Assignment 4

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Please note that

- **Released date: Mar. 10th**
- **Due date: Mar. 17th, by 11:59 pm.**
- Late submission is **NOT** accepted.
- Please hand in your solution in PDF format titled "student number + HW4.pdf". Note that the file size is at most 10MB.

Question 1. Consider the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

(a) Show that $\mathbf{x}_1$ and $\mathbf{x}_2$ form a basis of $\mathbb{R}^2$.

(b) Why must $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ be linearly dependent?

(c) What is the dimension of $\text{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$?

Solution

(a) Solving the equation $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = \mathbf{0}$ by reducing the corresponding argumented matrix

$$\left[\begin{array}{cc|c} 2 & 4 & 0 \\ 1 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

gives $c_1 = c_2 = 0$. Therefore $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent. Since $\mathbb{R}^2$ has dimension 2, $\mathbf{x}_1$ and $\mathbf{x}_2$ form a basis of $\mathbb{R}^2$

(b) $\mathbb{R}^2$ has dimension 2, and $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are 3 vectors in $\mathbb{R}^2$. As $3 > 2$, $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ must be linearly dependent.

(c) Because $\text{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ is a subspace of $\mathbb{R}^2$, it has dimension at most 2; Since $\mathbf{x}_1$ and $\mathbf{x}_2$ form a basis of $\mathbb{R}^2$, the dimension of $\text{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ is at least 2. Therefore the dimension of $\text{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ is 2.

Question 2. Let

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$$

(a) Show that $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{x}_3$ are linearly dependent.

(b) Show that $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent.

(c) What is the dimension of $\mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$?

(d) Give a geometric description of $\mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$.

Solution

(a) Show that $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{x}_3$ are linearly dependent.

Solving the euqation $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = \mathbf{0}$ by reducing the argumented matrix form gives:

$$\left[\begin{array}{ccc|c} 2 & 3 & 2 & 0 \\ 1 & -1 & 6 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since there are only two pivot columns, the above equation has non-trivial solutions, therefore, $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{x}_3$ are linearly dependent.

(b) Notice that the first and second columns of reduced row echelon form in (a) are pivot columns, which means the first and second columns of the matrix before reduing, i.e. $\mathbf{x}_1$ and $\mathbf{x}_2$, are linearly independent.

(c) By (a) we have $\mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2)$. Because $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent, the dimension of $\mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ is 2.

(d) The subspace $\mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ is a plane through the origin in $\mathbb{R}^3$.

Question 3. Find a basis of subspace S of $\mathbb{R}^4$ consisting of all vectors of the form $(a+b, a-b+2c, b, c)^T$, where a, b, and c are all real numbers. What is the dimension of S?

Solution

Let $\mathbf{x}$ be an element of S. Then there exist real numbers a, b, and c such that

$$\mathbf{x} = \begin{bmatrix} a+b \\ a-b+2c \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

Let $B = \{(1,1,0,0)^T, (1,-1,1,0)^T, (0,2,0,1)^T\}$. Since B spans S, and since B is a linearly independent set (check by definition), it follows that B is a basis for S and dimension of S is 3.

Question 4. Let V be the set of all polynomials of degree less than 3. Let S be the subspace of V consisting of all polynomials of the form $ax^2 + bx + 2a + 3b$. Find a basis of S.

Solution

Let $f \in S$. Then there exist reals a and b such that

$$f(x) = ax^2 + bx + 2a + 3b = a(x^2+2) + b(x+3).$$

Since $f(x) = 0, \forall x$ if and only if $a = b = 0$, $\{x^2+2, x+3\}$ is a linearly independent set. Therefore $\{x^2+2, x+3\}$ is a basis of S.

Question 5. In $C[-\pi, \pi]$, find the dimension of the subspace spanned by 1, $\cos 2x$, $\cos^2 x$.

