Assignment 6

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Please note that

- **Released date**: 31 Mar., 2022.
- **Due date**:
- Late submission is **NOT** accepted.
- Please submit your answers as a PDF file with a name like `"118010XXX_HW1.pdf"` (Your student ID + HW No.). You may either typeset you answers directly using computers, or scan your handwritten answers. (We recommend you use the printers on campus to scan. If you use your smartphone to scan, please limit the file size $\leq$ 10MB).
- Please make sure that your submitted file is clear and readable. Submitted file that can not be opened or not readable will get 0 point.

1. Let L be the linear operator on $\mathbb{R}^2$ defined by $L(\mathbf{x}) = (x_1\cos\alpha - x_2\sin\alpha, x_1\sin\alpha + x_2\cos\alpha)^T$ Express x_1, x_2, and $L(\mathbf{x})$ in terms of polar coordinates. Describe geometrically the effect of the linear transformation.

Solution

$x_1 = r\cos\theta, x_2 = r\sin\theta$ where $r = \left(x_1^2 + x_2^2\right)^{1/2}$ and θ is the angle between $\mathbf{x}$ and $\mathbf{e}_1$.

$$\begin{aligned} L(\mathbf{x}) &= (r\cos\theta\cos\alpha - r\sin\theta\sin\alpha, r\cos\theta\sin\alpha + r\sin\theta\cos\alpha)^T \\ &= (r\cos(\theta+\alpha), r\sin(\theta+\alpha))^T \end{aligned}$$

The linear transformation L has the effect of rotating a vector by an α in the counterclockwise direction.

2. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator. If $L\left((1,2)^T\right) = (-2,3)^T$ and $L\left((1,-1)^T\right) = (5,2)^T$ find the value of $L\left((7,5)^T\right)$.

Solution

Let

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

To determine $L(\mathbf{x})$ we must first express $\mathbf{x}$ as a linear combination

$$\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2$$

To do this we must solve the system $U\mathbf{c} = \mathbf{x}$ for $\mathbf{c}$. The solution is $\mathbf{c} = (4, 3)^T$ and it follows that

$$L(\mathbf{x}) = L(4\mathbf{u}_1 + 3\mathbf{u}_2) = 4L(\mathbf{u}_1) + 3L(\mathbf{u}_2) = 4\begin{pmatrix} -2 \\ 3 \end{pmatrix} + 3\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 18 \end{pmatrix}$$

3. Determine whether the following are linear transformations from $\mathbb{R}^3$ into $\mathbb{R}^2$.

   (a) $L(\mathbf{x}) = (x_2, x_3)^T$

   (b) $L(\mathbf{x}) = (0, 0)^T$

Solution

(a) $L(\mathbf{x})$ is a linear transformation, since

$$\begin{aligned} L(\alpha\mathbf{x} + \beta\mathbf{y}) &= \begin{pmatrix} \alpha x_2 + \beta y_2 \\ \alpha x_3 + \beta y_3 \end{pmatrix} = \begin{pmatrix} \alpha x_2 \\ \alpha x_3 \end{pmatrix} + \begin{pmatrix} \beta y_2 \\ \beta y_3 \end{pmatrix} \\ &= \alpha L(\mathbf{x}) + \beta L(\mathbf{y}) \end{aligned}$$

(b) $L(\mathbf{x})$ is a linear transformation, since

$$\begin{aligned} L(\alpha\mathbf{x} + \beta\mathbf{y}) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \alpha\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &= \alpha L(\mathbf{x}) + \beta L(\mathbf{y}) \end{aligned}$$

4. Determine whether the following are linear transformations from $\mathbb{R}^2$ into $\mathbb{R}^3$.

   (a) $L(\mathbf{x}) = (x_1, x_2, 1)^T$

   (b) $L(\mathbf{x}) = (x_1, x_2, x_1 + 2x_2)^T$

Solution

(a) $L(\mathbf{x})$ is not a linear transformation, since

$$L(\alpha\mathbf{x}) = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ 1 \end{pmatrix}, \alpha L(\mathbf{x}) = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha \end{pmatrix}$$

it follows that $L(\alpha\mathbf{x}) \neq \alpha L(\mathbf{x})$

(b) $L(\mathbf{x})$ is a linear transformation, since

$$\begin{aligned} L(\alpha\mathbf{x} + \beta\mathbf{y}) &= \begin{pmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \\ \alpha x_1 + \beta y_1 + 2(\alpha x_2 + \beta y_2) \end{pmatrix} = \alpha \begin{pmatrix} x_1 \\ x_2 \\ x_1 + 2x_2 \end{pmatrix} + \beta \begin{pmatrix} y_1 \\ y_2 \\ y_1 + 2y_2 \end{pmatrix} \\ &= \alpha L(\mathbf{x}) + \beta L(\mathbf{y}) \end{aligned}$$

