$f:\{1,2,3\} \to \underline{\mathbb{R}}$, $f(x)=x^2$

codomain 陪域

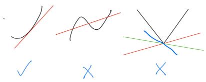
subset 子集

piecewise-defined function 分段函数

secant line 割线   tangent line 切线

复习定理成立条件

<u>Rough definition</u>: A **tangent line** to a curve at point P is the unique line that "just touches" the curve at only P.



极限

微分中值定理

- Identity function: $f(x) = x,\ D \in \mathbb{R}$.
- Natural exponential function: $f(x) = e^x,\ \ D \in \mathbb{R}$.
- Natural logarithmic function: $f(x) = \ln x,\ \ D = \{x \in \mathbb{R} : x > 0\}$.

THEOREM 2.2.4 –The Sandwich Theorem Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L.$$

Then $\lim_{x\to c} f(x) = L$.

THEOREM 2.4.6 A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

<u>Theorem (limits involving bounded functions)</u>

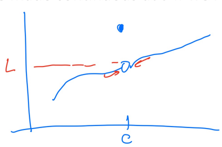
Suppose that f and g are functions defined on some open interval D containing c (except possibly at c). If $\lim_{x\to c} f(x) = 0$ and $g(x)$ is bounded on $D\backslash\{c\}$, then

$$\lim_{x\to c} f(x)g(x) = 0$$

Note: $D\backslash\{c\}$ = all elements in D but not c = $\{x \in D: x \notin c\}$

A discontinuity is said to be **removable** if $\lim_{x\to c} f(x) = L$ for some $L \in \mathbb{R}$ (this means $L \neq \pm\infty$).

In this case, f can be made continuous at c if we redefine $f(c) = L$.

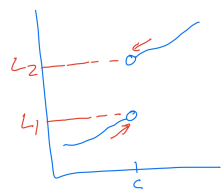


A discontinuity is called a **jump discontinuity** if

$$\lim_{x\to c^-} f(x) = L_1$$

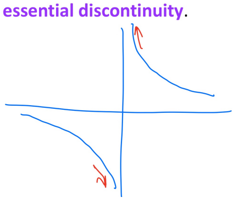
$$\lim_{x\to c^+} f(x) = L_2$$

$$L_1 \neq L_2, \qquad L_1, L_2 \in \mathbb{R}$$

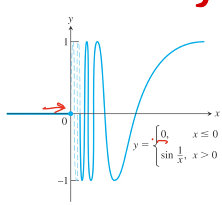


If $\lim_{x\to c^-} f(x)$ or $\lim_{x\to c^+} f(x)$ does not exist as a real number, then c is called an **essential discontinuity**.

infinity   oscillating

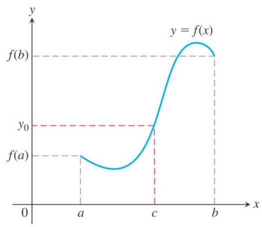


E.g.



(c) $f(x)$

THEOREM 2.5.11—The Intermediate Value Theorem for Continuous Functions If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



The Intermediate Value Theorem

$$\frac{ax^n}{bx^n}.$$

极限的证明方法

Polynomials 多项式 Rational function 有理函数

asymptote 渐近线
- horizontal 水平
- vertical 垂直
- oblique 倾斜

a classic problem

$$\lim_{x\to-\infty}\frac{(2x^5-6x^4+1)\left(\frac{1}{x^2}\right)}{(3x^2+x-7)\left(\frac{1}{x^2}\right)}$$

$$= \lim_{x\to-\infty}\frac{2x^3-6x^2-\cancel{\frac{1}{x^2}}^{\to 0}}{3+\cancel{\frac{1}{x}}^{\to 0}-\cancel{\frac{7}{x^2}}^{\to 0}}$$

$$= \lim_{x\to-\infty}\frac{\overset{\infty}{2x^2}\,\overset{-\infty}{(x-3)}}{3} = -\infty$$

Definition:

- f is said to be **differentiable on D** if f is differentiable at every interior point of D, and f is one-sided differentiable at every end-point.
- Note that open intervals contain no end-point.

