Note on the mutipole expansion of a heaving hemisphere on the free surface

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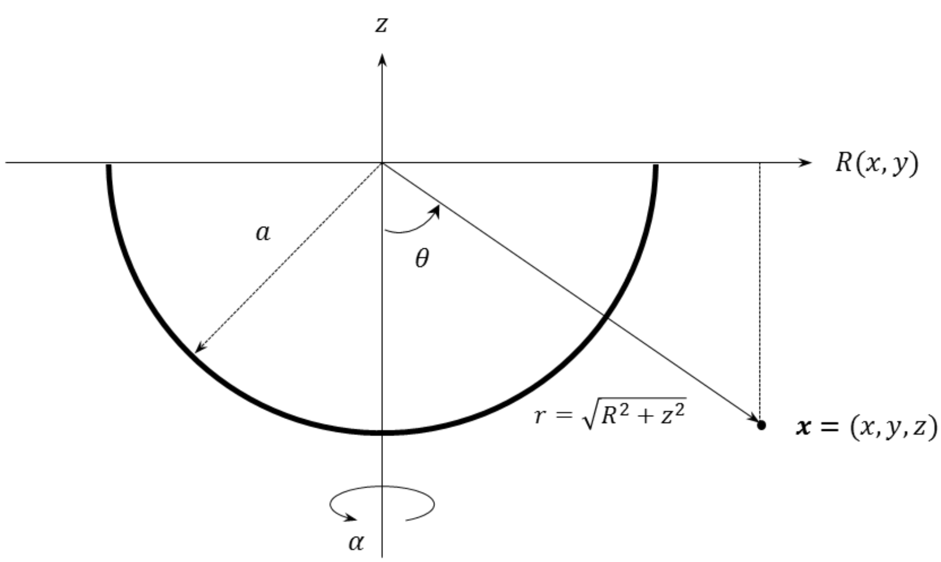
Heaving hemisphere problem

An heaving hemisphere problem was first solved by Havelock(1955). Havelock uses the multipole expansion method similarly to what done by Ursell(1949). He expressed the velocity potential with multipoles composed of wave source and wave free terms. Later, Hulme(1982) modified the expression of Havelock to be more rigorous.

The coordinates for the heaving hemisphere are defined in the figure below.



$$x = R\cos\alpha, \quad y = R\sin\alpha, \quad z = -r\cos\theta, \quad R = r\,sin\theta$$

$$r = \sqrt{R^2 + z^2} = \sqrt{x^2 + y^2 + z^2}, \qquad r^2 = x^2 + y^2 + z^2, \qquad R^2 = x^2 + y^2$$

Multipole expansion of heaving hemisphere

The multipole expansion for the heaving hemisphere is given by Hulme (1982). The multipoles satisfy the Laplace equation, the free surface boundary condition, the bottom boundary condition and the radiation condition. As the heave motion is symmetric, the velocity potential is expressed as follows:

$$\phi^S = c_0 a^2 \left(\psi_0^S + \sum_{n=1}^{\infty} c_n a^{2n} \psi_{2n}^S \right) ,$$

where

$$\psi_0^S = \frac{1}{r} + \pi i K \sum_{n=0}^{\infty} \frac{(-Kr)^n}{n!} P_n(\mu) - K \sum_{n=0}^{\infty} (-1)^n \frac{\partial}{\partial \nu} \left\{ \frac{(Kr)^\nu}{\nu!} P_\nu(\mu) \right\}_{\nu=n} ,$$

and

$$\psi_{2n}^S = \frac{P_{2n}(\mu)}{r^{2n+1}} + \frac{K}{2n} \frac{P_{2n-1}(\mu)}{r^{2n}} .$$

$$a : radius, \quad \mu = \cos\theta$$

Wave-pole is expressed in an alternative form. Please remind that the expression of di-gamma function in the following form is correct (different with Hulme original notation).

$$\psi_0^S = \frac{1}{r} + K \sum_{n=0}^{\infty} \frac{(-Kr)^n}{n!} \left[\{\psi(n+1) - \ln(Kr) + \pi i\} P_n(\mu) - \left. \frac{\partial P_\nu(\mu)}{\partial \nu} \right|_{\nu=n} \right]$$

As it can be derived as follows:

$$\frac{\partial}{\partial \nu} \left\{ \frac{(Kr)^\nu}{\nu!} P_\nu(\mu) \right\}_{\nu=n} = \frac{\partial}{\partial \nu} \left\{ \frac{(Kr)^\nu}{\Gamma(\nu+1)} P_\nu(\mu) \right\}_{\nu=n} = \left\{ \frac{(Kr)^\nu \{\ln(Kr) - \psi(\nu+1)\}}{\Gamma(\nu+1)} P_\nu(\mu) + \frac{(Kr)^\nu}{\Gamma(\nu+1)} \frac{\partial P_\nu(\mu)}{\partial \nu} \right\}_{\nu=n}$$

$$= \frac{(Kr)^n}{n!} \left[\{\ln(Kr) - \psi(n+1)\} P_n(\mu) + \left. \frac{\partial P_\nu(\mu)}{\partial \nu} \right|_{\nu=n} \right]$$

Multipole amplitudes

The body boundary condition is applied to calculate the multipole amplitudes.

$$\frac{\partial \phi^S}{\partial r} = \cos\theta = \mu = P_1(\mu)$$

$$\frac{\partial \phi^S}{\partial r} = c_0\left(a^2\frac{\partial \psi_0^S}{\partial r} + \sum_{n=1}^{\infty} c_n a^{2n}\left(a^2\frac{\partial \psi_{2n}^S}{\partial r}\right)\right) = P_1(\mu)$$

$$a^2\frac{\partial \psi_0^S}{\partial r} + \sum_{n=1}^{\infty} c_n a^{2n}\left(a^2\frac{\partial \psi_{2n}^S}{\partial r}\right) = \frac{P_1(\mu)}{c_0}$$

