# Assignment 2 Solutions

Jiawei Yao, Yannan Chen, Lianxin Zhang
jiaweiyao@link.cuhk.edu.cn, yannanchen@link.cuhk.edu.cn, lianxinzhang@link.cuhk.edu.cn

Please note that

- **Released date**: 17th Feb., 2022.
- **Due date**: 27th Feb., 2022. by 11:59pm.
- Late submission is **NOT** accepted.
- Please submit your answers as a PDF file with a name like `"118010XXX_HW2.pdf"` (Your student ID + HW No.). You may either typeset you answers directly using computers, or scan your handwritten answers. (We recommend you use the printers on campus to scan. If you use your smartphone to scan, please limit the file size $\leq$ 10MB).
- Please make sure that your submitted file is clear and readable. Submitted file that can not be opened or not readable will get 0 point.

1. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Show that if $d = a_{11}a_{22} - a_{21}a_{12} \neq 0$, then

$$A^{-1} = \frac{1}{d}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

**Solution**

If $d = a_{11}a_{22} - a_{21}a_{12} \neq 0$ then $\frac{1}{d}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{a_{11}a_{22}-a_{12}a_{21}}{d} & 0 \\ 0 & \frac{a_{11}a_{22}-a_{12}a_{21}}{d} \end{pmatrix} = I$

$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\left[\frac{1}{d}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}\right] = \begin{pmatrix} \frac{a_{11}a_{22}-a_{12}a_{21}}{d} & 0 \\ 0 & \frac{a_{11}a_{22}-a_{12}a_{21}}{d} \end{pmatrix} = I$

Therefore

$$\frac{1}{d}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} = A^{-1}$$

2. Prove that if $A$ is nonsingular then $A^T$ is nonsingular and

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

Hint: $(AB)^T = B^T A^T$.

**Solution**

Since

$$\begin{aligned} A^T\left(A^{-1}\right)^T &= \left(A^{-1}A\right)^T = I \\ \left(A^{-1}\right)^T A^T &= \left(AA^{-1}\right)^T = I \end{aligned}$$

it follows that

$$\left(A^{-1}\right)^T = \left(A^T\right)^{-1}$$

3. Let $A$ be an $n \times n$ matrix. Show that if $A^2 = O$, then $I - A$ is nonsingular and $(I - A)^{-1} = I + A$.

**Solution**

At first, we can construct two matrices $B$ and $C$ such that $B = I - A$ and $C = I + A$. Then, we can compute $BC = (I - A)(I + A) = I^2 - A^2 = I$. Therefore, $C$ is a multiplicative inverse of $B$, then $(I - A) = B$ is nonsingular. In addition, $(I - A)^{-1} = C = I + A$.

4. Let $A$ be an $n \times n$ matrix. Show that if $A^{k+1} = O$, then $I - A$ is nonsingular and

$$(I - A)^{-1} = I + A + A^2 + \cdots + A^k$$

**Solution**

At first, we can construct two matrices $B$ and $C$ such that $B = I - A$ and $C = I + A + A^2 + \cdots + A^k$. Then, we can compute $BC = (I - A)(I + A + A^2 + \cdots + A^k) = I^2 + A + A^2 + \cdots + A^k - A - A^2 - A^3 - \cdots - A^k - A^{k+1} = I$. Therefore, $C$ is a multiplicative inverse of $B$, then $(I - A) = B$ is nonsingular. In addition, $(I - A)^{-1} = C = I + A + A^2 + \cdots + A^k$.

5. For each of the following pairs of matrices, find an elementary matrix $E$ such that $EA = B$.

(a) $A = \begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix}, B = \begin{pmatrix} -4 & 2 \\ 5 & 3 \end{pmatrix}$

(b) $A = \begin{pmatrix} 2 & 1 & 3 \\ -2 & 4 & 5 \\ 3 & 1 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ -2 & 4 & 5 \end{pmatrix}$

(c) $A = \begin{pmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ -2 & 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ 0 & 3 & 5 \end{pmatrix}$

**Solution**

(a) $E = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$

(b) $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

(c) $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$

6. For each of the following pairs of matrices, find an elementary matrix $E$ such that $AE = B$.

(a)

$$A = \begin{bmatrix} 4 & 1 & 3 \\ 2 & 1 & 4 \\ 1 & 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 & 4 \\ 4 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix}, B = \begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix}$$

(c)

$$A = \begin{bmatrix} 4 & -2 & 3 \\ -2 & 4 & 2 \\ 6 & 1 & -2 \end{bmatrix}, B = \begin{bmatrix} 2 & -2 & 3 \\ -1 & 4 & 2 \\ 3 & 1 & -2 \end{bmatrix}$$

**Solution**

(a)

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

7. Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{bmatrix}$$

(a) Find elementary matrices $E_1, E_2, E_3$ such that

$$E_3 E_2 E_1 A = U$$

where $U$ is an upper triangular matrix.

