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Lecture 5

高斯噪声下的多样本检测

LECTURE4

- ✓ 假设检验方法：假设建立、概率映射、测量样本、统计判决
- ✓ 单样本参量检测：
  - Bayes平均风险最小准则：先验概率、代价因子已知
  - 极大极小准则：先验概率未知
  - NP准则：先验概率、代价因子均未知
- ✓检测性能指标：虚警概率、发现概率、错误概率等

检测场景

- 多样本
- 噪声高斯分布
- 白、非白
- 同分布、不同分布

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多样本

- 实信号$y_i=s_{ji}+n_i$, j=0,1; i=1….M
- 复信号$y_i=\alpha e^{-i\beta}u_{ji}+z_i$, j=0,1; i=1….M
- $Y=[y_1,y_2,\ldots y_M]^T$

✓时间采样

✓空间采样

✓频率采样

希尔伯特变换

$$\hat{s}(t)=\mathcal{H}\{s(t)\}=\frac{1}{\pi}\int\frac{s(\tau)}{t-\tau}d\tau=s(t)*\frac{1}{\pi t}$$

$$\Rightarrow \mathcal{F}\{s(t)\}\cdot\mathcal{F}\left\{\frac{1}{\pi t}\right\}=-jS(f)\cdot\mathrm{sgn}(f)$$

其中$\mathrm{sgn}(f)=\begin{cases}1 & f>0\\ 0 & f=0\\ -1 & f<0\end{cases}$

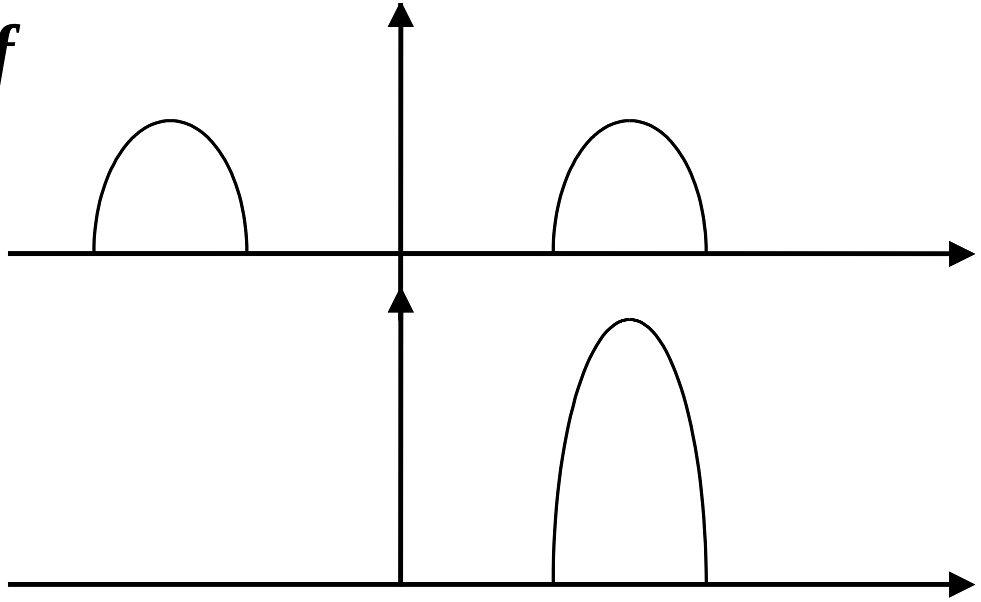
$$s(t)=\mathcal{H}^{-1}\{\hat{s}(t)\}=-\frac{1}{\pi}\int\frac{\hat{s}(\tau)}{t-\tau}d\tau=-\hat{s}(t)*\frac{1}{\pi t}$$

