Note on the mutipole expansion of a heaving hemisphere on the free surface

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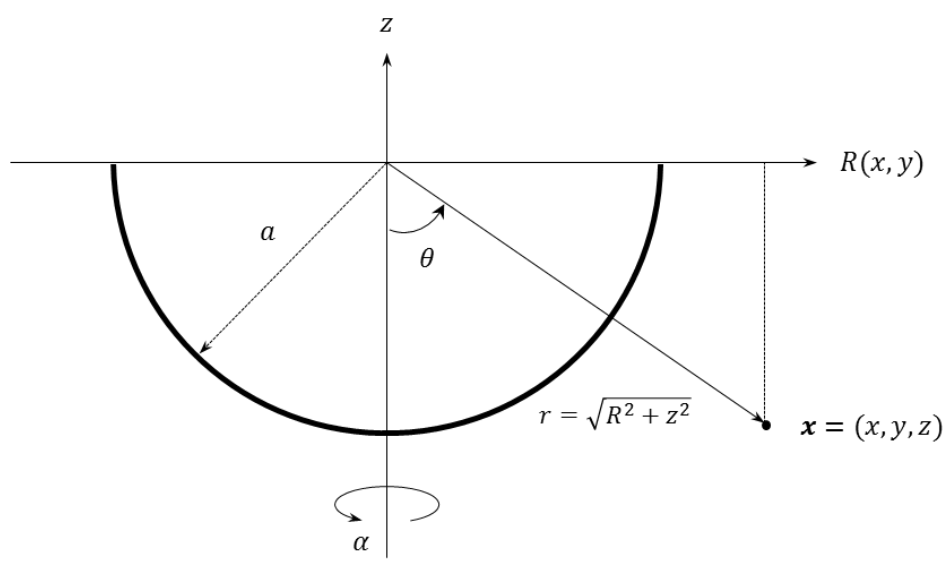
Heaving hemisphere problem

An heaving hemisphere problem was first solved by Havelock(1955). Havelock uses multipole expansion which is similar methodology of Ursell(1949). He expressed the velocity potential with multipoles composed of wave source and wave free terms. Later, Hulme(1982) modified the expression of Havelock in a more rigorous way.

The coordinates for heaving hemisphere is defined in below figure.



$$x = R\cos\alpha\,, \quad y = R\sin\alpha\,, \quad z = -r\cos\theta\,, \quad R = r\,sin\theta$$

$$r = \sqrt{R^2+z^2} = \sqrt{x^2+y^2+z^2}, \qquad r^2 = x^2+y^2+z^2, \qquad R^2 = x^2+y^2$$

Multipole expansion of heaving hemisphere

The multipole expansion for heaving hemisphere is given by Hulme (1982). Because heave is symmetric motion, the velocity potential can be expressed as following. The multipoles, satisfying the Laplace equation, the free surface boundary condition, bottom boundary condition and radiation condition, are used.

$$\phi^S = c_0 a^2 \left(\psi_0^S + \sum_{n=1}^{\infty} c_n a^{2n} \psi_{2n}^S \right)$$

where

$$\psi_0^S = \frac{1}{r} + \pi i K \sum_{n=0}^{\infty} \frac{(-Kr)^n}{n!} P_n(\mu) - K \sum_{n=0}^{\infty} (-1)^n \frac{\partial}{\partial \nu} \left\{ \frac{(Kr)^\nu}{\nu!} P_\nu(\mu) \right\}_{\nu=n}$$

$$\psi_{2n}^S = \frac{P_{2n}(\mu)}{r^{2n+1}} + \frac{K}{2n} \frac{P_{2n-1}(\mu)}{r^{2n}}$$

$$a : radius, \quad \mu = \cos\theta$$

Wave-pole can be expressed with alternative form. Please remind that the expression of di-gamma function in the following form is correct.

$$\psi_0^S = \frac{1}{r} + K \sum_{n=0}^{\infty} \frac{(-Kr)^n}{n!} \left[\{\psi(n+1) - \ln(Kr) + \pi i\} P_n(\mu) - \left. \frac{\partial P_\nu(\mu)}{\partial \nu} \right|_{\nu=n} \right]$$

cf)

$$\frac{\partial}{\partial \nu} \left\{ \frac{(Kr)^\nu}{\nu!} P_\nu(\mu) \right\}_{\nu=n} = \frac{\partial}{\partial \nu} \left\{ \frac{(Kr)^\nu}{\Gamma(\nu+1)} P_\nu(\mu) \right\}_{\nu=n} = \left\{ \frac{(Kr)^\nu \{\ln(Kr) - \psi(\nu+1)\}}{\Gamma(\nu+1)} P_\nu(\mu) + \frac{(Kr)^\nu}{\Gamma(\nu+1)} \frac{\partial P_\nu(\mu)}{\partial \nu} \right\}_{\nu=n}$$

$$= \frac{(Kr)^n}{n!} \left[\{\ln(Kr) - \psi(n+1)\} P_n(\mu) + \left. \frac{\partial P_\nu(\mu)}{\partial \nu} \right|_{\nu=n} \right]$$

Multipole amplitudes

The body boundary condition is applied to calculate the multipole amplitudes.

$$\frac{\partial \phi^S}{\partial r} = \cos\theta = \mu = P_1(\mu)$$

$$\frac{\partial \phi^S}{\partial r} = c_0\left(a^2\frac{\partial \psi_0^S}{\partial r} + \sum_{n=1}^{\infty} c_n a^{2n}\left(a^2\frac{\partial \psi_{2n}^S}{\partial r}\right)\right) = P_1(\mu)$$

$$a^2\frac{\partial \psi_0^S}{\partial r} + \sum_{n=1}^{\infty} c_n a^{2n}\left(a^2\frac{\partial \psi_{2n}^S}{\partial r}\right) = \frac{P_1(\mu)}{c_0}$$

