1 + 2x_3 = 2$ get $x_1 = 2 - 2\alpha$
Hence the least squares solution is
$(x_1, x_2, x_3)^T = \left\{(2 - 2\alpha, 1 - \alpha, \alpha)^T \mid \alpha \in \mathbb{R}\right\}$.

Question 10. Let $P = A\left(A^T A\right)^{-1} A^T$, where A is an $m \times n$ matrix of rank n.

(a) Show that $P^2 = P$.

(b) Prove that $P^k = P$ for $k = 1, 2, \ldots$

(c) Show that P is symmetric. [Hint: If B is non-singular, then $\left(B^{-1}\right)^T = \left(B^T\right)^{-1}$.]

Solution

(a)

$$\begin{aligned}
&P^2 = A\left(A^T A\right)^{-1} A^T \cdot A\left(A^T A\right)^{-1} A^T \\
&P^2 = A\left[\left(A^T A\right)^{-1}\left(A^T A\right)\right]\left(A^T A\right)^{-1} A^T \\
&P^2 = A\left[(B)^{-1}(B)\right]\left(A^T A\right)^{-1} A^T \quad \left(\text{ let } B = A^T A\right) \\
&P^2 = A[I]\left(A^T A\right)^{-1} A^T \quad \left(\text{ Since } BB^{-1} = I\right) \\
&P^2 = A\left(A^T A\right)^{-1} A^T \quad (\text{ Since } A \cdot I = A) \\
&P^2 = P
\end{aligned}$$

(b) The objective is to prove that $P^k = P$, for $k = 1, 2, \ldots$ We prove this by using induction on k. If $k = 1$ then we have $P^1 = P$ Assume $P^m = P$, for some m. Consider $P^{m+1} = P \cdot P^m$

$$\begin{aligned}
&= P \cdot P \\
&= P^2 \\
&= P \\
P^k &= P
\end{aligned}$$

Hence the result is true by induction on k.

(c) The objective is to show that P is symmetric. It is enough to prove $P^T = P$

$$\begin{aligned}
&\text{Consider } P^T = \left[A\left(A^T A\right)^{-1} A^T\right]^T \\
&= \left(A^T\right)^T\left[\left(A^T A\right)^{-1}\right]^T A^T \\
&= A\left[\left(A^T A\right)^T\right]^{-1} A^T \quad \left(\begin{array}{l}\text{Since } A \text{ is non-singular matrix.} \\ \text{Here } A \text{ is } m \times n \text{ matrix rank } n, \text{ so} \\ A \text{ is non-singular matrix}\end{array}\right) \\
&= A\left[\left(A^T\left(A^T\right)^T\right)\right]^{-1} A^T \\
&= A\left[\left(A^T A\right)\right]^{-1} A^T \\
&= P
\end{aligned}$$

Therefore, P is symmetric.

Question 11. Let $\mathbf{x} = (-1, -1, 1, 1)^T$ and $\mathbf{y} = (1, 1, 5, -3)^T$. Show that $\mathbf{x} \perp \mathbf{y}$. Calculate $\|\mathbf{x}\|_2, \|\mathbf{y}\|_2, \|\mathbf{x} + \mathbf{y}\|_2$ and verify that the Pythagorean law holds.

Solution

First we show $x \perp y$.

For this consider $x^T y = \left[(-1,-1,1,1)^T\right](1,1,5,-3)^T$

$$
\begin{aligned}
&= (-1,-1,1,1)\begin{bmatrix} 1 \\ 1 \\ 5 \\ -3 \end{bmatrix} \\
&= -1-1+5-3 \\
&= 5-5 \\
&= 0 \\
x^T y &= 0 \\
&\Rightarrow \quad x \perp y
\end{aligned}
$$

Second,

$$
\begin{aligned}
\|x\|_2 &= \left(\sum_{i=1}^{4} |x_i|^2\right)^{1/2} \\
&= \sqrt{(-1)^2+(-1)^2+1^2+1^2} \\
&= \sqrt{4} \\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
\|y\|_2 &= \left[\sum_{i=1}^{y} |y_i|^2\right]^{1/2} \\
&= \sqrt{(1)^2+(1)^2+5^2+(-3)^2} \\
&= \sqrt{1+1+25+9} \\
&= \sqrt{36} \\
&= 6 \\
x+y &= (0,0,6,-2)^T
\end{aligned}
$$

$$
\begin{aligned}
\|x+y\| &= \sqrt{(0)^2+(0)^2+6^2+(-2)^2} \\
&= \sqrt{36+4} \\
&= \sqrt{40} \\
&= 2\sqrt{10}
\end{aligned}
$$

The Pythagorean law is

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$
$$(2\sqrt{10})^2 = 2^2 + 6^2$$
$$\Rightarrow \quad 40 = 4 + 36$$

Hence the Pythagorean law is satisfied.

