Assignment 5

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Please note that

- **Released date: Mar. 24th**
- **Due date: Apr. 7th, by 11:59 pm.**
- Late submission is **NOT** accepted.
- Please hand in your solution in PDF format titled "student number + HW5.pdf". Note that the file size is at most 10MB.

Question 1. Let

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 1 & -2 & 3 \\ 2 & 3 & 2 \end{pmatrix}$$

(a) Find the values of $\det(M_{21})$, $\det(M_{22})$, and $\det(M_{23})$.

(b) Find the values of A_{21}, A_{22}, and A_{23}.

(c) Use your answers from part (b) to compute $\det(A)$.

Solution

(a) $\det(M_{21}) = -8, \det\big(M_{22)}\big) = -2$ and $\det(M_{23}) = 5$.

(b) $A_{21} = 8, A_{22} = -2$ and $A_{23} = -5$.

(c) $\det(A) = 1A_{21} + (-2)A_{22} + 3A_{23} = -3$.

Question 2. Evaluate the following determinant. Write your answer as a polynomial in x :

$$\begin{vmatrix} a-x & b & c \\ 1 & -x & 0 \\ 0 & 1 & -x \end{vmatrix}$$

Solution

$$\begin{vmatrix} a-x & b & c \\ 1 & -x & 0 \\ 0 & 1 & -x \end{vmatrix} = (-1)^{(1+1)} \cdot (a-x) \cdot \begin{vmatrix} -x & 0 \\ 1 & -x \end{vmatrix} + (-1)^{2+1} \cdot 1 \cdot \begin{vmatrix} b & c \\ 1 & -x \end{vmatrix}$$
$$= -x^3 + ax^2 + bx + c$$

Question 3. Evaluate each of the following determinants by inspection.

(a) $\begin{vmatrix} 0 & 0 & 3 \\ 0 & 4 & 1 \\ 2 & 3 & 1 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ -1 & -1 & -1 & 2 \end{vmatrix}$

(c) $\begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}$

Solution

(a)

$$\begin{vmatrix} 0 & 0 & 3 \\ 0 & 4 & 1 \\ 2 & 3 & 1 \end{vmatrix} = (-1)^{1+3} \cdot 3 \cdot \begin{vmatrix} 0 & 4 \\ 2 & 3 \end{vmatrix}$$
$$= 1 \cdot 3 \cdot (-8)$$
$$= -24$$

(b)

$$\begin{vmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ -1 & -1 & -1 & 2 \end{vmatrix} = 1 \begin{vmatrix} 3 & 1 & 1 \\ 0 & 2 & 2 \\ -1 & -1 & 2 \end{vmatrix} - 0 + 0 + 1 \begin{vmatrix} 1 & 1 & 3 \\ 3 & 1 & 1 \\ 0 & 2 & 2 \end{vmatrix}$$
$$= 1 \times 3 \begin{vmatrix} 2 & 2 \\ -1 & 2 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} + 1 \times \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}$$
$$+ 1 \times (-3) \times \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix}$$
$$= 3(4+2) + 1(2-2) + 1(2-2) - 3(2-6)$$
$$= 30$$

(c)

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} = (-1)^{4+1} \cdot 1 \cdot \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$
$$= -1 \cdot 1 \cdot 1$$
$$= -1$$

Question 4. Let

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \end{pmatrix}$$

(a) Use the elimination method to evaluate $\det(A)$.

(b) Use the value of $\det(A)$ to evaluate

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \\ 1 & 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 4 & 4 \\ 2 & 3 & -1 & -2 \end{vmatrix}$$

Solution

(a)

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 1 & -3 & -4 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & -5 & -7 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & -2 \end{vmatrix}$$

$$= -(-10)$$

$$= 10$$

Solution

(b)