(b) Determine the inverses of $E_1, E_2, E_3$ and set $L = E_1^{-1} E_2^{-1} E_3^{-1}$. What type of matrix is $L$ ? Verify that $A = LU$.

**Solution**

(a)

$$\begin{bmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - (-1)R_2} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} = U$$

So

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

(b)

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

$$L = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}.$$

8. Compute the LU factorization of each of the following matrices.

(a)

$$\begin{bmatrix} 3 & 1 \\ 9 & 5 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 2 & 4 \\ -2 & 1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 6 \\ -2 & 2 & 7 \end{bmatrix}$$

(d)

$$\begin{bmatrix} -2 & 1 & 2 \\ 4 & 1 & -2 \\ -6 & -3 & 4 \end{bmatrix}$$

**Solution**

(a)

$$\begin{bmatrix}3 & 1\\ 9 & 5\end{bmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{bmatrix}3 & 1\\ 0 & 2\end{bmatrix}$$

So

$$U = \begin{bmatrix}3 & 1\\ 0 & 2\end{bmatrix}, L = \begin{bmatrix}1 & 0\\ 3 & 1\end{bmatrix}.$$

(b)

$$\begin{bmatrix}2 & 4\\ -2 & 1\end{bmatrix} \xrightarrow{R_2 \to R_2 - (-1)R_1} \begin{bmatrix}2 & 4\\ 0 & 5\end{bmatrix}$$

So

$$U = \begin{bmatrix}2 & 4\\ 0 & 5\end{bmatrix}, L = \begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}.$$

(c)

$$\begin{bmatrix}1 & 1 & 1\\ 3 & 5 & 6\\ -2 & 2 & 7\end{bmatrix} \xrightarrow[R_3 \to R_3 - (-2)R_1]{R_2 \to R_2 - 3R_1} \begin{bmatrix}1 & 1 & 1\\ 0 & 2 & 3\\ 0 & 4 & 9\end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix}1 & 1 & 1\\ 0 & 2 & 3\\ 0 & 0 & 3\end{bmatrix}$$

So

$$U = \begin{bmatrix}1 & 1 & 1\\ 0 & 2 & 3\\ 0 & 0 & 3\end{bmatrix}, L = \begin{bmatrix}1 & 0 & 0\\ 3 & 1 & 0\\ -2 & 2 & 1\end{bmatrix}.$$

(d)

$$\begin{bmatrix}-2 & 1 & 2\\ 4 & 1 & -2\\ -6 & -3 & 4\end{bmatrix} \xrightarrow{R_2 \to R_2 - (-2)R_1} \begin{bmatrix}-2 & 1 & 2\\ 0 & 3 & 2\\ -6 & -3 & 4\end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_1} \begin{bmatrix}-2 & 1 & 2\\ 0 & 3 & 2\\ 0 & 0 & 2\end{bmatrix}$$

So

$$U = \begin{bmatrix}-2 & 1 & 2\\ 0 & 3 & 2\\ 0 & 0 & 2\end{bmatrix}, L = \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 3 & -2 & 1\end{bmatrix}.$$

9. Let $U$ and $R$ be $n \times n$ upper triangular matrices and set $T = UR$. Show that $T$ is also upper triangular and that $t_{jj} = u_{jj} r_{jj}$ for $j = 1, ..., n$ .