Integrating the above expression with respect to μ over (0, 1), and with the Hulme(1982) assumption that c_0 is not zero:

$$\int_0^1 a^2\frac{\partial \psi_0^S}{\partial r}d\mu + \sum_{n=1}^{\infty} c_n a^{2n}\int_0^1 a^2\frac{\partial \psi_{2n}^S}{\partial r}d\mu = \frac{I_{0,1}}{c_0} = \frac{1}{2c_0}$$

where

$$I_{m,n} = \int_0^1 P_m(\mu)P_n(\mu)d\mu$$

Let

$$a^2\frac{\partial \psi_0^S}{\partial r} = F(\mu, Ka)$$

$$\int_0^1 a^2\frac{\partial \psi_{2n}^S}{\partial r}d\mu = -\frac{1}{a^{2n}}\left\{(2n+1)\int_0^1 P_{2n}(\mu)d\mu + Ka\int_0^1 P_{2n-1}(\mu)d\mu\right\} = -\frac{Ka}{a^{2n}}I_{0,2n-1}$$

$$\int_0^1 P_{2n}(\mu)P_0(\mu)d\mu = 0 \quad if \quad n > 0$$

Substituting

$$\int_0^1 F(\mu, Ka)\,d\mu - Ka\sum_{n=1}^{\infty} c_n I_{0,2n-1} = \frac{1}{2c_0}$$

Recalling the body boundary condition

$$F(\mu, Ka) - \sum_{n=1}^{\infty} c_n\{(2n+1)P_{2n}(\mu) + KaP_{2n-1}(\mu)\} = \frac{P_1(\mu)}{c_0}$$
$$= 2P_1(\mu)\int_0^1 F(\mu, Ka)\, d - 2P_1(\mu)Ka\sum_{n=1}^{\infty} c_n I_{0,2n-1}$$

The following equations are obtained after some manipulations:

$$\sum_{n=1}^{\infty} c_n\left[(2n+1)P_{2n}(\mu) + Ka\left\{P_{2n-1}(\mu) - 2P_1(\mu)I_{0,2n-1}\right\}\right] = F(\mu, Ka) - 2P_1(\mu)\int_0^1 F(\mu, Ka)\, d\mu$$

Then multiplying $P_{2m}(\mu)$ and integrating with respect to μ over (0, 1).

$$\sum_{n=1}^{\infty} c_n\left[(2n+1)\int_0^1 P_{2n}(\mu)P_{2m}(\mu)d\mu + Ka\left\{\int_0^1 P_{2n-1}(\mu)P_{2m}(\mu)d\mu - 2\int_0^1 P_1(\mu)P_{2m}(\mu)d\mu\, I_{0,2n-1}\right\}\right]$$
$$= \int_0^1 F(\mu, Ka)P_{2m}(\mu)d\mu - 2\int_0^1 P_1(\mu)P_{2m}(\mu)d\mu\int_0^1 F(\mu, Ka)\, d\mu$$

Using the orthogonality of Legendre functions

$$\int_0^1 P_{2n}(\mu)P_{2m}(\mu)d\mu = \frac{\delta_{mn}}{4m+1}$$

The system of equations of the heaving hemisphere expressed with the multipoles is given as :

$$\frac{2m+1}{4m+1}c_m + Ka\sum_{n=1}^{\infty} c_n\left\{I_{2m,2n-1} - 2I_{2m,1}I_{0,2n-1}\right\} = J_{2m} - 2J_0 I_{2m,1}$$

where

$$J_{2m} = \int_0^1 F(\mu, Ka) P_{2m}(\mu) d\mu$$

An algebraic form is given as:

$$[\boldsymbol{\lambda} + \boldsymbol{M}][\boldsymbol{c}] = [\boldsymbol{\beta}]$$

$$\lambda_{mn} = \frac{2m+1}{4m+1}\delta_{mn}, \quad M_{mn} = Ka\{I_{2m,2n-1} - 2I_{2m,1}I_{0,2n-1}\}, \quad \beta_m = J_{2m} - 2J_0 I_{2m,1}$$

The value of c_0 is given as:

$$c_0 = \left[2\int_0^1 F(\mu, Ka)\, d\mu - Ka \sum_{n=1}^{\infty} c_n I_{0,2n-1}\right]^{-1}$$

Force acting on the heaving hemisphere

$$
\begin{aligned}
f &= \iint_S p\boldsymbol{n}dS = -\rho\iint_S \frac{\partial\phi}{\partial t}\boldsymbol{n}dS = i\omega\rho\iint_S \phi(-cos\theta)dS \\
&= i\omega\rho\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\phi^S(-\cos\theta)a^2d\theta d\alpha \\
&= -2\pi a^2(i\omega\rho)\int_0^{\frac{\pi}{2}} c_0a^2\left(\psi_0^S(a,\theta)+\sum_{n=1}^{\infty}c_na^{2n}\,\psi_{2n}^S(a,\theta)\right)\cos\theta\,d\theta \\
&= 2\pi a^3(i\omega\rho)\int_0^1 c_0a\left(\psi_0^S(a,\theta)+\sum_{n=1}^{\infty}c_na^{2n}\,\psi_{2n}^S(a,\theta)\right)P_1(\mu)d\mu
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 a\,\psi_0^S(a,\theta)P_1(\mu)d\mu &= \int_0^1 a\left[\frac{1}{a}+K\sum_{n=0}^{\infty}\frac{(-Ka)^n}{n!}\left[\{\psi(n+1)-\ln(Ka)+\pi i\}P_n(\mu)-\left.\frac{\partial P_\nu(\mu)}{\partial\nu}\right|_{\nu=n}\right]\right]P_1(\mu)d\mu \\
&= I_{0,1}+Ka\sum_{n=0}^{\infty}\frac{(-Ka)^n}{n!}\left[\{\psi(n+1)-\ln(Ka)+\pi i\}I_{1,n}-\left.\frac{\partial I_{1,\nu}}{\partial\nu}\right|_{\nu=n}\right]
\end{aligned}
$$