Solution

Let f be a vector in $\text{Span}(1, \cos 2x, \cos^2 x)$. Then there exist reals a, b, c such that

$$f(x) = a + b\cos 2x + c\cos^2 x.$$

Since $f = \mathbf{0}$ if and only if $f(x) = 0, \forall x$, and because

$$\begin{aligned} f(x) &= a + b\cos 2x + c\cos^2 x \\ &= a + b\cos 2x + c\frac{1-\cos 2x}{2} \\ &= (a + \frac{c}{2}) + (b_1 - \frac{c}{2})\cos 2x, \end{aligned}$$

we have $f = \mathbf{0}$ if and only if $(a + \frac{c}{2}) = 0$ and $(b - \frac{c}{2}) = 0$, or in matrix form:

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

Notice that the coefficient matrix is already in reduced row echelon form, picking the independent variable a, b, we see the any vector $f \in \text{Span}(1, \cos 2x, \cos^2 x)$ can be uniquely represented by linear combination of 1 and $\cos 2x$. Therefore, the subspace has dimension 2.

Question 6. Show that if U and V are subspaces of $\mathbb{R}^n$ and $U \cap V = \{\mathbf{0}\}$, then

$$\dim(U + V) = \dim U + \dim V$$

Solution

For either $U = \{\mathbf{0}\}$ or $V = \{\mathbf{0}\}$, the proof is trivial. In the following we consider $\dim U > 0$ and $\dim V > 0$.

Let $\{\mathbf{u}_1, \dots \mathbf{u}_m\}$ be a basis of U and $\{\mathbf{v}_1, \dots \mathbf{v}_n\}$ be a basis of V, where $m = \dim U$ and $n = \dim V$. We have $\text{Span}(\mathbf{u}_1, \dots \mathbf{u}_m, \mathbf{v}_1, \dots \mathbf{v}_n) = U + V$.

In the following we show that $\mathbf{u}_1, \dots \mathbf{u}_m, \mathbf{v}_1, \dots \mathbf{v}_n$ are indeed linearly independent. Consider the equation

$$c_1\mathbf{u}_1 + \cdots + c_m\mathbf{u}_m + c_{m+1}\mathbf{v}_1 + \cdots + c_{m+n}\mathbf{v}_n = \mathbf{0},$$

let $\mathbf{u} = c_1\mathbf{u}_1 + \cdots + c_m\mathbf{u}_m$ and $\mathbf{v} = c_{m+1}\mathbf{v}_1 + \cdots + c_{m+n}\mathbf{v}_n$, the above equation is equivalent to $\mathbf{u} + \mathbf{v} = \mathbf{0}$.

Notice that $\mathbf{u} \in U$ and $\mathbf{u} = -\mathbf{v} \in V$, by $U \cap V = \{\mathbf{0}\}$ we have $\mathbf{u} = \mathbf{v} = \mathbf{0}$.

Since $\{\mathbf{u}_1, \dots \mathbf{u}_m\}$ and $\{\mathbf{v}_1, \dots \mathbf{v}_n\}$ are linearly independent by themselves, the only solution to the above equation is therefore $c_i = 0, \forall i = 1, \dots, m + n$.

Therefore $\{\mathbf{u}_1, \dots \mathbf{u}_m, \mathbf{v}_1, \dots \mathbf{v}_n\}$ form a basis of $U + V$ and,

$$\dim(U + V) = m + n = \dim U + \dim V.$$

Question 7. Let $\mathbf{e}_1 = (1, 0)^T$ and $\mathbf{e}_2 = (0, 1)^T$. For each of the following, find the transition matrix corresponding to the change of basis from $\{\mathbf{u}_1, \mathbf{u}_2\}$ to $\{\mathbf{e}_1, \mathbf{e}_2\}$.

(a) $\mathbf{u}_1 = (1, 1)^T, \mathbf{u}_2 = (-1, 1)^T$

(b) $\mathbf{u}_1 = (1, 2)^T, \mathbf{u}_2 = (2, 5)^T$

(c) $\mathbf{u}_1 = (0, 1)^T, \mathbf{u}_2 = (1, 0)^T$

Solution

Let

$$U = [\mathbf{u}_1, \mathbf{u}_2]$$

and

$$E = [\mathbf{e}_1, \mathbf{e}_2]$$

.Transition matrix S corresponding to the change of basis from $\{\mathbf{u}_1, \mathbf{u}_2\}$ to $\{\mathbf{e}_1, \mathbf{e}_2\}$ is given by:

$$S = E^{-1}U$$

(a) $S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

(b) $S = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

(c) $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Question 8. Let $\mathbf{v}_1 = (3, 2)^T$ and $\mathbf{v}_2 = (4, 3)^T$. For each ordered basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ given in Question 7, find the transition matrix from $\{\mathbf{v}_1, \mathbf{v}_2\}$ to $\{\mathbf{u}_1, \mathbf{u}_2\}$.