Question 12. Let $\mathbf{x} = (1,1,1,1)^T$ and $\mathbf{y} = (8,2,2,0)^T$.

(a) Determine the angle θ between $\mathbf{x}$ and $\mathbf{y}$.

(b) Find the vector projection $\mathbf{p}$ of $\mathbf{x}$ onto $\mathbf{y}$.

(c) Verify that $\mathbf{x}-\mathbf{p}$ is orthogonal to $\mathbf{p}$.

(d) Compute $\|\mathbf{x}-\mathbf{p}\|_2$, $\|\mathbf{p}\|_2$, $\|\mathbf{x}\|_2$ and verify that the Pythagorean law is satisfied.

Solution

(a) we know that $\cos\theta = \frac{\langle xy\rangle}{\|X\|\|Y\|} = \frac{X^TY}{\|X\|\|Y\|} = \frac{12}{\sqrt{4}\sqrt{72}} = \frac{6}{\sqrt{72}} = \frac{6}{2\sqrt{6}} = \frac{1}{\sqrt{2}}$ $\Rightarrow \theta = \pi/4$

(b) The vector projection P of X onto Y is

$$P = \frac{X^TY}{Y^TY}Y$$
$$X^TY = 12, Y^TY = 72$$
$$P = \frac{12}{72}Y$$
$$= \frac{12}{72}(8,2,2,0)^T$$
$$= \left(\frac{4}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)^T$$

(c) $X - P = (1,1,1,1)^T - \left(\frac{4}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)^T$

$$= \left(\frac{-1}{3}, \frac{2}{3}, \frac{2}{3}, 1\right)^T$$

$$(X-P)^TP = \frac{-4}{9} + \frac{2}{9} + \frac{2}{9} + 0$$
$$= 0$$

(d)

$$X - P = \left(\frac{-1}{3}, \frac{2}{3}, \frac{2}{3}, 1\right)^T, P = \left(\frac{4}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)^T$$

$$\begin{aligned} \|X - P\|_2 &= \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9} + 1} \\ &= \sqrt{1+1} \\ &= \sqrt{2} \end{aligned}$$

$$\begin{aligned} \|P\|_2 &= \sqrt{\frac{16}{9} + \frac{1}{9} + \frac{1}{9} + 0} \\ &= \sqrt{2} \end{aligned}$$

$$\begin{aligned} \|X\|_2 &= \sqrt{1+1+1+1} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

$$\begin{aligned} &\text{Clearly} \quad \|X - P\|^2 + \|P\|^2 = 4 \\ &= \|X\|^2 \end{aligned}$$

Pythagorean law is satisfied.

Question 13. Which of the following sets of vectors form an orthonormal basis for $\mathbb{R}^2$?

(a) $\{(1,0)^T, (0,1)^T\}$

(b) $\left\{\left(\frac{3}{5}, \frac{4}{5}\right)^T, \left(\frac{5}{13}, \frac{12}{13}\right)^T\right\}$

(c) $\{(1,-1)^T, (1,1)^T\}$

(d) $\left\{\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)^T, \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)^T\right\}$

Solution

(a) Take $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ The inner product of these vectors is calculated as follows:

$$\begin{aligned} \langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= \mathbf{v}_2{}^T \mathbf{v}_1 \\ &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= 0(1) + 1(0) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Therefore, the inner product of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$.