$$
\int_0^1 a\,\psi_{2n}^S(a,\theta)P_1(\mu)d\mu = \int_0^1\left[\frac{P_{2n}(\cos\theta)}{a^{2n}}+\frac{Ka}{2n}\frac{P_{2n-1}(\cos\theta)}{a^{2n}}\right]P_1(\mu)d\mu = \frac{1}{a^{2n}}\left[I_{2n,1}+\frac{Ka}{2n}I_{2n-1,1}\right]
$$

The acting force is then given as

$$
f = 2\pi a^3(i\omega\rho)c_0\left[\int_0^1 a\,\psi_0^S(a,\theta)P_1(\mu)d\mu+\sum_{n=1}^{\infty}c_n\left\{I_{2n,1}+\frac{Ka}{2n}I_{2n-1,1}\right\}\right]
$$

The nondimensionalized force is given as:

$$
f' = \frac{f}{\omega M} = \frac{f}{\frac{2}{3}\pi\rho a^3\omega} = 3ic_0\left[\int_0^1 a\,\psi_0^S(a,\theta)P_1(\mu)d\mu+\sum_{n=1}^{\infty}c_n\left\{I_{2n,1}+\frac{Ka}{2n}I_{2n-1,1}\right\}\right]
$$

Wave elevation and fluids velocity

Wave elevation

$$\zeta = -\frac{1}{g}\frac{\partial \Phi}{\partial t} = \frac{i\omega}{g}\phi^S$$

$$\Phi = \mathrm{Re}\left[\phi(r,\theta,\alpha)e^{-i\omega t}\right] \qquad and \qquad \phi(r,\theta,\alpha) = \phi^S(r,\theta) = c_0 a^2\left(\psi_0^S(r,\theta) + \sum_{n=1}^{\infty} c_n a^{2n}\,\psi_{2n}^S(r,\theta)\right)$$

Velocity

$$u = \frac{\partial \Phi}{\partial x} = \mathrm{Re}\left[\frac{\partial \phi}{\partial x}e^{-i\omega t}\right], \qquad v = \frac{\partial \Phi}{\partial y} = \mathrm{Re}\left[\frac{\partial \phi}{\partial y}e^{-i\omega t}\right], \qquad w = \frac{\partial \Phi}{\partial z} = \mathrm{Re}\left[\frac{\partial \phi}{\partial z}e^{-i\omega t}\right]$$

$$\begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial \theta}{\partial x} & \frac{\partial \alpha}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial \theta}{\partial y} & \frac{\partial \alpha}{\partial y} \\ \frac{\partial r}{\partial z} & \frac{\partial \theta}{\partial z} & \frac{\partial \alpha}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial \phi}{\partial r} \\ \frac{\partial \phi}{\partial \theta} \\ \frac{\partial \phi}{\partial \alpha} \end{bmatrix}$$

$$\nabla\phi(x,y,z) = J(x,y,z;r,\theta,\alpha)\nabla\phi(r,\theta,\alpha)$$

As the heave motion is symmetric,

$$\frac{\partial \phi}{\partial \alpha} = 0$$

The components to be evaluated for the velocity computation are:

$$\frac{\partial \phi}{\partial r}, \qquad \frac{\partial \phi}{\partial \theta}, \qquad J(x,y,z;r,\theta,\alpha)$$

Derivatives with respect to r (Wave-term)

$$\begin{aligned}\frac{\partial \psi_0^S}{\partial r} &= -\frac{1}{r^2} + K\sum_{n=0}^{\infty}(-K)n\frac{(-Kr)^{n-1}}{n!}\left[\{\psi(n+1)-\ln(Kr)+\pi i\}P_n(\mu) - \left.\frac{\partial P_\nu(\mu)}{\partial \nu}\right|_{\nu=n}\right] + K\sum_{n=0}^{\infty}\frac{(-Kr)^n}{n!}\left[\left\{-\frac{1}{r}\right\}P_n(\mu)\right] \\ &= -\frac{1}{r^2} - K^2\sum_{n=0}^{\infty}n\frac{(-Kr)^{n-1}}{n!}\left[\{\psi(n+1)-\ln(Kr)+\pi i\}P_n(\mu) - \left.\frac{\partial P_\nu(\mu)}{\partial \nu}\right|_{\nu=n}\right] + K^2\sum_{n=0}^{\infty}\frac{(-Kr)^{n-1}}{n!}P_n(\mu) \\ &= -\frac{1}{r^2} - K^2\sum_{n=0}^{\infty}\frac{(-Kr)^{n-1}}{n!}\left[[n\{\psi(n+1)-\ln(Kr)+\pi i\}-1]P_n(\mu) - n\left.\frac{\partial P_\nu(\mu)}{\partial \nu}\right|_{\nu=n}\right]\end{aligned}$$

considering

$$\begin{aligned}a^2\left.\frac{\partial \psi_0^S}{\partial r}\right|_{r=a} &= -1 + Ka\sum_{n=0}^{\infty}\frac{(-Ka)^n}{n!}\left[[n\{\psi(n+1)-\ln(Ka)+\pi i\}-1]P_n(\mu) - n\left.\frac{\partial P_\nu(\mu)}{\partial \nu}\right|_{\nu=n}\right] \\ &= -1 - Ka\sum_{n=1}^{\infty}\frac{(-Ka)^n}{(n-1)!}\left.\frac{\partial P_\nu(\mu)}{\partial \nu}\right|_{\nu=n} + Ka\sum_{n=0}^{\infty}\frac{(-Ka)^n}{n!}[n\{\psi(n+1)-\ln(Ka)+\pi i\}-1]P_n(\mu)\end{aligned}$$

