Delta Kinematics for the Delta Robot Project

grzesiek2201

January 2022

Inverse Kinematics

As inverse kinematics is easier with parallel robots, we'll start from them. They are also essential for robot control, unlike forward kinematics. The inverse kinematics problem states as follows: given a set of coordinates, find motor angles that correspond to them.

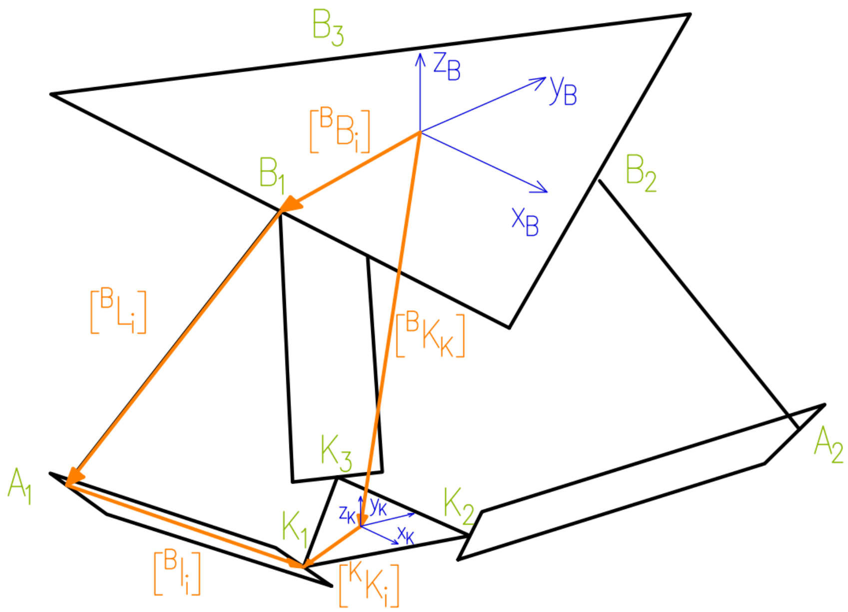

Figure 1: Kinematic structure of the robot. You can see two coordinate frames, one for the base(B) and one for the effector(K). Vectors drawn will be used in the vector-loop closure equations.

Vector-loop closure equation is as follows:

$$\left[{}^{B}B_i\right] + \left[{}^{B}L_i\right] + \left[{}^{B}l_i\right] = \left[{}^{B}K_K\right] + \left[{}^{B}_{K}R\right]\left[{}^{K}K_i\right] \tag{1}$$

but there is no rotation in the effector

$$\left[{}^{B}_{K}R\right] = \left[I_3\right],$$

so

$$\begin{aligned}\left[{}^{B}B_i\right] + \left[{}^{B}L_i\right] + \left[{}^{B}l_i\right] &= \left[{}^{B}K_K\right] + \left[{}^{K}K_i\right] \\ \left[{}^{B}l_i\right] &= \left[{}^{B}K_K\right] + \left[{}^{K}K_i\right] - \left[{}^{B}B_i\right] - \left[{}^{B}L_i\right]\end{aligned} \tag{2}$$

and

$$^{B}K_K = [x, y, z]^T \tag{3}$$

$$^{K}K_1 = \begin{bmatrix} 0 \\ -u_K \\ 0 \end{bmatrix} \; ^{K}K_2 = \begin{bmatrix} \frac{s_K}{2} \\ w_K \\ 0 \end{bmatrix} \; ^{K}K_3 = \begin{bmatrix} -\frac{s_K}{2} \\ w_K \\ 0 \end{bmatrix}, \tag{4}$$

$$^{B}B_1 = \begin{bmatrix} 0 \\ -w_B \\ 0 \end{bmatrix} \; ^{B}B_2 = \begin{bmatrix} \frac{\sqrt{3}}{2}w_B \\ \frac{1}{2}w_B \\ 0 \end{bmatrix} \; ^{B}B_3 = \begin{bmatrix} -\frac{\sqrt{3}}{2}w_B \\ \frac{1}{2}w_B \\ 0 \end{bmatrix} \tag{5}$$

$$^{B}L_1 = \begin{bmatrix} 0 \\ -L\cos(\theta_1) \\ -L\sin(\theta_1) \end{bmatrix} \; ^{B}L_2 = \begin{bmatrix} \frac{\sqrt{3}}{2}L\cos(\theta_2) \\ \frac{1}{2}L\cos(\theta_2) \\ -L\sin(\theta_2) \end{bmatrix} \; ^{B}L_3 = \begin{bmatrix} -\frac{\sqrt{3}}{2}L\cos(\theta_3) \\ \frac{1}{2}L\cos(\theta_3) \\ -L\sin(\theta_3) \end{bmatrix}, \tag{6}$$

where u_K - radius of the described circle in the triangle K, w_K - radius of the inscribed circle in the triangle K, s_K - length of the side of the triangle K, w_B - radius of the inscribed circle in the triangle B.