The derivatives of the other trigonometric functions:

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Implicit differentiation

Note:

In order for the approach above to work, y has to be a function of x at the required point "locally", (i.e. on a small scale), and it cannot have a vertical tangent.

normal line 法线

1. A function $f: [-10,10) \to \mathbb{R}$ is given by:

$$f(x) = \begin{cases} \sin x, & -10 \le x \le -\frac{\pi}{2} \\ |x|, & -\frac{\pi}{2} < x \le 0 \\ x^2, & 0 < x < 10,\ x \ne 2 \\ 0, & x = 2 \end{cases}$$

(a) For what values of x is $f(x)$ continuous? 5 marks

$-10 < x < -\pi/2$

$-\pi/2 < x < 2$

$2 < x < 10$

(b) For what values of c does $\lim_{x\to c} f(x)$ exist? 5 marks

$-10 < c < -\pi/2$

$-\pi/2 < c < 10$

(c) For what values of x is $f(x)$ differentiable? 5 marks

$-10 < x < -\pi/2$

$-\pi/2 < x < 0$

$0 < x < 2$

$2 < x < 10$

DEFINITIONS If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a) = L_a(x)$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

Error of standard linear approximation

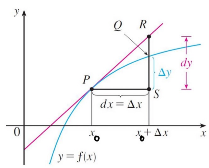
Q: What is $f(x_0 + \Delta x) - L(x_0 + \Delta x)$?

$$f(x_0 + \Delta x) - L(x_0 + \Delta x)$$

$$= f(x_0 + \Delta x) - f(x_0) - f'(x_0)\Delta x$$

$$= \left(\frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} - f'(x_0)\right)\Delta x$$

$$= \varepsilon \Delta x$$ （误差补偿）



THEOREM 4.1.1 –The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$.

Definition:

Let $f : D \to \mathbb{R}$, and let c be an interior point of D.

Then c is a **critical point** of f if:

(i) $f'(c) = 0$; or

(ii) $f'(c)$ does not exist (in $\mathbb{R}$)

THEOREM 4.1.2 –The First Derivative Theorem for Local Extreme Values If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

Rolle's Theorem

Suppose that a function f is continuous on $[a, b]$ and differentiable at every point in (a, b), and it satisfies $f(a) = f(b)$.

Then there exists c in (a, b) such that $f'(c) = 0$.

Mean Value Theorem (MVT)

Suppose that a function f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

COROLLARY 1 If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

monotonicity 单调性

DEFINITION A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**. 拐点

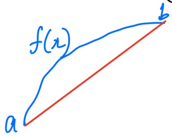
Graph sketching

1. Domain D and symmetry (even or odd function)
2. Critical points
3. Intervals of monotonicity
4. Points of inflection and intervals of concavity
5. Asymptotes 渐近线
6. x- and y-intercepts 交点

Theorem: Concavity and secant lines

Let f be continuous on $[a,b]$ and differentiable on (a,b).

(i) If f is concave down on (a,b), then the graph of f lies above the secant line joining $\left(a,f(a)\right)$ and $\left(b,f(b)\right)$ on (a,b).



Suppose $x \in (a,b)$, by Mean Value Theorem,

$\exists\ c_1 \in (a,x),\quad c_2 \in (x,b),\quad$ such that

$f(x) = f(a) + f'(c_1)(x-a)$

$f(b) = f(x) + f'(c_2)(b-x) = f(a) + f'(c_1)(x-a) + f'(c_2)(b-x)$

$\because f''(x) < 0 \quad \therefore f'(c_1) > f'(c_2)$ ☆ concave down ✓

$\therefore f(a) + f'(c_1)(b-a) > f(b) > f(a) + f'(c_2)(b-a)$

$$f'(c_1) > \frac{f(b)-f(a)}{b-a} > f'(c_2)$$ ☆

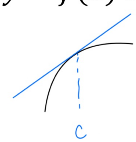
$g(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$

$f(x) = f(a) + f'(c_1)(x-a) > g(x)$ Q.E.D.

Theorem: Concavity and tangent lines

Let f be continuous on $[a,b]$ and differentiable on (a,b).

(i) If f is concave down on (a,b), then for any $c \in (a,b)$ the tangent line to $y = f(x)$ at c lies above the graph of $y = f(x)$.



