Notes EESL

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1 Analytical Calculation

Let us first remind ourselves of the task at hand. Given a total recorded signal $J(E)$ with a known zero-loss peak $Z(E)$, we want to recover the single scattering distribution $S(E)$ via deconvolution. Note that these are related by

$$\begin{aligned} J(E) &= Z(E) + J^1(E) + J^2(E) + J^3(E) + \dots \\ &= Z(E) + \frac{Z(E)}{I_0} * S(E) + \frac{Z(E)}{2! I_0^2} * S(E) * S(E) + \dots \\ &\equiv Z(E) + J_{\text{in}}(E), \end{aligned} \tag{1}$$

where we have defined $J_{\text{in}}(E)$ to absorb all the inelastic scattering contributions. Upon transforming eq. (1) to Fourier space, the convolutions become products and we obtain

$$\begin{aligned} j(\nu) &= z(\nu) + \frac{z(\nu)s(\nu)}{I_0} + \frac{z(\nu)s(\nu)^2}{2! I_0^2} + \dots \\ &= z(\nu)\left[1 + \frac{s(\nu)}{I_0} + \frac{1}{2!}\left(\frac{s(\nu)}{I_0}\right)^2 + \dots\right] \\ &= z(\nu)\exp\left[\frac{s(\nu)}{I_0}\right] \end{aligned} \tag{2}$$

From the definition of $J_{\text{in}}(E)$ in eq. (1) we also have

$$j(\nu) = z(\nu) + j_{\text{in}}(\nu). \tag{3}$$

Combining eq. (2) and (3) therefore gives

$$j_{\text{in}}(\nu) + z(\nu) = z(\nu)\exp\left[\frac{s(\nu)}{I_0}\right] \implies s(\nu) = I_0 \log\left[1 + \frac{j_{\text{in}}(\nu)}{z(\nu)}\right]. \tag{4}$$

In our toy model we take Gaussians for both $J(E)$ and $Z(E)$, that is

$$J_{\text{in}}(E) = \frac{I_s}{\sqrt{2\pi\sigma_s^2}} \exp\left[-\frac{(E-E_s)^2}{2\sigma_s^2}\right] \tag{5}$$

$$Z(E) = \frac{I_0}{\sqrt{2\pi\sigma_B^2}} \exp\left[-\frac{E^2}{2\sigma_B^2}\right]. \tag{6}$$

Here, the zero-loss peak $Z(E)$ and the inelastic scattering distribution J_{in} are centered about zero and E_s respectively. In addition, we have $\sigma_B \ll \sigma_s$, meaning that the background has a much tighter peak than the signal by various orders of magnitude.

In the following we will need the Fourier transform of eq. (5) and (6). They are

$$j_{\text{in}}(\nu) = I_s \exp[2\pi i\nu E_s] \exp[-2\pi^2\sigma_s^2\nu^2] \tag{7}$$

$$z(\nu) = I_0 \exp[-2\pi^2\sigma_B^2\nu^2]. \tag{8}$$

The idea is now to find $S(E)$ by expanding eq. (4) and taking the inverse Fourier transform of each term in the expansion. Let us first show how to expand eq. (4). After defining $g(\nu) \equiv j_{\text{in}}(\nu)/z(\nu)$, we get

$$s(\nu) = I_0\left(g(\nu) - \frac{g(\nu)^2}{2!} + \frac{g(\nu)^3}{3!} - \dots\right). \tag{9}$$

