Fundamentals of Filtering Techniques in Macroeconomics

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Abstract

This lecture note introduces key filtering techniques used in macroeconomic time series analysis, with a particular focus on methods for extracting cyclical components. In business cycle research, isolating short-term fluctuations from long-term trends is essential, yet there is no universal consensus on the best filtering approach. We present the theoretical foundations, frequency-domain representations, and basic implementation steps for several widely used filters: the Hodrick-Prescott, Hamilton, Baxter-King, and Christiano-Fitzgerald filters, as well as wavelet decompositions. While this version emphasizes the conceptual and mathematical principles underlying each method, future versions will include comparative analyses, empirical case studies, and practical guidance for applied research. Our goal is to clarify the assumptions, strengths, and limitations of each approach, thereby promoting transparency and robustness in macroeconomic analysis involving detrended data.

1 Introduction

Macroeconomic time series often exhibit a combination of long-term trends and short-term fluctuations. Disentangling these components is essential for understanding cyclical dynamics, particularly in the context of output gaps, policy evaluation, and business cycle synchronization. The need to extract the cyclical component has led to the development and widespread use of various filtering techniques in applied macroeconomics.

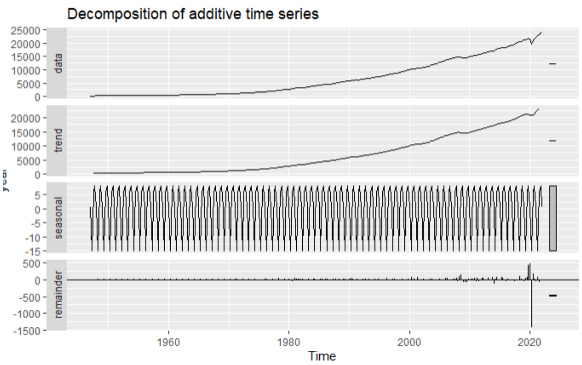
Among these techniques, the Hodrick-Prescott (HP) filter has long been a standard tool. However, recent literature has raised concerns about its statistical properties and the distortions it can introduce. Alternative approaches (such as the Hamilton regression-based filter, band-pass filters developed by Baxter and King (1999) and Christiano and Fitzgerald (2003), and wavelet-based decompositions) aim to address some of these limitations by offering more transparent or frequency-specific analyses. Spectral methods also provide additional insights into the time-frequency characteristics of macroeconomic fluctuations.

This document presents an overview of the theoretical foundations and practical implementation of several widely used filters, including the HP, Hamilton, Baxter-King, and Christiano-Fitzgerald filters, as well as wavelet approaches. While the focus here is primarily on definitions and mathematical formulations, future versions will expand to include comparative analyses, empirical illustrations, and practical guidance using U.S. GDP as a case study.

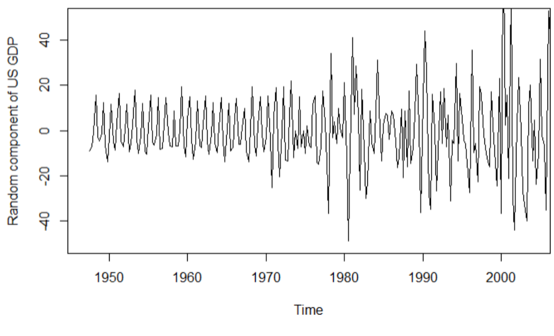
Our goal is not to prescribe a single best method, but to highlight the importance of understanding the assumptions, strengths, and limitations of each technique. By clarifying the mechanics of these filtering methods, we aim to promote transparency and robustness in macroeconomic research involving detrended data.

2 Cyclic components in macroeconomic time series

A key assumption in business cycle analysis is that the economy follows a natural growth path, around which short-term cyclical fluctuations occur. These fluctuations are typically viewed as transitory, while the long-term trend reflects more persistent structural factors. The figure below illustrates a decomposition of U.S. real GDP (quarterly frequency, 1947–2022), where a linear trend and seasonal pattern have been removed:



Focusing on the residual (random component) before 2004, we observe that the magnitude of fluctuations varies over time, with more pronounced volatility in recent years:



This variability suggests that a simple linear trend removal may be inadequate for capturing the underlying cyclical behavior. Moreover, such a decomposition does not account for potential structural breaks in the data, which can distort parameter estimates in linear regression models.

In macroeconomic research, extracting the cyclical component of a time series is often essential. Doing so allows for the analysis of business cycle dynamics, such as persistence, amplitude, and phase. It also enables the study of relationships between cycles across different aggregates; such as the correlation between output and fiscal variables in the literature on fiscal cyclicality. A prominent application is estimating the output gap, defined as the difference between potential and actual output.

Unlike seasonal cycles, business cycles do not exhibit fixed periodicities. This lack of unique frequencies makes their identification particularly challenging, as macroeconomic time series often reflect a mixture of trend, cyclical, and irregular components. One common approach for isolating the cycle is filtering, which leverages the idea that a time series can be decomposed into a sum of sine and cosine functions; each representing different frequencies and amplitudes. In the sections that follow, we will explore key filtering methods used for detrending macroeconomic data and examine empirical studies that apply these techniques in macroeconomic research.

3 The Hodrick-Prescott filter

One of the most widely used filters in macroeconomic research is the Hodrick-Prescott (HP) filter, introduced by Hodrick and Prescott (1980). The HP filter provides an estimate of the smooth trend component, denoted $\hat{g}_t$, which is uncorrelated with the cyclical component of the series. The key idea is that an observed time series can be decomposed into two parts: a trend (or growth) component and a cyclical component (after removing any seasonal effects):

$$y_t = g_t + c_t$$

The trend component, g_t, is assumed to evolve smoothly over time. Its smoothness is measured by penalizing the variability in its second difference (i.e., its acceleration). The cyclical component, c_t, captures deviations from this smooth path and is assumed to have a zero mean in the long run.

The HP filter defines the trend g_t as the solution to the following minimization problem:

$$\min_{g_t} \sum_{t=1}^{T} \left[(y_t - g_t)^2 + \lambda \left\{ (g_{t+1} - g_t) - (g_t - g_{t-1}) \right\}^2 \right]$$

The first term, $(y_t - g_t)^2$, measures the squared deviations of the observed data from the trend—it represents the cyclical component. The second term, $\{(g_{t+1} - g_t) - (g_t - g_{t-1})\}^2$, penalizes rapid changes in the growth rate of the trend. This term is minimized when the trend path is linear (i.e., has no acceleration).[1] The parameter λ is the smoothing parameter: it controls the trade-off between the fit to the data and the smoothness of the trend.

The larger the value of λ, the smoother the trend component will be. As $\lambda \to \infty$, the trend becomes perfectly smooth (a linear trend). As $\lambda \to 0$, no penalty is applied, and the

[1]Hodrick and Prescott apply the filter to the log of the series, so that $(g_{t+1} - g_t)$ approximates a growth rate.

trend simply equals the original series (i.e., no cycle is extracted).

This smoothing approach is not unique to economics. Similar methods have been used in actuarial science, notably in the Whittaker-Henderson Type A method for smoothing mortality rates in life tables (Whittaker, 1923).

From a probabilistic perspective, if we assume that the cyclical component c_t and the second differences of the trend are independent and identically distributed (i.i.d.) with zero means and variances σ_c^2 and $\sigma_{\Delta^2 y_t}^2$, respectively, then the optimal value of the smoothing parameter is given by:

$$\lambda = \frac{\sigma_c^2}{\sigma_{\Delta^2 y_t}^2}$$

This relationship illustrates that the choice of λ directly affects the smoothness of the estimated trend.

Hodrick and Prescott argued that for quarterly macroeconomic data, the typical variance of the cyclical component is about 5%, and the variance of the second difference of the trend is around (1/8)%. Based on these assumptions, they proposed $\lambda = 1600$ as a standard choice for quarterly data. They also suggested $\lambda = 100$ for annual data. In practice, the following rule-of-thumb values are often used:

- $\lambda = 1600$ for quarterly data
- $\lambda = 100$ for annual data
- $\lambda = 14400$ for monthly data

However, the choice of λ is not without controversy. Some researchers, such as Ravn and Uhlig (2002), argue that λ should scale with the fourth power of the frequency ratio. For example, if annual data are sampled four times less frequently than quarterly data, the corresponding λ would be $\lambda = 1600/4^4 = 6.25$. Moreover, the appropriate value of λ may vary across countries and variables, depending on their specific characteristics and data-generating processes.

3.1 Properties and Limitations of the HP Filter

In practice, many filters—including the HP filter—can be written as moving average filters, where a series x_t is obtained from a series y_t as:

$$x_t = \sum_{j=-k}^{l} \omega_j y_{t-j}$$

This representation can also be expressed compactly using the lag operator L, defined as $Ly_t = y_{t-1}$ and $L^{-j} y_t = y_{t+j}$.[2]

[2]Defined such that $Ly_t = y_{t-1}$ and $L^{-j} y_t = y_{t+j}$.

