DDA2001: Assignment 4

1. The assignment is due at Friday **11:59 pm, April 15, 2022**.
2. Please submit your solution in **PDF** form. **Any other forms of solution will not be accepted and will be graded as 0.** Please leave enough time to make sure you have uploaded your solution as requirement before due.
3. If you submit the assignment late, you will get 0 for this assignment. **No excuses will be accepted for any late submission.**
4. **Please make sure that your file could be downloaded successfully from BB after uploading your solution file.**

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1. **(15 points)** A firm specializes in making five types of spare automobile parts. Each part is first cast from iron in the casting shop and then sent to the finishing shop where holes are drilled, surfaces are turned, and edges are ground. The required worker-hours (per 100 units) for each part in the two shops are shown below:

Part	1	2	3	4	5
Casting	2	1	3	3	1
Fishing	3	2	2	1	1

The profits from the five parts are \$30, \$20, \$40, \$25, and \$10 (per 100 units), respectively. The capacities of the casting and finishing shops over the next month will be 700 and 1,000 worker-hours, respectively. Treat the quantities of each spare part to be made over the next month as decision variables, formulate the optimization problem to maximize profit. Justify whether it is a convex problem or not.

Solution: Assume the number of part 1, 2, 3, 4, 5 are x_1, x_2, x_3, x_4, x_5 (100 units) respectively. Then we can formulate the problem as following:

$$\begin{aligned} \max\ & 30x_1 + 20x_2 + 40x_3 + 25x_4 + 10x_5 \\ s.t.\ & 2x_1 + x_2 + 3x_3 + 3x_4 + x_5 \le 700 \\ & 3x_1 + 2x_2 + 2x_3 + x_4 + x_5 \le 1000 \end{aligned}$$

The objective is equivalent to min $f(x) = -(30x_1 + 20x_2 + 40x_3 + 25x_4 + 10x_5)$. And $f(x)$ is linear so it's convex. However, since the feasible set is set of integers and thus not convex.
Therefore, it's not a convex optimization problem. □

2. **(16 points)** Are the following sets convex? Give your reasons.

   (a) The set $\{x = (x_1, \ldots, x_n) \in \mathbf{R}^n : \alpha \le a^T x \le \beta\}$, where $a^T x = \sum_{i=1}^n a_i x_i$.

   (b) A rectangle set $\{x = (x_1, \ldots, x_n) \in \mathbf{R}^n : \alpha_i \le x_i \le \beta_i, i = 1, \ldots, n\}$. (Here $\alpha_i, \beta_i \in \mathbf{R}$ are constants, and $\alpha_i \le \beta_i$)

(c) The set $\{x = (x_1, \dots, x_n) \in \mathbf{R}^n : a_1^T x \le b_1, a_2^T x \le b_2\}$.

(d) For two given finite sets $S, T \subseteq \mathbf{R}^n$, the set of points closer to the set S than T, i.e., the set contains points

$$\{x = (x_1, \dots, x_n) \in \mathbf{R}^n : \mathbf{dist}(x, S) \le \mathbf{dist}(x, T)\},$$

where $\mathbf{dist}(x, S) = \min\{\|x - z\|_2 : z \in S\}$ is the smallest distance between the point x and points in the set S.

Solution:

(a) A slab is an intersection of two halfspaces, hence it is a convex set (and a polyhedron).

(b) As in (1), a rectangle is a convex set and a polyhedron because it is a finite intersection of halfspaces.

(c) A wedge is an intersection of two halfspaces, so it is convex set. It is also a polyhedron. It is a cone if $b_1 = 0$ and $b_2 = 0$.

(d) In general this set is not convex, as the following example in $\mathbb{R}$ shows. With $S = \{-1, 1\}$ and $T = \{0\}$, we have

$$\{x \mid \mathbf{dist}(x, S) \le \mathbf{dist}(x, T)\} = (\infty, -\frac{1}{2}] \cup [\frac{1}{2}, \infty).$$

which is not convex.

$\square$

3. (a) **(5 points)**Assume $\mathcal{I}$ is a finite nonempty index set. Suppose for any $i \in \mathcal{I}$, $f_i : \mathbb{R} \to \mathbb{R}$ is convex. Show that the function $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = \max_{i \in \mathcal{I}} f_i(x)$ is convex.

(b) **(5 points)**Suppose $\mathcal{I}$ is a nonempty index set[may be infinite]. Then the function $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = \max_{i \in \mathcal{I}} f_i(x)$ is convex if $\max_{i \in \mathcal{I}} f_i(x)$ exists for each x. Now, if $f : \mathbb{R} \to \mathbb{R}_+ (f \ge 0)$ is convex and $g : \mathbb{R} \to \mathbb{R}_{++} (g > 0)$ is concave. Show that $h : \mathbb{R} \to \mathbb{R}$ with $h(x) = \frac{f(x)^2}{g(x)}$ is convex.