**Solution**

Let $U = [u_{ij}]$ and $R = [r_{ij}]$ be upper triangular matrices. This implies that for every $i \geq j, u_{ij} = 0$ and $r_{ij} = 0$. Since $T = UR$, it follows that

$$t_{ij} = \sum_{n}^{k=1} u_{ik} r_{kj}, \quad i, j = 1, 2, ..., n.$$

when $i > j$: for each $k = 1, 2, ..., n$, we have that either $k < i$ or $k \geq i$. This implies that either $k < i$ or $k > j$. Therefore, either $u_{ik} = 0$ or $r_{kj} = 0$, for each $k = 1, 2, ..., n$. Thus, for every $i > j$, it follows that $t_{ij} = 0$. This is equivalent to the matrix T being upper triangular. when $i = j$: if $i = k$, then either $i > k$ or $i < k$. In the first case we have that $u_{ik} = 0$, while in the second we have that $r_{ki} = 0$. Therefore, we can conclude the following:

$$\mathbf{t_{ii}} = \sum_{n}^{k=1} u_{ik} r_{ki} = \mathbf{u_{ii} r_{ii}}, i = 1, 2, ..., n.$$

10. In general, matrix multiplication is not commutative (i.e., $AB \neq BA$). However, in certain special cases the commutative property does hold. Show that

(a) if $D_1$ and $D_2$ are $n \times n$ diagonal matrices, then $D_1 D_2 = D_2 D_1$.

(b) if $A$ is an $n \times n$ matrix and

$$B = a_0 I + a_1 A + a_2 A^2 + \ + a_k A^k$$

where $a_0, a_1, ..., a_k$ are scalars, then $AB = BA$.

**Solution**

(a)

$$\begin{aligned}
(D_1D_2)_{ij} &= \sum_{k=1}^{n} (D_1)_{ik} (D_2)_{kj} \\
&= \sum_{k=1}^{n} \delta_{ik} (D_1)_{ik} \delta_{kj} (D_2)_{kj} \\
&= \delta_{ij} (D_1)_{ii} \delta_{kj} (D_2)_{ii} \\
&= \delta_{ij} (D_2)_{ii} \delta_{kj} (D_1)_{ii} \\
&= (D_2D_1)_{ij}
\end{aligned}$$

where $\delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases}$

(b)

$$\begin{aligned}
AB &= A\left(a_0I + a_1A + a_2A^2 + \cdots + a_kA^k\right) \\
&= A\left(a_0I\right) + A\left(a_1A\right) + A\left(a_2A^2\right) + \cdots + A\left(a_kA^k\right) \\
&= a_0AI + a_1AA + a_2AA^2 + \cdots + a_kAA^k \\
&= a_0A + a_1A^2 + a_2A^3 + \cdots + a_kA^{k+1} \\
&= a_0IA + a_1AA + a_2A^2A + \cdots + a_kA^kA \\
&= \left(a_0I\right)A + \left(a_1A\right)A + \left(a_2A^2\right)A + \cdots + \left(a_kA^k\right)A \\
&= \left(a_0I + a_1A + a_2A^2 + \cdots + a_kA^k\right)A \\
&= BA
\end{aligned}$$

11. Perform each of the following block multiplications:

(a)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 2 & 1 & 2 & -1 \end{array}\right] \left[\begin{array}{ccc} 4 & -2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \\ \hline 1 & 2 & 3 \end{array}\right]$$

(b)

$$\left[\begin{array}{cc|cc} \frac{3}{5} & -\frac{4}{5} & 0 & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \end{array}\right] \left[\begin{array}{cc|c} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

**Solution**

(a)

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 4 & -2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 1 \\ 11 & -1 & 4 \end{pmatrix}$$

(b) Let

$$A_{11} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \quad A_{12} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$A_{21} = \begin{pmatrix} 0 & 0 \end{pmatrix} \quad A_{22} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

The block multiplication is performed as follows:

$$\begin{aligned} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{pmatrix} &= \begin{pmatrix} A_{11}A_{11}^T + A_{12}A_{12}^T & A_{11}A_{21}^T + A_{12}A_{22}^T \\ A_{21}A_{11}^T + A_{22}A_{12}^T & A_{21}A_{21}^T + A_{22}A_{22}^T \end{pmatrix} \\ &= \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \end{aligned}$$

12. Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad and \quad A^T = \begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{bmatrix}$$

Is it possible to perform the block multiplications of $AA^T$ and $A^TA$? Explain.

