MEP Introduction Report

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Abstract

A short revision on the continious Fourier transform, and the discrete approximation of it.

Contents

0.1 Approximating continious Fourier transform with dicrete Fourier transform

Lets start with a small refresher on Fourier transforms, and evaluate how to approximate the continious Fourier transform with the discrete Fourier transform.

0.1.1 Original definitions

The continous Fourier transfrom (CFT) of function $f(x)$ is defined as:

$$\mathcal{F}\left\{f(x)\right\} = F(\nu) = \int_{-\infty}^{\infty} \mathrm{e}^{-i2\pi\nu x} f(x) dx. \tag{0.1}$$

The inverse of the CFT is given by:

$$\mathcal{F}^{-1}\left\{F(\nu)\right\} = f(x) = \int_{-\infty}^{\infty} \mathrm{e}^{i2\pi\nu x} F(\nu) d\nu. \tag{0.2}$$

The discrete Fourtier transform (DFT) of discrete function $f[n]$ defined on $n \in \{0, ..., N-1\}$ is given by:

$$\mathrm{DFT}\left\{f[n]\right\} = F[k] = \sum_{n=0}^{N-1} \mathrm{e}^{-i2\pi kn} f[n], \forall k \in \{0, ..., N-1\}. \tag{0.3}$$

The inverse of the CFT is given by:

$$\mathrm{DFT}^{-1}\left\{F[k]\right\} = f[n] = \sum_{k=0}^{N-1} \mathrm{e}^{i2\pi kn} F[k], \forall n \in \{0, ..., N-1\}. \tag{0.4}$$

0.1.2 Discrete approximation CFT

If one approximates function $f(x)$ by $f[x_0 + n\Delta x], \forall n \in \{0, ..., N-1\}$, and ν by $k\Delta\nu, \forall k \in \{0, ..., N-1\}$, where:

$$\Delta x \Delta \nu = \frac{1}{N}, \tag{0.5}$$

starting from eq. (0.1), one can obtain:

$$\begin{aligned} F(\nu) \approx F[k\Delta\nu] &= \sum_{n=0}^{N-1} \Delta x f[x_0 + n\Delta x] \exp\left[-i2\pi k\Delta\nu(x_0 + n\Delta x)\right], \\ &= \Delta x \exp\left[-i2\pi k\Delta\nu x_0\right] \sum_{n=0}^{N-1} \exp\left[-i2\pi nk/N\right] f[x_0 + n\Delta x], \\ &= \Delta x \exp\left[-i2\pi k\Delta\nu x_0\right] \mathrm{DFT}\left\{f[n]\right\}. \end{aligned} \tag{0.6}$$

Similary, we can reobtain the original function $f(x) \approx f[n\Delta x + x_0]$ from the approximation of $F(\nu) \approx F[k\Delta\nu]$, using eq. (0.2):

$$\begin{aligned} f(x) \approx f[n\Delta x + x_0] &= \sum_{k=0}^{N-1} \Delta\nu F[k\Delta\nu] \exp\left[i2\pi k\Delta\nu(n\Delta x + x_0)\right] \\ &= \Delta\nu \sum_{k=0}^{N-1} F[k\Delta\nu] \exp\left[i2\pi k\Delta\nu x_0\right] \exp\left[i2\pi kn/N\right]. \end{aligned} \tag{0.7}$$

Defining:

$$G[k\Delta\nu] = \exp\left[i2\pi k\Delta\nu x_0\right] F[k\Delta\nu], \tag{0.8}$$

we find:

$$\begin{aligned} f(x) \approx f[n\Delta x + x_0] &= \Delta\nu \sum_{k=0}^{N-1} G[k\Delta\nu] \exp\left[i2\pi kn/N\right] \\ &= \Delta\nu \, \mathrm{DFT}^{-1} \left\{G[k\Delta\nu]\right\} \end{aligned} \tag{0.9}$$