$g(x) = f'(c)(x-c) + f(c) \geq f(x)$

① $x \in (a,c)$ $\quad f(c) = f(x) + f'(x_1)(c-x)$

$f(x) = f(c) + f'(x_1)(x-c)$

$g(x) \geq f(x) \iff f'(c) < f'(x_1)$ ✓

Newton's method

Note that L_i is given by $y = f(x_i) + f'(x_i)(x - x_i)$,

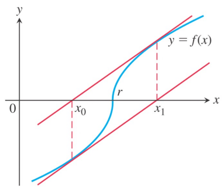
so

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, \qquad \text{if } f'(x_i) \neq 0}$$

Newton's method

$$f(x) = \begin{cases} \sqrt{x-r} & \text{if } x \geq r \\ -\sqrt{r-x} & \text{if } x < r \end{cases}$$

- x_n does not converge to any single number, and so does not converge to r.



antiderivative 不定积分 (原函数)

Antiderivatives

Definitions:

$$\int \underline{f(x)\,dx}$$ 被积表达式

$\int f(x)\,dx$ is called the **indefinite integral of f w.r.t. x**. f关于x的不定积分

$f(x)$ is called the **integrand**. 被积函数

dx is called the **variable of integration**. x→积分变量？

Riemann Sums

Definition:

Given a function $f: [a, b] \to \mathbb{R}$ with a partition P of $[a, b]$, a **Riemann sum** of f (w.r.t. P) is a sum of the form

$$S_P = \sum_{k=1}^{n} f(c_k)\Delta x_k = f(c_1)\Delta x_1 + \cdots + f(c_n)\Delta x_n$$

where $c_k \in [x_{k-1}, x_k]$ and $\Delta x_k = x_k - x_{k-1}$

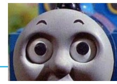
for each $k \in \{1, \dots, n\}$.

Definite Integrals (Riemann Integrals)

DEFINITION Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the **definite integral of f over $[a, b]$** and that J is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - J \right| < \epsilon.$$



Non-integrability

证明该函数的积分不存在

E.g. Consider the Dirichlet function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \quad \text{rational number} \\ 0 & \text{if } x \notin \mathbb{Q} \quad \text{irrational number} \end{cases}$$

Here $\mathbb{Q} = \left\{ \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{Z}, n \neq 0 \right\}$

Riemann Sums

Definition:

A **partition** of the interval $[a, b]$ is a set

分割

$$P = \{x_0, x_1, \dots, x_{n-1}, x_n\}$$

such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

A little more on Riemann sums

Definite integral 定积分
Infinite integral 不定积分

Definition

Let $P := \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$. The norm of P, denoted by $\|P\|$, is defined by

$$\|P\| = \max_{k:1 \le k \le n} \Delta x_k.$$

That is, $\|P\|$ is the length of the largest subinterval given by P.

THEOREM 5.3.1 —Integrability of Continuous Functions If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there,* then the definite integral $\int_a^b f(x)\,dx$ exists and f is integrable over $[a, b]$.

- Thus for any choice of $c_k \in [x_{k-1}, x_k]$, $m_k \le f(c_k) \le M_k$
- So

$$L_P(f) = \sum_{k=1}^{n} m_k \Delta x_k \le \sum_{k=1}^{n} f(c_k)\Delta x_k \le \sum_{k=1}^{n} M_k \Delta x_k = U_P(f)$$

- As $\|P\| \to 0$, $\quad U_P(f) - L_P(f) \to 0$

证明积分的存在

（黎曼和的极限存在）

- More formally, for any given ε, choose all Δx_k small enough such that $M_k - m_k < \frac{\varepsilon}{b-a}, \forall k$. (This is possible for continuous functions).

THEOREM 5.4.3 —The Mean Value Theorem for Definite Integrals If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

积分中值定理

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Mean Value Theorem for definite integrals

<u>Note</u>: Continuity condition cannot be skipped.

$\int_0^2 f(x)dx$

$= \int_0^1 f(x)dx + \int_1^2 f(x)dx$

$= 0 + 1 = 1$

$\text{av}(f) = \frac{1}{2}$

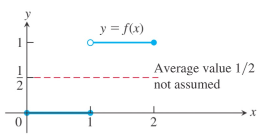


FIGURE 5.17 A discontinuous function need not assume its average value.

THEOREM 5.4.4 –The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

FTC 1

THEOREM 5.4.4 (Continued)—The Fundamental Theorem of Calculus, Part 2 If f is continuous over $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

FTC 2

THEOREM 5.5.6 –The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

应用题

Volumes using cross-sections

Definition

Let S be a solid that lies between the planes $x = a$ and $x = b$. The volume V of S is defined by

$$V := \int_a^b A(x)\,dx,$$

provided that the cross-section area function $A(x)$ is integrable.