**Solution**

It is possible to perform both block multiplications. To see this suppose that $A_{11}$ is a $k \times r$ matrix, $A_{12}$ is a $k \times (n-r)$ matrix, $A_{21}$ is an $(m-k) \times r$ matrix and $A_{22}$ is $(m-k) \times (n-r)$. It is possible to perform the block multiplication of $AA^T$ since the matrix multiplication $A_{11}A_{11}^T, A_{11}A_{21}^T, A_{12}A_{12}^T, A_{12}A_{22}^T$, $A_{21}A_{11}^T, A_{21}A_{21}^T, A_{22}A_{12}^T, A_{22}A_{22}^T$ are all possible. It is possible to perform the block multiplication of $A^TA$ since the matrix multiplications $A_{11}^TA_{11}$, $A_{11}^TA_{12}, A_{21}^TA_{21}, A_{21}^TA_{11}, A_{12}^TA_{12}, A_{22}^TA_{21}, A_{22}^TA_{22}$ are all possible.

13. Let $A$ be an $m \times n$ matrix, $X$ an $n \times r$ matrix, and $B$ an $m \times r$ matrix. Show that

$$AX = B$$

if and only if

$$A\mathbf{x}_j = \mathbf{b}_j, \quad j = 1, \ldots, r$$

**Solution**

$$\begin{aligned} AX &= A(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r) = (A\mathbf{x}_1, A\mathbf{x}_2, \ldots, A\mathbf{x}_r) \\ B &= (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r) \end{aligned}$$

$AX = B$ if and only if the column vectors of $AX$ and $B$ are equal

$A\mathbf{x}_j = \mathbf{b}_j \quad j = 1, \ldots, r$

14. Let $U$ be an $m \times m$ matrix, let $V$ be an $n \times n$ matrix, and let

$$\Sigma = \begin{pmatrix} \Sigma_1 \\ O \end{pmatrix}$$

where $\Sigma_1$ is an $n \times n$ diagonal matrix with diagonal entries $\sigma_1, \sigma_2, \ldots, \sigma_n$ and $O$ is the $(m-n) \times n$ zero matrix.

(a) Show that if $U = (U_1, U_2)$, where $U_1$ has $n$ columns, then

$$U\Sigma = U_1\Sigma_1$$

(b) Show that if $A = U\Sigma V^T$, then $A$ can be expressed as an outer product expansion of the form

$$A = \sigma_1\mathbf{u}_1\mathbf{v}_1^T + \sigma_2\mathbf{u}_2\mathbf{v}_2^T + \cdots + \sigma_n\mathbf{u}_n\mathbf{v}_n^T$$

**Solution**

(a)

$$U\Sigma = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma_1 \\ O \end{pmatrix} = U_1\Sigma_1 + U_2 O = U_1\Sigma_1$$

(b) If we let $X = U\Sigma$, then

$$X = U_1\Sigma_1 = (\sigma_1\mathbf{u}_1, \sigma_2\mathbf{u}_2, \ldots, \sigma_n\mathbf{u}_n)$$

and it follows that

$$A = U\Sigma V^T = XV^T = \sigma_1\mathbf{u}_1\mathbf{v}_1^T + \sigma_2\mathbf{u}_2\mathbf{v}_2^T + \cdots + \sigma_n\mathbf{u}_n\mathbf{v}_n^T$$

15. Let $A$ and $B$ be $n \times n$ matrices and define $2n \times 2n$ matrices $S$ and $M$ by

$$S = \begin{pmatrix} I & A \\ O & I \end{pmatrix}, \quad M = \begin{pmatrix} AB & O \\ B & O \end{pmatrix}$$

Determine the block form of $S^{-1}$ and use it to compute the block form of the product $S^{-1}MS$.

**Solution**

The block form of $S^{-1}$ is given by

S$^{-1} = \begin{pmatrix} I & -A \\ O & I \end{pmatrix}$It follows that

$$\begin{aligned} S^{-1}MS &= \begin{pmatrix} I & -A \\ O & I \end{pmatrix} \begin{pmatrix} AB & O \\ B & O \end{pmatrix} \begin{pmatrix} I & A \\ O & I \end{pmatrix} \\ &= \begin{pmatrix} I & -A \\ O & I \end{pmatrix} \begin{pmatrix} AB & ABA \\ B & BA \end{pmatrix} \\ &= \begin{pmatrix} O & O \\ B & BA \end{pmatrix} \end{aligned}$$