For the first part:

$$
\begin{aligned}
|A| &= \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \end{vmatrix} \\
&= -\begin{vmatrix} 0 & 1 & 2 & 3 \\ -2 & -2 & 3 & 3 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & -2 & -3 \end{vmatrix} R_2 \leftrightarrow R_3 \\
&= (-1)(-1)\begin{vmatrix} 0 & 1 & 2 & 3 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \\ 1 & 1 & 1 & 1 \end{vmatrix} \quad R_3 \leftrightarrow R_4 \\
&= \begin{vmatrix} 0 & 1 & 2 & 3 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \\ 1 & 1 & 1 & 1 \end{vmatrix} \\
&= |B|
\end{aligned}
$$

For the second part:

The matrix

$$
C = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 4 & 4 \\ 2 & 3 & -1 & -2 \end{pmatrix}
$$

is obtained by replacing R_3 by $R_3 \rightarrow R_2 + R_3$, R_4 by $R_4 \rightarrow R_4 + R_2$ of the matrix A. Under this operation, the value of determinant will not change. Therefore,

$$
|C| = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 4 & 4 \\ 2 & 3 & -1 & -2 \end{vmatrix} = |A|
$$

Therefore,

$$
|B| + |C| = |A| + |A| = 10 + 10 = 20
$$

Question 5. For each of the following, compute the determinant and state whether the matrix is singular or nonsingular:

(a) $\begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$

(e) $\begin{pmatrix} 2 & -1 & 3 \\ -1 & 2 & -2 \\ 1 & 4 & 0 \end{pmatrix}$

Solution

(a)

$$\begin{aligned} |A| &= \begin{vmatrix} 3 & 1 \\ 6 & 2 \end{vmatrix} \\ &= 3 \times 2 - 6 \times 1 \\ &= 6 - 6 \\ &= 0 \end{aligned}$$

Since $|A| = 0$, matrix A is singular.

(e)

$$\begin{aligned} |A| &= \begin{vmatrix} 2 & -1 & 3 \\ -1 & 2 & -2 \\ 1 & 4 & 0 \end{vmatrix} \\ &= 2\begin{vmatrix} 2 & -2 \\ 4 & 0 \end{vmatrix} + 1\begin{vmatrix} -1 & -2 \\ 1 & 0 \end{vmatrix} + 3\begin{vmatrix} -1 & 2 \\ 1 & 4 \end{vmatrix} \\ &= 2(0+8) + 1(0+2) + 3(-4-2) \\ &= 2(8) + 1(2) + 3(-6) \\ &= 0 \end{aligned}$$

Since $|A| = 0, \quad A$ is singular.

Question 6. Find all possible choices of c that would make the following matrix singular:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{pmatrix}$$

Solution

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 8 & c-1 \\ 0 & c-1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 8 & c-1 \\ 0 & 0 & 2-\frac{(c-1)^2}{8} \end{vmatrix} = 16-(c-1)^2$$

To make the matrix singular, the determinant should be 0.

Therefore, $16-(c-1)^2 = 0$, $c = 5$ or -3.

(*Tips*: Cofactor expansion is also ok here.)

Question 7. Let A be an $n \times n$ matrix and α a scalar. Show that

$$\det(\alpha A) = \alpha^n \det(A)$$

Solution

αA means multiplying a constant α to each row/column of A.

And EA means multiplying a constant α to a row of A, if E is a type II elementary matrix with $\det(E) = \alpha$.

Therefore,

$\alpha A = E_1 E_2 ... E_i ... E_n A$, if E_i is a type II elementary matrix with the nonzero entry in ith row is α.

Then,

$\det(\alpha A) = \det(E_1 E_2 ... E_i ... E_n A) = \det(E_1)\det(E_2)...\det(E_i)...\det(E_n)\det(A) = \alpha^n \det(A)$

Question 8. Let A be a nonsingular matrix. Show that

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Solution

$$\det(A^{-1})\det(A) = \det(A^{-1}A) = \det(I) = 1$$

Therefore,

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Question 9. Let E_1, E_2, and E_3 be 3×3 elementary matrices of types I, II, and III, respectively, and let A be a 3×3 matrix with $\det(A) = 6$. Assume, additionally, that E_2 was formed from I by multiplying its second row by 3 . Find the values of each of the following:

(a) $\det(E_1A)$ (b) $\det(E_2A)$ (c) $\det(E_3A)$

(d) $\det(AE_1)$ (e) $\det(E_1^2)$ (f) $\det(E_1E_2E_3)$

Solution

(a) -6; (b) 18; (c) 6; (d) -6; (e) 1; (f) -3

Question 10. Consider the 3×3 Vandermonde matrix

$$V = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix}$$

(a) Show that $\det(V) = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$.