If the solid S is generated by rotating the region

$$\{(x, y) : 0 \le r(x) \le y \le R(x),\ a \le x \le b\}$$

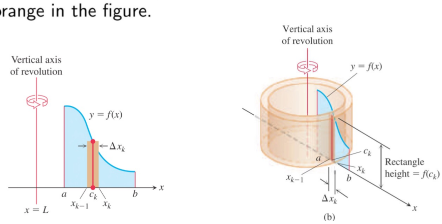
around the x-axis, then similarly,

$$V = \int_a^b \pi\left(R(x)^2 - r(x)^2\right)dx.$$

同理积分变量可换为 y.

Volumes using cylindrical shells

Consider revolving the following region in blue about the y-axis to generate a solid. Its volume can be computed by adding the volumes of all the "cylindrical shells", one of which is displayed in orange in the figure.



Arc length

Definition

Let f be a function such that f' is continuous on $[a, b]$. The length (or arc length) L of the curve $y = f(x)$ between the points $(a, f(a))$ and $(b, f(b))$ is defined by

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

Note:

If the curve is given by $x = g(y)$, $c \le y \le d$, and g' is continuous, then the arc length can be computed by

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + (g'(y))^2}\, dy.$$

$y = (\frac{x}{2})^{\frac{2}{3}}$, $0 \le x \le 2$

ans: $\frac{2}{27}(10^{\frac{3}{2}} - 1)$

Areas of surfaces of revolution

DEFINITION If the function $f(x) \ge 0$ is continuously differentiable on $[a, b]$, the **area of the surface** generated by revolving the graph of $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx. \qquad (3)$$

求面积需要两个参数.

Surface Area for Revolution About the y-Axis

If $x = g(y) \ge 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the graph of $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\, dy. \qquad (4)$$

物理题

Work

EXAMPLE 5 The conical tank in Figure 6.39 is filled to within 2 m of the top with olive oil weighing 0.9 g/cm^3 or 8820 N/m^3. How much work does it take to pump the oil to the rim of the tank?

$W = Fd$

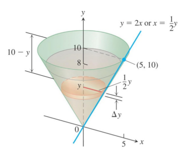


FIGURE 6.39 The olive oil and tank in Example 5.

Fluid forces

For a **static liquid**, the pressure p at depth h is given by

$$p = wh$$

where w is the **weight-density** of the fluid. SI units N/m^3

$w = \rho g$

ρ = mass density kg/m^3

$g = 9.8$ m/s^2

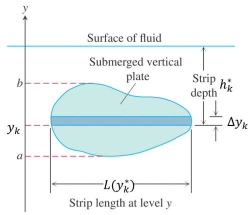
Fluid forces

Force exerted on S_k:

$$F_k = whA \approx w\overset{(S)}{h_k^*}L(y_k^*)\Delta y_k$$

Total force exerted on plate is :

$$F = \int_a^b wh(y)L(y)dy$$



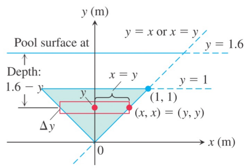
Fluid forces

$$F = \int_a^b wh(y)L(y)dy$$

$$= \int_0^1 (9.8 \times 10^3)(1.6 - y)(2y)dy$$

$$= 9.8\times10^3 \int_0^1 (3.2y - 2y^2)\,dy$$

$$= 9.8\times10^3 \left[1.6y^2 - \tfrac{2}{3}y^3\right]_0^1 = 9.8\times10^3\left(1.6 - \tfrac{2}{3}\right) \approx 9147\ \text{N}$$

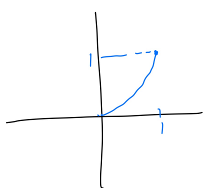


Inverse functions and differentiation

> **Definition**
>
> Let $f : D \to Y$ be a function.
>
> - We say that f is one-to-one (or injective) if $f(x_1) \neq f(x_2)$ for all distinct x_1 and x_2 in D (that is, $x_1 \neq x_2$). 单射的
> - We say that f is onto (or surjective) if, for every $y \in Y$, there exists $x \in D$ such that $f(x) = y$. 满射的
> - We say that f is bijective if it is both one-to-one and onto. A bijective function is called a bijection. 双射的

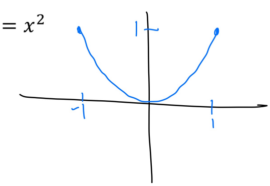
E.g. Is the following function one-to-one, onto or bijective? ✓

$f: [0,1] \to \mathbb{R},\ f(x) = x^2$



E.g. Is the following function one-to-one, onto or bijective? ✗

$f: [-1,1] \to \mathbb{R},\ f(x) = x^2$



Definition

Let $f : D \to \text{range}(f)$ be one-to-one (and hence bijective).