After substituting (3), (4), (5), (6) into (2):

$$ {}^{B}l_1 = \begin{bmatrix} x \\ y + L\cos(\theta_1) + a \\ z + L\sin(\theta_1) \end{bmatrix} \quad {}^{B}l_2 = \begin{bmatrix} x - \frac{\sqrt{3}}{2}L\cos(\theta_2) + b \\ y - \frac{1}{2}L\cos(\theta_2) + c \\ z + L\sin(\theta_2) \end{bmatrix} $$

$$ {}^{B}l_3 = \begin{bmatrix} x + \frac{\sqrt{3}}{2}L\cos(\theta_3) - b \\ y - \frac{1}{2}L\cos(\theta_3) + c \\ z + L\sin(\theta_3) \end{bmatrix}, $$

where

$$ a = w_B - u_K, b = \frac{s_K}{2} - \frac{\sqrt{3}}{2}w_B, c = w_K - \frac{1}{2}w_B. $$

It will be easier to square the vector's modulus

$$ l_i^2 = \left|\left[{}^{B}l_i\right]\right|^2 = l_{ix}^2 + l_{iy}^2 + l_{iz}^2. $$

After substitution we get

$$ \begin{cases} x^2 + y^2 + z^2 + L^2 + a^2 + 2ya - l^2 + 2L(y+a)\cos\theta_1 + 2zL\sin\theta_1 = 0 \\ x^2 + y^2 + z^2 + L^2 + b^2 + c^2 + 2xb + 2yc - l^2 - L(\sqrt{3}(x+b) + y + c)\cos\theta_2 + 2zL\sin\theta_2 = 0 \\ x^2 + y^2 + z^2 + L^2 + b^2 + c^2 - 2xb + 2yc - l^2 + L(\sqrt{3}(x-b) - y - c)\cos\theta_3 + 2zL\sin\theta_3 = 0. \end{cases} $$

The above system of equations can be simplified into

$$ E_i\cos\theta_i + F_i\sin\theta_i + G_i = 0, $$

where:

$$ E_1 = 2L(y+a), $$

$$ E_2 = -L(\sqrt{3}(x+b) + y + c), $$

$$ E_3 = L(\sqrt{3}(x - y - b - c), $$

$$ F_1 = 2zL, \quad F_2 = 2zL, \quad F_3 = 2zL, $$

$$ G_1 = x^2 + y^2 + z^2 + L^2 + a^2 + 2ya - l^2, $$

$$ G_2 = x^2 + y^2 + z^2 + L^2 + b^2 + c^2 + 2xb + 2yc - l^2, $$

$$ G_3 = x^2 + y^2 + z^2 + L^2 + b^2 + c^2 - 2xb + 2yc - l^2. $$

The equations can be solved using Weierstrass substitution

$$ E_i\left(\frac{1-t_i^2}{1+t_i^2}\right) + F_i\left(\frac{2t_i}{1+t_i^2}\right) + G_i = 0, $$

$$ E_i(1-t_i^2) + F_i(2t_i) + G_i(1+t_i^2) = 0, $$

$$ t_i^2(G_i - E_i) + t_i(2F_i) + (G_i + E_i) = 0. $$

Solution to the quadratic equation:

$$ t_{1,2} = \frac{-F_i \pm \sqrt{F_i^2 + E_i^2 - G_i^2}}{G_i - E_i}, $$

这里的t是要取正还是负

We get the solution to the inverse kinematics problem:

$$ \Theta_i = 2arctg(t_i). \tag{7} $$

Forward Kinematics

The forward kinematics problem states as follows: given a set of motor angles, find effector coordiantes that correspond to them. In order to find a solution to this problem, we must find points of intersection of three spheres, centers of which are located at the elbow joints transposed by the vectors $\left[{}^{B}A_{ih}\right]$ and radia of which are equal to the length of the forearms. Intersection points are the solution to the forward kinematics problem. The diagram is shown in the Figure 2.