Integrate above expression with respect to μ over (0, 1), Hulme(1982) assume c_0 is not zero.

$$\int_0^1 a^2\frac{\partial \psi_0^S}{\partial r}d\mu + \sum_{n=1}^{\infty} c_n a^{2n}\int_0^1 a^2\frac{\partial \psi_{2n}^S}{\partial r}d\mu = \frac{I_{0,1}}{c_0} = \frac{1}{2c_0}$$

where

$$I_{m,n} = \int_0^1 P_m(\mu)P_n(\mu)d\mu$$

Let

$$a^2\frac{\partial \psi_0^S}{\partial r} = F(\mu, Ka)$$

$$\int_0^1 a^2\frac{\partial \psi_{2n}^S}{\partial r}d\mu = -\frac{1}{a^{2n}}\left\{(2n+1)\int_0^1 P_{2n}(\mu)d\mu + Ka\int_0^1 P_{2n-1}(\mu)d\mu\right\} = -\frac{Ka}{a^{2n}}I_{0,2n-1}$$

$$\int_0^1 P_{2n}(\mu)P_0(\mu)d\mu = 0 \quad if \quad n > 0$$

Substitute

$$\int_0^1 F(\mu, Ka)\,d\mu - Ka\sum_{n=1}^{\infty} c_n I_{0,2n-1} = \frac{1}{2c_0}$$

Recall the body boundary condition

$$F(\mu, Ka) - \sum_{n=1}^{\infty} c_n\{(2n+1)P_{2n}(\mu) + KaP_{2n-1}(\mu)\} = \frac{P_1(\mu)}{c_0}$$
$$= 2P_1(\mu)\int_0^1 F(\mu, Ka)\, d - 2P_1(\mu)Ka\sum_{n=1}^{\infty} c_n I_{0,2n-1}$$

Following equations are obtained after some manipulation :

$$\sum_{n=1}^{\infty} c_n\left[(2n+1)P_{2n}(\mu) + Ka\left\{P_{2n-1}(\mu) - 2P_1(\mu)I_{0,2n-1}\right\}\right] = F(\mu, Ka) - 2P_1(\mu)\int_0^1 F(\mu, Ka)\, d\mu$$

Multiply $P_{2m}(\mu)$ and integrate with respect to μ over (0, 1).

$$\sum_{n=1}^{\infty} c_n\left[(2n+1)\int_0^1 P_{2n}(\mu)P_{2m}(\mu)d\mu + Ka\left\{\int_0^1 P_{2n-1}(\mu)P_{2m}(\mu)d\mu - 2\int_0^1 P_1(\mu)P_{2m}(\mu)d\mu\, I_{0,2n-1}\right\}\right]$$
$$= \int_0^1 F(\mu, Ka)P_{2m}(\mu)d\mu - 2\int_0^1 P_1(\mu)P_{2m}(\mu)d\mu \int_0^1 F(\mu, Ka)\, d\mu$$

Using the orthogonality of Legendre functions

$$\int_0^1 P_{2n}(\mu)P_{2m}(\mu)d\mu = \frac{\delta_{mn}}{4m+1}$$

The system equation of heaving hemisphere expressed with the multipoles are given as :

$$\frac{2m+1}{4m+1}c_m + Ka\sum_{n=1}^{\infty} c_n\left\{I_{2m,2n-1} - 2I_{2m,1}I_{0,2n-1}\right\} = J_{2m} - 2J_0 I_{2m,1}$$

where

$$J_{2m} = \int_0^1 F(\mu, Ka)P_{2m}(\mu)d\mu$$

An alternative form is given as :

$$[\boldsymbol{\lambda} + \boldsymbol{M}][\boldsymbol{c}] = [\boldsymbol{\beta}]$$

$$\lambda_{mn} = \frac{2m+1}{4m+1}\delta_{mn}, \quad M_{mn} = Ka\{I_{2m,2n-1} - 2I_{2m,1}I_{0,2n-1}\}, \quad \beta_m = J_{2m} - 2J_0 I_{2m,1}$$

The value of c_0 is given as :

$$c_0 = \left[2\int_0^1 F(\mu, Ka)\, d\mu - Ka\sum_{n=1}^{\infty} c_n I_{0,2n-1}\right]^{-1}$$

Force acting on the heaving hemisphere

$$
\begin{aligned}
f &= \iint_S p\boldsymbol{n}dS = -\rho \iint_S \frac{\partial \phi}{\partial t}\boldsymbol{n}dS = i\omega\rho \iint_S \phi(-cos\theta)dS \\
&= i\omega\rho \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \phi^S(-\cos\theta)a^2 d\theta d\alpha \\
&= -2\pi a^2(i\omega\rho)\int_0^{\frac{\pi}{2}} c_0 a^2\left(\psi_0^S(a,\theta) + \sum_{n=1}^{\infty} c_n a^{2n}\,\psi_{2n}^S(a,\theta)\right)\cos\theta\, d\theta \\
&= 2\pi a^3(i\omega\rho)\int_0^1 c_0 a\left(\psi_0^S(a,\theta) + \sum_{n=1}^{\infty} c_n a^{2n}\,\psi_{2n}^S(a,\theta)\right)P_1(\mu)d\mu
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 a\,\psi_0^S(a,\theta)P_1(\mu)d\mu &= \int_0^1 a\left[\frac{1}{a} + K\sum_{n=0}^{\infty}\frac{(-Ka)^n}{n!}\left[\{\psi(n+1) - \ln(Ka) + \pi i\}P_n(\mu) - \left.\frac{\partial P_\nu(\mu)}{\partial \nu}\right|_{\nu=n}\right]\right]P_1(\mu)d\mu \\
&= I_{0,1} + Ka\sum_{n=0}^{\infty}\frac{(-Ka)^n}{n!}\left[\{\psi(n+1) - \ln(Ka) + \pi i\}I_{1,n} - \left.\frac{\partial I_{1,\nu}}{\partial \nu}\right|_{\nu=n}\right]
\end{aligned}
$$

$$
\int_0^1 a\,\psi_{2n}^S(a,\theta)P_1(\mu)d\mu = \int_0^1 \left[\frac{P_{2n}(\cos\theta)}{a^{2n}} + \frac{Ka}{2n}\frac{P_{2n-1}(\cos\theta)}{a^{2n}}\right]P_1(\mu)d\mu = \frac{1}{a^{2n}}\left[I_{2n,1} + \frac{Ka}{2n}I_{2n-1,1}\right]
$$