Derivatives with respect to r (Wave-free-term)

$$\begin{aligned}\frac{\partial \psi_{2n}^S}{\partial r} &= -(2n+1)\frac{P_{2n}(\mu)}{r^{2n+2}} - K\frac{P_{2n-1}(\mu)}{r^{2n+1}} \\ &= -\frac{1}{r^{2n+2}}[(2n+1)P_{2n}(\mu) + KrP_{2n-1}(\mu)]\end{aligned}$$

considering

$$a^2\left.\frac{\partial \psi_{2n}^S}{\partial r}\right|_{r=a} = -\frac{1}{a^{2n}}\{(2n+1)P_{2n}(\mu) + KaP_{2n-1}(\mu)\}$$

Derivatives with respect to theta (Wave-term)

$$\psi_0^S = \frac{1}{r} + K\sum_{n=0}^{\infty}\frac{(-Kr)^n}{n!}\left[\{\psi(n+1) - \ln(Kr) + \pi i\}P_n(\mu) - \left.\frac{\partial P_\nu(\mu)}{\partial \nu}\right|_{\nu=n}\right]$$

$$\frac{\partial \psi_0^S}{\partial \theta} = \frac{\partial \mu}{\partial \theta}\frac{\partial \psi_0^S}{\partial \mu}, \quad and \quad \mu = \cos\theta$$

$$\frac{\partial \psi_0^S}{\partial \mu} = K\sum_{n=0}^{\infty}\frac{(-Kr)^n}{n!}\left[\{\psi(n+1) - \ln(Kr) + \pi i\}\frac{\partial P_n(\mu)}{\partial \mu} - \left.\frac{\partial^2 P_\nu(\mu)}{\partial \nu \partial \mu}\right|_{\nu=n}\right], \qquad \frac{\partial \mu}{\partial \theta} = -\sin\theta$$

Derivatives with respect to theta (Wave-free-term)

$$\psi_{2n}^S = \frac{P_{2n}(\mu)}{r^{2n+1}} + \frac{K}{2n}\frac{P_{2n-1}(\mu)}{r^{2n}} = \frac{1}{r^{2n+1}}\left[P_{2n}(\mu) + \frac{Kr}{2n}P_{2n-1}(\mu)\right]$$

$$\frac{\partial \psi_{2n}^S}{\partial \mu} = \frac{1}{r^{2n+1}}\left[\frac{\partial P_{2n}(\mu)}{\partial \mu} + \frac{Kr}{2n}\frac{\partial P_{2n-1}(\mu)}{\partial \mu}\right]$$

Jacobian (Gasirowicz 1974, pp: 167-168, Arfket 1985; p.108)

: See Wolfram alpha (θ definition is opposite , $\theta = \pi - \theta'$;

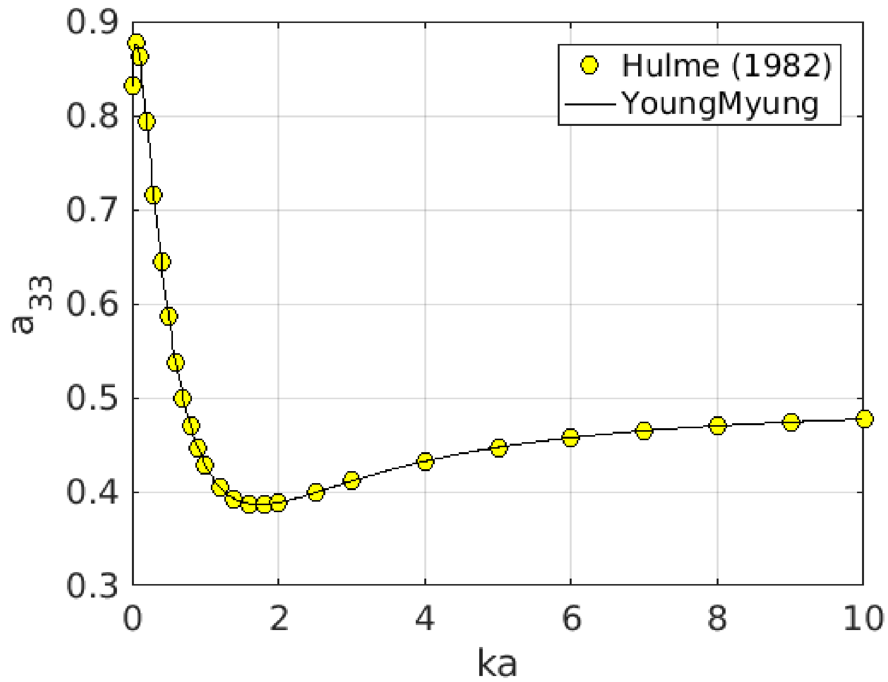
$$x = r\sin\theta\cos\alpha\,, \quad y = r\sin\theta\sin\alpha\,, \quad z = -r\cos\theta\,, \quad R = r\ sin\theta$$

$$r = \sqrt{R^2 + z^2} = \sqrt{x^2 + y^2 + z^2}, \quad r^2 = x^2 + y^2 + z^2, \quad R^2 = x^2 + y^2$$