For example, consider the following filter, which smooths the series y_t:

$$x_t = \frac{1}{8}y_{t-2} + \frac{1}{4}y_{t-1} + \frac{1}{4}y_t + \frac{1}{4}y_{t+1} + \frac{1}{8}y_{t+2}$$

This can be rewritten in lag operator notation as:

$$x_t = \left(\frac{1}{8}L^{-2} + \frac{1}{4}L^{-1} + \frac{1}{4}L^0 + \frac{1}{4}L^1 + \frac{1}{8}L^2\right) y_t$$

In the context of the HP filter, when $y_t = g_t + c_t$, the trend component can be written as a moving average of y_t:

$$g_t = G(L)y_t$$

where $G(L)$ is a function of the lag operator and the filter weights. Since $c_t = y_t - g_t$, the cyclical component is also a moving average of y_t:

$$c_t = (1 - G(L))y_t = C(L)y_t$$

The cyclical component obtained from the HP filter has a specific moving average representation:

$$HP(L) = \frac{\lambda(1-L)^2(1-L^{-1})^2}{1+\lambda(1-L)^2(1-L^{-1})^2}$$

King and Rebelo (1993) show that this expression can be derived from the first-order conditions of the HP minimization problem:

$$\min_{g_t} \sum_{t=1}^{T} \left[(y_t - g_t)^2 + \lambda \left\{(g_{t+1} - g_t) - (g_t - g_{t-1})\right\}^2\right]$$

The corresponding first-order condition is:

$$\begin{aligned} -2(y_t - g_t) + 2\lambda \left[(g_t - g_{t-1}) - (g_{t-1} - g_{t-2})\right] \\ -4\lambda \left[(g_{t+1} - g_t) - (g_t - g_{t-1})\right] + 2\lambda \left[(g_{t+2} - g_{t+1}) - (g_{t+1} - g_t)\right] = 0 \end{aligned}$$

Simplifying, we obtain the operator form:

$$y_t = g_t \left[\lambda L^{-2} - 4\lambda L^{-1} + (1 + 6\lambda) - 4\lambda L + \lambda L^2\right]$$

This can be written compactly as:

$$F(L)g_t = y_t$$

where:

$$F(L) = 1 + \lambda(1-L)^2(1-L^{-1})^2$$

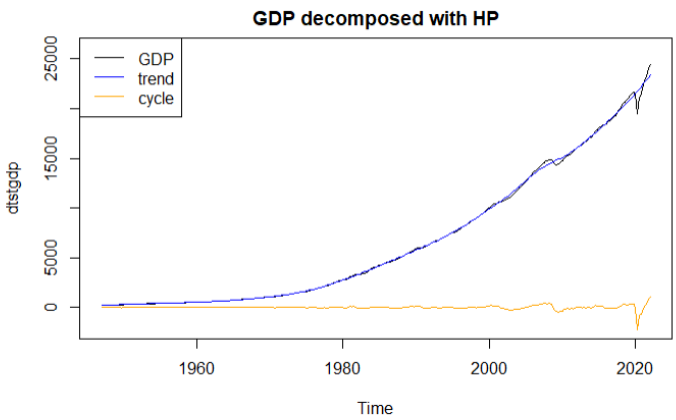
Thus, the filter $G(L)$ is the inverse of $F(L)$: $G(L) = F(L)^{-1}$. Consequently, the cyclical filter can be written as:

$$C(L) = 1 - F(L)^{-1} = [F(L) - 1]F(L)^{-1}$$

which corresponds to the moving average form given earlier.

3.2 Example: Application to US GDP

We can apply the HP filter to U.S. GDP data (1947 Q1–2022 Q1) using the `hpfilter()` function from the R package `mFilter`:




```
hpf <- hpfilter(dtstgdp, freq = 1600, type = "lambda")
```

3.3 Advantages and Limitations

Advantages:

- Simple to use and widely implemented.
- The minimization problem has a unique solution.
- The filtered trend g_t has the same length as the original series y_t.
- The filter can stationarize time series integrated of order four or less.

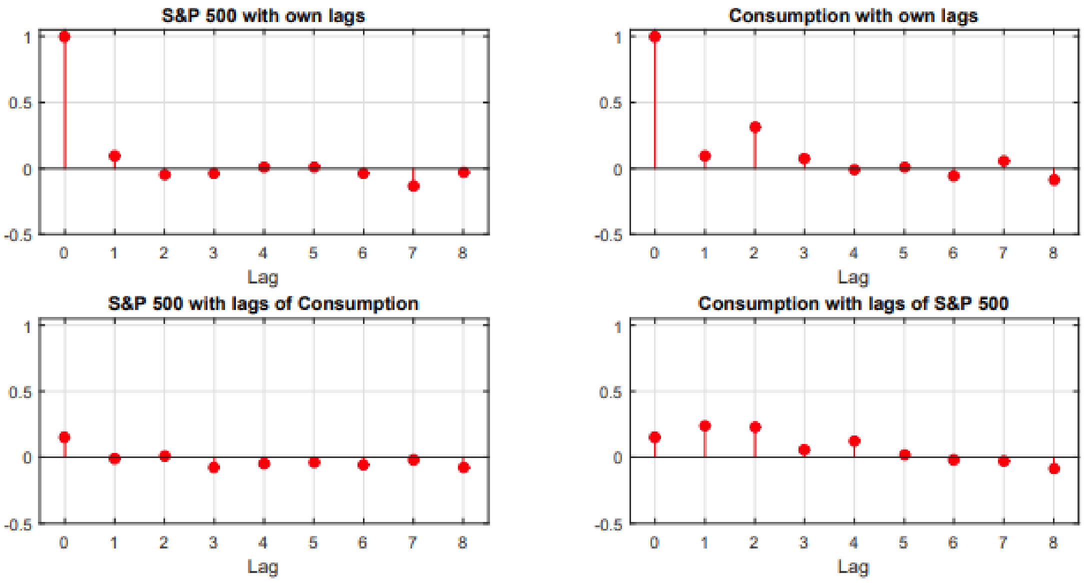
Limitations:

- The HP filter suffers from an *end-of-sample* issue: because it uses both forward and backward differences, estimates near the start and end of the sample are biased. The trend tends to converge towards the first and last observations, making it sensitive to transitory shocks at the sample edges.[3]

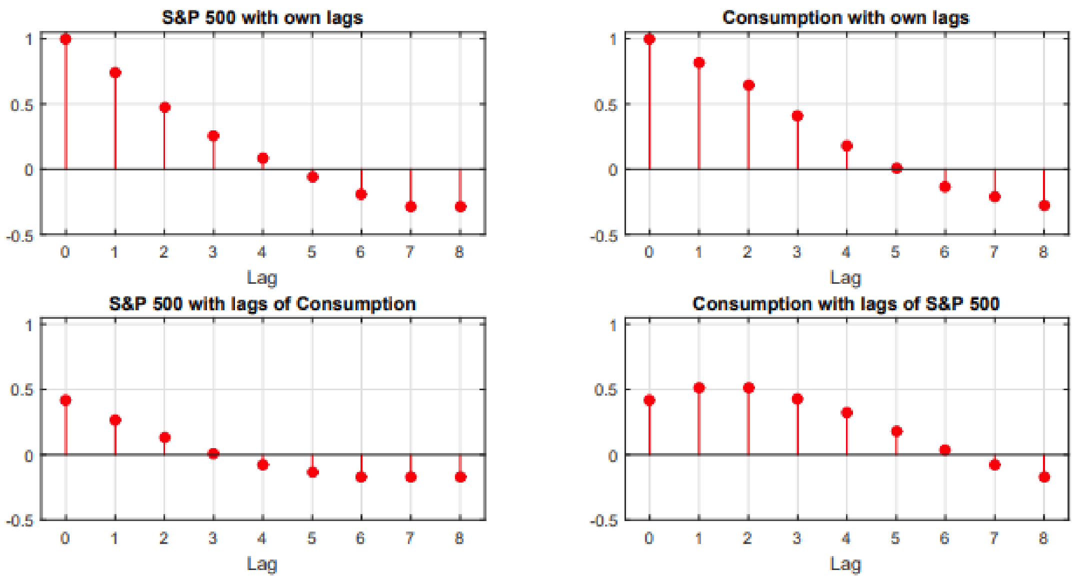
[3]One proposed solution is to use a one-sided HP filter that relies only on past data.

- The choice of λ is ad-hoc and can induce spurious cyclical patterns, especially if the data-generating process is a random walk.

Hamilton (2017) argues against the use of the HP filter in his paper *Why You Should Never Use the Hodrick-Prescott Filter*. He shows that applying the HP filter to a random walk can create artificial cyclical patterns, solely driven by the choice of λ. This can lead to misleading conclusions when comparing filtered series, as shown below:



First-difference data (autocorrelation)



HP-cyclical components (autocorrelation), Hamilton (2017)

Moreover, Hamilton demonstrates—using maximum likelihood estimation—that the commonly used value $\lambda = 1600$ is not optimal for quarterly data.