The acting forces are given as

$$
f = 2\pi a^3(i\omega\rho)c_0\left[\int_0^1 a\,\psi_0^S(a,\theta)P_1(\mu)d\mu + \sum_{n=1}^{\infty} c_n\left\{I_{2n,1} + \frac{Ka}{2n}I_{2n-1,1}\right\}\right]
$$

Nondimensionalized force are given as

$$
f' = \frac{f}{\omega M} = \frac{f}{\frac{2}{3}\pi\rho a^3\omega} = 3ic_0\left[\int_0^1 a\,\psi_0^S(a,\theta)P_1(\mu)d\mu + \sum_{n=1}^{\infty} c_n\left\{I_{2n,1} + \frac{Ka}{2n}I_{2n-1,1}\right\}\right]
$$

Wave elevation and fluids velocity

Wave elevation

$$\zeta = -\frac{1}{g}\frac{\partial \Phi}{\partial t} = \frac{i\omega}{g}\phi^S$$

$$\Phi = \mathrm{Re}\left[\phi(r,\theta,\alpha)e^{-i\omega t}\right] \qquad and \qquad \phi(r,\theta,\alpha) = \phi^S(r,\theta) = c_0 a^2\left(\psi_0^S(r,\theta) + \sum_{n=1}^{\infty} c_n a^{2n}\,\psi_{2n}^S(r,\theta)\right)$$

Velocity

$$u = \frac{\partial \Phi}{\partial x} = \mathrm{Re}\left[\frac{\partial \phi}{\partial x}e^{-i\omega t}\right], \qquad v = \frac{\partial \Phi}{\partial y} = \mathrm{Re}\left[\frac{\partial \phi}{\partial y}e^{-i\omega t}\right], \qquad w = \frac{\partial \Phi}{\partial z} = \mathrm{Re}\left[\frac{\partial \phi}{\partial z}e^{-i\omega t}\right]$$

$$\begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial \theta}{\partial x} & \frac{\partial \alpha}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial \theta}{\partial y} & \frac{\partial \alpha}{\partial y} \\ \frac{\partial r}{\partial z} & \frac{\partial \theta}{\partial z} & \frac{\partial \alpha}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial \phi}{\partial r} \\ \frac{\partial \phi}{\partial \theta} \\ \frac{\partial \phi}{\partial \alpha} \end{bmatrix}$$

$$\nabla\phi(x,y,z) = J(x,y,z;r,\theta,\alpha)\nabla\phi(r,\theta,\alpha)$$

Because heave motion is symmetry,

$$\frac{\partial \phi}{\partial \alpha} = 0$$

The components to be evaluate for the velocity computation

$$\frac{\partial \phi}{\partial r}, \qquad \frac{\partial \phi}{\partial \theta}, \qquad J(x,y,z;r,\theta,\alpha)$$

Derivatives with respect to r (Wave-term)

$$\begin{aligned}
\frac{\partial \psi_0^S}{\partial r} &= -\frac{1}{r^2} + K\sum_{n=0}^{\infty}(-K)n\frac{(-Kr)^{n-1}}{n!}\left[\{\psi(n+1)-\ln(Kr)+\pi i\}P_n(\mu)-\left.\frac{\partial P_\nu(\mu)}{\partial \nu}\right|_{\nu=n}\right] + K\sum_{n=0}^{\infty}\frac{(-Kr)^n}{n!}\left[\left\{-\frac{1}{r}\right\}P_n(\mu)\right] \\
&= -\frac{1}{r^2} - K^2\sum_{n=0}^{\infty}n\frac{(-Kr)^{n-1}}{n!}\left[\{\psi(n+1)-\ln(Kr)+\pi i\}P_n(\mu)-\left.\frac{\partial P_\nu(\mu)}{\partial \nu}\right|_{\nu=n}\right] + K^2\sum_{n=0}^{\infty}\frac{(-Kr)^{n-1}}{n!}P_n(\mu) \\
&= -\frac{1}{r^2} - K^2\sum_{n=0}^{\infty}\frac{(-Kr)^{n-1}}{n!}\left[[n\{\psi(n+1)-\ln(Kr)+\pi i\}-1]P_n(\mu)-n\left.\frac{\partial P_\nu(\mu)}{\partial \nu}\right|_{\nu=n}\right]
\end{aligned}$$

Cf)

$$\begin{aligned}
a^2\left.\frac{\partial \psi_0^S}{\partial r}\right|_{r=a} &= -1 + Ka\sum_{n=0}^{\infty}\frac{(-Ka)^n}{n!}\left[[n\{\psi(n+1)-\ln(Ka)+\pi i\}-1]P_n(\mu)-n\left.\frac{\partial P_\nu(\mu)}{\partial \nu}\right|_{\nu=n}\right] \\
&= -1 - Ka\sum_{n=1}^{\infty}\frac{(-Ka)^n}{(n-1)!}\left.\frac{\partial P_\nu(\mu)}{\partial \nu}\right|_{\nu=n} + Ka\sum_{n=0}^{\infty}\frac{(-Ka)^n}{n!}[n\{\psi(n+1)-\ln(Ka)+\pi i\}-1]P_n(\mu)
\end{aligned}$$