0.1.3 Convolutions

Definition of concolution The concolution of two functions $f(x)$ and $g(x)$ is defined as te integral:

$$h(x) = \int_{-\infty}^{\infty} f(\bar{x})g(x-\bar{x})d\bar{x}. \tag{0.10}$$

Assuming $f(x)$ and $g(x)$ have a limited domain, $h(n)$ will also have a limited domain, given by:

$$\begin{aligned} f(x) &= \begin{cases} f(x), & x_{0,f} \leq x \leq x_{1,f} \\ 0, & x < x_{0,f} \vee x > x_{1,f} \end{cases}, \\ g(x) &= \begin{cases} g(x), & x_{0,g} \leq x \leq x_{1,g} \\ 0, & x < x_{0,g} \vee x > x_{1,g} \end{cases}, \\ h(x) &= \begin{cases} h(x), & x_{0,f} + x_{0,g} \leq x \leq x_{1,f} + x_{1,g} \\ 0, & x < x_{0,f} + x_{0,f} \vee x > x_{1,f} + x_{1,g} \end{cases}. \end{aligned} \tag{0.11}$$

This simplifies eq. (0.10) to:

$$h(x) = \int_{x_{0,f}}^{x_{1,f}} f(\bar{x})g(x-\bar{x})d\bar{x}. \tag{0.12}$$

Futhermore, the Fourier transform of the convolution has the beautiful property that, for the limited domain functions, but this can be extended into infinity:

$$\begin{aligned} H(\nu) &= \int_{x_{0,1}+x_{0,g}}^{x_{1,f}+x_{1,g}} h(x) \exp\left[-i2\pi\right] dx, \\ &= \int_{x_{0,1}+x_{0,g}}^{x_{1,f}+x_{1,g}} \int_{x_{0,1}}^{x_{1,f}} f(\bar{x})g(x-\bar{x})d\bar{x} \exp\left[-i2\pi\nu x\right] dx, \\ &= \int_{x_{0,1}}^{x_{1,f}} \int_{x_{0,1}+x_{0,g}}^{x_{1,f}+x_{1,g}} g(x-\bar{x}) \exp\left[-i2\pi\nu x\right] dx \quad f(\bar{x})d\bar{x}. \end{aligned} \tag{0.13}$$

Defining $\hat{x} = x - \bar{x}$, evaluation the limits on the integrals and realising $\hat{d(x)} = dx$, you obtain:

$$\begin{aligned} H(\nu) &= \int_{x_{0,f}}^{x_{1,f}} \int_{x_{0,g}}^{x_{1,g}} g(\hat{x}) \exp\left[-i2\pi\nu(\hat{x}+\bar{x})\right] d\hat{x} \quad f(\bar{x}) d\bar{x}, \\ &= \int_{x_{0,g}}^{x_{1,g}} g(\hat{x}) \exp\left[-i2\pi\nu\hat{x}\right] d\hat{x} \int_{x_{0,f}}^{x_{1,f}} f(\bar{x}) \exp\left[-i2\pi\nu\bar{x}\right] d\bar{x}, \\ &= F(\nu)G(\nu). \end{aligned} \tag{0.14}$$

Discretisation of the convolution and its CFT Again, we can approximate $f(x)$, $g(x)$ and $h(n)$ by:

$$\begin{aligned} f(x) &\approx f[n_f(\Delta x)_f + x_{0,f}], \forall n_f \in \{0, N_f - 1\}, (\Delta x)_f = (x_{1,f} - x_{0,f})/N_f, \\ g(x) &\approx g[n_g(\Delta x)_g + x_{0,g}], \forall n_g \in \{0, N_g - 1\}, (\Delta x)_g = (x_{1,g} - x_{0,g})/N_g. \end{aligned} \tag{0.15}$$