Hint: Make use of row operation III.

(b) What conditions must the scalars x_1, x_2, and x_3 satisfy in order for V to be nonsingular?

Solution

(a)

$$\det(\text{V}) = \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = \begin{vmatrix} 1 & x_1 & x_1^2 \\ 0 & x_2 - x_1 & x_2^2 - x_1^2 \\ 0 & x_3 - x_1 & x_3^2 - x_1^2 \end{vmatrix} = \begin{vmatrix} x_2 - x_1 & x_2^2 - x_1^2 \\ x_3 - x_1 & x_3^2 - x_1^2 \end{vmatrix}$$

$= (x_2 - x_1)(x_3^2 - x_1^2) - (x_3 - x_1)(x_2^2 - x_1^2)$

$= x_2x_3^2 - x_1x_3^2 - x_2x_1^2 + x_1^3 - x_3x_2^2 + x_1x_2^2 + x_3x_1^2 - x_1^3$

$= (x_2x_3^2 - x_1x_3^2 + x_3x_1^2 - x_1x_2x_3) + (x_1x_2^2 - x_3x_2^2 - x_2x_1^2 + x_1x_2x_3)$

$= x_3(x_2x_3 - x_1x_3 + x_1^2 - x_1x_2) + x_2(x_1x_2 - x_3x_2 - x_1^2 + x_1x_3)$

$= (x_3 - x_2)(x_2x_3 - x_1x_3 + x_1^2 - x_1x_2)$

$= (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$

(b)

To make V nonsingular,

$\det(V) = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2) \neq 0$.

Then the scalars x_1, x_2, and x_3 should satisfy $x_1 \neq x_2 \neq x_3$.

Question 11. A matrix A is said to be *skew symmetric* if $A^T = -A$. For example,

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is skew symmetric, since

$$A^T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -A$$

If A is an $n \times n$ skew-symmetric matrix and n is odd, show that A must be singular.

Solution

Since $A^T = -A$ and n is odd,

$\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A) = -\det(A)$

Therefore,

$\det(A) = 0$, which means A must be singular.

Question 12. Let A be a $k \times k$ matrix and let B be an $(n-k) \times (n-k)$ matrix. Let

$$E = \begin{bmatrix} I_k & O \\ O & B \end{bmatrix}, F = \begin{bmatrix} A & O \\ O & I_{n-k} \end{bmatrix}, C = \begin{bmatrix} A & O \\ O & B \end{bmatrix}, \tag{1}$$

where I_k and I_{n-k} are the $k \times k$ and $(n-k) \times (n-k)$ identity matrices.

(a) Show that $\det(E) = \det(B)$.

(b) Show that $\det(F) = \det(A)$.

(c) Show that $\det(C) = \det(A)\det(B)$.

Solution

(a) The matrix E can be written as

$$E = \begin{bmatrix} 1 & O & O \\ O & I_{k-1} & O \\ O & 0 & B \end{bmatrix}, \tag{2}$$

By the cofactor expansion formula, we have

$$|E| = e_{11}(-1)^{1+1}\det\left(\begin{bmatrix} I_{k-1} & O \\ O & B \end{bmatrix}\right) = \det\left(\begin{bmatrix} I_{k-1} & O \\ O & B \end{bmatrix}\right) \tag{3}$$

By further applying the above derivations recursively, it will be seen that $|E| = |B|$.

(b) The proof is similar to that in (a).

(c) As

$$C = \begin{bmatrix} A & O \\ O & B \end{bmatrix} = EF, \ |C| = |E||F| = |B||A| = |A||B|. \tag{4}$$

.