Derivatives with respect to r (Wave-free-term)

$$\begin{aligned}
\frac{\partial \psi_{2n}^S}{\partial r} &= -(2n+1)\frac{P_{2n}(\mu)}{r^{2n+2}} - K\frac{P_{2n-1}(\mu)}{r^{2n+1}} \\
&= -\frac{1}{r^{2n+2}}[(2n+1)P_{2n}(\mu)+KrP_{2n-1}(\mu)]
\end{aligned}$$

Cf)

$$a^2\left.\frac{\partial \psi_{2n}^S}{\partial r}\right|_{r=a} = -\frac{1}{a^{2n}}\{(2n+1)P_{2n}(\mu)+KaP_{2n-1}(\mu)\}$$

Derivatives with respect to theta (Wave-term)

$$\psi_0^S = \frac{1}{r} + K\sum_{n=0}^{\infty}\frac{(-Kr)^n}{n!}\left[\{\psi(n+1) - \ln(Kr) + \pi i\}P_n(\mu) - \frac{\partial P_\nu(\mu)}{\partial \nu}\bigg|_{\nu=n}\right]$$

$$\frac{\partial \psi_0^S}{\partial \theta} = \frac{\partial \mu}{\partial \theta}\frac{\partial \psi_0^S}{\partial \mu}, \quad and \quad \mu = \cos\theta$$

$$\frac{\partial \psi_0^S}{\partial \mu} = K\sum_{n=0}^{\infty}\frac{(-Kr)^n}{n!}\left[\{\psi(n+1) - \ln(Kr) + \pi i\}\frac{\partial P_n(\mu)}{\partial \mu} - \frac{\partial^2 P_\nu(\mu)}{\partial \nu \partial \mu}\bigg|_{\nu=n}\right], \qquad \frac{\partial \mu}{\partial \theta} = -\sin\theta$$

Derivatives with respect to theta (Wave-free-term)

$$\psi_{2n}^S = \frac{P_{2n}(\mu)}{r^{2n+1}} + \frac{K}{2n}\frac{P_{2n-1}(\mu)}{r^{2n}} = \frac{1}{r^{2n+1}}\left[P_{2n}(\mu) + \frac{Kr}{2n}P_{2n-1}(\mu)\right]$$

$$\frac{\partial \psi_{2n}^S}{\partial \mu} = \frac{1}{r^{2n+1}}\left[\frac{\partial P_{2n}(\mu)}{\partial \mu} + \frac{Kr}{2n}\frac{\partial P_{2n-1}(\mu)}{\partial \mu}\right]$$

Jacobian (Gasirowicz 1974, pp: 167-168, Arfket 1985; p.108)

: See Wolfram alpha (θ definition is opposite , $\theta = \pi - \theta'$;

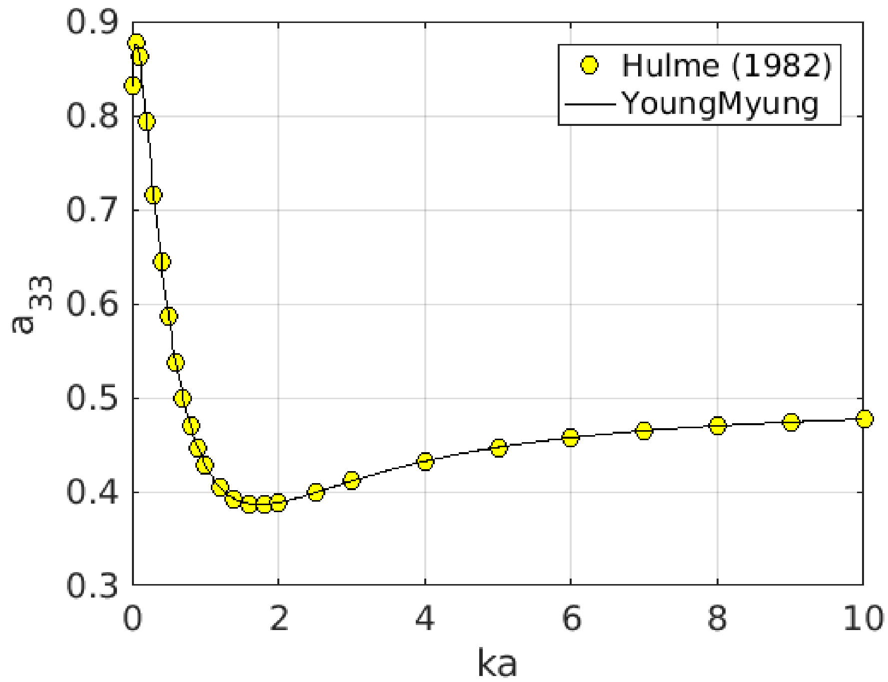
$$x = r\sin\theta\cos\alpha, \quad y = r\sin\theta\sin\alpha, \quad z = -r\cos\theta, \quad R = r\ sin\theta$$

$$r = \sqrt{R^2 + z^2} = \sqrt{x^2 + y^2 + z^2}, \qquad r^2 = x^2 + y^2 + z^2, \qquad R^2 = x^2 + y^2$$