If we ensure that $(\Delta x)_f = (\Delta x)_g \equiv \Delta x$, we can approximate $h(n)$:

$$h(x) \approx h[n_h \Delta x + x_{0,f} + x_{0,g}], \forall n_h \in \{0, N_f + N_g - 1\}, \tag{0.16}$$

where

$$h[n_h \Delta x + x_{0,f} + x_{0,g}] = \sum_{k=0}^{N_f - 1} f[k\Delta x + x_{0,f}] g[n\Delta x + x_{0,g} - k\Delta x] \Delta x. \tag{0.17}$$

To discretisize the continious fourier transfrom, we once again need to define $\Delta\nu$, but mind the difference for $\Delta\nu$ for the discretisations of $f(x)$ or $g(x)$ and $h(x)$:

$$\begin{aligned} (\Delta\nu)_f &= \frac{1}{N_f \Delta x} \\ (\Delta\nu)_g &= \frac{1}{N_g \Delta x} \\ (\Delta\nu)_h &= \frac{1}{(N_f + N_g)\Delta x} \\ &= \left((\Delta\nu)_f^{-1} + (\Delta\nu)_g^{-1}\right)^{-1} \end{aligned} \tag{0.18}$$

Now the discrete approximation of the CFT of the convolution (given by eq. (0.13)) is given by:

$$\begin{aligned} H(\nu) \approx H[k(\Delta\nu)_h] &= \sum_{n=0}^{N_f+N_g-1} h[n\Delta x + x_{0,f} + x_{0,g}] \exp\left[-i2\pi k(\Delta\nu)_h (n\Delta x + x_{0,f} + x_{0,g}\right] \Delta x, \\ = \sum_{n=0}^{N_f+N_g-1} \sum_{l=0}^{N_f-1} & f[l\Delta x + x_{0,f}] g[(n-l)\Delta x + x_{0,g}] \exp\left[-i2\pi k(\Delta\nu)_h (n\Delta x + x_{0,f} + x_{0,g})\right] (\Delta x)^2. \end{aligned} \tag{0.19}$$

Defining $m = n - l$, and swapping summation order, you obtain:

$$H[k(\Delta\nu)_h] = (\Delta x)^2 \exp[-2\pi ik(\Delta\nu)_h x_{0,f}] \sum_{l=0}^{N_f-1} f[l(\Delta\nu)_h + x_{0,f}]$$
$$\exp[-2\pi ik(\Delta\nu)_h x_{0,g}] \sum_{m=-l}^{N_f+N_g-l-1} g[m\Delta x + x_{0,g}] \exp[-2\pi ik(\Delta\nu)_h(m+l)\Delta x], \tag{0.20}$$

reevaluating the boundries on the second summation, inputing eq. (0.18) and shuffling above result leads to:

$$H[k(\Delta\nu)_h] = \exp[-2\pi ik(\Delta\nu)_h x_{0,f}] \sum_{l=0}^{N_f-1} f[l(\Delta\nu)_h + x_{0,f}] \exp[-2\pi ikl/(N_f+N_g)]$$
$$\exp[-2\pi ik(\Delta\nu)_h x_{0,g}] \sum_{m=0}^{N_g-1} g[m\Delta x + x_{0,g}] \exp[-2\pi ikhm/(N_f+N_g)]. \tag{0.21}$$

Looking at the above equation, one almost recognizes a multiplication of two approximate CFT's (see eq. (0.6)). However, there is a significant difference, which means it is not a simple multiplication of the two CFT's: where in the CFT of the singular function, you see that $\Delta\nu = 1/N_f$ or $\Delta\nu = 1/N_g$ resectivily, whereas in eq. (0.21), $\Delta\nu = 1/(N_f + N_g)$, as given by eq. (0.18). Nor is it a simple factor with which the CFT's could be multiplied: the factor is present in the exponential within the summations. Each single exponential term in the summations need to raised to the power $(N_f + Ng)/N_f$ and $(N_f + Ng)/N_g$ respectivily. THIS IS THE PROBLEM.

0.1.4 Other calculations on CFTs

For the deconvolution of the spectra, we will need the devision and logaritmic values of two CFT's.