4 The Hamilton Filter (2017)

Hamilton (2017) proposes an alternative to the HP filter, based on an OLS regression of the observed (potentially non-stationary) time series y_t on a constant and its four most recent lagged values. Specifically, the model is:

$$y_{t+h} = \beta_0 + \beta_1 y_t + \beta_2 y_{t-1} + \beta_3 y_{t-2} + \beta_4 y_{t-3} + v_{t+h}$$

where h is a chosen forecast horizon. The cyclical (or stationary) component is then obtained from the residuals of this regression:

$$\hat{v}_{t+h} = y_{t+h} - \hat{\beta}_0 - \hat{\beta}_1 y_t - \hat{\beta}_2 y_{t-1} - \hat{\beta}_3 y_{t-2} - \hat{\beta}_4 y_{t-3}$$

This residual series $\hat{v}_{t+h}$ is stationary, provided that the fourth difference of y_t is stationary.

When applied to a random walk process, the Hamilton filter effectively reduces to a difference filter. In this case, the OLS estimates converge (in large samples) to $\beta_1 = 1$ and $\beta_j = 0$ for $j = 0, 2, 3, \ldots$, and the forecast error simplifies to:

$$\tilde{v}_{t+h} = y_{t+h} - y_t$$

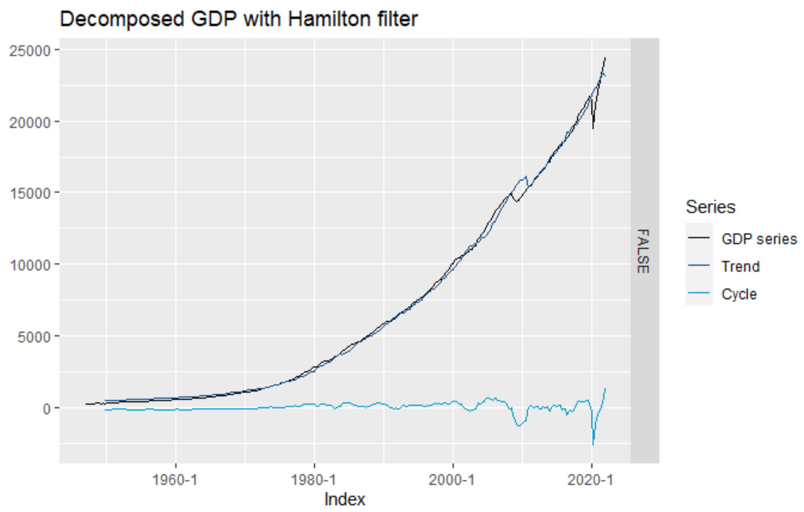
Hamilton recommends setting $h = 8$ for quarterly data, aligning with typical business cycle frequencies.

Compared to the HP filter, the Hamilton approach has several advantages:

- It avoids the end-of-sample bias inherent in the HP filter, as it relies solely on past information (aside from y_{t+h}).
- It does not require the ad-hoc specification of a smoothing parameter like λ.

However, there are also limitations. As Schüler (2021) notes, the Hamilton filter can suffer from small sample biases and may be sensitive to structural breaks. Moreover, by construction, it imposes a particular structure on the detrended component, which may affect the interpretation of cyclical fluctuations.

Example: We can apply the Hamilton filter to U.S. GDP data (1947 Q1–2022 Q1) using the function `yth_filter()` from the R package `neverhpfilter`:



`ham <- yth_filter(xdtstgdp)`

5 Generalities on Spectral Analysis

Some filtering methods are based on spectral analysis, which decomposes a time series into components associated with different frequencies of fluctuations. In this framework, a time series is viewed as the sum of periodic functions, typically sines and cosines.[4]

5.1 Properties of Sine and Cosine Functions

Sine and cosine functions are characterized by the following properties:

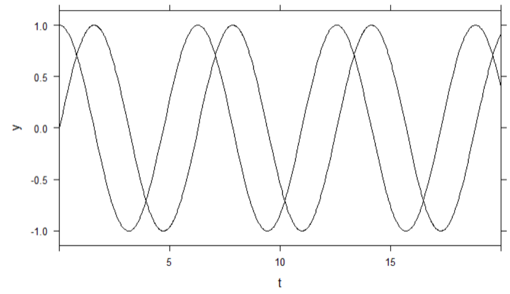
- The frequency (f): the fraction of a full cycle completed per unit of time.
- The period (T): the time required to complete one full cycle, with $T = \frac{1}{f}$.

We also distinguish between:

- The common frequency: $f = \frac{\omega}{2\pi}$.
- The angular frequency ω, expressed in radians per unit of time.

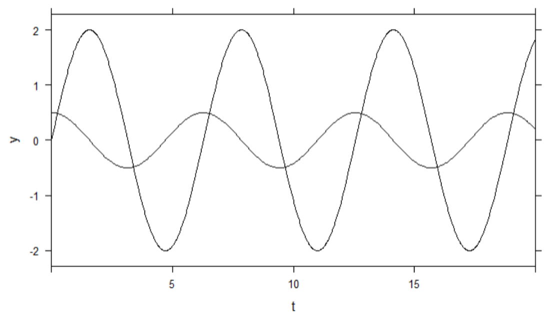
The following figure illustrates these properties: sine and cosine functions have the same period and amplitude, but they differ by a phase shift. This means they are at different points in the cycle for a given time t.

[4] These functions are periodic, e.g., $\sin(t) = \sin(t + 2\pi)$.



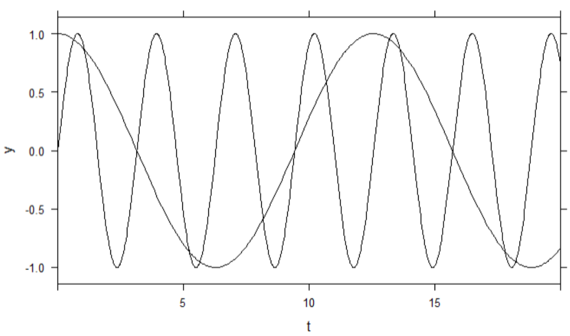
`sin1 <- makeFun(A*sin(w*t) ~ t, A=1, w=1)` `cos1 <- makeFun(A*cos(w*t) ~ t, A=1, w=1)`

For example, when $\omega = 1$, the sine function completes its first cycle at $t = 2\pi$. The amplitude of the fluctuations can be adjusted by multiplying the function by a constant A:



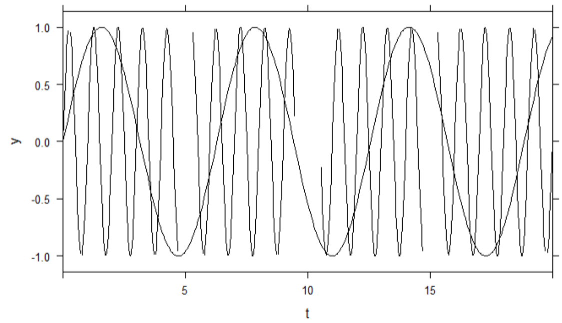
`sin2 <- makeFun(A*sin(w*t) ~ t, A=2, w=1)` `cos2 <- makeFun(A*cos(w*t) ~ t, A=0.5, w=1)`

In the example, the amplitude increases for $A = 2$ and decreases for $A = 0.5$. The frequency is controlled by ω: a higher ω means more cycles per unit time, while a lower ω stretches the cycle.



`sin3 <- makeFun(A*sin(w*t) ~ t, A=1, w=2)` `cos3 <- makeFun(A*cos(w*t) ~ t, A=1, w=0.5)`

For example, with $\omega = 2$, there are two cycles between t_0 and $t_{2\pi}$, while $\omega = 1$ corresponds to a single cycle. Conversely, for $\omega = 0.5$, it takes twice as long to complete one cycle. When $\omega = 2\pi$, a full cycle completes at $t = 1$. The figure below compares $\sin(2\pi t)$ and $\sin(t)$:



sin1 <- makeFun(A*sin(w*t) ~ t, A=1, w=1)   sin4 <- makeFun(A*sin(w*t) ~ t, A=1, w=2π)

In general, for two points in time t_1 and t_2 at the same phase of the cycle, the relationship is:

$$t_2\omega - t_1\omega = 2\pi$$

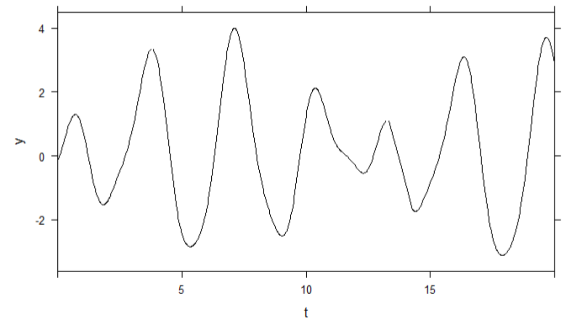
Dividing by ω, we obtain the period of oscillation:

$$t_2 - t_1 = \frac{2\pi}{\omega}$$

For example, with $\omega = 1$, the period is 2π, while for $\omega = 2\pi$, the period is 1. We can also include a phase term ϕ in the general form of the function:

$$A \sin(\omega t + \phi)$$

where ϕ represents the phase shift or starting point of the cycle. Finally, by summing sine and cosine functions with different amplitudes and frequencies, we can model complex cyclical patterns:



comp <- makeFun(a1*sin(t*w1) + a2*cos(t*w2) + a3*sin(t*w3) + a4*cos(t*w4) ~ t, a1=2, a2=0.25, a3=-1.5, a4=-0.4, w1=2, w2=0.1, w3=1.5, w4=4)