Vectors $\left[{}^{B}A_{ih}\right]$ are equal to:

$$
{}^{B}A_{1h} = \begin{bmatrix} 0 \\ -w_B - L\cos(\theta_1) + u_K \\ -L\sin(\theta_1) \end{bmatrix}, {}^{B}A_{2h} = \begin{bmatrix} \frac{\sqrt{3}}{2}(w_B + L\cos(\theta_2)) - \frac{s_K}{2} \\ \frac{1}{2}(w_B + L\cos(\theta_2)) - w_K \\ -L\sin(\theta_2) \end{bmatrix},
$$

$$
{}^{B}A_{3h} = \begin{bmatrix} -\frac{\sqrt{3}}{2}(w_B + L\cos(\theta_3)) + \frac{s_K}{2} \\ \frac{1}{2}(w_B + L\cos(\theta_3)) - w_K \\ -L\sin(\theta_3) \end{bmatrix}. \tag{8}
$$

There are three spheres: $S_1(P_1, r_1)$ $S_2(P_2, r_2)$ $S_3(P_3, r_3)$, where points P_i are known and equal to:

$$
P_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, P_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, P_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}.
$$

The system of equations describing the presented issue looks as follows:

$$
\begin{cases} (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r_1^2 \\ (x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2 = r_2^2 \\ (x - x_3)^2 + (y - y_3)^2 + (z - z_3)^2 = r_3^2 \end{cases}, \quad K = \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{9}
$$

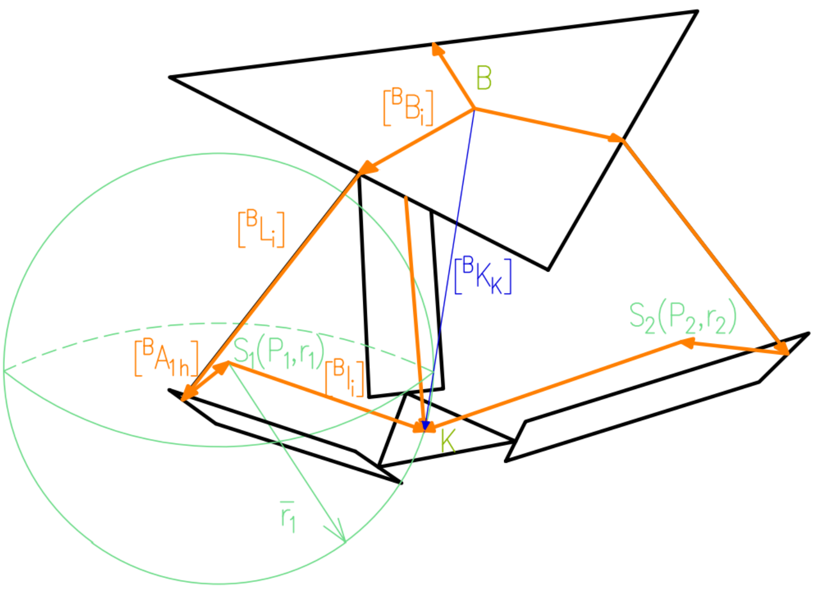

Figure 2: Kinematic diagram with vectors and spheres drawn in order to solve forward kinematics.

After expanding we must substract third equation from the first and second equation, result of which is the system of equations:

$$
\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \end{cases},
$$

gdzie

$$
a_{11} = 2(x_3 - x_1), \quad a_{12} = 2(y_3 - y_1), \quad a_{13} = 2(z_3 - z_1),
$$

$$
a_{21} = 2(x_3 - x_2), \quad a_{22} = 2(y_3 - y_2), \quad a_{23} = 2(z_3 = z_2),
$$

$$b_1 = r_1^2 - r_3^2 - x_1^2 - y_1^2 - z_1^2 + x_3^2 + y_3^2 + z_3^2, \quad b_2 = r_2^2 - r_3^2 - x_2^2 - y_2^2 - z_2^2 + x_3^2 + y_3^2 + z_3^2.$$

The above system of equations has been solved by determining z:

$$\begin{cases} z = \frac{b_1}{a_{13}} - \frac{a_{12}}{a_{13}}y - \frac{a_{11}}{a_{13}}x \\ \\ z = \frac{b_2}{a_{23}} - \frac{a_{22}}{a_{23}}y - \frac{a_{21}}{a_{23}}x \end{cases} \tag{10}$$