Solution:

(a) Let $x, y \in \mathbf{R}$ and $\lambda \in [0, 1]$, we have:

$$\forall i \in \mathcal{I}, f_i(\lambda x + (1 - \lambda) y) \le \lambda f_i(x) + (1 - \lambda) f_i(x)$$

Now take the supreme on both sides:

$$\max_{i \in \mathcal{I}} f_i(\lambda x + (1 - \lambda) y) \le \max_{i \in \mathcal{I}} (\lambda f_i(x) + (1 - \lambda) f_i(x)) \le \lambda \max_{i \in \mathcal{I}} f_i(x) + (1 - \lambda) \max_{i \in \mathcal{I}} f_i(y)$$

which proves the convexity of F by definition.

(b) Define that

$$h(x) = \max_{t \ge 0} \tilde{h}_t(x),$$
$$\tilde{h}_t(x) = -g(x) t^2 + 2 f(x) t.$$

Since

$$\frac{\mathrm{d}\tilde{h}_t(x)}{\mathrm{d}t} = -2g(x)t + 2f(x) = 0 \;\Rightarrow\; t = \frac{f(x)}{g(x)},$$
$$\frac{\mathrm{d}^2\tilde{h}_t(x)}{\mathrm{d}t^2} = -2g(x) < 0,$$

we can see that

$$h(x) = \max_{t\geq 0} \tilde{h}_t(x) = \tilde{h}_t(x)|_{t=f(x)/g(x)} = \frac{f(x)^2}{g(x)}.$$

Since $f(x)$ and $-g(x)$ are both convex, and $t \geq 0$, the nonnegative weighted sum $\tilde{h}_t(x) = -g(x)t^2 + 2f(x)t$ is also convex. Thus, we can apply the result in sub-problem (a) and show that $h(x)$ is also convex.

□

4. **(24 points)** Prove the following functions are convex.

(a) Exponential. $f(x) = e^{ax}$ is convex on $\mathbf{R}$, for any $a \in \mathbf{R}$.

(b) Powers. $f(x) = x^a$ is convex on $\mathbf{R}_+$ when $a \geq 1$ or $a \leq 0$, and concave on $\mathbf{R}_+$ for $0 \leq a \leq 1$.

(c) Powers of absolute value. $f(x) = |x|^p$, for $p \geq 1$, is convex on $\mathbf{R}$.

(d) Logarithm. $f(x) = -\log x$ is convex on $\mathbf{R}_+ = \{x > 0\}$.

(e) Weighted sum. $f(x) = w_1 \exp(x) + w_2(ax+b)^2$ for $x \in \mathbf{R}$ and $w_1, w_2 \geq 0$ and $a, b \in \mathbf{R}$.

(f) Maximum. $f(x) = \max\{\exp(x), ax^2\}$ for $x \in \mathbf{R}$ and $a > 0$.

Solution:

(a) $f(x) = e^{ax}$ then we can get $f^{''} = a^2 e^{ax} > 0$ so $f(x)$ is convex.

(b) $f(x) = x^a$ then we can get $f^{''} = a * (a-1)x^{a-2}$. When $a \geq 1$ or $a \leq 0 \rightarrow f^{''} \geq 0$, and when $0 \leq a \leq 1 \rightarrow f^{''} \leq 0$. Thus, we can get x^a is convex on $\mathbf{R}_{++ \text{when } a\geq 1 \text{ or } a\leq 0, \text{ and concave for } 0\leq a\leq 1}$.

(c) We prove by definition here. For $\forall x_1, x_2 \in \mathbf{R}$ and $\forall \lambda \in [0,1]$:

$$|\lambda x_1 + (1-\lambda)x_2|^p < \lambda |x_1|^p + (1-\lambda)|x_2|^p$$

(d) $f(x) = \log x$ then we can get $f^{''} = -\frac{1}{x^2} < 0$ so $f(x)$ is concave.

(e) Since $\exp(x)$ and $(ax+b)^2$ are both convex, $w_1, w_2 \geq 0$, then $f(x)$ is also convex.

(f) Since $\exp(x)$ and ax^2 for $a > 0$ are both convex, then $f(x)$ is also convex.