5.2 Spectral Decomposition and Periodograms

Consider a time series y_t with $t = 1, 2, \ldots, T$, where T is an even number. We can decompose y_t into $T/2$ possible periodic components, each associated with a different frequency ω. Specifically, we define:

$$\omega_j = \frac{2\pi j}{T} \quad \text{for } j = 1, 2, \ldots, T/2$$

and express y_t as a sum of sine and cosine functions:

$$y_t = a_1 \cos(t\omega_1) + b_1 \sin(t\omega_1) + \cdots + a_{T/2} \cos(t\omega_{T/2}) + b_{T/2} \sin(t\omega_{T/2})$$

If the frequencies ω_j are known, we could estimate the coefficients a_j and b_j by running a regression of y_t on the corresponding sine and cosine terms. In matrix form, the model is:

$$X = \begin{bmatrix} \cos(\omega_1) & \sin(\omega_1) & \ldots & \cos(\omega_{T/2}) & \sin(\omega_{T/2}) \\ \cos(2\omega_1) & \sin(2\omega_1) & \ldots & \cos(2\omega_{T/2}) & \sin(2\omega_{T/2}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cos(T\omega_1) & \sin(T\omega_1) & \ldots & \cos(T\omega_{T/2}) & \sin(T\omega_{T/2}) \end{bmatrix}$$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{bmatrix} \quad \beta = \begin{bmatrix} a_1 \\ b_1 \\ \vdots \\ a_{T/2} \\ b_{T/2} \end{bmatrix}$$

However, the number of possible frequencies is theoretically infinite, and for a sample of size T, the matrix X has dimensions $T \times T$. This makes direct estimation of such a model computationally intensive. The main objective of spectral analysis is to determine how much of the variability in y_t can be explained by cycles of different frequencies. The spectral decomposition of y_t can be written as:

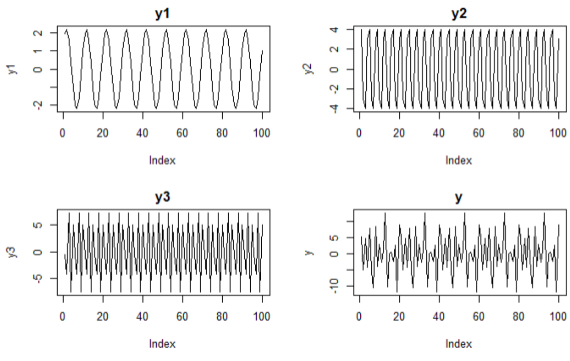
$$y_t = \sum_{j=1}^{T/2} [a_j \cos(\omega_j t) + b_j \sin(\omega_j t)]$$

In practice, a periodogram is used to identify the dominant frequencies in a time series. This is especially useful for detecting cyclical behavior that may not correspond to typical seasonal frequencies (e.g., monthly or quarterly). The first step in constructing a periodogram is estimating the coefficients a_j and b_j. This is typically done using the Fast Fourier Transform (FFT). Once these coefficients are estimated, the periodogram value at each frequency is calculated as:

$$P\left(\frac{\omega_j}{2\pi}\right) = \hat{a}_j^2 + \hat{b}_j^2$$

This value represents the sum of squared coefficients at frequency $\omega_j/2\pi$. A large value of P indicates that this frequency plays a significant role in the oscillations of y_t. Identifying dominant frequencies can help both describe the series and guide the fitting of sine or cosine functions to the data.

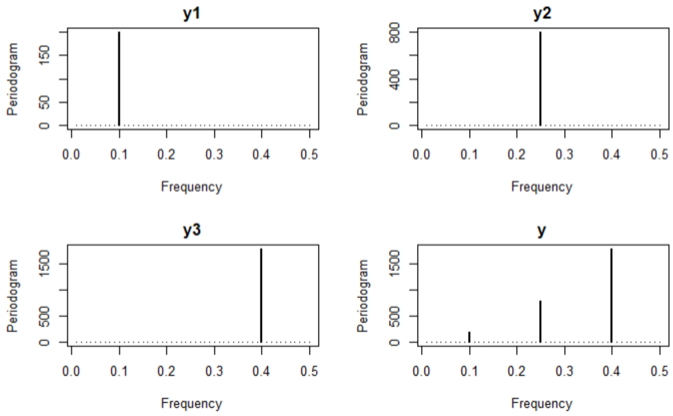
Example: We simulate a dataset composed of three series y_1, y_2, and y_3, each with different frequencies 10/100, 25/100, and 40/100, respectively. The combined series is given by $y = y_1 + y_2 + y_3$:




```
t = 1:100;   w = c(10/100, 25/100, 40/100)
y1 = 1 * cos(2 * pi * t * w[1]) + 2 * sin(2 * pi * t * w[1])
y2 = 3 * cos(2 * pi * t * w[2]) + 4 * sin(2 * pi * t * w[2])
y3 = 5 * cos(2 * pi * t * w[3]) + 6 * sin(2 * pi * t * w[3])

y = y1 + y2 + y3
```

We then use the `periodogram()` function from the `TSA` package in R to estimate the periodogram and identify the dominant frequencies in the series:

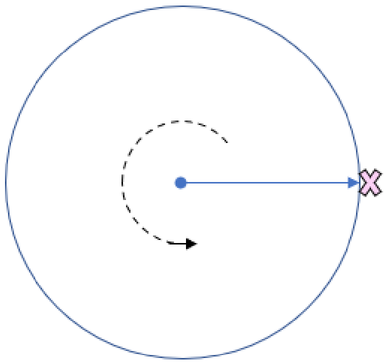


The periodogram correctly highlights the dominant frequencies corresponding to 10/100, 25/100, and 40/100.

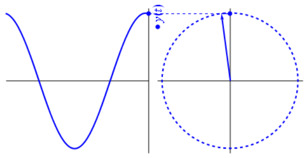
6 Complex Numbers and Fourier Analysis: Some Definitions

6.1 Rotation and Sine Waves

Imagine a rotating arrow centered at the origin, initially aligned along the positive x-axis (i.e., the angle from the positive x-axis is zero):



As this arrow rotates counterclockwise around the circle, the vertical position (the y-coordinate of the tip of the arrow) traces out a sine wave over time. This dynamic is visualized in the following animation:



The properties of this wave depend on the characteristics of the rotation:

- A faster rotation creates more cycles per unit of time, corresponding to a higher frequency.
- A smaller circle produces a smaller amplitude.
- Starting the rotation at a different point in time shifts the phase of the wave.

As discussed earlier, by combining multiple sine waves—each with its own frequency, amplitude, and phase—we can generate more complex wave patterns. This is the foundation of Fourier analysis: the idea that any sufficiently smooth function can be approximated over a finite interval by a weighted sum of sine and cosine waves with different frequencies. These linear combinations are known as Fourier series.

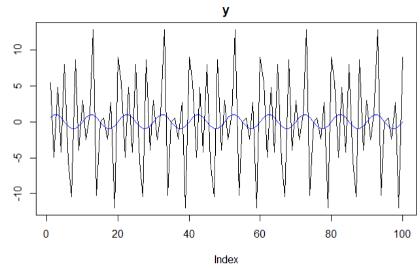
For a visual illustration, see the following resource: `https://bilimneguzellan.net/en/follow-up-to-fourier-series-2/`

6.2 Decomposing a Process with a Sine Function

As discussed earlier, a time series can be constructed by summing sine and cosine waves of various frequencies. Conversely, it is also possible to decompose a given time series into distinct waves of different frequencies. This process is known as Fourier decomposition and is the foundation for extracting frequency bands of interest from a signal. Devices that perform this operation are known as filters.

The simplest filter isolates a specific frequency from a time series. To extract the contribution of a particular frequency ω in a series y_t, we compute the following projection (or weight):

$$x(\omega) = \frac{1}{T}\sum_{t=0}^{T} \sin(\omega t) y_t$$



If the sine wave and the series y_t are well-aligned (i.e., they oscillate at the same frequency and phase), the value of $x(\omega)$ will be large and positive.

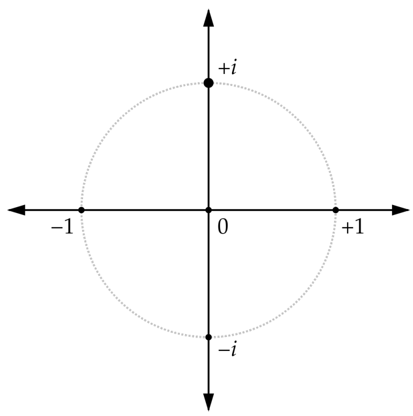
6.3 Complex Numbers

A complex number can be written as:

$$z = A + Bi$$

where $i = \sqrt{-1}$. Complex numbers can be represented as points on a plane, with the real part (A) along the x-axis and the imaginary part (B) along the y-axis. Alternatively, they can be visualized on the unit circle in the complex plane, where the magnitude (modulus) of the number is the distance from the origin (often normalized to 1), and the angle from the positive x-axis represents the phase.

