Assignment 1

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Please note that

- **Released date**: 20 Jan., 2022.
- **Due date**: 13 Feb., 2022. by 11:59pm.
- Late submission is **NOT** accepted.
- Please submit your answers as a PDF file with a name like `"118010XXX_HW1.pdf"` (Your student ID + HW No.). You may either typeset you answers directly using computers, or scan your handwritten answers. (We recommend you use the printers on campus to scan. If you use your smartphone to scan, please limit the file size $\leq$ 10MB).
- Please make sure that your submitted file is clear and readable. Submitted file that can not be opened or not readable will get 0 point.

1. In each of the following systems, interpret each equation as a line in the plane. For each system, graph the lines and determine geometrically the number of solutions.

   (a)
$$\begin{aligned} x_1 + x_2 &= 4 \\ x_1 - x_2 &= 2 \end{aligned}$$

   (b)
$$\begin{aligned} x_1 + 2x_2 &= 4 \\ -2x_1 - 4x_2 &= 4 \end{aligned}$$

   (c)
$$\begin{aligned} 2x_1 - x_2 &= 3 \\ -4x_1 + 2x_2 &= -6 \end{aligned}$$

   (d)
$$\begin{aligned} x_1 - x_2 &= 1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

Solution

(a) One solution. The two lines intersect at the point $(3, 1)$.
(b) No solution. The lines are parallel.
(c) Infinitely many solutions. Both equations represent the same line.
(d) One solution. The two lines intersect at the point $(3, 2)$.

2. Write out the system of equations that corresponds to the following augmented matrix:

$$\left(\begin{array}{rrr|r} 5 & -2 & 1 & 3 \\ 2 & 3 & -4 & 0 \end{array}\right)$$

Solution

$$\begin{aligned} 5x_1 - 2x_2 + x_3 &= 3 \\ 2x_1 + 3x_2 - 4x_3 &= 0 \end{aligned}$$

3. Solve each of the following systems.

(a)

$$\begin{aligned} 2x_1 + x_2 &= 8 \\ 4x_1 - 3x_2 &= 6 \end{aligned}$$

(b)

$$\begin{aligned} 4x_1 + 3x_2 &= 4 \\ \frac{2}{3}x_1 + 4x_2 &= 3 \end{aligned}$$

(c)

$$\begin{aligned} x_2 + x_3 + x_4 &= 0 \\ 3x_1 + 3x_3 - 4x_4 &= 7 \\ x_1 + x_2 + x_3 + 2x_4 &= 6 \\ 2x_1 + 3x_2 + x_3 + 3x_4 &= 6 \end{aligned}$$

Solution

(a) $(3, 2)$
(b) $(1/2, 2/3)$
(c) $(4, -3, 1, 2)$

4. Given a system of the form

$$-m_1x_1 + x_2 = b_1$$
$$-m_2x_1 + x_2 = b_2$$

where m_1, m_2, b_1, and b_2 are constants:

(a) Show that the system will have a unique solution if $m_1 \neq m_2$.

(b) Show that if $m_1 = m_2$, then the system will be consistent only if $b_1 = b_2$.

(c) Give a geometric interpretation of parts (a) and (b).

Solution

(a) When $m_1 = 0$, $m_2 \neq 0$, the augmented matrix for the system is

$$\left(\begin{array}{cc|c} 0 & 1 & b_1 \\ -m_2 & 1 & b_2 \end{array}\right)$$

By substitution, we could obtain a unique solution$(\frac{b_1-b_2}{m_2}, b_1)$.
When $m_2 = 0$, $m_1 \neq 0$, the augmented matrix for the system is

$$\left(\begin{array}{cc|c} -m_1 & 1 & b_1 \\ 0 & 1 & b_2 \end{array}\right)$$

By substitution, we could obtain a unique solution$(\frac{b_2-b_1}{m_1}, b_2)$.
When $m_1 \neq 0$, $m_2 \neq 0$, the augmented matrix for the system is

$$\left(\begin{array}{cc|c} -m_1 & 1 & b_1 \\ -m_2 & 1 & b_2 \end{array}\right) \xrightarrow{R_2 \to R_2 - \frac{m_2}{m_1}R_1} \left(\begin{array}{cc|c} -m_1 & 1 & b_1 \\ 0 & 1 - \frac{m_2}{m_1} & b_2 - \frac{b_1m_2}{m_1} \end{array}\right)$$

By substitution, we could obtain a unique solution$(\frac{b_2-b_1}{m_1-m_2}, \frac{m_1b_2-b_1m_2}{m_1-m_2})$.