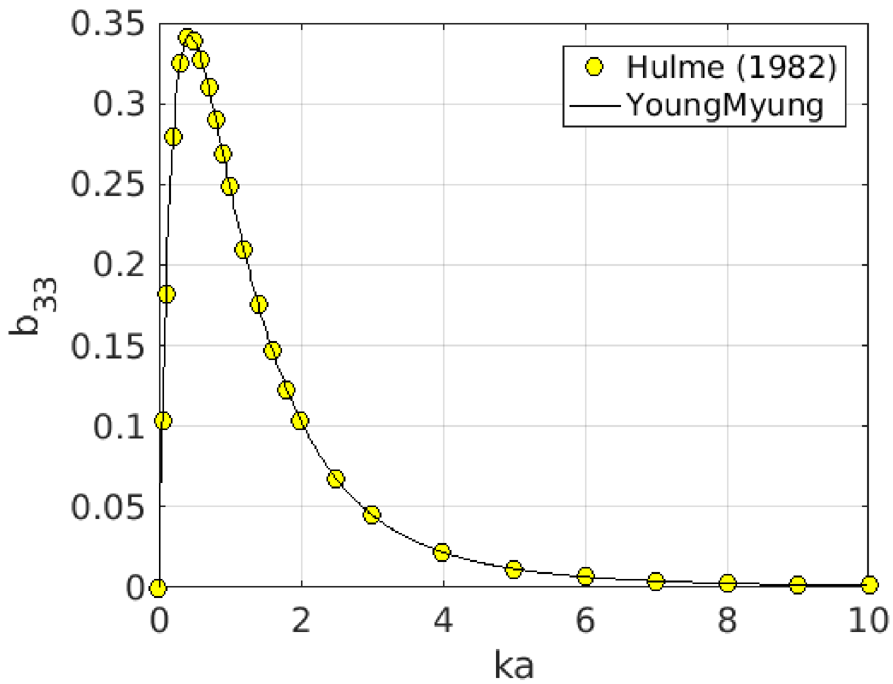
$$\frac{\partial r}{\partial x} = \sin\theta\cos\alpha, \qquad \frac{\partial r}{\partial y} = \sin\theta\sin\alpha, \qquad \frac{\partial r}{\partial z} = -\cos\theta$$

$$\frac{\partial \theta}{\partial x} = -\frac{\cos\theta\cos\alpha}{r}, \qquad \frac{\partial \theta}{\partial y} = -\frac{\cos\theta\sin\alpha}{r}, \qquad \frac{\partial \theta}{\partial z} = -\frac{\sin\theta}{r}$$