Solution

Let

$$V = [\mathbf{v}_1, \mathbf{v}_2]$$

and

$$U = [\mathbf{u}_1, \mathbf{u}_2]$$

.Transition matrix S corresponding to the change of basis from $\{\mathbf{v}_1, \mathbf{v}_2\}$ to $\{\mathbf{u}_1, \mathbf{u}_2\}$ is given by:

$$S = U^{-1}V$$

(a) $S = \begin{bmatrix} 2.5 & 3.5 \\ -0.5 & -0.5 \end{bmatrix}$

(b) $S = \begin{bmatrix} 11 & 14 \\ -4 & -5 \end{bmatrix}$

(c) $S = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

Question 9. Let $\mathbf{v}_1 = (4, 6, 7)^T$,$\mathbf{v}_2 = (0, 1, 1)^T$, and $\mathbf{v}_3 = (0, 1, 2)^T$, and let $\mathbf{u}_1 = (1, 1, 1)^T$,$\mathbf{u}_2 = (1, 2, 2)^T$, $\mathbf{u}_3 = (2, 3, 4)^T$.

(a) Find the transition matrix from $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

(b) If $\mathbf{x} = 2\mathbf{v}_1 + 3\mathbf{v}_2 - 4\mathbf{v}_3$, determine the coordinates of $\mathbf{x}$ with respect to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

Solution

(a) Let

$$V = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$$

and

$$U = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$$

.Transition matrix S corresponding to the change of basis from $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is given by:

$$S = U^{-1}V = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(b) $[\mathbf{x}]_U = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ -2 \end{bmatrix}$

Question 10. Let V be the set of all polynomials of degree less than 3. Find the transition matrix representing the change of coordinates on V from the ordered basis $[1, x, x^2]$ to the ordered basis

$$[1, 1+x, 1+x+x^2]$$

Solution

It's easy to calculate the transition matrix from $[1, 1+x, 1+x+x^2]$ to $[1, x, x^2]$:

$$1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$$

$$1 + x = 1 \cdot 1 + 1 \cdot x + 0 \cdot x^2$$

$$1 + x + x^2 = 1 \cdot 1 + 1 \cdot x + 1 \cdot x^2$$

The transition matrix is:

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The inverse of S gives the transition matrix from $[1, x, x^2]$ to $[1, 1{+}x, 1{+}x{+}x^2]$:

$$S^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Question 11. Let $E = \{\mathbf{u}_1, ..., \mathbf{u}_n\}$ and $F = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ be two ordered bases for $\mathbb{R}^n$, and set

$$U = (\mathbf{u}_1, ..., \mathbf{u}_n)$$

$$V = (\mathbf{v}_1, ..., \mathbf{v}_n)$$

Show that the transition matrix from E to F can be determined by calculating the reduced row echelon form of $(V|U)$.

Solution

The transition matrix from E to F is $V^{-1}U$.
On the other hand, the reduced row echelon form of $(V|U)$ involves executing row operations until V becomes the identity matrix. This is equivalent to multiplying both left and right hand sides by V^{-1}, i.e.

$$V^{-1}V|V^{-1}U = I|V^{-1}U$$

Thus the transition matrix from E to F is the result (i.e the right hand side) of the reduced row echelon form of $(V|U)$.

Question 12. For each of the following matrices, find a basis for the row space, a basis for the column space, and a basis for the null space.

(a)

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix}$$

(b)

$$\begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix}$$

Solution

(a) The matrix can be reduced to the row echelon form as $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

So a basis for the row space can be

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

.

A basis for the column space can be

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix} \right\}$$

.