Question 13. For each of the following, compute (i) $\det(A)$, (ii) $\mathrm{adj}(A)$, and (iii) A^{-1}.

(a)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \tag{5}$$

(b)

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{bmatrix} \tag{6}$$

Solution

(a)

$$|A| = -7, \mathrm{adj}(A) = \begin{bmatrix} -1 & -2 \\ -3 & 1 \end{bmatrix}, A^{-1} = \begin{bmatrix} 1/7 & 2/7 \\ 3/7 & -1/7 \end{bmatrix}. \tag{7}$$

(b)

$$|A| = 3, \mathrm{adj}(A) = \begin{bmatrix} -3 & 5 & 2 \\ 0 & 1 & 1 \\ 6 & -8 & -5 \end{bmatrix}, A^{-1} = \frac{1}{3}\mathrm{adj}(A). \tag{8}$$

Question 14. Use Cramer's rule to solve the following system.

$$\begin{aligned} 2x_1 + x_2 - 3x_3 &= 0 \\ 4x_1 + 5x_2 + x_3 &= 8 \\ -2x_1 - x_2 + 4x_3 &= 2 \end{aligned} \tag{9}$$

Solution

As $A = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 5 & 1 \\ -2 & -1 & 4 \end{bmatrix} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$, and $\mathbf{b} = [0, 8, 2]^T$, by the Cramer's rule, we have

$$x_1 = \frac{|A_1|}{|A|} = \frac{|[\mathbf{b}, \mathbf{a}_2, \mathbf{a}_3]|}{|A|} = \frac{24}{6} = 4. \tag{10}$$

Similarly,

$$x_2 = \frac{|A_2|}{|A|} = \frac{|[\mathbf{a}_1, \mathbf{b}, \mathbf{a}_3]|}{|A|} = \frac{-12}{6} = -2, \tag{11a}$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{|[\mathbf{a}_1, \mathbf{a}_2, \mathbf{b},]|}{|A|} = \frac{12}{6} = 2. \tag{11b}$$

Question 15. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \tag{12}$$

(a) Compute $\det(A)$. Is A nonsingular?

(b) Compute $\operatorname{adj}(A)$ and the product $A\operatorname{adj}(A)$.

Solution

(a) As

$$\begin{aligned} |A| &= \sum_{j=1}^{3} a_{1j}(-1)^{1+j}|M_{1j}| && \text{(13a)} \\ &= 1|M_{11}| - 2|M_{12}| + 3|M_{13}| && \text{(13b)} \\ &= 1 \times (-1) - 2 \times (-2) + 3 \times (-1) && \text{(13c)} \\ &= 0, && \text{(13d)} \end{aligned}$$

A is singular.

(b) $\operatorname{adj}(A) = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix}$, and $A\operatorname{adj}(A) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

Question 16 Show that if A is nonsingular, then $\operatorname{adj}(A)$ is nonsingular and

$$[\operatorname{adj}(A)]^{-1} = \det(A^{-1})A = \operatorname{adj}(A^{-1}). \tag{14}$$

Solution

For a nonsingular matrix A, we have

$$|A| \neq 0 \tag{15a}$$

$$|A^{-1}| = \frac{1}{|A|} \tag{15b}$$

$$A^{-1} = \frac{\mathrm{adj}(A)}{|A|} \tag{15c}$$

(a) Therefore, $\mathrm{adj}(A) = |A|A^{-1}$ is also nonsingular;

(b) $\mathrm{adj}(A) = |A|A^{-1} \Rightarrow [\mathrm{adj}(A)]^{-1} = \frac{1}{|A|}(A^{-1})^{-1} = |A^{-1}|A$;

(c) $A = \frac{\mathrm{adj}(A^{-1})}{|A^{-1}|} \Rightarrow \mathrm{adj}(A^{-1}) = |A^{-1}|A$.

Combining (b) and (c), we have

$$[\mathrm{adj}(A)]^{-1} = |A^{-1}|A = \mathrm{adj}(A^{-1}). \tag{16}$$