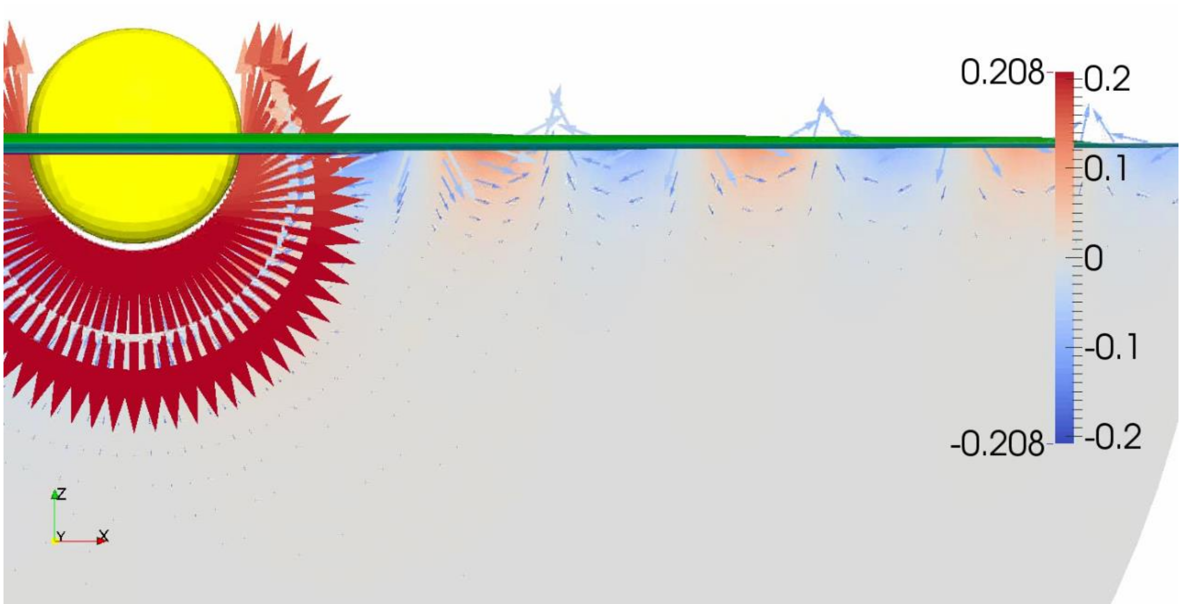
Results



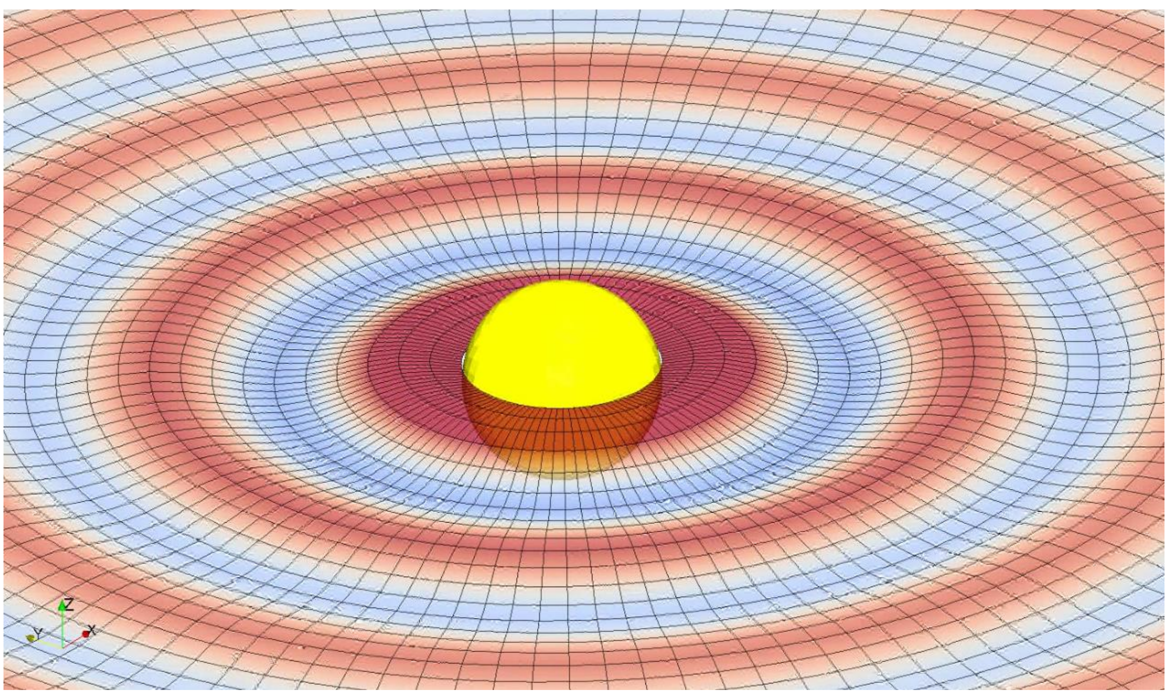
Nondimensionalized Added Mass



Nondimensionalized Damping



Velocity field of heaving hemisphere ($A_{heave} = 0.3m, \omega = 2.5\ rad/s$)



Wave field of heaving hemisphere ($A_{heave} = 0.3m, \omega = 2.5\ rad/s$)

References

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Havelock T. 1955. Waves due to a floating hemi-sphere making periodic heaving oscillations. *Proc. R. Soc. Lond*. A 231. pp.1-7.

Appendix A. Legendre Function and Multipoles

$$\frac{\partial}{\partial\nu}\left\{\frac{(Kr)^\nu}{\nu!}P_\nu(\cos\theta)\right\}_{\nu=n} = \frac{\partial}{\partial\nu}\left\{\frac{(Kr)^\nu}{\Gamma(\nu+1)}P_\nu(\cos\theta)\right\}_{\nu=n}$$

$$= \left[\frac{\ln(Kr)\,(Kr)^\nu\Gamma(\nu+1) - (Kr)^\nu\Gamma(\nu+1)\psi(\nu+1)}{\Gamma(\nu+1)^2}P_\nu(\cos\theta) + \frac{(Kr)^\nu}{\Gamma(\nu+1)}\frac{\partial}{\partial\nu}P_\nu(\cos\theta)\right]_{\nu=n}$$

$$= \frac{(Kr)^n}{n!}\left[\{\ln(Kr) - \psi(n+1)\}P_n(\cos\theta) + \left\{\frac{\partial}{\partial\nu}P_\nu(\cos\theta)\right\}_{\nu=n}\right]$$

$$\frac{\partial}{\partial\nu}\left\{\frac{(Kr)^{\nu-1}}{(\nu-1)!}P_\nu(\cos\theta)\right\}_{\nu=n} = \left[\frac{\ln(Kr)\,(Kr)^{\nu-1}\Gamma(\nu) - (Kr)^{\nu-1}\Gamma(\nu)\psi(\nu)}{\Gamma(\nu)^2}P_\nu(\cos\theta) + \frac{(Kr)^{\nu-1}}{\Gamma(\nu)}\frac{\partial}{\partial\nu}P_\nu(\cos\theta)\right]_{\nu=n}$$

$$= \frac{(Kr)^{n-1}}{(n-1)!}\left[\{\ln(Kr) - \psi(n)\}P_n(\cos\theta) + \left\{\frac{\partial}{\partial\nu}P_\nu(\cos\theta)\right\}_{\nu=n}\right]$$

$$\frac{\partial}{\partial\nu}\left\{\frac{(Kr)^\nu}{\nu!}\frac{dP_\nu(\cos\theta)}{d\theta}\right\}_{\nu=n} = \frac{(Kr)^n}{n!}\left[\{\ln(Kr) - \psi(n+1)\}\frac{dP_n(\cos\theta)}{d\theta} + \frac{\partial}{\partial\theta}\left\{\frac{\partial}{\partial\nu}P_\nu(\cos\theta)\right\}_{\nu=n}\right]$$

Evaluate

$$P_n(\cos\theta), \qquad \frac{dP_n(\cos\theta)}{d\theta}, \qquad \left\{\frac{\partial}{\partial\nu}P_\nu(\cos\theta)\right\}_{\nu=n}, \qquad \frac{\partial}{\partial\theta}\left\{\frac{\partial}{\partial\nu}P_\nu(\cos\theta)\right\}_{\nu=n}$$

⇨ Velocity and Potential, Wave Elevation

Derivatives with respect to order (n)

The derivaties with respect to order of Legendre function is given in Szmytkowski(2011).

$$\left.\frac{\partial P_\nu(z)}{\partial \nu}\right|_{\nu=n} = P_n(z)\ln\frac{z+1}{2} + R_n(z)$$

$$\frac{\partial}{\partial z}\left(\left.\frac{\partial P_\nu(z)}{\partial \nu}\right|_{\nu=n}\right) = \frac{dP_n(z)}{dz}\ln\frac{z+1}{2} + P_n(z)\frac{1}{z+1} + \frac{dR_n(z)}{dz}$$

where

Residual Form (Safe and Efficient)

$$R_n(z) = 2\{\psi(2n+1) - \psi(n+1)\}P_n(z) + 2(-1)^n \sum_{k=0}^{n-1} (-1)^k \frac{2k+1}{(n-k)(k+n+1)} P_k(z)$$

$$\frac{dR_n(z)}{dz} = 2\{\psi(2n+1) - \psi(n+1)\}\frac{dP_n(z)}{dz} + 2(-1)^n \sum_{k=0}^{n-1} (-1)^k \frac{2k+1}{(n-k)(k+n+1)} \frac{dP_k(z)}{dz}$$