A key property is that multiplying a complex number by i corresponds to a 90-degree counterclockwise rotation on the complex plane:



6.4 Euler's Formula

Suppose a quantity changes over time at a rate proportional to its current value. This is expressed mathematically as:

$$\frac{\partial x(t)}{\partial t} = kx(t)$$

The solution to this differential equation is:

$$x(t) = x(0)e^{kt}$$

Similarly, consider a point rotating around a circle. The change in the position of the radius vector $r(t)$ is proportional to the angular velocity ω and the current position, but it is rotated 90 degrees relative to $r(t)$. This is captured by the differential equation:

$$\frac{\partial r(t)}{\partial t} = i\omega r(t)$$

where i represents a 90-degree counterclockwise rotation. The solution is:

$$r(t) = r(0)e^{i\omega t}$$

If we start at $r(0) = 1$ (a unit vector pointing along the positive x-axis) and let ω be the angular velocity in radians per second, we obtain the expression:

$$r(t) = e^{i\omega t}$$

This is the core of Euler's formula, which elegantly connects exponential functions, trigonometry, and complex numbers. A famous special case is Euler's identity:

$$e^{i\pi} + 1 = 0$$

which holds when $\omega = \pi$ and $t = 1$.

6.5 Filtering with Complex Numbers and the Fourier Transform

The term $e^{i\omega t}$ is often referred to as a phasor. A phasor is a complex number representation of a sinusoidal function whose amplitude, angular frequency ω, and initial phase θ remain constant over time. As described earlier, a phasor can be visualized as a point rotating counterclockwise around the unit circle at a rate of ω radians per second. Returning to our earlier filtering equation:

$$x(\omega) = \frac{1}{T}\sum_{t=0}^{T} \sin(\omega t) y_t$$

we can generalize this by replacing the sine wave with the phasor, yielding the Fourier Transform:

$$F(\omega) = \frac{1}{T}\sum_{t=0}^{T} e^{i\omega t} y_t$$

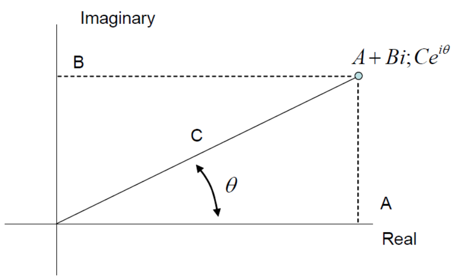
The product $e^{i\omega t} y_t$ represents scaling the length of a complex-valued vector (the phasor) by the real-valued observation y_t. Summing these scaled phasors across t produces the complex-valued Fourier Transform $F(\omega)$, which quantifies the contribution of each frequency ω to the series. The connection between complex numbers and trigonometric functions is highlighted by Euler's formula:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Thus, any complex number $z = A + Bi$ can be rewritten in polar form as:

$$z = Ce^{i\theta}$$

where $C = |z| = \sqrt{A^2 + B^2}$ is the magnitude (or amplitude) and $\theta = \tan^{-1}(B/A)$ is the phase angle. Geometrically, multiplying a vector by $e^{i\theta}$ corresponds to rotating it by an angle θ in the complex plane:



Fourier Transform Definitions

For a continuous-time signal $\{x_t\}$, the Fourier Transform is defined as:

$$x(\omega) = \int_{-\infty}^{\infty} x_t\, e^{-i\omega t} dt$$

where $i = \sqrt{-1}$ is the imaginary unit and ω is the angular frequency, measured in radians per unit of time.[5] By Euler's identity, we have:

$$e^{-i\omega t} = \cos(\omega t) - i\sin(\omega t)$$

showing that each periodic component of a time series can be represented as a sum of

[5]This definition extends to discrete-time filters as well. For a filter defined by $x(L) = \sum_{h=-\infty}^{\infty} x_h L^h$, where $t = h$, the Fourier Transform captures its frequency response.

sine and cosine functions in the Fourier domain. Given the Fourier Transform $x(\omega)$, we can recover the original time series x_t using the Inverse Fourier Transform:

$$x_t = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\omega t} x(\omega) d\omega$$

6.6 Spectral Representation: Definitions and Formulation

A time series or sequence $\{y_t\}$ can be viewed as a function mapping from the set of integers into the real line. For a finite series y_t, with $t = 0, \ldots, T-1$, the Fourier decomposition is given by:

$$y_t = \sum_{j=0}^{n} \{\alpha_j \cos(\omega_j t) + \beta_j \sin(\omega_j t)\}$$

where $\omega_j = \frac{2\pi j}{T}$ is a multiple of the fundamental frequency $\omega_1 = \frac{2\pi}{T}$, and α_j, β_j are the Fourier coefficients. Allowing $n \to \infty$, the series can be rewritten as:

$$y_t = \sum_{j} \{\cos(\omega_j t) dA(\omega_j) + \sin(\omega_j t) dB(\omega_j)\}$$

As the coefficients vanish for higher n, they are replaced by differentials: $\alpha_j = dA(\omega_j)$ and $\beta_j = dB(\omega_j)$. In the limit as $n \to \infty$, y_t can be represented in integral form (a Fourier-Stieltjes integral):

$$y_t = \int_0^{\pi} [\cos(\omega t) dA(\omega) + \sin(\omega t) dB(\omega)]$$

Spectral Assumptions:

- The functions $A(\omega)$ and $B(\omega)$ are stochastic processes indexed by ω with zero mean: $E\{dA(\omega)\} = E\{dB(\omega)\} = 0$.
- The processes are uncorrelated, and non-overlapping increments are uncorrelated:

$$E\{dA(\omega)dB(\lambda)\} = E\{dA(\omega)dA(\lambda)\} = E\{dB(\omega)dB(\lambda)\} = 0 \quad (\omega \neq \lambda).$$

- The variances of the increments are:

$$V\{dA(\omega)\} = V\{dB(\omega)\} = 2dF(\omega) = 2f(\omega)d\omega.$$

- $F(\omega)$ is the spectral distribution function, and $f(\omega)$ is the spectral density function.
- It follows that:

$$E\{dZ(\omega)dZ^*(\lambda)\} = 0 \quad (\omega \neq \lambda), \quad E\{dZ(\omega)dZ^*(\omega)\} = f(\omega)d\omega.$$

We define the following conjugate complex stochastic processes:

$$dZ(\omega) = \frac{1}{2}\left\{dA(\omega) - idB(\omega)\right\}, \quad dZ^*(\omega) = \frac{1}{2}\left\{dA(\omega) + idB(\omega)\right\}.$$

Using the properties of $A(\omega)$ (even function: $A(-\omega) = A(\omega)$) and $B(\omega)$ (odd function: $B(-\omega) = -B(\omega)$), we can extend the interval of integration from $[0, \pi]$ to $[-\pi, \pi]$, with:

$$dZ^*(\omega) = dZ(-\omega).$$

Returning to the integral representation:

$$y_t = \int_0^\pi \left[\cos(\omega t)dA(\omega) + \sin(\omega t)dB(\omega)\right],$$

and using Euler's formula $e^{-i\omega t} = \cos(\omega t) - i\sin(\omega t)$,[6] we can rewrite the expression as:

$$y_t = \int_0^\pi \left[\frac{e^{i\omega t} + e^{-i\omega t}}{2}dA(\omega) - i\frac{e^{i\omega t} - e^{-i\omega t}}{2}dB(\omega)\right],$$

which simplifies to:

$$y_t = \int_0^\pi \left[e^{i\omega t}dZ(\omega) + e^{-i\omega t}dZ^*(\omega)\right].$$

Extending the integral over $[-\pi, \pi]$, we obtain the spectral representation of y_t:

$$y_t = \int_{-\pi}^\pi e^{i\omega t}dZ(\omega).$$

6.7 The Hodrick-Prescott Filter in the Frequency Domain

The HP filter was originally developed in the time domain. King and Rebelo (1993) showed that for $T \to \infty$, the frequency response function of the HP cyclical filter, $C(\omega)$, is given by:

$$C(\omega) = \frac{4\lambda[1 - \cos(\omega)]^2}{1 + 4\lambda[1 - \cos(\omega)]^2}.$$

The moving average weights γ_j for the HP filter can be derived via the inverse Fourier transform:

$$\gamma_j = \frac{1}{2\pi}\int_{-\pi}^\pi \frac{4\lambda[1 - \cos(\omega)]^2}{1 + 4\lambda[1 - \cos(\omega)]^2}e^{i\omega j}d\omega.$$

The HP filter acts as a high-pass filter, allowing through only components with frequencies above a cutoff value χ. The relationship between the cutoff frequency and the parameter λ is:

$$\chi = \frac{\pi}{\arcsin\left(\frac{1}{2}\lambda^{-1/4}\right)},$$

[6] $\cos(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$, $\sin(\omega t) = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$.

which implies:

$$\lambda = \left[2\sin\left(\frac{\pi}{\chi}\right)\right]^{-4}.$$

Thus, the HP filter retains the part of the fluctuation band above the cutoff frequency χ.