□

5. **(20 points)** Let X be a discrete random variable which takes values in $\{a_1, a_2, \ldots, a_n\}$ where $a_1 < a_2 < \ldots < a_n$, with probability $\mathbb{P}(X = a_i) = p_i$, $i = 1, \ldots, n$. The probability mass vector $p = (p_1, \ldots, p_n)$ obviously belongs to the set $P = \{p = (p_1, \ldots, p_n) \in \mathbf{R}^n : \sum_{i=1}^n p_i = 1, 0 \leq p_i \leq 1, \ i = 1, \ldots, n\}$. For each of the following functions of the vector $p = (p_1, \ldots, p_n)$, determine if the function is convex, concave, quasiconvex, or quasiconcave.

(a) $\mathbb{E}[X]$.

(b) $\mathbb{P}(X \geq \alpha)$.

(c) $\mathbb{P}(\alpha \leq X \leq \beta)$.

(d) $Var(X)$.

Solution:

(a) $\mathbb{E}x = p_1 a_1 + \cdots + p_n a_n$ is linear, hence convex, concave, quasiconvex and quasiconcave.

(b) Let $j = \min\{i \mid a_i \geq \alpha\}$. Then $\mathbb{P}(x \geq \alpha) = \sum_{i=j}^{n} p_i$ is a linear funciton of p, hence convex, concave, quasiconvex, and quasiconcave.

(c) Let $j = \min\{i \mid a_i \geq \alpha\}$ and $k = \max\{i \mid a_i \leq \beta\}$. Then $\mathbb{P}(\alpha \leq x \leq \beta) = \sum_{i=j}^{k} p_i$. This is a linear function of p, hence convex, concave, quasiconvex and quasiconcave.

(d) We have

$$Var(x) = \mathbb{E}x^2 - (\mathbb{E}x)^2 = \sum_{i=1}^{n} p_i a_i^2 - (\sum_{i=1}^{n} p_i a_i)^2.$$

So $Var(x)$ is a concave quadratic function of p. The function is not convex or quasiconvex. Consider the example with $n = 2$, $a_1 = 0$, $a_2 = 1$. Both $(p_1.p_2) = (1/4, 3/4)$ and $(p_1, p_2) = (3/4, 1/4)$ lie in the set P and have $Var(x) = 3/16$, but the convex combination $(p_1, p_2) = (1/2, 1/2)$ has a variance $Var(x) = 1/4 > 3/16$. This shows that the sublevel sets are not convex.

□

6. (a) **(5 points)** If $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ are convex, and f is a nondecreasing function, show that the function $h(x) = f(g(x))$ is convex.

   (b) **(5 points)** If $f : \mathbf{R} \to \mathbf{R}_+$ and $g : \mathbf{R} \to \mathbf{R}_+$ are convex, both are nondecreasing functions, show that the function $h(x) = f(x)g(x)$ is convex.

   (c) **(5 points)** If $f : \mathbf{R} \to \mathbf{R}_+$ is concave and nondecreasing, $g : \mathbf{R} \to \mathbf{R}_+$ is concave and nonincreasing, show that the function $h(x) = f(x)g(x)$ is concave.

Solution:

(a) For $\forall x_1, x_2 \in \mathbf{R}$ and $\forall \lambda \in [0, 1]$

$$\begin{aligned} h\left(\lambda x_1 + (1-\lambda)x_2\right) &= f\left(g\left(\lambda x_1 + (1-\lambda)x_2\right)\right) \\ &\leqslant f\left(\lambda g\left(x_1\right) + (1-\lambda) g\left(x_2\right)\right) \\ &\leqslant \lambda f\left(g\left(x_1\right)\right) + (1-\lambda) f\left(g\left(x_2\right)\right) \\ &= \lambda h\left(x_1\right) + (1-\lambda) h\left(x_2\right) \end{aligned}$$

The second inequality is due to f is non decareasing and g is convex. The third inequality is due to f is convex.

(b) We prove the result by verifying Jensen's inequality. f and g are positive and convex, hence for $0 \leq \theta \leq 1$,

$$\begin{aligned} f(\theta x + (1-\theta)y)g(\theta x + (1-\theta)y) &\leq (\theta f(x) + (1-\theta)f(y))\,(\theta g(x) + (1-\theta)g(y)) \\ &= \theta f(x)g(x) + (1-\theta)f(y)g(y) + \theta(1-\theta)(f(y) - f(x))(g(x) - g(y)). \end{aligned}$$

The third term is less than or equal to zero if f and g are both increasing or both decreasing. Therefore

$$f(\theta x + (1-\theta)y)g(\theta x + (1-\theta)y) \le \theta f(x)g(x) + (1-\theta)f(y)g(y)$$

(c) Reverse the inequalities in the solution of subproblem (b).

$\square$