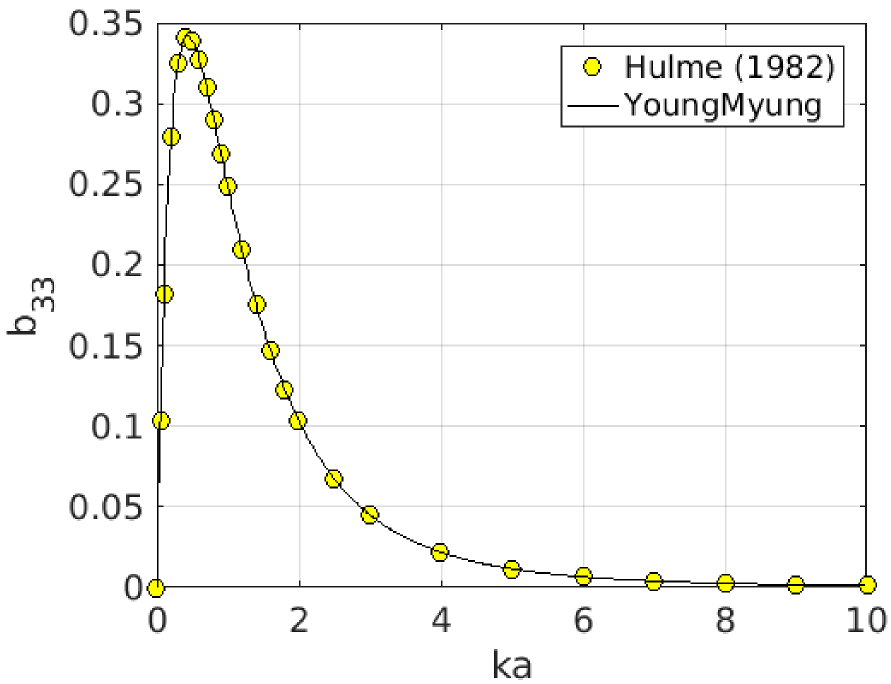
$$\frac{\partial r}{\partial x} = \sin\theta\cos\alpha, \quad \frac{\partial r}{\partial y} = \sin\theta\sin\alpha, \quad \frac{\partial r}{\partial z} = -\cos\theta$$

$$\frac{\partial \theta}{\partial x} = -\frac{\cos\theta\cos\alpha}{r}, \quad \frac{\partial \theta}{\partial y} = -\frac{\cos\theta\sin\alpha}{r}, \quad \frac{\partial \theta}{\partial z} = -\frac{\sin\theta}{r}$$