解析信号（预包络信号）

$$s(t)=\int_0^{\infty} S(f)e^{j2\pi ft}df+\int_{-\infty}^{0} S(f)e^{j2\pi ft}df$$

$$=\int_0^{\infty} S(f)e^{j2\pi ft}df+\int_0^{\infty}\left[S(f)e^{j2\pi ft}\right]^{*}df$$

$$=Re\int_{-\infty}^{\infty} 2S(f)U(f)e^{j2\pi ft}df$$



$$\boxed{S_P(f)=2S(f)U(f)\Rightarrow s(t)=Re\left[s_P(t)\right]}$$

$$\Downarrow$$

$$\boxed{s_P(t)=s(t)*\left[\delta(t)+j\frac{1}{\pi t}\right]=s(t)+j\hat{s}(t)}$$

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检验准则

- Bayes平均风险最小准则

$$\frac{f\left(\vec{Y}\setminus H_1\right)}{f\left(\vec{Y}\setminus H_0\right)}\geq\frac{P\left(H_0\right)\left(C_{10}-C_{00}\right)}{P\left(H_1\right)\left(C_{01}-C_{11}\right)},\quad 判为H_1;$$

$$\frac{f\left(\vec{Y}\setminus H_1\right)}{f\left(\vec{Y}\setminus H_0\right)}<\frac{P\left(H_0\right)\left(C_{10}-C_{00}\right)}{P\left(H_1\right)\left(C_{01}-C_{11}\right)},\quad 判为H_0。$$

- 最小平均错误概率

$$\frac{f\left(\vec{Y}\setminus H_1\right)}{f\left(\vec{Y}\setminus H_0\right)}\geq\frac{P\left(H_0\right)}{P\left(H_1\right)}=\frac{P\left(H_0\right)}{1-P\left(H_0\right)}=\frac{1-P\left(H_1\right)}{P\left(H_1\right)},\quad 判为H_1;$$

$$\frac{f\left(\vec{Y}\setminus H_1\right)}{f\left(\vec{Y}\setminus H_0\right)}<\frac{P\left(H_0\right)}{P\left(H_1\right)}=\frac{P\left(H_0\right)}{1-P\left(H_0\right)}=\frac{1-P\left(H_1\right)}{P\left(H_1\right)},\quad 判为H_0。$$

检验准则

- **MAP**

$$\frac{P\left(H_1 \backslash \vec{Y}\right)}{P\left(H_0 \backslash \vec{Y}\right)} \geq 1，\text{判为} H_1;$$

$$\frac{P\left(H_1 \backslash \vec{Y}\right)}{P\left(H_0 \backslash \vec{Y}\right)} < 1，\text{判为} H_0。$$

- **ML准则**

$$\frac{f\left(Y \backslash H_1\right)}{f\left(Y \backslash H_0\right)} \geq 1，\text{判为} H_1;$$

否则，判为H_0。

检验准则

- **极大极小**

$$\frac{f\left(\vec{Y}\setminus H_1\right)}{f\left(\vec{Y}\setminus H_0\right)} > \tau，\text{判为}H_1;$$

$$\frac{f\left(\vec{Y}\setminus H_1\right)}{f\left(\vec{Y}\setminus H_0\right)} = \tau,\text{以概率 }\eta\text{判为}H_1;$$

$$\frac{f\left(\vec{Y}\setminus H_1\right)}{f\left(\vec{Y}\setminus H_0\right)} < \tau，\text{判为}H_0\text{。}$$

门限的确定：

$$C_{10}\alpha\left(q_0\right)+C_{00}\left[1-\alpha\left(q_0\right)\right]=C_{01}\beta\left(q_0\right)+C_{11}\left[1-\beta\left(q_0\right)\right]$$

检验准则

- **NP**

$$\frac{f\left(\vec{Y} \backslash H_1\right)}{f\left(\vec{Y} \backslash H_0\right)} > \tau\text{，判为}H_1\text{；}$$

$$\frac{f\left(\vec{Y} \backslash H_1\right)}{f\left(\vec{Y} \backslash H_0\right)} = \tau\text{，以概率 }\eta\text{判为}H_1\text{；}$$

$$\frac{f\left(\vec{Y} \backslash H_1\right)}{f\left(\vec{Y} \backslash H_0\right)} < \tau\text{，判为}H_0\text{。}$$