Find the inner product with the same vectors.
$\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \mathbf{v}_1{}^T \mathbf{v}_1 = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1(1)+0(0) = 1+0 = 1$ $\langle \mathbf{v}_2, \mathbf{v}_2 \rangle = \mathbf{v}_2{}^T \mathbf{v}_2$
$= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0(0) + 1(1) = 0 + 1 = 1$
Since the inner products $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0, \langle \mathbf{v}_1, \mathbf{v}_1 \rangle = 1, \langle \mathbf{v}_2, \mathbf{v}_2 \rangle = 1$, so by the definition of an orthonormal set, $\{(1,0)^T, (0,1)^T\}$ is an orthonormal.
And, no vector of the set $\{(1,0)^T, (0,1)^T\}$ is a scalar multiple of the other, so it is a linearly independent set and hence it is an orthonormal basis in $\mathbb{R}^2$.
(b)
The inner product of these vectors is calculated as follows:

$$\begin{aligned}
&\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \mathbf{v}_2{}^T \mathbf{v}_1 \\
&= \begin{pmatrix} \frac{5}{13} & \frac{12}{13} \end{pmatrix} \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} \\
&= \frac{5}{13}\left(\frac{3}{5}\right) + \frac{12}{13}\left(\frac{4}{5}\right) \\
&= \frac{3}{13} + \frac{48}{65} \\
&= \frac{15+48}{65} \\
&= \frac{63}{65} \\
&\neq 0
\end{aligned}$$

Since the inner product of $\mathbf{v}_1 = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 5/13 \\ 12/13 \end{pmatrix}$ is not equal to 0 , that is $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle \neq 0$. Therefore, by the definition of an orthonormal set, the set $\left\{\left(\frac{3}{5}, \frac{4}{5}\right)^T, \left(\frac{5}{13}, \frac{12}{13}\right)^T\right\}$ is not an orthonormal. Hence, the set $\left\{\left(\frac{3}{5}, \frac{4}{5}\right)^T, \left(\frac{5}{13}, \frac{12}{13}\right)^T\right\}$ is not an orthonormal basis for $\mathbb{R}^2$.
(c) The inner product is 0, and the the inner product with the same vectors is caculcated as follows: $\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \mathbf{v}_1^T \mathbf{v}_1 = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1(1) + (-1)(-1) = 1 + 1 = 2 \neq 1$
Since the inner product $\langle \mathbf{v}_1, \mathbf{v}_1 \rangle \neq 1$, so by the definition of an orthonormal set, the set $\{(1,0)^T, (0,1)^T\}$ is not an orthonormal.
Hence, the set $\{(1,-1)^T, (1,1)^T\}$ is not an orthonormal basis for $\mathbb{R}^2$.
(d) Similarly, we can get that this vector set is an orthonormal basis in $\mathbb{R}^2$.

Question 14. Let

$$\mathbf{u}_1 = \begin{pmatrix} \frac{1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} \\ -\frac{4}{3\sqrt{2}} \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

(a) Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

(b) Let $\mathbf{x} = (1,1,1)^T$. Write $\mathbf{x}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2$, and $\mathbf{u}_3$ using Theorem 5.5.2 and use Parseval's formula to compute $\|\mathbf{x}\|$.

Solution

(a) We show $\{u_1, u_2, u_3\}$ is an orthonormal basis for R^3, for this we show $u_1^T u_1 = 1, u_1^T u_2 = 1,$

$$u_3^T u_3 = 1, u_1^T u_2 = 0, u_1^T u_3 = 0, u_2^T u_3 = 0$$

Consider

$$\begin{aligned}
u_1^T u_1 &= \frac{1}{18} + \frac{1}{18} + \frac{16}{18} = 1 \\
u_2^T u_2 &= \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1 \\
u_3^T u_2 &= \frac{1}{2} + \frac{1}{2} + 0 = 1 \\
u_1^T u_2 &= \frac{\sqrt{2}}{9} + \frac{\sqrt{2}}{9} - \frac{2\sqrt{2}}{9} = 0 \\
u_1^T u_3 &= \frac{1}{6} - \frac{1}{6} + 0 = 0 \\
u_2^T u_3 &= \frac{\sqrt{2}}{3} - \frac{\sqrt{2}}{3} + 0 = 0
\end{aligned}$$