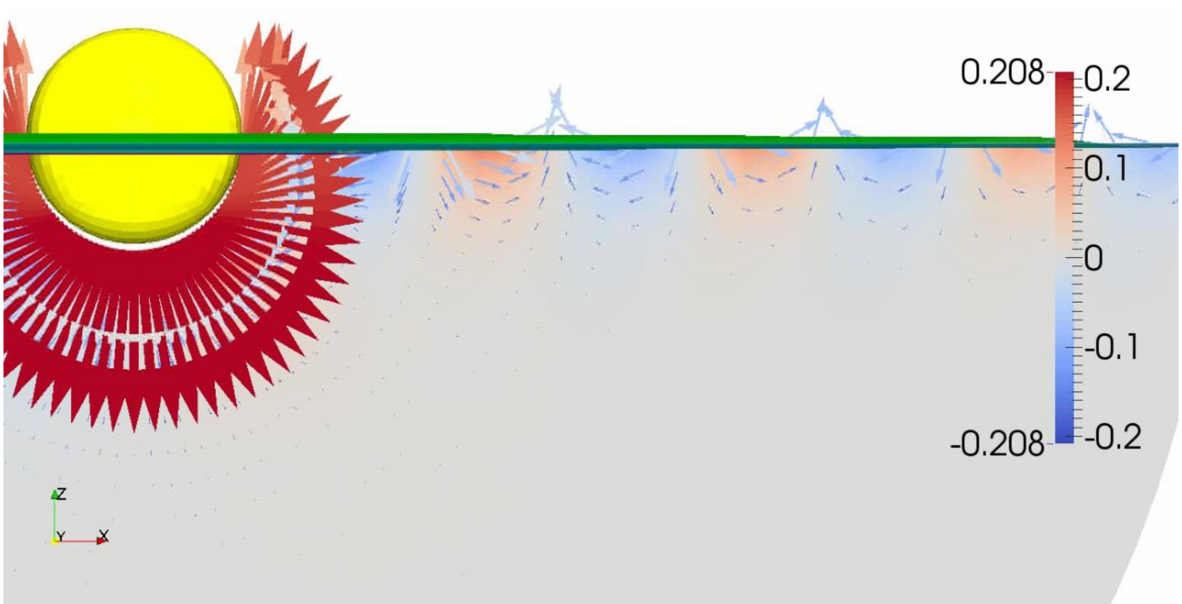
Results



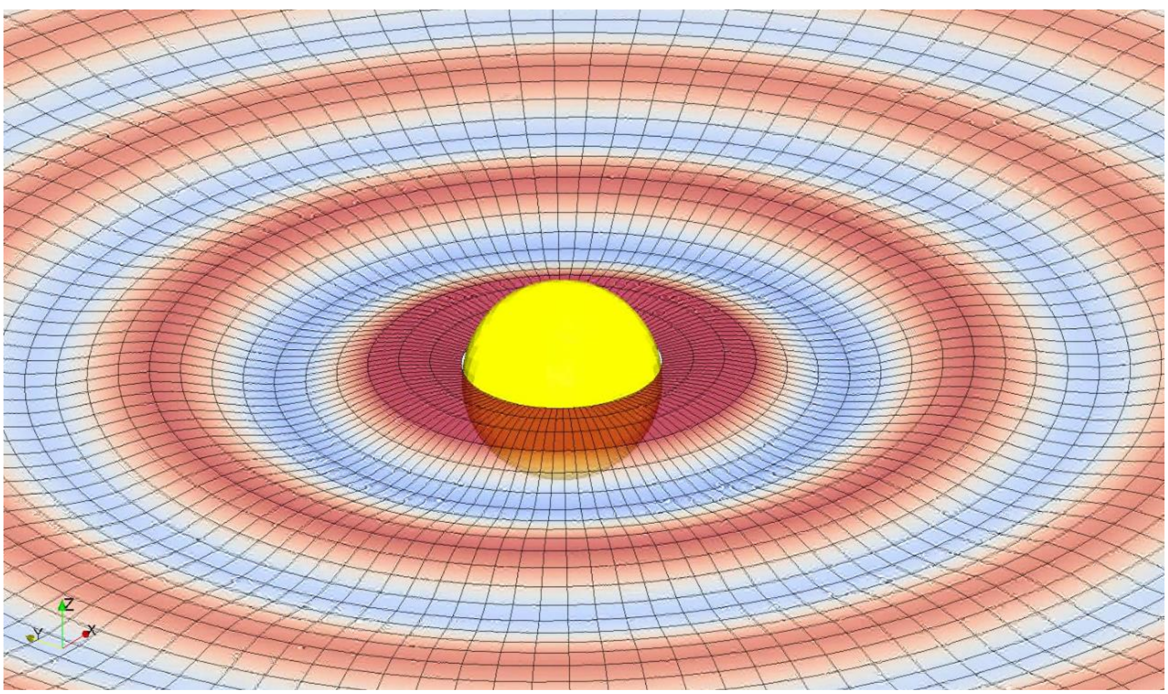
Nondimensionalized Added Mass



Nondimensionalized Damping



Velocity field of heaving hemisphere ($A_{heave} = 0.3m, \omega = 2.5\ rad/s$)



Wave field of heaving hemisphere ($A_{heave} = 0.3m, \omega = 2.5\ rad/s$)

References

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Appendix A. Legendre Function and Multipoles

$$\frac{\partial}{\partial \nu}\left\{\frac{(Kr)^{\nu}}{\nu!}P_{\nu}(\cos\theta)\right\}_{\nu=n} = \frac{\partial}{\partial \nu}\left\{\frac{(Kr)^{\nu}}{\Gamma(\nu+1)}P_{\nu}(\cos\theta)\right\}_{\nu=n}$$

$$= \left[\frac{\ln(Kr)\,(Kr)^{\nu}\Gamma(\nu+1) - (Kr)^{\nu}\Gamma(\nu+1)\psi(\nu+1)}{\Gamma(\nu+1)^2}P_{\nu}(\cos\theta) + \frac{(Kr)^{\nu}}{\Gamma(\nu+1)}\frac{\partial}{\partial \nu}P_{\nu}(\cos\theta)\right]_{\nu=n}$$

$$= \frac{(Kr)^n}{n!}\left[\{\ln(Kr) - \psi(n+1)\}P_n(\cos\theta) + \left\{\frac{\partial}{\partial \nu}P_{\nu}(\cos\theta)\right\}_{\nu=n}\right]$$

$$\frac{\partial}{\partial \nu}\left\{\frac{(Kr)^{\nu-1}}{(\nu-1)!}P_{\nu}(\cos\theta)\right\}_{\nu=n} = \left[\frac{\ln(Kr)\,(Kr)^{\nu-1}\Gamma(\nu) - (Kr)^{\nu-1}\Gamma(\nu)\psi(\nu)}{\Gamma(\nu)^2}P_{\nu}(\cos\theta) + \frac{(Kr)^{\nu-1}}{\Gamma(\nu)}\frac{\partial}{\partial \nu}P_{\nu}(\cos\theta)\right]_{\nu=n}$$

$$= \frac{(Kr)^{n-1}}{(n-1)!}\left[\{\ln(Kr) - \psi(n)\}P_n(\cos\theta) + \left\{\frac{\partial}{\partial \nu}P_{\nu}(\cos\theta)\right\}_{\nu=n}\right]$$

$$\frac{\partial}{\partial \nu}\left\{\frac{(Kr)^{\nu}}{\nu!}\frac{dP_{\nu}(\cos\theta)}{d\theta}\right\}_{\nu=n} = \frac{(Kr)^n}{n!}\left[\{\ln(Kr) - \psi(n+1)\}\frac{dP_n(\cos\theta)}{d\theta} + \frac{\partial}{\partial \theta}\left\{\frac{\partial}{\partial \nu}P_{\nu}(\cos\theta)\right\}_{\nu=n}\right]$$

Evaluate

$$P_n(\cos\theta), \qquad \frac{dP_n(\cos\theta)}{d\theta}, \qquad \left\{\frac{\partial}{\partial \nu}P_{\nu}(\cos\theta)\right\}_{\nu=n}, \qquad \frac{\partial}{\partial \theta}\left\{\frac{\partial}{\partial \nu}P_{\nu}(\cos\theta)\right\}_{\nu=n}$$

⇨ Velocity and Potential, Wave Elevation

Derivatives with respect to order (n)

The derivaties with respect to order of Legendre function is given in Szmytkowski(2011).

$$\left.\frac{\partial P_\nu(z)}{\partial \nu}\right|_{\nu=n} = P_n(z)\ln\frac{z+1}{2} + R_n(z)$$

$$\frac{\partial}{\partial z}\left(\left.\frac{\partial P_\nu(z)}{\partial \nu}\right|_{\nu=n}\right) = \frac{dP_n(z)}{dz}\ln\frac{z+1}{2} + P_n(z)\frac{1}{z+1} + \frac{dR_n(z)}{dz}$$

where

Residual Form (Safe and Efficient)

$$R_n(z) = 2\{\psi(2n+1) - \psi(n+1)\}P_n(z) + 2(-1)^n\sum_{k=0}^{n-1}(-1)^k\frac{2k+1}{(n-k)(k+n+1)}P_k(z)$$

$$\frac{dR_n(z)}{dz} = 2\{\psi(2n+1) - \psi(n+1)\}\frac{dP_n(z)}{dz} + 2(-1)^n\sum_{k=0}^{n-1}(-1)^k\frac{2k+1}{(n-k)(k+n+1)}\frac{dP_k(z)}{dz}$$