门限的确定：

$$P_{fa} = \int_{\tau}^{\infty} f\left(L \backslash H_0\right) dL = \alpha$$

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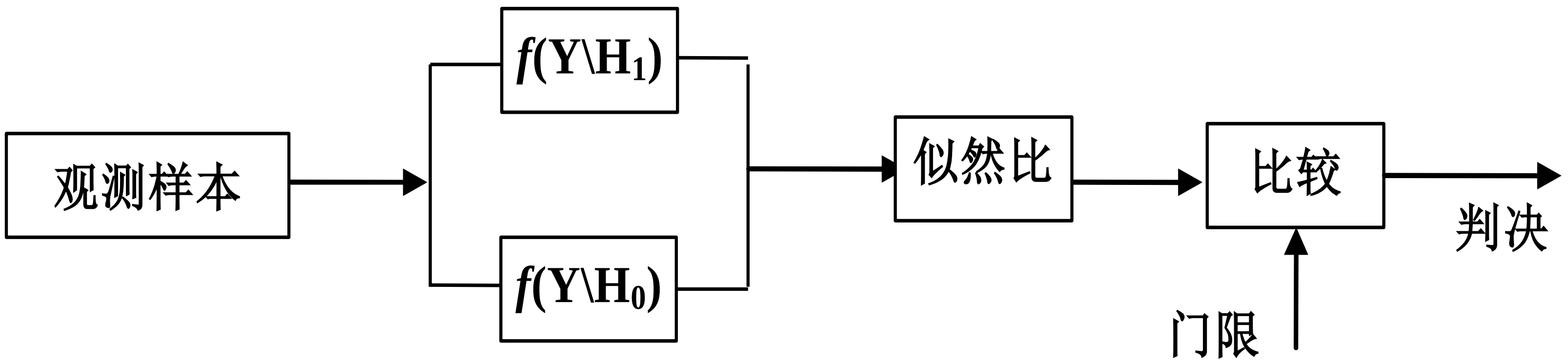
最佳检测

- **二元假设检验：**

  H_1:Y=S_1+N

  H_0:Y=S_0+N

- $Y=[y_1,y_2,\ldots y_M]^T$；　$S_j=[s_{j1},s_{j2},\ldots s_{jM}]^T$
- $N=[n_1,n_2,\ldots n_M]^T$，**白噪声，均值为零**



白噪声

- 噪声各分量独立同分布（IID）
- 两种假设下观测值Y为高斯分布
- 均值:

$$H_1:E(Y \backslash H_1)=S_1$$

$$H_0:E(Y \backslash H_0)=S_0$$

- 方差:

$$E\left\{\left(y_i-s_{1i}\right)^2 \backslash H_1\right\}=E\left\{\left(y_i-s_{0i}\right)^2 \backslash H_0\right\}=\sigma^2$$

似然表达

- **似然比**

$$L(Y)=\frac{f(Y\backslash H_1)}{f(Y\backslash H_0)}=\frac{\prod_{i=1}^{M} f(y_i\backslash H_1)}{\prod_{i=1}^{M} f(y_i\backslash H_0)}=\prod_{i=1}^{M}\frac{f(y_i\backslash H_1)}{f(y_i\backslash H_0)}$$

$$=\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{M}\left[2y_i s_{0i}-2y_i s_{1i}-\left(s_{0i}^2-s_{1i}^2\right)\right]\right\}$$