5. Determine whether the following are linear operators on $\mathbb{R}^{n\times n}$.

   (a) $L(A) = 2A$

   (b) $L(A) = A^T$

Solution

(a) $L(A) = 2A$ is a linear operator, since

$$\begin{aligned} L(\alpha A + \beta B) &= 2(\alpha A + \beta B) = \alpha(2A) + \beta(2B) \\ &= \alpha L(\mathbf{x}) + \beta L(\mathbf{y}) \end{aligned}$$

(b)$L(A) = A^T$ is a linear operator, since

$$\begin{aligned} L(\alpha A + \beta B) &= (\alpha A + \beta B)^T = \alpha(A^T) + \beta(B^T) \\ &= \alpha L(\mathbf{x}) + \beta L(\mathbf{y}) \end{aligned}$$

6. Determine whether the following are linear transformations from P_2 to P_3.

   (a) $L(p(x)) = xp(x)$

   (b) $L(p(x)) = x^2 + p(x)$

Solution

Let $p_1(x) = ax + b$ is a polynomial in P_2, and $p_2(x) = cx + d$ is also a polynomial in P_2, we assume α and β are scalars.

(a) $L(\alpha p_1(x) + \beta p_2(x)) = x(\alpha p_1(x) + \beta p_2(x))$

$= x\alpha p_1(x) + x\beta p_2(x)$

$= \alpha L(p_1(x)) + \beta L(p_2(x))$

$= \alpha a x^2 + \alpha b x + \beta c x^2 + \beta d x$

$= (\alpha a + \beta c)x^2 + (\alpha c + \beta d)x$

Thus, L is a linear transformation from P_2 to P_3.

(b) $L(\alpha p_1(x) + \beta p_2(x)) = x^2 + \alpha p_1(x) + \beta p_2(x)$

$\alpha L(p_1(x)) + \beta L(p_2(x)) = \alpha x^2 + \alpha p_1(x) + \beta x^2 + \beta p_2(x) = (\alpha + \beta)x^2 + \alpha p_1(x) + \beta p_2(x)$

$L(\alpha p_1(x) + \beta p_2(x)) \neq \alpha p_1(x) + \beta p_2(x)$

Thus, L is not a linear transformation from P_2 to P_3.

7. For each $f \in C[0,1]$, define $L(f) = F$, where $F(x) = \int_0^x f(t)dt \quad 0 \leq x \leq 1$

   Show that L is a linear operator on $C[0,1]$ and then find $L(e^x)$ and $L(x^2)$.

Solution

If $f, g \in C[0,1]$ then

$$\begin{aligned} L(\alpha f + \beta g) &= \int_0^x (\alpha f(t) + \beta g(t))dt \\ &= \alpha \int_0^x f(t)dt + \beta \int_0^x g(t)dt \\ &= \alpha L(f) + \beta L(g) \end{aligned}$$

Thus L is a linear transformation from $C[0,1]$ to $C[0,1]$.

$$L(e^x) = \int_0^x (e^t)dt \quad = e^x - 1$$

$$L(x^2) = \int_0^x (x^2)dt \quad = \frac{1}{3}x^3$$

8. Find the kernel and range of each of the following linear operators on P_3 :

   (a) $L(p(x)) = xp'(x)$

   (b) $L(p(x)) = p(x) - p'(x)$

   (c) $L(p(x)) = p(0)x + p(1)$

Solution

(a) If $p(x) = ax^2 + bx + c$ is in $\ker(L)$, then

$$L(p) = \left(2ax^2 + bx\right)$$

must equal the zero polynomial $z(x) = 0x^2 + 0x$. Equating coefficients we see that $a = b = 0$ and hence $\ker(L) = P_1$. The range of L is $Span(x^2, x)$.