After substracting second equation from the first equation:

$$x = \frac{a_2}{a_1}y + \frac{a_3}{\frac{a_{11}}{a_{13}} - \frac{a_{21}}{a_{23}}},$$

where

$$a_1 = \frac{a_{11}}{a_{13}} - \frac{a_{21}}{a_{23}}, \quad a_2 = \frac{a_{22}}{a_{23}} - \frac{a_{12}}{a_{13}}, \quad a_3 = \frac{b_1}{a_{13}} - \frac{b_2}{a23}.$$

We get

$$x = f(y) = a_4 y + a_5,$$

where

$$a_4 = \frac{a_2}{a_1}, \quad a_5 = \frac{a_3}{a_1}.$$

The dependency of $x(y)$ has been determined. Next we need to substitute it into one of the equations in the (10). We get:

$$z = f(y) = a_6 y + a_7,$$

where

$$a_6 = \frac{-a_{21}a_4 - a22}{a_{23}}, \quad a_7 = \frac{b_2 - a_{21}a_5}{a_{23}}.$$

The dependency $z(y)$ has been determined, so we need to substitue it into one of the sphere equations. We get

$$ay^2 + by + c = 0, \tag{11}$$

where

$$a = a_4^2 + 1 + a_6^2, \quad b = 2a_4(a_5 - a_1) + 2a_6(a_7 - z_1) - 2y_1,$$
$$c = a_5(a_5 - 2x_1) + a_7(a_7 - 2z_1) + x_1^2 + y_1^2 + z_1^2 - r_1^2.$$

The equation (11) is an quadratic formula, solution to which is as follows:

$$\begin{cases} x_+ = a_4 y_+ + a_5 \\ \\ y_+ = \frac{-b+\sqrt{b^2-4ac}}{2a} \\ \\ z_+ = a_6 y_+ + a_7 \end{cases}, \qquad \begin{cases} x_- = a_4 y_- + a_5 \\ \\ y_- = \frac{-b-\sqrt{b^2-4ac}}{2a} \\ \\ z_- = a_6 y_- + a_7 \end{cases},$$

$$K_+ = \begin{bmatrix} x_+ \\ y_+ \\ z_+ \end{bmatrix}, K_- = \begin{bmatrix} x_- \\ y_- \\ z_- \end{bmatrix}.$$

In order for the real solutions to exist, the $\Delta \geq 0$ must exist. If calculated Δ will be smaller than zero, it suggests that an error occured when reading the motor angles, as motor angles cannot have irrational values.

Now we need to consider singularities of the given solution. It has no solutions, when at least one of the below condition is met:

$$a_{13} = 0, \tag{12}$$

$$a_{23} = 0, \tag{13}$$

$$a_1 = 0, \tag{14}$$

$$a = 0. \tag{15}$$

Condition (14) is met, when

$$a_1 = \frac{a_{11}}{a_{13}} - \frac{a_{21}}{a_{23}} = \frac{(x_3 - x_1)(z_3 - z_2) - (x_3 - x_2)(z_3 - z_1)}{(z_3 - z_1)(z_3 - z_2)} = 0$$

$$\Updownarrow$$

$$\frac{x_3 - x_1}{z_3 = z_1} = \frac{x_3 - x_2}{z_3 - z_2}.$$

Let $z_3 = x_3 = 0$, so the point 3 lies in the middle of the coordinate system. Then

$$\frac{x_1}{z_1} = \frac{x_2}{z_2}.$$

This condition is only met when three points, which are the centers of the spheres, are collinear in the XZ plane. That is why for a valid geometry, this singularity never occurs.

Condition (15) is met, when

$$a = a_4^2 + a_6^2 + 1 = 0 \Leftrightarrow a_4^2 + a_6^2 = -1.$$

For real a_4 and a_6 this condition cannot be met, that is why it does not have to be considered in the overall solution.

Conditions (12) and (13) are met, when $z_3 - z_1 = 0$ and $z_3 - z_2 = 0$. It is only possible in a configuration in which the angle of the first arm is equal to the angle of the third arm and when angle of the second arm is equal to the angle of the third arm. This is possible under typical operation and has to be accounted for in the general solution. In order to get rid of the singularities the solution to the problem, when $z_1 = z_2 = z_3 =: z_n$, has been found. We start from the same equations for the intersection of three spheres as before

$$\begin{cases} (x - x_1)^2 + (y - y_1)^2 + (z - z_n)^2 = r_1^2 \\ (x - x_2)^2 + (y - y_2)^2 + (z - z_n)^2 = r_2^2 \\ (x - x_3)^2 + (y - y_3)^2 + (z - z_n)^2 = r_3^2. \end{cases}$$