(b) When $m_1 = m_2$, the augmented matrix for the system is

$$\left(\begin{array}{cc|c} -m_1 & 1 & b_1 \\ -m_2 & 1 & b_2 \end{array}\right) \xrightarrow{R_2 \to R_2 - R_1} \left(\begin{array}{cc|c} -m_1 & 1 & b_1 \\ 0 & 0 & b_2 - b_1 \end{array}\right)$$

So only if $b_2 - b_1 = 0$, that is $b_1 = b_2$, the system will be consistent.
(c) For part (a):
If $m_1 \neq m_2$, the two lines will intersect at one point, that is a unique solution.
For part(b):
If $b_1 \neq b_2$, the two lines will be parallel, that is no solution.
Only if $b_1 = b_2$,both equations will represent the same line, that is infinite solutions.

5. Consider a system of the form

$$a_{11}x_1 + a_{12}x_2 = 0$$
$$a_{21}x_1 + a_{22}x_2 = 0$$

where a_{11}, a_{12}, a_{21}, and a_{22} are constants. Explain why a system of this form must be consistent.

Solution

When $a_{11} \neq 0$, the augmented matrix for the system is

$$\left(\begin{array}{cc|c} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \end{array}\right) \xrightarrow{R_2 \to R_2 - \frac{a_{21}}{a_{11}} R_1} \left(\begin{array}{cc|c} a_{11} & a_{12} & 0 \\ 0 & \frac{a_{22}a_{11} - a_{12}a_{21}}{a_{11}} & 0 \end{array}\right)$$

If $a_{22}a_{11} - a_{12}a_{21} = 0$, x_2 will be free variable, then the system is consistent.
If $a_{22}a_{11} - a_{12}a_{21} \neq 0$, the system will have a unique solution $(0, 0)$.
Obviously, the results above can be expand to the situation that at least one of the four constants $\neq 0$.
And when all four constant $= 0$, the system will have infinite solutions.
So a system of this form must be consistent.

6. For each of the following systems, use Gaussian elimination to obtain an equivalent system whose coefficient matrix is in row echelon form. Indicate whether the system is consistent. Compute the solution set if the system is consistent.

(a)

$$\begin{aligned} x_1 - 2x_2 &= 3 \\ 2x_1 - x_2 &= 9 \end{aligned}$$

(b)

$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= 4 \\ x_1 - 2x_2 + 2x_3 &= 1 \\ 11x_1 + 2x_2 + x_3 &= 14 \end{aligned}$$

(c)

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ 2x_1 + 3x_2 - x_3 - x_4 &= 2 \\ 3x_1 + 2x_2 + x_3 + x_4 &= 5 \\ 3x_1 + 6x_2 - x_3 - x_4 &= 4 \end{aligned}$$

Solution

(a)

$$\left(\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 1 & 1 \end{array}\right)$$

The system is consistent, the solution is (5,1).

(b)

$$\left(\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 1 & -\frac{7}{8} & \frac{1}{8} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The system is consistent, the solution is $(\frac{5-x_3}{4}, \frac{1+7x_3}{8}, x_3)$.

(c)

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -3 & -3 & 0 \\ 0 & 0 & 1 & 1 & -\frac{2}{5} \\ 0 & 0 & 0 & 0 & -\frac{21}{5} \end{array}\right)$$

The system is not consistent.