(b) If $p(x) = ax^2 + bx + c$ is in $\ker(L)$, then

$$L(p) = \left(ax^2 + bx + c\right) - (2ax + b) = ax^2 + (b - 2a)x + (c - b)$$

must equal the zero polynomial $z(x) = 0x^2 + 0x + 0$. Equating coefficients we see that $a = b = c = 0$ and hence $\ker(L) = \{\mathbf{0}\}$. The range of L is all of P_3. To see this note that if $p(x) = ax^2 + bx + c$ is any vector in P_3 and we define $q(x) = ax^2 + (b + 2a)x + c + b + 2a$ then

$$L(q(x)) = \left(ax^2 + (b + 2a)x + c + b + 2a\right) - (2ax + b + 2a) = ax^2 + bx + c = p(x)$$

(c) If $p(x) = ax^2 + bx + c$ is in $\ker(L)$, then

$$L(p) = (cx + a + b + c)$$

must equal the zero polynomial $z(x) = 0x + 0 + 0 + 0$. Equating coefficients we see that $a = b = c = 0$ and hence $\ker(L) = Span(x^2 - x)$. The range of L is P_2.

9. For each of the following linear transformations L mapping $\mathbb{R}^3$ into $\mathbb{R}^2$, find a matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for every $\mathbf{x}$ in $\mathbb{R}^3$:

   (a) $L\left((x_1, x_2, x_3)^T\right) = (x_1 + x_2, 0)^T$

   (b) $L\left((x_1, x_2, x_3)^T\right) = (x_1, x_2)^T$

   (c) $L\left((x_1, x_2, x_3)^T\right) = (x_2 - x_1, x_3 - x_2)^T$

Solution

(a) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(c) $A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

10. Let

$$\mathbf{y}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{y}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{y}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and let $\mathcal{I}$ be the identity operator on $\mathbb{R}^3$. (a) Find the coordinates of $\mathcal{I}(\mathbf{e}_1), \mathcal{I}(\mathbf{e}_2)$, and $\mathcal{I}(\mathbf{e}_3)$ with respect to $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$. (b) Find a matrix A such that $A\mathbf{x}$ is the coordinate vector of $\mathbf{x}$ with respect to $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$.

Solution

(a) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \mathcal{I}(\mathbf{e}_1) = 0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0\mathbf{y}_1 + 0\mathbf{y}_2 + 1\mathbf{y}_3$

$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \mathcal{I}(\mathbf{e}_2) = 0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (-1) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0\mathbf{y}_1 + 1\mathbf{y}_2 - 1\mathbf{y}_3$

$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathcal{I}(\mathbf{e}_1) = 1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (-1) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1\mathbf{y}_1 - 1\mathbf{y}_2 + 0\mathbf{y}_3$

(b) $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$

11. Let $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ be defined as in the last exercise, and let L be the linear operator on $\mathbb{R}^3$ defined by

$$\begin{aligned} &L\left(c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3\right) \\ &= \left(c_1 + c_2 + c_3\right)\mathbf{y}_1 + \left(2c_1 + c_3\right)\mathbf{y}_2 - \left(2c_2 + c_3\right)\mathbf{y}_3 \end{aligned}$$

(a) Find a matrix representing L with respect to the ordered basis $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$.

(b) For each of the following, write the vector $\mathbf{x}$ as a linear combination of $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ and use the matrix from part (a) to determine $L(\mathbf{x})$:

(i) $\mathbf{x} = (7, 5, 2)^T$

(ii) $\mathbf{x} = (3, 2, 1)^T$

(iii) $\mathbf{x} = (1, 2, 3)^T$

Solution

(a) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & -2 & -1 \end{bmatrix}$

(b)

(a) $\mathbf{x} = \begin{pmatrix} 7 \\ 5 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

Let $S = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, then $S^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$.