After multiplication and substitution of the third equation for the first and the second equation we get

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases} \Rightarrow \begin{cases} x = \frac{c}{a} - \frac{b}{a}y \\ y = \frac{af-cd}{ae-bd} \end{cases} \Rightarrow \begin{cases} x = \frac{ce-bf}{ae-bd} \\ y = \frac{af-cd}{ae-bd}, \end{cases}$$

where

$$\begin{aligned} a &= 2(x_3 - x_1), & b &= 2(y_3 = y_1), & c &= r_1^2 - r_3^2 - x_1^2 - y_1^2 + x_3^2 + y_3^2, \\ d &= 2(x_3 - x_2), & e &= 2(y_3 - y_2), & f &= r_2^2 - r_3^2 - x_2^2 - y_2^2 + x_3^2 + y_3^2. \end{aligned}$$

The calculated x and y have been substituted into the first equation from the 9 system of equations:

$$Az^2 + Bz + C = 0,$$

where

$$\begin{gathered} A = 1, \\ B = -2z_n, \\ C = x^2 + y^2 + x_1^2 + y_1^2 - 2xx_1 - r_1^2 + z_n^2. \end{gathered}$$

The solution to the forward kinematics problem is as follows:

$$\begin{cases} x = \frac{ce-bf}{ad-bd} \\ y = \frac{af-cd}{ae-bd} \\ z = \frac{-B\pm\sqrt{B^2-4C}}{2}. \end{cases}$$

We also have to investigate the singularities of the solution:

$$ae - bd = 0,$$
$$\Delta < 0 \Leftrightarrow B^2 - 4AC = 0.$$

The first of the singularities has been expanded below:

$$ae - bd = 0 \Leftrightarrow \tfrac{x_3 - x_1}{y_3 - y_1} = \tfrac{x_3 - x_2}{y_3 - y_2}.$$

The above equality is met only when centers of the spheres lay on the XY plane and are collinear. For a symmetric configuration of the robot, this singularity does not have to be considered in the general solution.

The second of the singularities is met when

$$B^2 - 4AC = 0 \Leftrightarrow r_1^2 < (x - x_1)^2 + (y - y_1)^2.$$

The above inequality cannot be met, as we know the the below equation is true

$$(x - x_1)^2 + (y - y_1)^2 = r_1^2.$$

There is also a situation in which two of the three spheres are position on the same height. Instead of deriving further equations, which would let us get rid of this singularity, it is possible to add a very small value to one of the sphere's height when the situation occurs. It does not influence the accuracy of the visualization and control in a visible way, but it does simplify the theoretical considerations.

Visualization

The calculation will based on the angular position of the motors measured by enoders as well as equations derived below. In order to create the visualization, we must know coordinates of all the points of a drawn line. Objects that will be drawn as a line are: each of the arms, each of the forearms, base, and effector. Each of the legs will be a simple prism, thus their representation will consist of two bases and four edges.

The base vertices are described with vectors

$$ {}^B b_1 = \begin{bmatrix} \frac{s_B}{2} \\ -w_B \\ 0 \end{bmatrix}, {}^B b_2 = \begin{bmatrix} 0 \\ u_K \\ 0 \end{bmatrix}, {}^B b_3 = \begin{bmatrix} -\frac{s_B}{2} \\ -w_B \\ 0 \end{bmatrix}. \tag{16} $$

The vectors below $\left[{}^B A_i\right]$ are describing position of the elbow joints in respect to the base B:

$$ {}^B A_1 = \begin{bmatrix} 0 \\ -w_B - L\cos(\theta_1) \\ -L\sin(\theta_1) \end{bmatrix}, {}^B A_2 = \begin{bmatrix} \frac{\sqrt{3}}{2}(w_B + L\cos(\theta_2)) \\ \frac{1}{2}(w_B + L\cos(\theta_2)) \\ -L\sin(\theta_2) \end{bmatrix}, $$

$$ {}^B A_3 = \begin{bmatrix} -\frac{\sqrt{3}}{2}(w_B + L\cos(\theta_3)) \\ \frac{1}{2}(w_B + L\cos(\theta_3)) \\ -L\sin(\theta_3) \end{bmatrix}. \tag{17} $$