The **inverse function** of f is the function $f^{-1} : \text{range}(f) \to D$ defined by

$$f^{-1}(y_0) = x_0, \quad \text{where } f(x_0) = y_0$$

$(f^{-1})'(y_0) = \frac{1}{f(x_0)}$

Inverse functions and differentiation

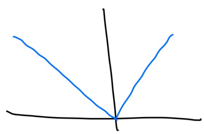
Facts (without proof):

1. If f is continuous and f^{-1} exists, then f^{-1} is also continuous.
2. If f is continuous and its domain is an interval, then $\text{range}(f)$ is also an interval.
3. If f is one-to-one and continuous on an interval I, then f is monotonic on I.

Natural logarithmic function

If we take $g(x) = |x|$ with $D = \mathbb{R} \backslash \{0\}$, then

$$g'(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \end{cases}$$



so

$$g'(x) = \frac{|x|}{x}$$

THEOREM 7.5.6 –Cauchy's Mean Value Theorem Suppose functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and also suppose $g'(x) \neq 0$ throughout (a, b). Then there exists a number c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

DEFINITION Let $f(x)$ and $g(x)$ be positive for x sufficiently large.

1. f **grows faster than** g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$$

or, equivalently, if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0.$$

We also say that g **grows slower than** f as $x \to \infty$.

2. f and g **grow at the same rate** as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

where L is finite and positive.

(f) $\log_a$ vs $\log_b$: For $a > 1$ and $b > 1$:

$$\lim_{x\to\infty} \frac{\log_a x}{\log_b x} = \lim_{x\to\infty} \frac{\ln x \,/\, \ln a}{\ln x \,/\, \ln b} = \frac{\ln b}{\ln a} > 0$$

Log functions with base > 1 all grow at the same rate.

E.g $\sqrt{x^2 + 2021}$ and $(98\sqrt{x} - 1)^2$ grow at the same rate since they both grow at the same rate as $f(x) = x$:

同阶量

Little-oh and big-oh notation

DEFINITION A function f is **of smaller order than** g as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. We indicate this by writing $\boldsymbol{f = o(g)}$ ("f is little-oh of g").

DEFINITION Let $f(x)$ and $g(x)$ be positive for x sufficiently large. Then f is **of at most the order of** g as $x \to \infty$ if there is a positive integer M for which

$$\frac{f(x)}{g(x)} \le M,$$

for x sufficiently large. We indicate this by writing $\boldsymbol{f = O(g)}$ ("f is big-oh of g").

$$\arctan x + \operatorname{arccot} x = \frac{\pi}{2}$$

$$\operatorname{arcsec} x + \operatorname{arccsc} x = \frac{\pi}{2}$$

- $\sin : (-\pi/2, \pi/2) \to (-1, 1)$, $\arcsin'(x) = \dfrac{1}{\sqrt{1-x^2}}$.
- $\cos : (0, \pi) \to (-1, 1)$, $\arccos'(x) = \dfrac{-1}{\sqrt{1-x^2}}$.
- $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$, $\arctan'(x) = \dfrac{1}{1+x^2}$.
- $\cot : (0, \pi) \to \mathbb{R}$, $\operatorname{arccot}'(x) = \dfrac{-1}{1+x^2}$.
- $\sec : (0, \pi/2) \cup (\pi/2, \pi) \to (-\infty, -1) \cup (1, \infty)$,

$$\operatorname{arcsec}'(x) = \frac{1}{|x|\sqrt{x^2-1}}.$$

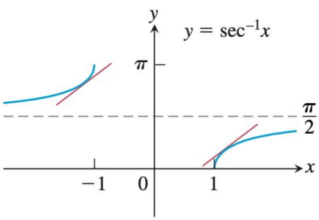
- $\csc : (-\pi/2, 0) \cup (0, \pi/2) \to (-\infty, -1) \cup (1, \infty)$,

$$\operatorname{arccsc}'(x) = \frac{-1}{|x|\sqrt{x^2-1}}.$$

分部积分法

$$\int u\, dv = uv - \int v\, du$$

E.g. Find $\int \underbrace{e^x}_{f(x)} \underbrace{\sin x}_{g'(x)}\, dx$



$$\int f_0(x)g_0(x)\,dx = f_0(x)g_1(x) - \int f_1(x)g_1(x)\,dx$$

$$= f_0(x)g_1(x) - f_1(x)g_2(x) + \int f_2(x)g_2(x)\,dx$$

$$= \cdots$$

$$= f_0(x)g_1(x) - f_1(x)g_2^{(1)}(x) + f_2(x)g_3^{(2)} - \cdots \pm \int f_m(x)g_m(x)\,dx$$