The set $\{u_1, u_2, u_3\}$ forms an orthogonal basis for R^3

(b) Let $x = (1,1,1)^T$ x as the linear combination of u_1, u_2 and u_3 is given by $x = c_1u_1 + c_2u_2 + c_3u_3$, where

$$c_1 = x^T u_1 = (1,1,1) \begin{bmatrix} \frac{1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} \\ -\frac{4}{3\sqrt{2}} \end{bmatrix} = \frac{1}{3\sqrt{2}} + \frac{1}{3\sqrt{2}} - \frac{4}{3\sqrt{2}} = \frac{-2}{3\sqrt{2}} = \frac{-\sqrt{2}}{3}$$

$$c_2 = x^T u_2 = (1,1,1) \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \frac{2}{3} + \frac{2}{3} + \frac{1}{3} = \frac{5}{3}$$

$$c_3 = x^T u_3 = (1,1,1) \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + 0 = 0$$

Therefore $x = -\frac{\sqrt{2}}{3}u_1 + \frac{5}{3}u_2, \|\mathbf{x}\| = \sqrt{3}$

Question 15. Let S be the subspace of $\mathbb{R}^3$ spanned by the vectors $\mathbf{u_2}$ and $\mathbf{u_3}$ of **Question 14**. Let $\mathbf{x} = (1,2,2)^T$. Find the projection $\mathbf{p}$ of $\mathbf{x}$ onto S. Show that $(\mathbf{p}-\mathbf{x}) \perp \mathbf{u_2}$ and $(\mathbf{p}-\mathbf{x}) \perp \mathbf{u_3}$.

Solution

Projection $\mathbf{p}$ of $\mathbf{x} = (1,2,2)^T$ onto S is given by $\mathbf{p} = UU^T\mathbf{x}$ where $U = \begin{bmatrix} u_1 & u_2 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/\sqrt{2} \\ 2/3 & -1/\sqrt{2} \\ 1/3 & 0 \end{bmatrix} \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} =$
$\begin{bmatrix} \frac{4}{9}+\frac{1}{2} & \frac{4}{9}-\frac{1}{2} & \frac{2}{9}+0 \\ \frac{4}{9}-\frac{1}{2} & \frac{4}{9}+\frac{1}{2} & \frac{2}{9}+0 \\ \frac{2}{9}+0 & \frac{2}{9}-0 & \frac{1}{9}+0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{17}{18} & -\frac{1}{18} & \frac{2}{9} \\ -\frac{1}{18} & \frac{17}{18} & \frac{2}{9} \\ \frac{2}{9} & \frac{2}{9} & \frac{1}{9} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} 23 \\ 41 \\ 16 \end{bmatrix}$

As $\mathbf{p} = \frac{1}{18} \begin{bmatrix} 23 \\ 41 \\ 16 \end{bmatrix}$ and $\mathbf{x} = (1,2,2)^T$, get

$$\begin{aligned} \mathbf{p}-\mathbf{x} &= \frac{1}{18} \begin{bmatrix} 23 \\ 41 \\ 16 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \\ &= \frac{1}{18} \begin{bmatrix} 5 \\ 5 \\ -20 \end{bmatrix} \end{aligned}$$

To show that $(\mathbf{p}-\mathbf{x}) \perp \mathbf{u_2}$, consider $(\mathbf{p}-\mathbf{x}) \cdot \mathbf{u_2}$

$$\begin{aligned} (\mathbf{p}-\mathbf{x}) \cdot \mathbf{u_2} &= \frac{1}{18} \begin{bmatrix} 5 \\ 5 \\ -20 \end{bmatrix} \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix} \\ &= \frac{1}{18} \left(\frac{10}{3} + \frac{10}{3} - \frac{20}{3} \right) \\ &= 0 \end{aligned}$$

From the above calculation, we observe that $(\mathbf{p}-\mathbf{x}) \cdot \mathbf{u_2} = 0$

Therefore, $(\mathbf{p}-\mathbf{x}) \perp \mathbf{u_2}$

Question 16. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ form an orthonormal basis for $\mathbb{R}^2$ and let $\mathbf{u}$ be a unit vector in $\mathbb{R}^2$. If $\mathbf{u}^T\mathbf{u}_1 = \frac{1}{2}$, determine the value of $\left|\mathbf{u}^T\mathbf{u}_2\right|$.

Solution

Let u_1 and u_2 form an orthonormal basis for R^2 and let u be a unit vector in R^2.