7 Band-Pass Filters

7.1 Prior Filtering Methods

As discussed in a previous document, applying a first-difference can help remove unit roots, but this approach has drawbacks. Differencing can alter the timing of relationships between variables (introducing phase shifts) and can distort the relative importance of different frequencies by amplifying higher-frequency components. The Hodrick-Prescott (HP) filter also has limitations, as previously discussed.

7.2 Band-Pass Filters

An alternative approach is the band-pass filter. In this method, the business cycle is identified as the component of a time series with fluctuations confined to a specific frequency band, defined by an upper and lower limit. To apply a band-pass filter, it is necessary to specify the periodicity of the business cycle. Two widely used filters are those proposed by Baxter and King (1999) and Christiano and Fitzgerald (2003). In these filters, the detrending and smoothing problem is formulated in the frequency domain. Prescott (1986) interprets the HP filter as an approximation to a high-pass filter that removes fluctuations with a period longer than 8 years.

7.3 Baxter and King (1999)

Baxter and King (1999) decompose a time series into three components: trend, cycle, and irregular fluctuations. Their method starts by specifying the characteristics of the cyclical component, following the definition from Burns and Mitchell (1946): business cycles are fluctuations with a duration between 1.5 and 8 years (or 6 to 32 quarters). The ideal band-pass filter retains periodic components with cycle lengths between 6 and 32 quarters, while filtering out higher and lower frequencies. BK construct an approximation to this ideal filter using a finite-order moving average. They outline six desirable properties:

- The filter should preserve the essential properties of the extracted component.
- The filter should introduce no phase shifts (requiring symmetric weights).
- The filter should closely approximate the ideal band-pass filter.
- The filtered series should be stationary when applied to trending data.

- The filter's weights should not depend on the sample size (time-invariant).
- The method should be practical and not require an infinite time series.

The ideal band-pass filter is derived from ideal low-pass filters. The ideal low-pass filter retains only frequencies in the range $-\underline{\omega} < |\omega| < \underline{\omega}$. Its time-domain representation is:

$$b(L) = \sum_{h=-\infty}^{\infty} b_h L^h,$$

where the filter weights are given by the inverse Fourier transform:

$$b_h = \frac{1}{2\pi} \int_{-\pi}^{\pi} B(\omega) e^{i\omega h} d\omega,$$

with:

$$B(\omega) = \begin{cases} 1 & \text{if } |\omega| \leq \underline{\omega}, \\ 0 & \text{otherwise.} \end{cases}$$

Evaluating the integral yields the weights:

$$b_0 = \frac{\underline{\omega}}{\pi}, \quad b_h = \frac{\sin(h\underline{\omega})}{h\pi}.$$

This filter requires an infinite-order moving average for exact frequency isolation. In practice, an approximation is used by truncating the moving average at lag K:

$$a(L) = \sum_{h=-K}^{K} a_h L^h,$$

with frequency response:

$$\alpha_K(\omega) = \sum_{h=-K}^{K} a_h e^{-i\omega h}.$$

The weights a_h are chosen to minimize the squared approximation error:

$$Q = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(B(\omega) - \alpha_K(\omega)\right)^2 d\omega.$$

For an optimal approximation, we set $a_h = b_h$ for $|h| \leq K$ and $a_h = 0$ otherwise.

Low-pass vs. High-pass Filters:

- A low-pass filter retains low-frequency fluctuations, removing high-frequency noise.
- A high-pass filter removes low-frequency trends, retaining high-frequency components.

The weights for a high-pass filter are $1 - b_0$ for $h = 0$ and $-b_h$ for $h \neq 0$.

The ideal band-pass filter retains frequencies in the interval $\underline{\omega} < |\omega| < \overline{\omega}$. It is constructed by combining two low-pass filters with cutoff frequencies $\underline{\omega}$ and $\overline{\omega}$:

$$B'(\omega) = \overline{B}(\omega) - \underline{B}(\omega),$$

where $B'(\omega)$ has a value of 1 within the target band and 0 elsewhere.
For quarterly data:

- A cycle of 1.5 years corresponds to $p = 6$ quarters per cycle, so:
$$\omega_H = \frac{2\pi}{6} = \frac{\pi}{3}.$$
- A cycle of 8 years corresponds to $p = 32$ quarters per cycle, so:
$$\omega_L = \frac{2\pi}{32} = \frac{\pi}{16}.$$

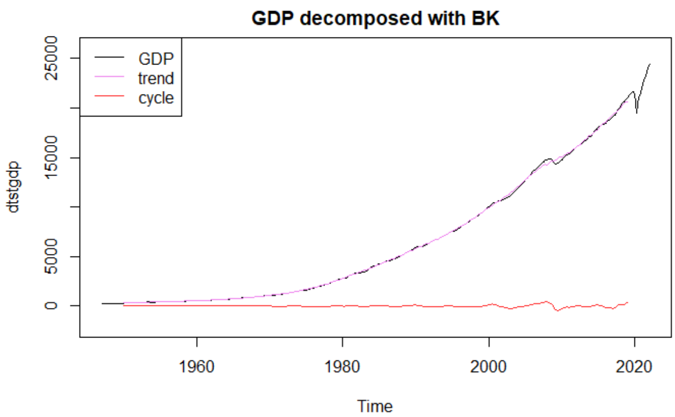
Thus, the band-pass filter is defined as:

$$B'(\omega) = \begin{cases} 1 & \text{for } \omega \in [\pi/16, \pi/3] \text{ or } [-\pi/3, -\pi/16], \\ 0 & \text{otherwise.} \end{cases}$$

This interval $[\pi/16, \pi/3]$ corresponds to business cycle frequencies. Periodic components within this range pass through the filter, while trends ($[0, \pi/16]$) and irregular fluctuations ($[\pi/3, \pi]$) are removed.

Example:

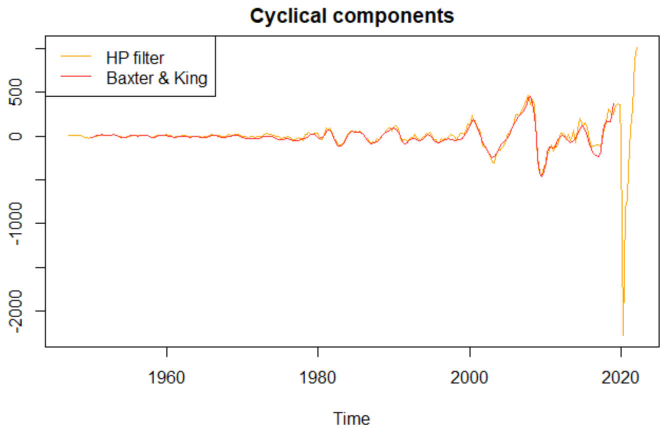
We can apply the BK filter to U.S. GDP data (1947Q1–2022Q1) using the `bkfilter()` function from the R package `mFilter`:




```
bkf <- bkfilter(dtstgdp, pl = 6, pu = 32)
```

Here, `pl` and `pu` specify the minimum and maximum cycle lengths (in quarters), set to 6 (1.5 years) and 32 (8 years), respectively.

Example: Comparing with the HP Filter



The results from both filters are similar, though the BK filter produces a smoother cyclical component, as it removes high-frequency noise. The filter can be adapted to different data frequencies by adjusting ω_L and ω_H accordingly.

7.4 Christiano and Fitzgerald (2003) Filter

Another widely used band-pass filter is the one proposed by Christiano and Fitzgerald (2003). Like the Baxter and King (BK) filter, the CF filter approximates an ideal band-pass filter designed for infinitely long time series. However, there are key differences between the two methods:

- The BK filter involves a trade-off between the length of the moving average (the trimming factor) and the precision of the filter approximation. Consequently, the BK filter discards some observations at the beginning and end of the sample.
- In contrast, the CF filter uses the entire sample when calculating each filtered data point, avoiding trimming.
- The CF filter is designed to perform well on a broader class of time series than the BK filter, converging asymptotically to the optimal filter and outperforming the BK filter in real-time applications.