A basis for the null space can be

$$\left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(b) The matrix can be reduced to the row echelon form as $\begin{bmatrix} 1 & 0 & 0 & -\frac{10}{7} \\ 0 & 1 & 0 & -\frac{2}{7} \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

So a basis for the row space can be

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -\frac{10}{7} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{2}{7} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

.

A basis for the column space can be

$$\left\{ \begin{pmatrix} -3 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} \right\}$$

.

A basis for the null space can be

$$\left\{ \begin{pmatrix} \frac{10}{7} \\ \frac{2}{7} \\ 0 \\ 1 \end{pmatrix} \right\}$$

(c) The matrix can be reduced to the row echelon form as $\begin{bmatrix} 1 & 0 & 0 & -\frac{13}{20} \\ 0 & 1 & 0 & \frac{21}{20} \\ 0 & 0 & 1 & \frac{3}{4} \end{bmatrix}$.

So a basis for the row space can be

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -\frac{13}{20} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{21}{20} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \frac{3}{4} \end{pmatrix} \right\}$$

.

A basis for the column space can be

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

.

A basis for the null space can be

$$\left\{ \begin{pmatrix} \frac{13}{20} \\ -\frac{21}{20} \\ -\frac{3}{4} \\ 1 \end{pmatrix} \right\}$$

Question 13. In each of the following, determine the dimension of the subspace of $\mathbb{R}^3$

spanned by the given vectors:

(a)

$$\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 6 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

Solution

(a) $\begin{bmatrix} 1 & 2 & -3 \\ -2 & -2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & -3 \\ 0 & 0 & 12 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

So the dimension is 3.

(b) $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

So the dimension is 3.

(c) $\begin{bmatrix} 1 & -2 & 3 & 2 \\ -1 & 2 & -2 & -1 \\ 2 & -4 & 5 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 8 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$

So the dimension is 3.

Question 14. For each of the following choices of A and $\mathbf{b}$, determine whether $\mathbf{b}$ is in the column space of A and state whether the system $A\mathbf{x} = \mathbf{b}$ is consistent:

(a)

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

(c)

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$$

Solution

Because we have the conclusion that a linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.

(a)

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & 4 & 8 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

So $\mathbf{b}$ is in the column space of A, and the system is consistent.

(b)

$$\left[\begin{array}{cc|c} 2 & 1 & 4 \\ 3 & 4 & 6 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 1 & 4 \\ 0 & \frac{5}{2} & 0 \end{array}\right]$$

So $\mathbf{b}$ is in the column space of A, and the system is consistent.

(c)

$$\left[\begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

So $\mathbf{b}$ is in the column space of A, and the system is consistent.

Question 15. Let A be a 4×3 matrix and suppose that the vectors $\mathbf{z}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{z}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ form a basis for $\mathrm{Null}(A)$. If $\mathbf{b} = \mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3$, find all solutions of the system $A\mathbf{x} = \mathbf{b}$.

Solution

Because $\mathbf{b} = \mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3$, we can get $\mathbf{b} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)(1, 2, 1)^T$. So the solutions of the system $A\mathbf{x} = \mathbf{b}$ is

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + Span \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

.

Question 16. Let A be a 4×5 matrix and let U be the reduced row echelon form of A.

If $\mathbf{a}_1 = \begin{bmatrix} 2 \\ 1 \\ -3 \\ -2 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$,

(a) find a basis for Null(A).

(b) given that $\mathbf{x}_0$ is a solution to $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 0 \\ 5 \\ 3 \\ 4 \end{bmatrix}$, and $\mathbf{x}_0 = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$

(1) find all solutions to the system.

(2) determine the remaining column vectors of A.