E.g. Evaluate $\int x^3 3^x\, dx$

i	Sign	$f_i(x)$	$g_i(x)$
0	+	x^3	3^x
1	-	$3x^2$	$(\ln 3)^{-1}\, 3^x$
2	+	$6x$	$(\ln 3)^{-2}\, 3^x$
3	−	6	$(\ln 3)^{-3}\, 3^x$
4	+	0	$(\ln 3)^{-4}\, 3^x$
5			

处理 $\int \sin^m x\cdot\cos^n x\,dx$ 的常用方法.

$$\int \sin^2 x\cdot\cos^5 x\,dx = \int \sin^2 x\cdot(1-\sin^2 x)^2\,d\sin x$$
$$= \int \sin^2 x - 2\sin^4 x + \sin^6 x\,\,d\sin x$$
$$= \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C$$

$$\int \sin^2 x\cos^4 x\,dx = \int \frac{1-\cos 2x}{2}\cdot\left(\frac{1+\cos 2x}{2}\right)^2 dx$$
$$= \frac{1}{8}\int -\cos^3 2x - \cos^2 2x + \cos 2x + 1\,\,dx$$
$$= -\frac{1}{16}\int 1-\sin^2 2x\,\,d\sin 2x - \frac{1}{16}\int 1+\cos 4x\,\,dx + \frac{1}{16}\sin 2x + \frac{x}{8}$$
$$= -\frac{1}{16}\sin 2x + \frac{1}{48}\sin^3 2x - \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{16}\sin 2x + \frac{x}{8} + C$$
$$= \frac{1}{48}\sin^3 2x - \frac{1}{64}\sin 4x + \frac{x}{16} + C$$

$$\int \tan^m x\,dx = \int \tan^{m-2} x\cdot(\sec^2 x - 1)\,dx$$
$$= \int \tan^{m-2} x\,\,d\tan x - \int \tan^{m-2} x\,dx$$
$$= \frac{1}{m-1}\tan^{m-1} - \int \tan^{m-2} x\,dx$$

$$\int \sec^m x\,dx = \int \sec^{m-2} x\,\,d\tan x$$
$$= \sec^{m-2} x\cdot\tan x - \int \tan x\,d\sec^{m-2} x$$
$$= \sec^{m-2} x\cdot\tan x - (m-2)\int \sec^{m-2}(\sec^2 - 1)\,dx$$
$$= \sec^{m-2} x\cdot\tan x - (m-2)\int \sec^m x\,dx + (m-2)\int \sec^{m-2} x\,\,dx$$
$$\Rightarrow \quad \int \sec^m x\,dx = \frac{1}{m-1}\sec^{m-2} x\cdot\tan x + \frac{m-2}{m-1}\int \sec^{m-2} x\,\,dx$$

$$\int \tan^3 x\cdot\sec^4 x\,\,dx = \int \tan^3 x\,\,(\tan^2 x + 1)\,\,d\tan x$$
$$= \frac{1}{6}\tan^6 x + \frac{1}{4}\tan^4 x + C$$

$$\int \tan^3 x\cdot\sec^3 x\,\,dx = \int (\sec^2 x - 1)\sec^2 x\,\,d\sec x$$
$$= \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$$

$$\int \tan^2 x\cdot\sec^2 x\,dx = \int (\sec^2 x - 1)\sec^2 x\,dx$$
$$= \int \sec^4 x - \sec^2 x\,dx$$

转化为上式.

Finding undetermined coefficients

Heaviside "cover-up" method

In Case 1 where

$$\frac{f(x)}{g(x)} = \frac{f(x)}{(x-r_1)\dots(x-r_n)} = \frac{A_1}{(x-r_1)} + \dots + \frac{A_n}{(x-r_n)}$$

there is a quick way for finding $A_1, \dots, A_n$.

Finding undetermined coefficients

$$\frac{f(x)}{(x-r_2)\dots(x-r_n)} = A_1 + (x-r_1)\left(\frac{A_2}{(x-r_2)} + \dots + \frac{A_n}{(x-r_n)}\right)$$

Substituting $x = r_1$ yields:

$$A_1 = \frac{f(x)}{(x-r_2)\dots(x-r_n)}$$

Cover up $(x-r_1)$, then sub. $x=r_1$

Finding undetermined coefficients

Differentiating method

Let's use an example to illustrate:

$$\frac{f(x)}{(x-r)^3} = \frac{A}{(x-r)} + \frac{B}{(x-r)^2} + \frac{C}{(x-r)^3}$$

where $\deg(f(x)) \le 2$.