$$\overset{H_1}{\geq} th$$

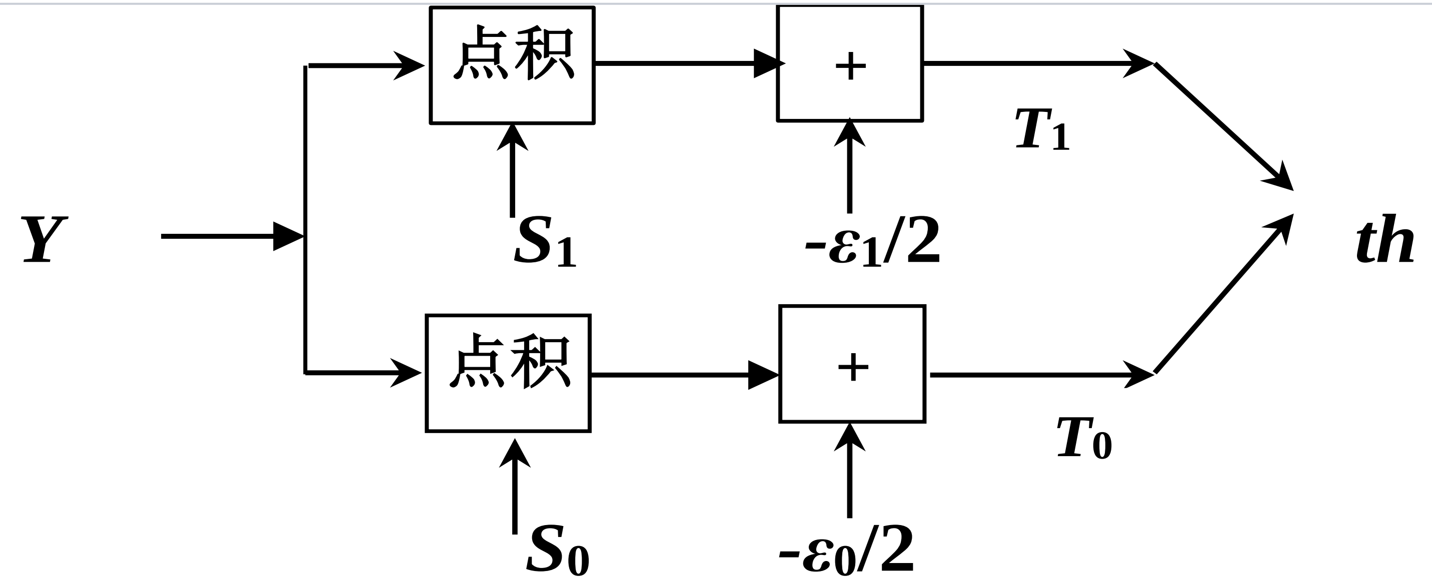
- **对数似然比**

$$\boxed{l(Y)=-\frac{1}{2\sigma^2}\sum_{i=1}^{M}\left[2y_i s_{0i}-2y_i s_{1i}-\left(s_{0i}^2-s_{1i}^2\right)\right]}$$

相关检测器

$$\boxed{T(Y)=\sigma^2\cdot l(Y)=\sum_{i=1}^{M}y_i\left(s_{1i}-s_{0i}\right)-\frac{1}{2}\left(\varepsilon_1-\varepsilon_0\right)=T_1(Y)-T_0(Y)}$$

其中 $\begin{cases}\varepsilon_1=\sum_{i=1}^{M}s_{1i}^2\\ \varepsilon_0=\sum_{i=1}^{M}s_{0i}^2\end{cases}$，$\begin{cases}T_1(Y)=\sum_{i=1}^{M}y_is_{1i}-\frac{1}{2}\varepsilon_1\\ T_0(Y)=\sum_{i=1}^{M}y_is_{0i}-\frac{1}{2}\varepsilon_0\end{cases}$



相关接收

- **最大似然—差异信号能量**

$$Y^T\left(S_1 - S_0\right) \overset{H_1}{\geq} \frac{1}{2}\left(S_1^T \cdot S_1 - S_0^T \cdot S_0\right)$$

$$\Leftrightarrow$$

$$\left(Y - S_0\right)^T\left(Y - S_0\right) \overset{H_1}{\geq} \left(Y - S_1\right)^T\left(Y - S_1\right)$$