Schelknuoff Representation

$$R_n(z) = 2(-1)^n \sum_{k=0}^{n} (-1)^k \frac{(k+n)!}{(k!)^2(n-k)!}\{\psi(k+n+1) - \psi(k+1)\}\left(\frac{z+1}{2}\right)^k$$

$$\frac{dR_n(z)}{dz} = (-1)^n \sum_{k=0}^{n} (-1)^k \frac{(k+n)!}{(k!)^2(n-k)!} k\{\psi(k+n+1) - \psi(k+1)\}\left(\frac{z+1}{2}\right)^{k-1}$$

$$\frac{(k+n)!}{(k!)^2(n-k)!} = \frac{(n+1)(n+2)\cdots(n+k)}{1\cdot 2\cdots k}\frac{(n+1-1)(n+1-2)\cdots(n+1-k)}{1\cdot 2\cdots k}$$

Appendix B. Auxiliary Integrals

***J**_m* Term

$$F(\mu, Ka) = -1 - Ka\sum_{n=1}^{\infty}\frac{(-Ka)^n}{(n-1)!}\frac{\partial P_\nu(\mu)}{\partial \nu}\bigg|_{\nu=n} + Ka\sum_{n=0}^{\infty}\frac{(-Ka)^n}{n!}[n\{\psi(n+1) - \ln(Ka) + \pi i\} - 1]P_n(\mu)$$

$$\begin{aligned} J(m, Ka) &= \int_0^1 F(\mu, Ka)P_m(\mu)d\mu \\ &= -I_{m,0} - (Ka)\sum_{n=1}^{\infty}\frac{(-Ka)^n}{(n-1)!}\frac{\partial I(m,\nu;0)}{\partial \nu}\bigg|_{\nu=n} + (Ka)\sum_{n=0}^{\infty}\frac{(-Ka)^n}{n!}[n\{\psi(n+1) - \ln(Ka) + \pi i\} - 1]I_{m,n} \end{aligned}$$

***I**_mn* term

$$I(2m, 2n+1; 0) = \frac{(-1)^{m+n+1}}{4^{m+n}}\frac{(2m)!\,(2n+1)!}{(2m-2n-1)(2m+2n+2)}\frac{1}{(m!\,n!)^2}$$

$$I(2m, 2n; 0) = \delta_{mn}\frac{1}{4n+1}$$

$$I(2m+1, 2n+1; 0) = \delta_{mn}\frac{1}{4n+3}$$

$$I(2m+1, 2n; 0) = I(2n, 2m+1; 0)$$

The derivatives of I_{mn} with respect to order

if $m = n$

$$\left.\frac{\partial I(m,\nu;0)}{\partial \nu}\right|_{\nu=n} = \int_0^1 P_m(x)\left.\frac{\partial P_\nu(x)}{\partial \nu}\right|_{\nu=n} dx$$

$$\left.\frac{\partial P_\nu(z)}{\partial \nu}\right|_{\nu=n} = P_n(z)\ln\frac{z+1}{2} + R_n(z)$$

where

$$R_n(z) = 2\{\psi(2n+1) - \psi(n+1)\}P_n(z) + 2(-1)^n \sum_{k=0}^{n-1}(-1)^k \frac{2k+1}{(n-k)(k+n+1)} P_k(z)$$

Therefore

$$\left.\frac{\partial I(m,\nu;0)}{\partial \nu}\right|_{\nu=n} = \int_0^1 P_m(x)P_n(z)\ln\frac{z+1}{2}dx + \int_0^1 P_m(x)R_n(z)dx$$

$$\int_0^1 P_m(x)R_n(z)dx = 2\{\psi(2n+1) - \psi(n+1)\}\int_0^1 P_m(x)P_n(z)dx + 2(-1)^n \sum_{k=0}^{n-1}(-1)^k \frac{2k+1}{(n-k)(k+n+1)}\int_0^1 P_m(x)P_k(z)dx$$

$$= 2\{\psi(2n+1) - \psi(n+1)\}I(m,n;0) + 2(-1)^n \sum_{k=0}^{n-1}(-1)^k \frac{2k+1}{(n-k)(k+n+1)} I(m,k;0)$$

Integral with log is integrated with adaptive numerical quadrature

$$\int_0^1 P_{2m}(x)P_n(z)\ln\frac{z+1}{2}dx$$

if m != n

$$\int_0^1 P_\mu(x) P_\nu(x) dx = \frac{A \sin\left(\frac{1}{2}\pi\nu\right)\cos\left(\frac{1}{2}\pi\mu\right) - A^{-1}\sin\left(\frac{1}{2}\pi\mu\right)\cos\left(\frac{1}{2}\pi\nu\right)}{\frac{1}{2}\pi(\nu-\mu)(\mu+\nu+1)}$$

$$A = \frac{\Gamma\left(\frac{1}{2}(\mu+1)\right)\Gamma\left(1+\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}(\nu+1)\right)\Gamma\left(1+\frac{1}{2}\mu\right)}$$