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = S^{-1}\mathbf{x} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 7 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$

$$L(\mathbf{x}) = L \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = 7\mathbf{y}_1 + 6\mathbf{y}_2 - 8\mathbf{y}_3$$

(b)

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = S^{-1}\mathbf{x} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$L(\mathbf{x}) = L \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 3\mathbf{y}_1 + 3\mathbf{y}_2 - 3\mathbf{y}_3$$

(c)

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = S^{-1}\mathbf{x} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$$

$$L(\mathbf{x}) = L \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = \mathbf{y}_1 + 5\mathbf{y}_2 + 3\mathbf{y}_3$$

12. The linear transformation L defined by

$$L(p(x)) = p'(x) + p(0)$$

maps P_3 into P_2. Find the matrix representation of L with respect to the ordered bases $[x^2, x, 1]$ and

$[2, 1-x]$. For each of the following vectors $p(x)$ in P_3, find the coordinates of $L(p(x))$ with respect to the ordered basis $[2, 1-x]$:

(a) $x^2 + 2x - 3$

(b) $x^2 + 1$

Solution

First, we will see the matrix representation of L. For that, $L\left(x^2\right) = 2x, L(x) = 1$ and $L(1) = 1$. Now we will express $L\left(x^2\right), L(x)$ and $L(1)$ in terms of the basis $\{2, 1-x\}$. So,

$$\mathrm{L}\left(\mathrm{x}^2\right) = 2\mathrm{x} = 1 \cdot 2 + 2 \cdot (1-x),$$

$$L(x) = 1 = \frac{1}{2} \cdot 2 + 0 \cdot (1-x),$$

$$L(1) = 1 = \frac{1}{2} \cdot 2 + 0 \cdot (1-x).$$

So the matrix representation of $L = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ -2 & 0 & 0 \end{bmatrix}$.

(a) Let $p(x) = x^2 + 2x - 3$, then $L(p(x)) = 2x + 2 - 3 = 2x - 1$. Now we can see that $2x - 1 = \frac{1}{2} \cdot 2 - 2 \cdot (1-x)$. So, the coordinates for $L(p(x))$ with respect to the basis $\{2, 1-x\}$ is $\left(\frac{1}{2}, -2\right)$.

(b) Let $p(x) = x^2 + 1$, then $L(p(x)) = 2x + 1$. Now we can see that $2x + 1 = \frac{3}{2} \cdot 2 - 2 \cdot (1-x)$ So, the coordinates for $L(p(x))$ with respect to the basis $(2, 1-x)$ is $\left(\frac{3}{2}, -2\right)$.

13. Let $E = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $F = \{\mathbf{b}_1, \mathbf{b}_2\}$, where

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

and

$$\mathbf{b}_1 = (1, -1)^T, \quad \mathbf{b}_2 = (2, -1)^T$$

For each of the following linear transformations L from $\mathbb{R}^3$ into $\mathbb{R}^2$, find the matrix representing L with respect to the ordered bases E and F :

$$L(\mathbf{x}) = (x_1 + x_2, x_1 - x_3)^T$$

Solution

We first compute the values of $L(\mathbf{u}_1), L(\mathbf{u}_2)$, and $L(\mathbf{u}_3)$. We have

$$\begin{aligned} L(\mathbf{u}_1) &= L\left((1,0,-1)^T\right) = (1+0, 1-(-1))^T = (1,2)^T \\ L(\mathbf{u}_2) &= L\left((1,2,1)^T\right) = (1+2, 1-1)^T = (3,0)^T \\ L(\mathbf{u}_3) &= L\left((-1,1,1)^T\right) = (-1+1, -1-1)^T = (0,-2)^T \end{aligned}$$

Next, we find the inverse of $B = (\mathbf{b}_1, \mathbf{b}_2)$. Note that B is nonsingular since $F = \{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis in $\mathbb{R}^2$. We have $B^{-1} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$. The matrix representation A of L is

$$A = B^{-1}(L(\mathbf{u}_1), L(\mathbf{u}_2), L(\mathbf{u}_3)) = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -5 & -3 & 4 \\ 3 & 3 & -2 \end{bmatrix}$$

14. Let $\{\mathbf{u}_1, \mathbf{u}_2\}$ and $\{\mathbf{v}_1, \mathbf{v}_2\}$ be ordered bases for $\mathbb{R}^2$, where

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

and

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Let L be the linear transformation defined by

$$L(\mathbf{x}) = (-x_1, x_2)^T$$

and let B be the matrix representing L with respect to $\{\mathbf{u}_1, \mathbf{u}_2\}$.

(a) Find the transition matrix S corresponding to the change of basis from $\{\mathbf{u}_1, \mathbf{u}_2\}$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$.

(b) Find the matrix A representing L with respect to $\{\mathbf{v}_1, \mathbf{v}_2\}$ by computing SBS^{-1}.