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n维Euclidean实空间

- **矢量的点积**

$$(X, Y) = \sum_{i=1}^{M} x_i y_i$$

- **范数**

$$\|X\| = \sqrt{(X, X)} = \sqrt{\sum_{i=1}^{M} x_i^2}$$

- **距离**

$$\|X\text{-}Y\| = \sqrt{\sum_{i=1}^{M} (x_i - y_i)^2}$$

似然表达

似然函数

$$f\left(Y \backslash H_j\right)=\prod f\left(y_i \backslash H_j\right)$$

$$=\frac{1}{\left(\sqrt{2\pi\sigma}\right)^M} exp\left\{-\frac{1}{2\sigma^2}\sum\left(y_i-s_{ji}\right)^2\right\}$$

$$=\frac{1}{\left(\sqrt{2\pi\sigma}\right)^M} exp\left\{-\frac{1}{2\sigma^2}\left\|Y-S_j\right\|^2\right\}$$

似然比

$$lnL=ln\frac{f\left(Y \backslash H_1\right)}{f\left(Y \backslash H_0\right)}=-\frac{1}{2\sigma^2}\left\{\left\|Y-S_1\right\|^2-\left\|Y-S_0\right\|^2\right\}$$

$$=\frac{1}{\sigma^2}\left[Y^T\cdot\left(S_1-S_0\right)\right]-\frac{1}{2\sigma^2}\left\|S_1\right\|^2+\frac{1}{2\sigma^2}\left\|S_0\right\|^2$$

检验统计量分布

$$T(Y)=\sigma^2\cdot l(Y)=\sum_{i=1}^{M} y_i\left(s_{1i}-s_{0i}\right)-\frac{1}{2}\left(\varepsilon_1-\varepsilon_0\right)\ge\sigma^2\cdot \ln th$$

$$E\left(T\setminus H_0\right)=E\left\{\sum_{i=1}^{M} y_i\left(s_{1i}-s_{0i}\right)\right\}-\frac{1}{2}\left(\varepsilon_1-\varepsilon_0\right)$$

$$=\sum_{i=1}^{M} s_{0i}s_{1i}-\frac{1}{2}\sum_{i=1}^{M} s_{0i}^2-\frac{1}{2}\sum_{i=1}^{M} s_{1i}^2$$

$$=-\frac{1}{2}\sum_{i=1}^{M}\left(s_{1i}-s_{0i}\right)^2$$

$$=-\frac{1}{2}\left\|S_1-S_0\right\|^2$$

检验统计量分布

$$Var\left(T \setminus H_0\right) = Var\left(\sum_{i=1}^{M} y_i\left(s_{1i} - s_{0i}\right) \setminus H_0\right)$$

$$= \sum_{i=1}^{M} Var\left(y_i\right)\left(s_{1i} - s_{0i}\right)^2$$

$$= \sigma^2 \left\| S_1 - S_0 \right\|^2$$

$$H_0: T \in N\left(-\frac{1}{2}\left\|S_1 - S_0\right\|^2, \sigma^2 \left\|S_1 - S_0\right\|^2\right)$$

同理$H_1: T \in N\left(\frac{1}{2}\left\|S_1 - S_0\right\|^2, \sigma^2 \left\|S_1 - S_0\right\|^2\right)$

虚警概率

$$\alpha = P(D_1 \setminus H_0) = \Pr\{T(Y) > th' \setminus H_0\}$$

$$= \int_{\sigma^2 \ln th}^{\infty} f(T \setminus H_0) dT$$

$$= Q\left[\frac{\ln th}{d} + \frac{d}{2}\right]$$

- 偏移系数

$$d^2 = \frac{\left[E(Y \setminus H_1) - E(Y \setminus H_0)\right]^2}{Var(Y \setminus H_0)} = \frac{\left\|S_1 - S_0\right\|^2}{\sigma^2}$$