Solution

(a) Solving the equation $Ax = \mathbf{0}$ gives:

$$\begin{cases} x_1 = -2x_3 + x_5 \\ x_2 = -3x_3 + 2x_5 \\ x_3 = x_3 \\ x_4 = -5x_5 \\ x_5 = x_5 \end{cases}$$

therefore a basis of Null(A) can be:

$$\left\{ \begin{pmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ -5 \\ 1 \end{pmatrix} \right\}$$

(b) (1) The solution to the system can be written as:

$$\begin{pmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 0 \end{pmatrix} + Span\left\{ \begin{pmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ -5 \\ 1 \end{pmatrix} \right\}$$

(2) Because $\mathbf{x}_0$ is a solution to $A\mathbf{x} = \mathbf{b}$, we can get $3\mathbf{a}_1 + 2\mathbf{a}_2 + 2\mathbf{a}_4 = \mathbf{b}$. So $\mathbf{a}_4 = \frac{1}{2}(\mathbf{b} - 3\mathbf{a}_1 - 2\mathbf{a}_2) = (-2, -1, 3, 4)^T$.
Based on (a), we know that $(-2, -3, 1, 0, 0)^T, (1, 2, 0, -5, 1)^T$ are the solutions to $Ax = \mathbf{0}$. Then we get:

$$\begin{cases} -2\mathbf{a}_1 - 3\mathbf{a}_2 + \mathbf{a}_3 = 0 \\ \mathbf{a}_1 + 2\mathbf{a}_2 - 5\mathbf{a}_4 + \mathbf{a}_5 = 0 \end{cases}$$

So, $\mathbf{a}_3 = (1, 8, 3, -1)^T, \mathbf{a}_5 = (-10, -10, 12, 20)^T$

Question 17. Let A and B be $m \times n$ matrices. Show that $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$.

Solution

Assuming $A = (\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n), B = (\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n)$.
Then we have $A + B = (\mathbf{a}_1 + \mathbf{b}_1, \mathbf{a}_2 + \mathbf{b}_2, ..., \mathbf{a}_n + \mathbf{b}_n)$. So, we have $\text{rank}(A) = \dim(Span(\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n))$,
$\text{rank}(B) = \dim(Span(\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n))$,
$\text{rank}(A + B) = \dim(Span(\mathbf{a}_1 + \mathbf{b}_1, \mathbf{a}_2 + \mathbf{b}_2, ..., \mathbf{a}_n + \mathbf{b}_n))$.
For any vector $\mathbf{x} \in Span(\mathbf{a}_1 + \mathbf{b}_1, \mathbf{a}_2 + \mathbf{b}_2, ..., \mathbf{a}_n + \mathbf{b}_n)$, and $\mathbf{x}$ can be denoted as $\mathbf{x} = c_1(\mathbf{a}_1 + \mathbf{b}_1) + c_2(\mathbf{a}_2 + \mathbf{b}_2) + ... + c_n(\mathbf{a}_n + \mathbf{b}_n)$.

$$\begin{aligned} \mathbf{x} &= c_1(\mathbf{a}_1 + \mathbf{b}_1) + c_2(\mathbf{a}_2 + \mathbf{b}_2) + ... + c_n(\mathbf{a}_n + \mathbf{b}_n) \\ &= (c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + ... + c_n\mathbf{a}_n) + (c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + ... + c_n\mathbf{b}_n) \\ &\in Span(\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n) + Span(\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n) \end{aligned}$$

So we get that $Span(\mathbf{a}_1 + \mathbf{b}_1, \mathbf{a}_2 + \mathbf{b}_2, ..., \mathbf{a}_n + \mathbf{b}_n) \in Span(\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n) + Span(\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n)$.
So $\text{rank}(A + B) = \dim(Span(\mathbf{a}_1 + \mathbf{b}_1, \mathbf{a}_2 + \mathbf{b}_2, ..., \mathbf{a}_n + \mathbf{b}_n)) \leq \dim(Span(\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n) + Span(\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n)) \leq \dim(Span(\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n)) + \dim(Span(\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n)) = \text{rank}(A) + \text{rank}(B)$.

Question 18. Show that if A and B are $n \times n$ matrices and $\mathrm{Null}(A - B) = \mathbb{R}^n$ then $A = B$.

Solution

Because $\mathrm{Null}(A - B) = \mathbb{R}^n$,
then for any vector $\mathbf{v} \in \mathbb{R}^n$, $(A - B)\mathbf{v} = \mathbf{0}$.
So $A - B = \mathbf{0}$, i.e. $A = B$.