7. Which of the matrices that follow are in row echelon form? Which are in reduced row echelon form?

(a)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(d)

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(e)

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

(f)

$$\begin{pmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

(g)

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & 6 \end{pmatrix}$$

(h)

$$\begin{pmatrix} 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution

Row echelon form: (a), (c), (d), (g), (h).
Reduced row echelon form: (c), (d), (g).

8. The augmented matrices that follow are in row echelon form. For each case, indicate whether the corresponding linear system is consistent. If the system has a unique solution, find it.

(a)

$$\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array}\right)$$

(b)

$$\left(\begin{array}{cc|r} 1 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$$

(c)

$$\left(\begin{array}{rrr|r} 1 & -2 & 4 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(d)

$$\left(\begin{array}{rrr|r} 1 & -2 & 2 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

(e)

$$\left(\begin{array}{rrr|r} 1 & 3 & 2 & -2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Solution

(a) Inconsistent;
(b) consistent, (4,-1);
(c) consistent, infinitely many solutions;
(d) consistent, (4, 5, 2)
(e) in consistent.

9. The augmented matrices that follow are in reduced row echelon form. In each case, find the solution set to the corresponding linear system.

(a)

$$\left(\begin{array}{rrr|r} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

(b)

$$\left(\begin{array}{rrr|r} 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(c)

$$\left(\begin{array}{rrrr|r} 1 & 5 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Solution

(a) The solution is $(-2, 5, 3)$.

(b) The solution is $\{(2 + 3\alpha, \alpha, 2) | \alpha \text{ real}\}$.

(c) The solution is $\{(3 - 5\alpha + 2\beta, \alpha, \beta, 6) | \alpha, \beta \text{ real}\}$.

10. Use Gauss-Jordan reduction to solve each of the following systems.

$$\text{(a)}\ \begin{aligned} x_1 + 3x_2 + x_3 + x_4 &= 3 \\ 2x_1 - 2x_2 + x_3 + 2x_4 &= 8 \\ 3x_1 + x_2 + 2x_3 - x_4 &= -1 \end{aligned}$$

$$\text{(b)}\ \begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ 2x_1 + x_2 - x_3 + 3x_4 &= 0 \\ x_1 - 2x_2 + x_3 + x_4 &= 0 \end{aligned}$$

Solution

(a) The solution is $\{(\frac{3}{4} - \frac{5}{8}\alpha, -\frac{1}{4} - \frac{1}{8}\alpha, \alpha, 3) | \alpha \text{ real}\}$.

(b) The solution is $\{\alpha(-\frac{4}{3}, 0, \frac{1}{3}, 1) | \alpha \text{ real}\}$.

11. Consider a linear system whose augmented matrix is of the form

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 1 \\ -1 & 4 & 3 & 2 \\ 2 & -2 & a & 3 \end{array}\right]$$

For what values of a will the system have a unique solution?

Solution

The REF can be obtained from the given augmented matrix by elementary row operations as

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 6 & 4 & 3 \\ 0 & 0 & a+2 & 4 \end{array}\right],$$

to make the system have an unique solution, $a + 2 \neq 0 \Rightarrow a \neq 2$.

12. Consider a linear system whose augmented matrix is of the form

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & a & b \end{array}\right]$$

(a) For what values of a and b will the system have infinitely many solutions?

(b) For what values of a and b will the system be inconsistent?

Solution

(a) A row-echelon form for the system is

$$\begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & a-5 & b-4 \end{bmatrix}$$

There will be infinitely many solutions if there is a free variable, and this can only happen here is the the bottom row is $\begin{bmatrix} 0 & 0 & u & v \end{bmatrix}$, where $u = 0$ and $v = 0$. Thus, $a = 5$ and $b = 4$.

(b) This can only happen here is the the bottom row is $\begin{bmatrix} 0 & 0 & u & v \end{bmatrix}$, where $u = 0$ and $v \neq 0$. Thus, $a = 5$ and $b \neq 4$.