Note that each term in this expansion corresponds to a Gaussian, with subsequent terms getting an increasingly smaller spread. To be specific, from eq. (7) and (8) we find

$$\begin{aligned} g(\nu) &= \frac{I_s}{I_0} \exp[-2\pi^2(\sigma_s^2-\sigma_B^2)\nu^2 + 2\pi i\nu E_s] \\ &= \frac{I_s}{I_0} \exp\left[-\frac{E_s^2}{2(\sigma_s^2-\sigma_B^2)}\right] \exp\left[-2\pi^2(\sigma_s^2-\sigma_B^2)\left(\nu - \frac{iE_s}{2\pi(\sigma_s^2-\sigma_B^2)}\right)^2\right], \end{aligned} \tag{10}$$

where we have completed the square on the last line. If we furthermore define $\sigma_g^2 \equiv \sigma_s^2 - \sigma_B^2$, the k^{th} order contribution to $s(\nu)$ from eq. (9) can be written as

$$g^k(\nu) = \left(\frac{I_s}{I_0}\right)^k \exp\left[-\frac{kE_s^2}{2\sigma_g^2}\right] \exp\left[-2k\pi^2\sigma_g^2\left(\nu - \frac{iE_s}{2\pi\sigma_g^2}\right)^2\right]. \tag{11}$$

Next, we take the inverse Fourier transform of $g^k(\nu)$ and shift $\nu \to \nu + iE_s/2\pi\sigma_g^2$:

$$
\begin{aligned}
G_k(E) &= \left(\frac{I_s}{I_0}\right)^k \exp\left[-\frac{kE_s^2}{2\sigma_g^2}\right] \mathcal{F}^{-1}\left\{\exp\left[-2k\pi^2\sigma_g^2\left(\nu - \frac{iE_s}{2\pi\sigma_g^2}\right)^2\right]\right\} \\
&= \left(\frac{I_s}{I_0}\right)^k \exp\left[-\frac{kE_s^2}{2\sigma_g^2}\right] \exp\left[\frac{E_sE}{\sigma_g^2}\right] \mathcal{F}^{-1}\left\{\exp[-2k\pi^2\sigma_g^2\nu^2]\right\} \\
&= \left(\frac{I_s}{I_0}\right)^k \exp\left[-\frac{kE_s^2}{2\sigma_g^2}\right] \exp\left[\frac{E_sE}{\sigma_g^2}\right] \frac{1}{\sqrt{2k\pi\sigma_g^2}} \exp\left[-\frac{E^2}{2k\sigma_g^2}\right]. \qquad (12)
\end{aligned}
$$

As can be seen from (12) we get another Gaussian for $G_k(E)$. This can be made explicit by completing the square:

$$
\begin{aligned}
G_k(E) &= \frac{1}{\sqrt{2k\pi\sigma_g^2}} \left(\frac{I_s}{I_0}\right)^k \exp\left[-\frac{kE_s^2}{2\sigma_g^2}\right] \exp\left[-\frac{(E-kE_s)^2}{2k\sigma_g^2}\right] \exp\left[\frac{kE_s^2}{2\sigma_g^2}\right] \\
&= \frac{1}{\sqrt{2k\pi\sigma_g^2}} \left(\frac{I_s}{I_0}\right)^k \exp\left[-\frac{(E-kE_s)^2}{2k\sigma_g^2}\right]. \qquad (13)
\end{aligned}
$$

Hence, the single scattering distribution $S(E)$ becomes

$$
\begin{aligned}
S(E) &= I_0\left(G_1(E) - \frac{G_2(E)}{2!} + \frac{G_3(E)}{3!} - \dots\right) \\
&= \frac{I_s}{\sqrt{2\pi(\sigma_s^2 - \sigma_B^2)}} \exp\left[-\frac{(E-E_s)^2}{2(\sigma_s^2-\sigma_B^2)}\right] - \frac{I_s^2/I_0}{2\sqrt{4\pi(\sigma_s^2-\sigma_B^2)}} \exp\left[-\frac{(E-2E_s)^2}{4(\sigma_s^2-\sigma_B^2)}\right] + \dots
\end{aligned}
\qquad (14)
$$

As a sanity check, we notice that $S(E) \to J_{\mathrm{in}}(E)$ if $I_0 \gg I_s$ and $\sigma_s \gg \sigma_B$.