Vectors $\left[{}^B K_i\right]$ describe the position of the effector's joints in respect to the base B:

$$ {}^B K_1 = \begin{bmatrix} x \\ -\frac{s_K}{2} + y \\ z \end{bmatrix}, {}^B K_2 = \begin{bmatrix} \frac{s_K}{2} + x \\ w_K + y \\ z \end{bmatrix}, {}^B K_3 = \begin{bmatrix} -\frac{s_K}{2} + x \\ w_K + y \\ z \end{bmatrix}. \tag{18} $$

Vectors $\left[{}^B B_i\right]$ describe the position of the arm joints in respect to the base B:

$$ {}^B B_1 = \begin{bmatrix} 0 \\ -w_B \\ 0 \end{bmatrix}, {}^B B_2 = \begin{bmatrix} \frac{\sqrt{3}}{2}w_B \\ \frac{1}{2}w_B \\ 0 \end{bmatrix}, {}^B B_3 = \begin{bmatrix} -\frac{\sqrt{3}}{2}w_B \\ \frac{1}{2}w_B \\ 0 \end{bmatrix}. \tag{19} $$

Vectors $\left[{}^B A_{ij}\right]$ describe the position of the elbow joints transposed by a vector equal to their width, in respect to the base B:

$$ {}^B A_{11} = \begin{bmatrix} -\frac{e_w}{2} \\ -w_B - L\cos(\theta_1) \\ -L\sin(\theta_1) \end{bmatrix}, {}^B A_{12} = \begin{bmatrix} \frac{e_w}{2} \\ -w_B - L\cos(\theta_1) \\ -L\sin(\theta_1) \end{bmatrix}, $$

$$ {}^B A_{21} = \begin{bmatrix} \frac{\sqrt{3}}{2}(w_B + L\cos(\theta_2)) + \frac{e_w}{4} \\ \frac{1}{2}(w_B + L\cos(\theta_2)) - \frac{\sqrt{3}}{4}e_w \\ -L\sin(\theta_2) \end{bmatrix}, {}^B A_{22} = \begin{bmatrix} \frac{\sqrt{3}}{2}(w_B + L\cos(\theta_2)) - \frac{e_w}{4} \\ \frac{1}{2}(w_B + L\cos(\theta_2)) + \frac{\sqrt{3}}{4}e_w \\ -L\sin(\theta_2) \end{bmatrix}, $$

$$ {}^B A_{31} = \begin{bmatrix} -\frac{\sqrt{3}}{2}(w_B + L\cos(\theta_3)) + \frac{e_w}{4} \\ \frac{1}{2}(w_B + L\cos(\theta_3)) + \frac{\sqrt{3}}{4}e_w \\ -L\sin(\theta_3) \end{bmatrix}, {}^B A_{32} = \begin{bmatrix} -\frac{\sqrt{3}}{2}(w_B + L\cos(\theta_3)) - \frac{e_w}{4} \\ \frac{1}{2}(w_B + L\cos(\theta_3)) - \frac{\sqrt{3}}{4}e_w \\ -L\sin(\theta_3) \end{bmatrix}. $$

Vectors $\left[{}^B K_{ij}\right]$ describe the position of the efector joints transpoed by the same vector as above, in respect to the base B:

$$ {}^B K_{11} = \begin{bmatrix} x - \frac{e_w}{2} \\ -\frac{s_K}{2} + y \\ z \end{bmatrix}, {}^B K_{12} = \begin{bmatrix} x + \frac{e_w}{2} \\ -\frac{s_K}{2} + y \\ z \end{bmatrix}, $$

$$ {}^B K_{21} = \begin{bmatrix} \frac{s_K}{2} + x + \frac{e_w}{4} \\ w_K + y - \frac{\sqrt{3}}{4}e_w \\ z \end{bmatrix}, {}^B K_{21} = \begin{bmatrix} \frac{s_K}{2} + x - \frac{e_w}{4} \\ w_K + y + \frac{\sqrt{3}}{4}e_w \\ z \end{bmatrix}, $$

$$ {}^B K_{31} = \begin{bmatrix} -\frac{s_K}{2} + x + \frac{e_w}{4} \\ w_K + y + \frac{\sqrt{3}}{4}e_w \\ z \end{bmatrix}, {}^B K_{32} = \begin{bmatrix} -\frac{s_K}{2} + x - \frac{e_w}{4} \\ w_K + y - \frac{\sqrt{3}}{4}e_w \\ z \end{bmatrix}. $$

We know all the points necessary to create the visualization of the kinematic structure of the robot.