注： $Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du$

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发现概率和漏警概率

$$P_d = P\left(D_1 \setminus H_1\right) = \Pr\left\{T\left(Y\right) \geq th' \setminus H_1\right\}$$

$$= \int_{\sigma^2 \ln th}^{\infty} f\left(T \setminus H_1\right) dT = Q\left[\frac{\ln th}{d} - \frac{d}{2}\right] = Q\left[Q^{-1}\left(\alpha\right) - d\right]$$

虚警概率的逆

$$\beta = P\left(D_0 \setminus H_1\right) = \Pr\left\{T\left(Y\right) < th' \setminus H_1\right\}$$

$$= \int_{-\infty}^{\sigma^2 \ln th} f\left(T \setminus H_1\right) dT = 1 - Q\left[\frac{\ln th}{d} - \frac{d}{2}\right]$$

错误概率

$$P_e = P(H_0)P(D_1 \setminus H_0) + P(H_1)P(D_0 \setminus H_1)$$

- **若先验概率相等，则**

$$P_e = P(D_1 \setminus H_0) = \Pr\{T(Y) > 0 \setminus H_0\}$$

$$= \int_0^{\infty} f(T \setminus H_0)\,dT$$

$$= Q\left(\frac{\frac{1}{2}\|S_1 - S_0\|^2}{\sqrt{\sigma^2\|S_1 - S_0\|^2}}\right) = Q\left(\frac{1}{2}\sqrt{\frac{\|S_1 - S_0\|^2}{\sigma^2}}\right)$$