The CF filter is computed as:

$$c_t = \beta_0 y_t + \sum_{j=1}^{T-1-t} \beta_j y_{t+j} + \tilde{\beta}_{T-t} y_T + \sum_{j=1}^{t-2} \beta_j y_{t-j} + \tilde{\beta}_{t-1} y_1$$

where the filter weights are defined as:

$$\beta_j = \frac{\sin(jb) - \sin(ja)}{\pi j} \quad \text{for } j \geq 1, \quad \beta_0 = \frac{b-a}{\pi}$$

and the boundary terms:

$$\tilde{\beta}_k = -\frac{1}{2}\beta_0 - \sum_{j=1}^{k-1} \beta_j$$

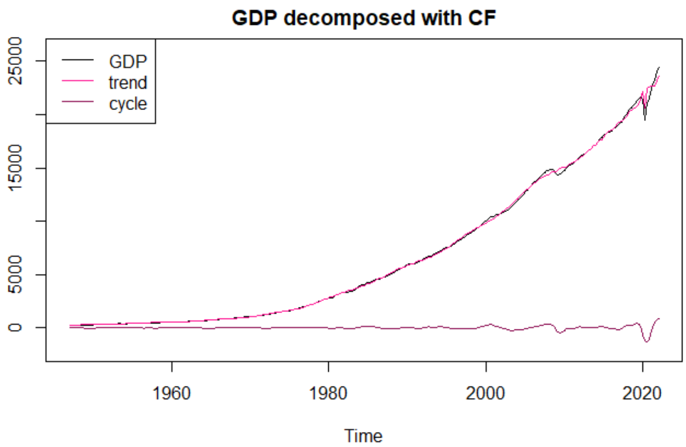
with the cutoff frequencies:

$$a = \frac{2\pi}{p_u}, \quad b = \frac{2\pi}{p_l},$$

where p_l and p_u are the lower and upper cycle lengths (in periods) defining the passband.

Example:

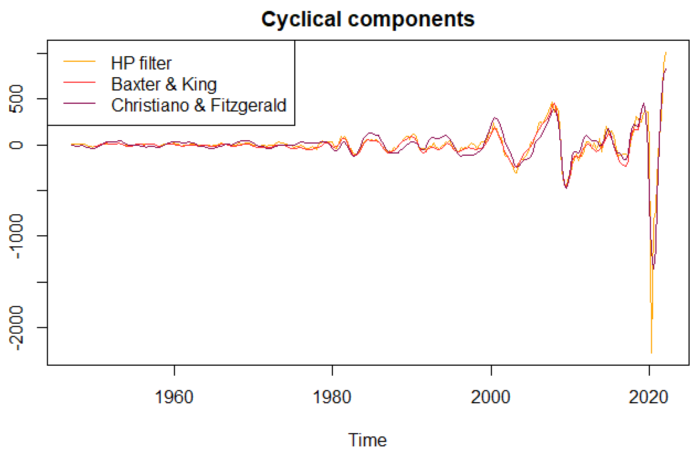
We can apply the CF filter to U.S. GDP data (1947Q1–2022Q1) using the `cffilter()` function from the R package `mFilter`:




```
cff <- cffilter(dtstgdp, pl = 6, pu = 32, root = TRUE)
```

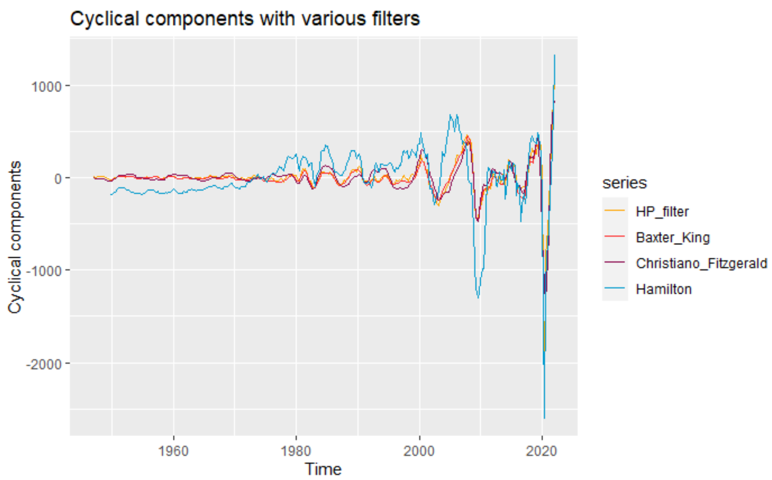
As before, `pl` is set to 6 quarters per cycle and `pu` to 32 quarters (8 years).

Example: Comparing Filters



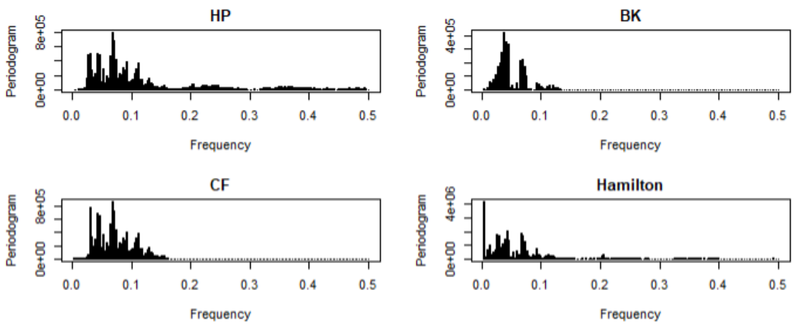
8 Comparison of Filters

Example: Combined Filter Comparison



8.1 Periodogram Analysis

Example: Periodograms for Different Filters



The periodogram reveals that the HP filter retains some high-frequency components that are removed by the band-pass filters. Specifically, the HP filter functions as a high-pass filter, passing through cycles with a frequency of eight years or less.

9 Wavelet Decompositions

A key limitation of the Fourier transform is that, while it provides a global frequency representation of a time series, it loses information about when specific events or patterns occur in time. The standard Fourier decomposition assumes stationarity—implying that the statistical properties of the series (such as its mean and variance) are constant over time. However, many economic and financial time series exhibit non-stationarity, structural breaks, or local changes in behavior that the Fourier approach cannot effectively capture.

To address this limitation, the Short-Time Fourier Transform (STFT) was developed. The STFT applies the Fourier transform to localized segments of the series using a fixed-length window. This process produces a time-frequency representation, mapping the series into a two-dimensional function of time and frequency. Specifically, at each time step, the STFT analyzes a window of length $N\delta t$, estimating frequencies in the range T^{-1} to $(2\delta t)^{-1}$. However, the STFT suffers from a key trade-off:

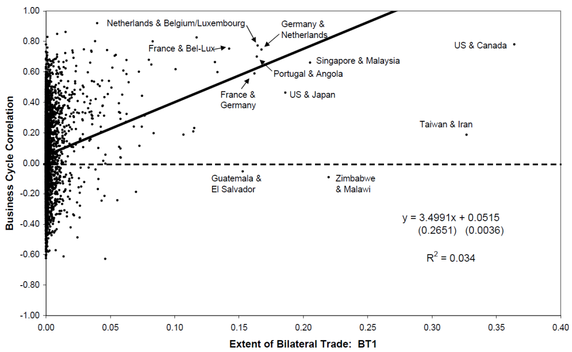
- A wide window provides better frequency resolution but poor time localization.
- A narrow window improves time resolution but reduces frequency precision.

This limitation is known as the uncertainty principle in time-frequency analysis. To overcome these trade-offs, wavelet transforms were developed. Wavelet analysis allows for a multi-resolution approach, providing high time resolution at high frequencies and high frequency resolution at low frequencies. Wavelet methods decompose a time series into components that vary both in time and in scale (frequency). They use wavelet functions; localized, oscillating functions that are stretched (scaled) and shifted (translated) to capture features across different time-frequency regions. This flexibility makes wavelet transforms especially useful for analyzing non-stationary time series, structural breaks, and transient phenomena. In R, wavelet decompositions can be performed using the `waveslim` package, which provides functions for both discrete and continuous wavelet transforms.

10 Examples from Macroeconomic Research

10.1 Determinants of Business Cycle Comovement (Baxter and Kouparitsas, 2004)

How are business cycle fluctuations transmitted across countries?
Baxter and Kouparitsas (2004) conduct an empirical study on over 100 countries to identify the main determinants of business cycle comovement. A key finding is that bilateral trade is a significant driver: countries with higher bilateral trade tend to have more highly correlated business cycles.



Scatter plot of business cycle correlation and bilateral trade, Baxter and Kouparitsas (2004)

In their methodology, the cyclical component of macroeconomic series is extracted using a band-pass filter. Regression analysis is then conducted using the extreme bounds analysis approach, with business cycle correlation across countries as the dependent variable. The study also examines other potential determinants of comovement, including:

- Similarity in industrial structure,
- Membership in a currency union,
- Extent of total trade in each country,
- Factor endowments,
- Gravity variables (distance, common language, adjacency).

Main conclusion: The most robust determinants of business cycle comovement are bilateral trade, distance, and membership in industrialized/advanced economies.

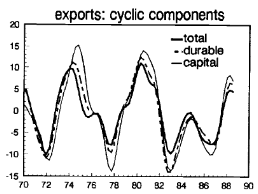
10.2 International Trade and Business Cycles (Baxter, 1995)

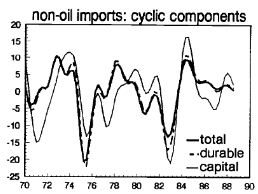
Baxter (1995) investigates the extent to which business cycles across countries share a global component. The study finds that economic aggregates in a group of advanced economies are positively correlated with those of the United States, suggesting a tendency for business cycles in advanced economies to move together.