Finding undetermined coefficients

$$\frac{f(x)}{(x-r)^3} = \frac{A}{(x-r)} + \frac{B}{(x-r)^2} + \frac{C}{(x-r)^3}$$

Multiply by $(x-r)^3$:

$$f(x) = A(x-r)^2 + B(x-r) + C \qquad (1)$$

Set $x = r$:

$$f(r) = C$$

Differentiate (1) :

$$f'(x) = 2A(x-r) \quad + B \qquad (2)$$

Set $x = r$:

$$f'(r) = B$$

Differentiate (2) :

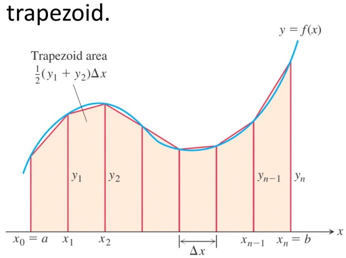
$$f''(x) = 2A$$

Since $\deg(f(x)) \le 2$, $f''(x)$ is a constant K, so

$$A = \frac{f''(x)}{2} = \frac{K}{2}$$

- **Numerical integration**
  - Trapezoidal rule
  - Simpson's rule

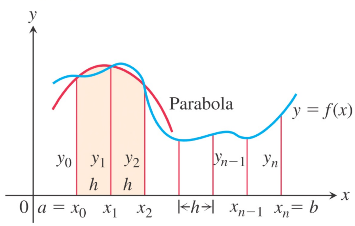
In the **trapezoidal rule**, we approximate each $\int_{x_{k-1}}^{x_k} f(x)$ with the area of a trapezoid.



$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} T_k = \frac{\Delta x}{2}\underbrace{(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n)}_{(y_0+y_1)+(y_1+y_2)+\cdots+(y_{n-1}+y_n)},$$

where $\Delta x = (b-a)/n$.

- We can take this approach a step further by approximating f with quadratic polynomials $Ax^2 + Bx + C$.



- Consider an evenly spaced partition $P := \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$, where n is even.

- For $k \in \{1, 3, 5, \ldots, n-1\}$, over the interval $[x_{k-1}, x_{k+1}]$, consider approximating f by the quadratic function $p_k(x) := A_k x^2 + B_k x + C_k$ whose graph passes through the three points (x_{k-1}, y_{k-1}), (x_k, y_k) and (x_{k+1}, y_{k+1}).

$$\int_a^b f(x)\,dx \approx \frac{h}{3}(y_0 + 4y_1 + 2y_2 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n),$$

where n is even, and the coefficients are 1, 4, 2, 4, 2, 4, 2, ..., 4, 2, 4, and 1.

Error bounds X

Then:

$|E_L| \le \frac{(b-a)^2}{2n}\max|f'|$ $\quad$ $|E_R| \le \frac{(b-a)^2}{2n}\max|f'|$

$|E_T| \le \frac{(b-a)^3}{12n^2}\max|f''|$ $\quad$ $|E_M| \le \frac{(b-a)^3}{24n^2}\max|f''|$

$$|E_S| \le \frac{(b-a)^5}{180n^4}\max|f^{(4)}|$$

Improper integrals: Type 1 敛散性

Definition

- An improper integral is said to be convergent if the corresponding limit exists, and is said to be divergent if the limit does not exist (as a real number). eg. $\pm\infty$
- We define

$$\int_{-\infty}^{\infty} f(x)\,dx := \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

  whenever both improper integrals on the right converge. We may use any real number c in this definition.

Improper integrals – Type 2

Definition

- If f is discontinuous at c, where $a < c < b$, and is continuous on $[a, b] \setminus \{c\}$, then

$$\int_{a}^{b} f(x)\,dx := \int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx,$$

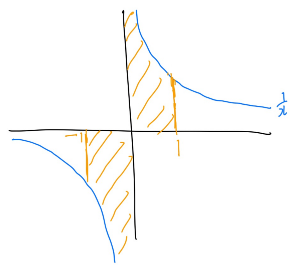
  provided that both improper integrals on the right converge.