Schelknuoff Representation

$$R_n(z) = 2(-1)^n\sum_{k=0}^{n}(-1)^k\frac{(k+n)!}{(k!)^2(n-k)!}\{\psi(k+n+1) - \psi(k+1)\}\left(\frac{z+1}{2}\right)^k$$

$$\frac{dR_n(z)}{dz} = (-1)^n\sum_{k=0}^{n}(-1)^k\frac{(k+n)!}{(k!)^2(n-k)!}k\{\psi(k+n+1) - \psi(k+1)\}\left(\frac{z+1}{2}\right)^{k-1}$$

$$\frac{(k+n)!}{(k!)^2(n-k)!} = \frac{(n+1)(n+2)\cdots(n+k)}{1\cdot 2\cdots k}\frac{(n+1-1)(n+1-2)\cdots(n+1-k)}{1\cdot 2\cdots k}$$

Appendix B. Auxiliary Integrals

J_m **Term**

$$F(\mu, Ka) = -1 - Ka\sum_{n=1}^{\infty}\frac{(-Ka)^n}{(n-1)!}\frac{\partial P_\nu(\mu)}{\partial \nu}\bigg|_{\nu=n} + Ka\sum_{n=0}^{\infty}\frac{(-Ka)^n}{n!}[n\{\psi(n+1) - \ln(Ka) + \pi i\} - 1]P_n(\mu)$$

$$\begin{aligned} J(m, Ka) &= \int_0^1 F(\mu, Ka)P_m(\mu)d\mu \\ &= -I_{m,0} - (Ka)\sum_{n=1}^{\infty}\frac{(-Ka)^n}{(n-1)!}\frac{\partial I(m,\nu;0)}{\partial \nu}\bigg|_{\nu=n} + (Ka)\sum_{n=0}^{\infty}\frac{(-Ka)^n}{n!}[n\{\psi(n+1) - \ln(Ka) + \pi i\} - 1]I_{m,n} \end{aligned}$$

I_{mn} **term**

$$I(2m, 2n+1; 0) = \frac{(-1)^{m+n+1}}{4^{m+n}}\frac{(2m)!\,(2n+1)!}{(2m-2n-1)(2m+2n+2)}\frac{1}{(m!\,n!)^2}$$

$$I(2m, 2n; 0) = \delta_{mn}\frac{1}{4n+1}$$

$$I(2m+1, 2n+1; 0) = \delta_{mn}\frac{1}{4n+3}$$

$$I(2m+1, 2n; 0) = I(2n, 2m+1; 0)$$

The derivatives of I_{mn} with respect to order

if $m = n$

$$\left.\frac{\partial I(m,\nu;0)}{\partial \nu}\right|_{\nu=n} = \int_0^1 P_m(x) \left.\frac{\partial P_\nu(x)}{\partial \nu}\right|_{\nu=n} dx$$

$$\left.\frac{\partial P_\nu(z)}{\partial \nu}\right|_{\nu=n} = P_n(z) \ln\frac{z+1}{2} + R_n(z)$$

where

$$R_n(z) = 2\{\psi(2n+1) - \psi(n+1)\}P_n(z) + 2(-1)^n \sum_{k=0}^{n-1} (-1)^k \frac{2k+1}{(n-k)(k+n+1)} P_k(z)$$

Therefore

$$\left.\frac{\partial I(m,\nu;0)}{\partial \nu}\right|_{\nu=n} = \int_0^1 P_m(x)P_n(z)\ln\frac{z+1}{2}dx + \int_0^1 P_m(x)R_n(z)dx$$

$$\int_0^1 P_m(x)R_n(z)dx = 2\{\psi(2n+1) - \psi(n+1)\}\int_0^1 P_m(x)P_n(z)dx + 2(-1)^n \sum_{k=0}^{n-1} (-1)^k \frac{2k+1}{(n-k)(k+n+1)} \int_0^1 P_m(x)P_k(z)dx$$

$$= 2\{\psi(2n+1) - \psi(n+1)\}I(m,n;0) + 2(-1)^n \sum_{k=0}^{n-1} (-1)^k \frac{2k+1}{(n-k)(k+n+1)} I(m,k;0)$$

Integral with log is integrated with adaptive numerical quadrature

$$\int_0^1 P_{2m}(x)P_n(z)\ln\frac{z+1}{2}dx$$

if m != n

$$\int_0^1 P_\mu(x)P_\nu(x)dx = \frac{A\sin\left(\frac{1}{2}\pi\nu\right)\cos\left(\frac{1}{2}\pi\mu\right) - A^{-1}\sin\left(\frac{1}{2}\pi\mu\right)\cos\left(\frac{1}{2}\pi\nu\right)}{\frac{1}{2}\pi(\nu-\mu)(\mu+\nu+1)}$$

$$A = \frac{\Gamma\left(\frac{1}{2}(\mu+1)\right)\Gamma\left(1+\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}(\nu+1)\right)\Gamma\left(1+\frac{1}{2}\mu\right)}$$