	US	Australia	Austria	Canada	France	Germany	Italy	Japan	Switzerland	UK
output	1.00	0.60	0.54	0.81	0.46	0.85	0.49	0.66	0.48	0.64
consumption	1.00	-0.13	0.45	0.46	0.42	0.64	0.04	0.49	0.48	0.42
investment	1.00	0.21	0.57	0.00	0.22	0.66	0.39	0.59	0.38	0.46
employment	1.00	-0.17	0.58	0.50	0.36	0.60	0.11	0.48	0.43	0.68
gov't purchases	1.00	0.46	0.31	0.08	-0.18	0.40	0.23	0.06	-0.01	-0.10
net exports	1.00	0.03	0.29	-0.10	-0.25	-0.23	-0.28	-0.59	-0.10	-0.11

The paper applies the Baxter and King (1999) band-pass filter (with parameters `pl = 6`, `pu = 32`) to extract cyclical components. Key findings include:

- Strong comovement between the cyclical components of net exports and the current account.
- Fluctuations in imports and exports are primarily driven by durable goods.





Key insights:

- Investment is positively correlated with output and negatively correlated with net exports.
- The high volatility of investment is a central feature of national business cycles.
- Investment booms tend to coincide with current account deficits.

Using a general equilibrium two-country model, the study examines the responses of macroeconomic aggregates to total factor productivity (TFP) and fiscal shocks:

- A permanent increase in productivity leads to an investment boom and a higher output path, but results in an initial trade deficit.
- When labor input is variable, the response depends on asset market structures.
- For transitory shocks, output and consumption booms are associated with current account surpluses, as the foreign country decumulates bonds to smooth consumption while awaiting shock transmission.

The study also addresses puzzles such as the **saving-investment correlation**:[7]

10.3 Cyclicality of Fiscal Policy

Should fiscal policy be procyclical or countercyclical? In this context, cyclicality refers specifically to the behavior of government spending relative to the business cycle. According to standard Keynesian models, fiscal policy should be countercyclical: governments are expected to increase spending during economic downturns to stimulate aggregate demand and support the economy. Public expenditures thus act as a stabilizing force. In contrast, neoclassical models typically treat government spending as exogenous, and do not prescribe an active cyclical role for fiscal policy.

Empirical studies have documented significant differences in the cyclicality of fiscal policy across country groups. We now review some key findings from the literature.

Measuring Cyclicality:

Cyclicality is typically measured by:

- Estimating the **cyclical components** of GDP and fiscal variables using filters (e.g., Hodrick-Prescott or Baxter-King).
- Calculating the **correlation coefficient** between these cyclical components (e.g., Kaminsky, Reinhart, and Végh, 2004).

[7] Theoretical models suggest that countries with high savings should invest in countries with higher returns, yet empirical studies show a strong correlation between national savings and investment rates. This puzzle has been widely discussed in the literature, with implications for capital mobility and international financial integration.

Alternative approaches include regression-based measures. Lane (2003) argues that regression-based measures of cyclicality are more precise than simple correlations. Rigobón (2004) warns that single-equation regressions may produce biased estimates due to endogeneity.
Interpretation: Fiscal policy is considered:

- **Countercyclical** if the correlation between cyclical components of government spending and GDP is negative. In this case, fiscal policy stabilizes the business cycle (expansionary in bad times, contractionary in good times).
- **Procyclical** if the correlation is positive, indicating that government spending rises during booms and falls during recessions, potentially amplifying business cycle fluctuations.

While many fiscal variables can be used, the literature generally recommends focusing on **policy instruments** (e.g., tax rates, government spending) rather than fiscal outcomes (e.g., tax revenues, budget balances). Ratios to GDP should also be used cautiously, as they can introduce bias—especially when both the numerator and denominator are influenced by the business cycle. Kaminsky, Reinhart, and Végh (2004) illustrate these challenges:

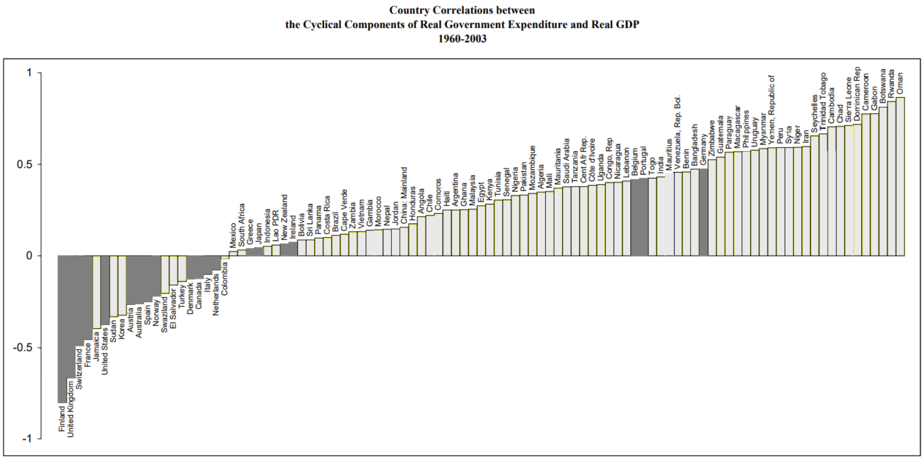
	g	τ	Tax Revenues	Primary Balance	g/GDP	Tax Revenues/GDP	Primary Balance/ GDP
Countercyclical	-	+	+	+	-	+/0/-	+/0/-
Procyclical	+	-	+/0/-	+/0/-	+/0/-	+/0/-	+/0/-
Acyclical	0	0	+	+	-	+/0/-	+/0/-

For example, using tax revenues as a share of GDP complicates interpretation: both tax revenues and GDP tend to increase over time, and procyclical tax policy (e.g., lower tax rates during booms) may obscure the true direction of cyclicality.

10.4 Cyclicality of Fiscal Policy: Kaminsky, Reinhart, and Végh (2004)

Kaminsky, Reinhart, and Végh (2004) decompose macroeconomic series using both the Hodrick-Prescott and Baxter-King (1999) filters. Their results show distinct patterns:

- Most **OECD countries** exhibit a **negative correlation** between the cyclical components of government spending and GDP—consistent with countercyclical fiscal policy.
- Most **developing countries** display a **positive correlation**, indicating procyclical fiscal behavior.



Dark bars represent OECD economies.

Moreover, in many developing countries, there is a positive correlation between the cyclical components of government spending and net capital inflows. This suggests that capital flow cycles and business cycles reinforce each other: fiscal policy tends to be expansionary when capital inflows are high and contractionary when they dry up.

Correlations between Fiscal Policy, Real GDP, and Net Capital Inflows

Countries	Central Government				General Government Expenditure	Consolidated Government Expenditure Minus Interest Payments	Inflation Tax
	Expenditure	Expenditure Minus Interest Payments	Expenditure on Goods and Services	Expenditure on Wages and Salaries			
HP Filter							
Correlation with Real GDP							
OECD	-0.13*	-0.05	-0.06	-0.15*	-0.06	-0.07	0.16*
Middle-High Income	0.38*	0.10	0.08	0.01	0.43*	0.10	-0.15*
Middle-Low Income	0.22*	0.13	0.07	0.03	0.20*	0.12	-0.09*
Low Income	0.38*	0.24*	0.54*	0.59*	0.37*	0.17*	-0.20*
Correlation with Net Capital Inflows							
OECD	0.03	0.05	0.04	0.04	0.09	0.03	0.04
Middle-High Income	0.25*	0.22*	0.28*	0.27*	0.25*	0.20*	-0.31*
Middle-Low Income	0.16*	0.11	0.13	0.12	0.18*	0.13	-0.14*
Low Income	0.20*	0.05	0.20	0.37	0.24*	-0.16	-0.09*
Band-Pass Filter							
Correlation with Real GDP							
OECD	-0.05	-0.15*	-0.11	-0.20*	-0.02	-0.12	0.15*
Middle-High Income	0.53*	0.19*	0.23*	0.13	0.44*	0.23*	-0.13*
Middle-Low Income	0.29*	0.29*	0.26*	0.23*	0.23*	0.23*	-0.10*
Low Income	0.46*	0.42*	0.53*	0.59*	0.34*	0.32*	-0.16*
Correlation with Net Capital Inflows							
OECD	0.07	0.08	0.05	0.04	0.14*	0.00	0.02
Middle-High Income	0.19*	0.12	0.28*	0.25*	0.16*	0.09	-0.25*
Middle-Low Income	0.14*	0.08	0.05	0.10	0.16*	0.11	-0.10*
Low Income	0.19*	0.25*	0.27*	0.39*	0.22*	0.13	-0.07*

11 Conclusion

This document has outlined the fundamental concepts, theoretical foundations, and mathematical formulations behind several filtering techniques commonly used in macroeconomic time series analysis. While no single method is universally optimal, understanding the assumptions, strengths, and limitations of each approach is essential for informed empirical work.

Future versions of this document will aim to enrich the theoretical material by adding comparative evaluations, empirical illustrations, and practical implementation guidance using macroeconomic datasets. Particular emphasis will be placed on exploring the implications of filter choice in applied research—such as estimating output gaps, studying fiscal cyclicality, and analyzing international business cycle synchronization. Ultimately, the goal is to provide a clear, useful reference for researchers and practitioners seeking to apply filtering techniques critically and transparently in the study of macroeconomic fluctuations.

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