$\int_{-1}^{1} \frac{1}{x}\,dx$ 不存在

$\int_{-1}^{-a} \frac{1}{x}\,dx + \int_{a}^{1} \frac{1}{x}\,dx = 0$

Remarks

- In the definition above, $\int_a^b f(x)\,dx$ may also be defined if the right-hand side is $\infty + \infty$, $-\infty - \infty$ or $a \pm \infty$ (where $a \in \mathbb{R}$).
- But it is undefined for " $\infty - \infty$ "



$$\int_0^3 \frac{1}{x-1}\,dx$$

is undefined according to our definition.

$\int_{-1}^{1} \frac{1}{x}\,dx$ is undefined

Improper integrals: convergence tests

Theorem (Direct Comparison Test)

Suppose that $a \in \mathbb{R}$, and suppose that f and g are continuous functions on $[a, \infty)$. If there exists $c \in [a, \infty)$ such that $0 \le f(x) \le g(x)$ for all $x \in [c, \infty)$, then the following statements hold:

(i) *If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.*

(ii) *If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges.*

Theorem (Limit Comparison Test)

Suppose that $a \in \mathbb{R}$, and suppose that f and g are positive continuous functions on $[a, \infty)$. If

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

f & g grow at the same rate

for some $L \in \mathbb{R}_+ := (0, \infty)$, then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

both converge or both diverge.

$$\lim_{x\to\infty} \frac{(1-e^{-x})/x}{1/x} = \lim_{x\to\infty} (1-e^{-x}) = 1$$

By limit comparison test (comparing with $\int_1^\infty \frac{1}{x}\,dx$),

$$\int_1^\infty \frac{1-e^{-x}}{x}\,dx$$

diverges.

Definition:

- A **first-order differential equation** (D.E.) is an equation of the form

$$\frac{dy}{dx} = F(x, y)$$

where y is the dependent variable and x is the independent variable.

Definition

A **first-order linear differential equation** is an equation of the form

$$\underbrace{\frac{dy}{dx} + P(x)y = Q(x)}_{\text{Standard form}}$$

Linear equations

In summary, to solve $y' + P(x)y = Q(x)$:

1. Let $v(x) := e^{\int P(x)dx}$ (an integrating factor), where $\int P(x)dx$ is ***any*** antiderivative of $P(x)$:
2. The general solution is given by:

$$y = \frac{1}{v(x)} \int v(x)Q(x)\, dx$$

1. Malthusian Growth Model

If $P = P(t)$ is the population at time t, then

$$\frac{dP}{dt} = kP \qquad \text{(separable eq}^{\text{n}}\text{)}$$

Hence the particular solution is:

$$P = P_0 e^{kt}$$

where P_0 is the "initial population".

2. Logistic Growth Model

In this model,

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) \qquad (k > 0)$$

where $M > 0$ is the carrying capacity.

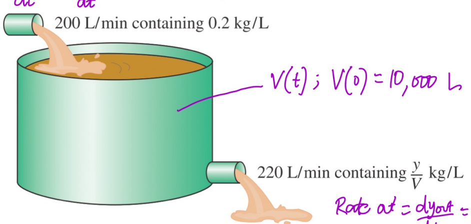
Let us assume $0 < P < M$.

$$P = \frac{M}{1 + Ce^{-kt}} \qquad C > 0$$

This is the general solution called the **logistic function**.

Mixture problem

Rate in $= \frac{dy_{in}}{dt} = \frac{dV_{in}}{dt} \cdot C_{in} = (200)(0.2)$



$V(t)$; $V(0) = 10{,}000$ L

Rate out $= \frac{dy_{out}}{dt} = \frac{dV_{out}}{dt} \cdot C = (220)\left(\frac{y}{V}\right)$

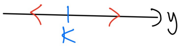
$\frac{dV}{dt} = -20$ L/min (net flow out)

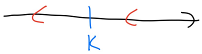
Let K be an equilibrium value of $\frac{dy}{dx} = f(y)$.

Consider a phase line analysis:

In this case, $y = K$ is a **stable equilibrium**,

For other cases of $y = K$ we may have an **unstable equilibrium**, e.g.:





Newton's law of cooling

Newton's law of cooling (or heating) states that the rate of change of the temperature of an object is proportional to the difference of temperatures between the object and its surroundings. In other words, if $H(t)$ is the temperature of an object at time t, then H satisfies the differential equation

$$\frac{dH}{dt} = k(H - R), \qquad\qquad (k < 0)$$

where k is some negative constant (why?) and R is the surrounding temperature, which is a constant.