If m is even

$$\begin{aligned}\frac{\partial}{\partial\nu}\left[\int_0^1 P_{2m}(x)P_\nu(x)dx\right] &= \frac{2}{\pi}\frac{\partial}{\partial\nu}\left[\frac{A(2m,\nu)\sin\left(\frac{1}{2}\pi\nu\right)\cos(m\pi) - A(2m,\nu)^{-1}\sin(m\pi)\cos\left(\frac{1}{2}\pi\nu\right)}{(\nu-2m)(2m+\nu+1)}\right] \\ &= \frac{2}{\pi}(-1)^m\frac{\partial}{\partial\nu}\left[\frac{A(2m,\nu)\sin\left(\frac{1}{2}\pi\nu\right)}{(\nu-2n)(2m+\nu+1)}\right]\end{aligned}$$

$$\begin{aligned}&\frac{\partial}{\partial\nu}\left[\frac{A(2m,\nu)\sin\left(\frac{1}{2}\pi\nu\right)}{(\nu-2n)(2m+\nu+1)}\right] \\ &= \frac{\left\{A'\sin\left(\frac{1}{2}\pi\nu\right) + \frac{1}{2}\pi A\cos\left(\frac{1}{2}\pi\nu\right)\right](\nu-2m)(2m+\nu+1) - A\sin\left(\frac{1}{2}\pi\nu\right)\{(2m+\nu+1)+(\nu-2m)\}}{(\nu-2m)^2(2m+\nu+1)^2} \\ &= \frac{\left\{A'\sin\left(\frac{1}{2}\pi\nu\right) + \frac{1}{2}\pi A\cos\left(\frac{1}{2}\pi\nu\right)\right](\nu-2m)(2m+\nu+1) - A\sin\left(\frac{1}{2}\pi\nu\right)(2\nu+1)}{(\nu-2m)^2(2m+\nu+1)^2}\end{aligned}$$

$$\begin{aligned}A'(2m,\nu) &= \frac{\partial}{\partial\nu}\left(\frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}(\nu+1)\right)\Gamma(1+m)}\right) \\ &= \frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(1+m)}\left\{\frac{\frac{1}{2}\Gamma\left(1+\frac{1}{2}\nu\right)\psi\left(1+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}(\nu+1)\right) - \frac{1}{2}\Gamma\left(1+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}(\nu+1)\right)\psi\left(\frac{1}{2}(\nu+1)\right)}{\Gamma\left(\frac{1}{2}(\nu+1)\right)^2}\right\} \\ &= \frac{1}{2}\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(1+n)}\frac{\Gamma\left(1+\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}(m+1)\right)}\left\{\psi\left(1+\frac{1}{2}\nu\right) - \psi\left(\frac{1}{2}(\nu+1)\right)\right\} \\ &= \frac{1}{2}A(2m,\nu)\left\{\psi\left(1+\frac{1}{2}\nu\right) - \psi\left(\frac{1}{2}(\nu+1)\right)\right\}\end{aligned}$$

If *m* is odd

$$\frac{\partial}{\partial\nu}\left[\int_0^1 P_{2m+1}(x)P_\nu(x)dx\right] = \frac{2}{\pi}\frac{\partial}{\partial\nu}\left[\frac{A(2m+1,\nu)\sin\left(\frac{1}{2}\pi\nu\right)\cos\left(m\pi+\frac{\pi}{2}\right) - A(2m+1,\nu)^{-1}\sin\left(m\pi+\frac{\pi}{2}\right)\cos\left(\frac{1}{2}\pi\nu\right)}{(\nu-2m-1)(2m+\nu+2)}\right]$$

$$= -\frac{2}{\pi}(-1)^m\frac{\partial}{\partial\nu}\left[\frac{B(2m+1,\nu)\cos\left(\frac{1}{2}\pi\nu\right)}{(\nu-2m-1)(2m+\nu+2)}\right]$$

where

$$B(2m+1,\nu) = A(2m+1,\nu)^{-1}$$

$$\frac{\partial}{\partial\nu}\left[\frac{B(2m+1,\nu)\cos\left(\frac{1}{2}\pi\nu\right)}{(\nu-2m-1)(2m+\nu+2)}\right]$$

$$= \frac{\left\{B'\cos\left(\frac{\pi}{2}\nu\right) - B\frac{\pi}{2}\sin\left(\frac{\pi}{2}\nu\right)\right\}(\nu-2m-1)(2m+\nu+2) - B\cos\left(\frac{\pi}{2}\nu\right)\{(2m+\nu+2)+(\nu-2m-1)\}}{(\nu-2m-1)^2(2m+\nu+2)^2}$$

$$= \frac{\left\{B'\cos\left(\frac{\pi}{2}\nu\right) - B\frac{\pi}{2}\sin\left(\frac{\pi}{2}\nu\right)\right\}(\nu-2m-1)(2m+\nu+2) - B\cos\left(\frac{\pi}{2}\nu\right)(2\nu+1)}{(\nu-2m-1)^2(2m+\nu+2)^2}$$

$$B(2m+1) = A^{-1} = \frac{\Gamma\left(\frac{1}{2}(\nu+1)\right)\Gamma\left(m+\frac{3}{2}\right)}{\Gamma(m+1)\Gamma\left(1+\frac{1}{2}\nu\right)}$$

$$\frac{\partial B(2m+1,\nu)}{\partial\nu} = \frac{1}{2}\frac{\Gamma\left(m+\frac{3}{2}\right)}{\Gamma(m+1)}\frac{\Gamma\left(\frac{1}{2}(\nu+1)\right)}{\Gamma\left(1+\frac{1}{2}\nu\right)}\left\{\psi\left(\frac{1}{2}(\nu+1)\right) - \psi\left(1+\frac{1}{2}\nu\right)\right\}$$

$$= \frac{1}{2}B(2m+1)\left\{\psi\left(\frac{1}{2}(\nu+1)\right) - \psi\left(1+\frac{1}{2}\nu\right)\right\}$$

$$\psi(n+1) = \frac{\Gamma'(n+1)}{\Gamma(n+1)} = -\gamma + \sum_{k=1}^{n}\frac{1}{k} \quad , \quad \psi(n) = -\gamma + \sum_{k=1}^{n-1}\frac{1}{k}$$