If $c_1 = u^T u_1 = \frac{1}{2}$ and $c_2 = u^T u_2$ then by the theorem let $\{u_1, u_2 \ldots u_n\}$ be an orthonormal basis for an inner product space v. If $v = \sum_{i=1}^{n} c_i u_i$ then $c_i = \langle v, u_i \rangle$

We have $u = c_1 u_1 + c_2 u_2$. It follows from parseval's formula that

$$\begin{aligned} 1 &= \|u\|^2 \\ &= c_1^2 + c_2^2 \\ &= \frac{1}{4} + c_2^2 \\ \Rightarrow \frac{3}{4} &= c_2^2 \end{aligned}$$

Hence $\left|u^T u_2\right| = |c_2| = \frac{\sqrt{3}}{2}$

Question 17. Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be an orthonormal basis for an inner product space V. If $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$ is a vector with the properties $\|\mathbf{x}\| = 5, \langle \mathbf{u}_1, \mathbf{x} \rangle = 4$, and $\mathbf{x} \perp \mathbf{u}_2$, then what are the possible values of c_1, c_2, c_3 ?

Solution

By Perceval's formula:

$$c_1^2 + c_2^2 + c_3^2 = \|x\|^2 = 5^2 = 25$$

Consider the following theorem: "Let $\{u_1, u_2, \ldots u_n\}$ be an orthonormal basis for an inner product space V. If $v = \sum_{i=1}^{n} c_i u_i$ then $c_i = \langle v_i u_i \rangle^m$

Use above theorem to find the value of constants as follows:

$$c_1 = \langle u_1, x \rangle = 4$$

And

$$c_2 = \langle u_2, x \rangle = 0$$

As $c_1^2 + c_2^2 + c_3^2 = 25$, therefore,

$$
\begin{aligned}
4^2 + 0^2 + c_3^2 &= 25 \\
c_3^2 &= 9 \\
c_3 &= \pm 3
\end{aligned}
$$

Therefore, the values are $c_1 = 4, c_2 = 0, c_3 = \pm 3$

Question 18. For each of the following, use the Gram-Schmidt process to find an orthonormal basis for $Row(A)$.

(a) $A = \begin{pmatrix} -1 & 3 \\ 1 & 5 \end{pmatrix}$ (b) $A = \begin{pmatrix} 2 & 5 \\ 1 & 10 \end{pmatrix}$

Solution

(a)

$$\left\{ \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)^T, \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)^T \right\}$$

(b)

$$\left\{ \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right)^T, \left(\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right)^T \right\}$$

Question 19. Given the basis $\left\{(1, 2, -2)^T, (4, 3, 2)^T, (1, 2, 1)^T\right\}$ for $\mathbb{R}^3$, use the Gram-Schmidt process to obtain an orthonormal basis.

Solution

$$\left\{ \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right)^T, \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right)^T, \left(-\frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right)^T \right\}$$

Question 20. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 12 \\ 6 \\ 18 \end{pmatrix}$

(a) Use the Gram-Schmidt process to find an orthonormal basis for the column space of A.

(b) Factor A into a product QR, where Q has an orthonormal set of column vectors and R is upper triangular.

(c) Solve the least squares problem $A\mathbf{x} = \mathbf{b}$

Solution

(a) The orthonormal basis is

$$\left\{ \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right)^T, \left(\frac{-\sqrt{2}}{6}, \frac{4\sqrt{2}}{6}, \frac{-\sqrt{2}}{6} \right)^T \right\}$$
$$= \left\{ \frac{1}{3}(2,1,2)^T, \frac{\sqrt{2}}{6}(-1,4,-1)^T \right\}$$

(b) We can factorize A into the product θR, where

$$R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} = \begin{bmatrix} 3 & 5/3 \\ 0 & \sqrt{2}/3 \end{bmatrix}$$

$$\theta = \begin{bmatrix} 2/3 & -\sqrt{2}/6 \\ 1/3 & 2\sqrt{2}/3 \\ 2/3 & -\sqrt{2}/6 \end{bmatrix}$$

(c) $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}, b = \begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix}$

The normal equations for this system are $A^T A x = A^T b$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 5 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 66 \\ 36 \end{bmatrix}$$

$$9x_1 + 5x_2 = 66$$
$$5x_1 + 3x_2 = 36$$

Multiply first equation with 5 and second equation with 9 and then subtract, we get $x_1 = 9, x_2 = -3$. Therefore $x = \begin{bmatrix} 9 \\ -3 \end{bmatrix}$