(c) Verify that

$$\begin{aligned} L(\mathbf{v}_1) &= a_{11}\mathbf{v}_1 + a_{21}\mathbf{v}_2 \\ L(\mathbf{v}_2) &= a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2 \end{aligned}$$

Solution

(a) The transition matrix corresponding to the change of basis from $\{\mathbf{u}_1, \mathbf{u}_2\}$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$ is

$$\begin{aligned} S = V^{-1}U &= [\mathbf{v}_1\mathbf{v}_2]^{-1}[\mathbf{u}_1\mathbf{u}_2] \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix} \end{aligned}$$

(b) By inspection

$$\begin{aligned} L(\mathbf{u}_1) &= L(1,1)^T = (-1,1)^T = 0\mathbf{u}_1 + 1\mathbf{u}_2 \\ L(\mathbf{u}_2) &= L(-1,1)^T = (1,1)^T = 1\mathbf{u}_1 + 0\mathbf{u}_2 \end{aligned}$$

So the matrix representation of L with respect to $\{\mathbf{u}_1, \mathbf{u}_2\}$ is

$$B = [L(\mathbf{u}_1)_U \, L(\mathbf{u}_2)_U] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Since the transition matrix from $\{\mathbf{u}_1, \mathbf{u}_2\}$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$ is S,

$$A = SBS^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -4 & -1 \end{bmatrix}$$

which is the matrix representation of L with respect to $\{\mathbf{v}_1, \mathbf{v}_2\}$.

(c) The elements of A are $a_{11} = 1, a_{12} = 0, a_{21} = -4, a_{22} = -1$. Therefore,

$$L(\mathbf{v}_1) = L\left((2,1)^T\right) = (-2,1)^T = 1 \cdot (2,1)^T - 4 \cdot (1,0)^T = a_{11}\mathbf{v}_1 + a_{21}\mathbf{v}_2$$

$$L(\mathbf{v}_2) = L\left((1,0)^T\right) = (-1,0)^T = 0 \cdot (2,1)^T - 1 \cdot (1,0)^T = a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2$$

15. Let L be the operator on P_3 defined by

$$L(p(x)) = xp'(x) + p''(x)$$

(a) Find the matrix A representing L with respect to $\left[1, x, x^2\right]$.

(b) Find the matrix B representing L with respect to $\left[1, x, 1+x^2\right]$.

(c) Find the matrix S such that $B = S^{-1}AS$.

(d) If $p(x) = a_0 + a_1x + a_2\left(1+x^2\right)$, calculate $L^n(p(x))$.

Solution

(a)

$$L(1) = x \cdot (1)' + (1)'' = 0 = 0 \cdot 1 + 0 \cdot x + 0^2 \cdot x$$
$$L(x) = x \cdot (x)' + (x)'' = x = 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2$$
$$L\left(x^2\right) = x \cdot \left(x^2\right)' + \left(x^2\right)'' = 2x^2 + 2 = 2 \cdot 1 + 0 \cdot x + 2 \cdot x^2$$
$$A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(b)

$$L(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot \left(1 + x^2\right)$$
$$L(x) = x = 0 \cdot 1 + 1 \cdot x + 0 \cdot \left(1 + x^2\right)$$
$$L\left(1 + x^2\right) = x \cdot \left(1 + x^2\right)' + \left(1 + x^2\right)'' = 2x^2 + 2 = 0 \cdot 1 + 0 \cdot x + 2\left(1 + x^2\right)$$
$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(c)

$$\begin{array}{ccc} \circ & \xrightarrow{A} & \circ \\ \uparrow S & & \downarrow S^{-1} \\ \circ & \xrightarrow{B} & \circ \end{array}$$

To represent B with respect to A, we have

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

such that $B = S^{-1}AS$.

(d) Let $\beta = \left[1, x, 1 + x^2\right]$.

$$\begin{aligned} \left[L^{\mathrm{n}}(p(x))\right]_\beta &= \left[L^{\mathrm{n}}\right]_\beta \left[p(x)\right]_\beta \\ &= [L]_\beta^{\mathrm{n}}[p(x)]_\beta = B^{\mathrm{n}}[p(x)]_\beta \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{\mathrm{n}} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ a_1 \\ 2^{\mathrm{n}} a_2 \end{bmatrix} \\ L^{\mathrm{n}}(p(x)) &= a_1 x + 2^{\mathrm{n}} a_2 \left(1 + x^2\right) \end{aligned}$$