13. Prove that the reduced row echelon form of any matrix $A \in \mathbb{R}^{m \times n}$ is unique.

Solution

Proof. Let A be a matrix and suppose it has two row reduced echelon forms say B and C. That means applying a sequence of row operations to A we got B and applying another sequence of row operations we got C. We need to show that $B = C$.
Note that A, B, C are row equivalent to each other since row operation gives a row equivalent matrix. That means every row in A is a linear combination of rows of B and vice versa. Similarly every row in A is a linear combination of rows of C and vice versa.
On the contrary let us assume that B and C are not equal. Then select the leftmost column where they differ and also select all pivot columns (leading 1 columns) to the left of this column giving rise to two matrices say R and S. Since B and C were row rquivalent the matrices R and S are row equivalent since deletion of columns does not affect row equivalence.
Note that after interchanging some rows (if requird) the matrices R and S look like:

$$R = \left[\begin{array}{c|c} I_{r\times r} & \mathbf{r} \\ \hline \mathbf{0} & 0 \end{array}\right] S = \left[\begin{array}{c|c} I_{r\times r} & \mathbf{s} \\ \hline \mathbf{0} & 0 \end{array}\right]$$

It follows that R and S are row equivalent since deletion of columns (variables simultaneously) does not affect row equivalence, and that they are reduced but not equal. Now we treat these matrices as augmented matrices of two linear systems. The system for R has a unique solution $\mathbf{r}$ or is inconsistent, respectively. Similarly, the system for S has a unique solution s or is inconsistent, respectively. Since the systems are equivalent, $r = s$ or both systems are inconsistent. Either way we have $R = S$, a contradiction.

14. Find non-zero 2×2 matrices A and B such that $AB = O$.

Solution

$$\text{Consider, } A = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 \times 1 + (-1) \times 1 & 1 \times 2 + (-1) \times 2 \\ 3 \times 1 + (-3) \times 1 & 3 \times 2 + (-3) \times 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

15. Let

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Compute A^2 and A^3. What will A^n turn out to be?

Solution

$$A^2 = AA = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = A$$

$$A^3 = A^2 A = AA = A$$

Thus, we can prove that for any $n > 2$,

$$A^n = A^{n-2}A^2 = A^{n-2}A = A^{n-1}$$

Iteratively and with all forementioned results, it holds true that for any $n > 0$,

$$A^n = A^{n-1} = ... = A^3 = A^2 = A$$

16. Let

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Compute A^2 and A^3. What will A^{2n} and A^{2n+1} turn out to be?

Solution

$$A^2 = AA = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

$$A^3 = A^2A = IA = A$$

Thus, iteratively,we can prove that for any $n > 1$,

$$A^{2n} = A^{2n-2}A^2 = A^{2n-2}I = A^{2(n-1)}$$

Iteratively and with all forementioned results, it holds true that for any $n > 0$,

$$A^{2n} = A^{2(n-1)} = ... = A^{2\times 2} = A^{2\times 1} = I$$

Thus, for any $n > 0$,

$$A^{2n+1} = A^{2n}A = IA = A$$

17. Let A be an $m \times n$ matrix. Show that A^TA and AA^T are both symmetric.

Solution

(i) $(A^TA)^T = A^T(A^T)^T = A^TA$.

(ii) $(AA^T)^T = (A^T)^TA^T = AA^T$.

18. Let A and B be symmetric $n \times n$ matrices. Prove that $AB = BA$ if and only if AB is also symmetric.

Solution

(i) $\Leftarrow$: If AB is symmetric, then

$$\begin{aligned} AB &= (AB)^T \\ &= B^T A^T \\ &= BA \end{aligned}$$

(ii) $\Rightarrow$: If $AB = BA$, then

$$\begin{aligned} (AB)^T &= (BA)^T \\ &= A^T B^T \\ &= AB. \end{aligned}$$

Therefore, AB is also symmetric.

19. Show that every $n \times n$ matrix can represented as a sum of a symmetric matrix and a skew-symmetric matrix.

Solution

$$\begin{aligned} A &= \frac{1}{2}(2A) \\ &= \frac{1}{2}(A + A^T + A - A^T) \\ &= \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) \\ &\triangleq A_1 + A_2, \end{aligned}$$

where A_1 is symmetric and A_2 is skew-symmetric.