If m is even

$$\frac{\partial}{\partial\nu}\left[\int_0^1 P_{2m}(x)P_\nu(x)dx\right] = \frac{2}{\pi}\frac{\partial}{\partial\nu}\left[\frac{A(2m,\nu)\sin\left(\frac{1}{2}\pi\nu\right)\cos(m\pi) - A(2m,\nu)^{-1}\sin(m\pi)\cos\left(\frac{1}{2}\pi\nu\right)}{(\nu-2m)(2m+\nu+1)}\right]$$
$$= \frac{2}{\pi}(-1)^m\frac{\partial}{\partial\nu}\left[\frac{A(2m,\nu)\sin\left(\frac{1}{2}\pi\nu\right)}{(\nu-2n)(2m+\nu+1)}\right]$$

$$\frac{\partial}{\partial\nu}\left[\frac{A(2m,\nu)\sin\left(\frac{1}{2}\pi\nu\right)}{(\nu-2n)(2m+\nu+1)}\right]$$
$$= \frac{\left\{A'\sin\left(\frac{1}{2}\pi\nu\right)+\frac{1}{2}\pi A\cos\left(\frac{1}{2}\pi\nu\right)\right](\nu-2m)(2m+\nu+1) - A\sin\left(\frac{1}{2}\pi\nu\right)\{(2m+\nu+1)+(\nu-2m)\}}{(\nu-2m)^2(2m+\nu+1)^2}$$
$$= \frac{\left\{A'\sin\left(\frac{1}{2}\pi\nu\right)+\frac{1}{2}\pi A\cos\left(\frac{1}{2}\pi\nu\right)\right](\nu-2m)(2m+\nu+1) - A\sin\left(\frac{1}{2}\pi\nu\right)(2\nu+1)}{(\nu-2m)^2(2m+\nu+1)^2}$$

$$A'(2m,\nu) = \frac{\partial}{\partial\nu}\left(\frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}(\nu+1)\right)\Gamma(1+m)}\right)$$
$$= \frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(1+m)}\left\{\frac{\frac{1}{2}\Gamma\left(1+\frac{1}{2}\nu\right)\psi\left(1+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}(\nu+1)\right) - \frac{1}{2}\Gamma\left(1+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}(\nu+1)\right)\psi\left(\frac{1}{2}(\nu+1)\right)}{\Gamma\left(\frac{1}{2}(\nu+1)\right)^2}\right\}$$
$$= \frac{1}{2}\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(1+n)}\frac{\Gamma\left(1+\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}(m+1)\right)}\left\{\psi\left(1+\frac{1}{2}\nu\right) - \psi\left(\frac{1}{2}(\nu+1)\right)\right\}$$
$$= \frac{1}{2}A(2m,\nu)\left\{\psi\left(1+\frac{1}{2}\nu\right) - \psi\left(\frac{1}{2}(\nu+1)\right)\right\}$$

If *m* is odd

$$\frac{\partial}{\partial \nu}\left[\int_0^1 P_{2m+1}(x)P_\nu(x)dx\right] = \frac{2}{\pi}\frac{\partial}{\partial \nu}\left[\frac{A(2m+1,\nu)\sin\left(\frac{1}{2}\pi\nu\right)\cos\left(m\pi+\frac{\pi}{2}\right) - A(2m+1,\nu)^{-1}\sin\left(m\pi+\frac{\pi}{2}\right)\cos\left(\frac{1}{2}\pi\nu\right)}{(\nu-2m-1)(2m+\nu+2)}\right]$$

$$= -\frac{2}{\pi}(-1)^m\frac{\partial}{\partial \nu}\left[\frac{B(2m+1,\nu)\cos\left(\frac{1}{2}\pi\nu\right)}{(\nu-2m-1)(2m+\nu+2)}\right]$$

where

$$B(2m+1,\nu) = A(2m+1,\nu)^{-1}$$

$$\frac{\partial}{\partial \nu}\left[\frac{B(2m+1,\nu)\cos\left(\frac{1}{2}\pi\nu\right)}{(\nu-2m-1)(2m+\nu+2)}\right]$$

$$= \frac{\left\{B'\cos\left(\frac{\pi}{2}\nu\right) - B\frac{\pi}{2}\sin\left(\frac{\pi}{2}\nu\right)\right\}(\nu-2m-1)(2m+\nu+2) - B\cos\left(\frac{\pi}{2}\nu\right)\{(2m+\nu+2)+(\nu-2m-1)\}}{(\nu-2m-1)^2(2m+\nu+2)^2}$$

$$= \frac{\left\{B'\cos\left(\frac{\pi}{2}\nu\right) - B\frac{\pi}{2}\sin\left(\frac{\pi}{2}\nu\right)\right\}(\nu-2m-1)(2m+\nu+2) - B\cos\left(\frac{\pi}{2}\nu\right)(2\nu+1)}{(\nu-2m-1)^2(2m+\nu+2)^2}$$

$$B(2m+1) = A^{-1} = \frac{\Gamma\left(\frac{1}{2}(\nu+1)\right)\Gamma\left(m+\frac{3}{2}\right)}{\Gamma(m+1)\Gamma\left(1+\frac{1}{2}\nu\right)}$$

$$\frac{\partial B(2m+1,\nu)}{\partial \nu} = \frac{1}{2}\frac{\Gamma\left(m+\frac{3}{2}\right)}{\Gamma(m+1)}\frac{\Gamma\left(\frac{1}{2}(\nu+1)\right)}{\Gamma\left(1+\frac{1}{2}\nu\right)}\left\{\psi\left(\frac{1}{2}(\nu+1)\right) - \psi\left(1+\frac{1}{2}\nu\right)\right\}$$

$$= \frac{1}{2}B(2m+1)\left\{\psi\left(\frac{1}{2}(\nu+1)\right) - \psi\left(1+\frac{1}{2}\nu\right)\right\}$$

$$\psi(n+1) = \frac{\Gamma'(n+1)}{\Gamma(n+1)} = -\gamma + \sum_{k=1}^{n}\frac{1}{k} \quad , \quad \psi(n) = -\gamma + \sum_{k=1}^{n-1}\frac{1}{k}$$