偏移系数

错误概率

- **约束:平均能量有限**

$$\bar{\varepsilon} = \frac{1}{2}\left(\varepsilon_1 + \varepsilon_0\right)$$

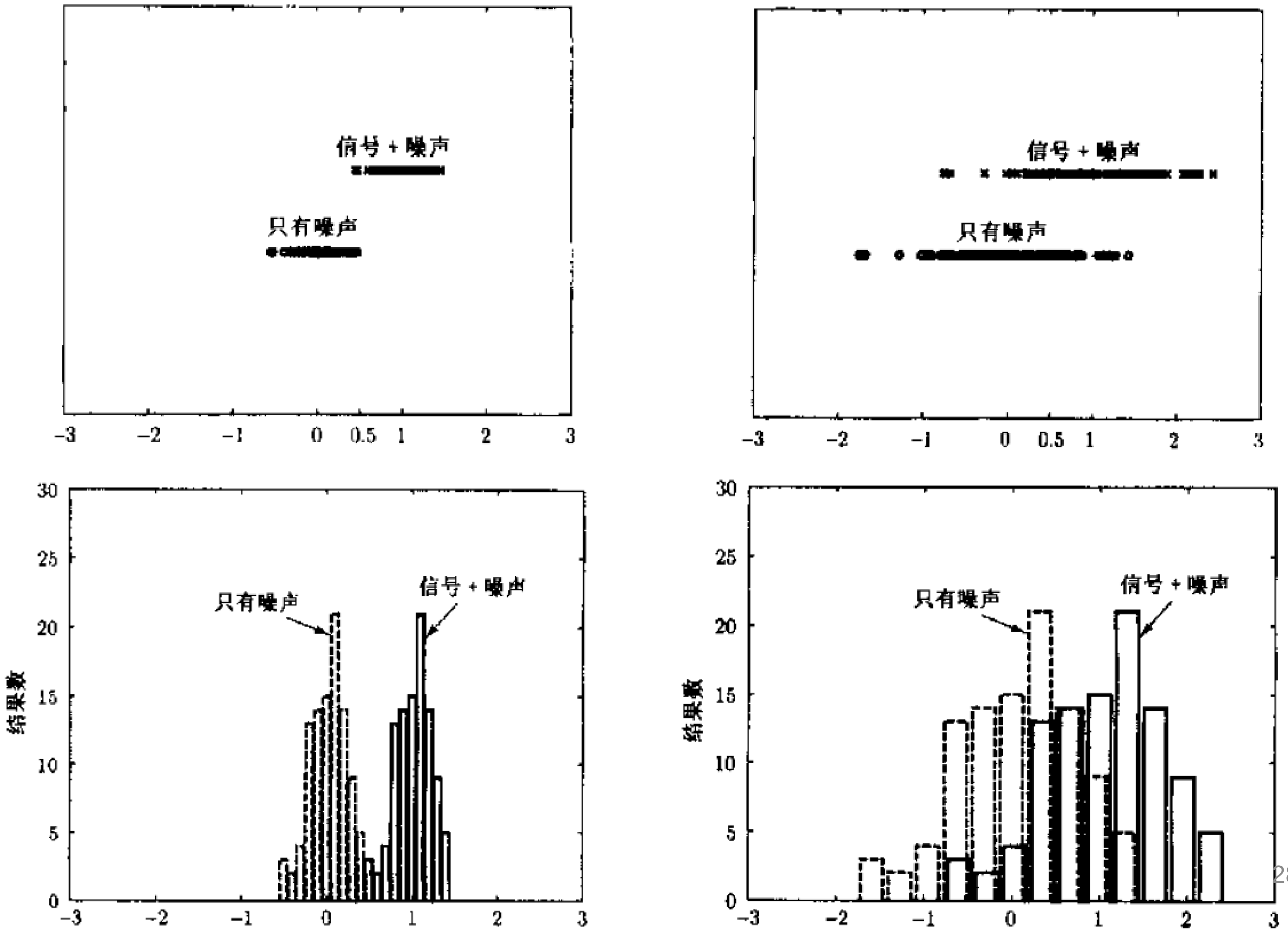
- **归一化**

$$\rho_s = \frac{S_1^T S_0}{\frac{1}{2}\left(S_1^T S_1 + S_0^T S_0\right)}$$

- **平均错误概率**

$$P_e = Q\left(\sqrt{\frac{\bar{\varepsilon}\left(1-\rho_s\right)}{2\sigma^2}}\right)$$

检测性能

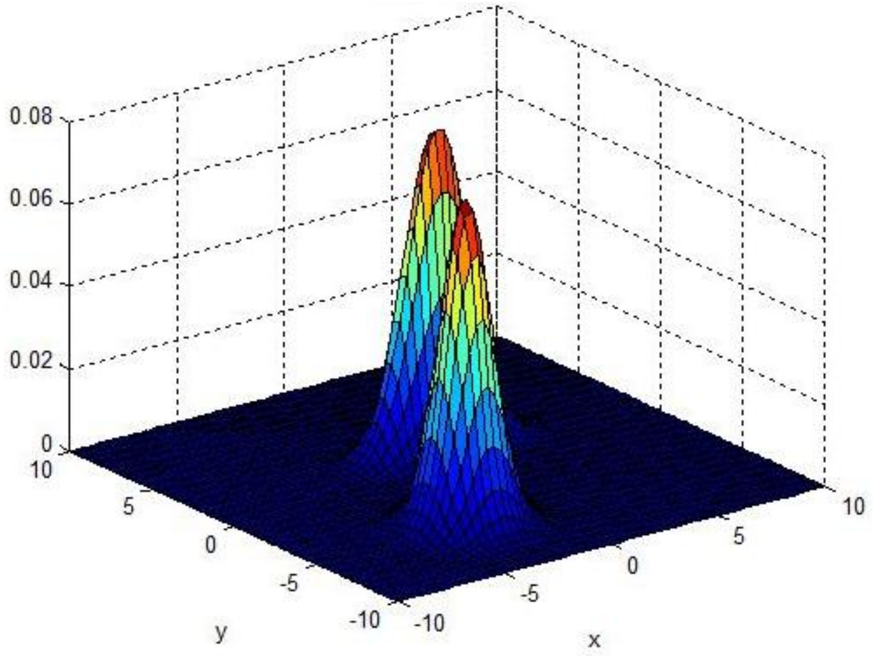


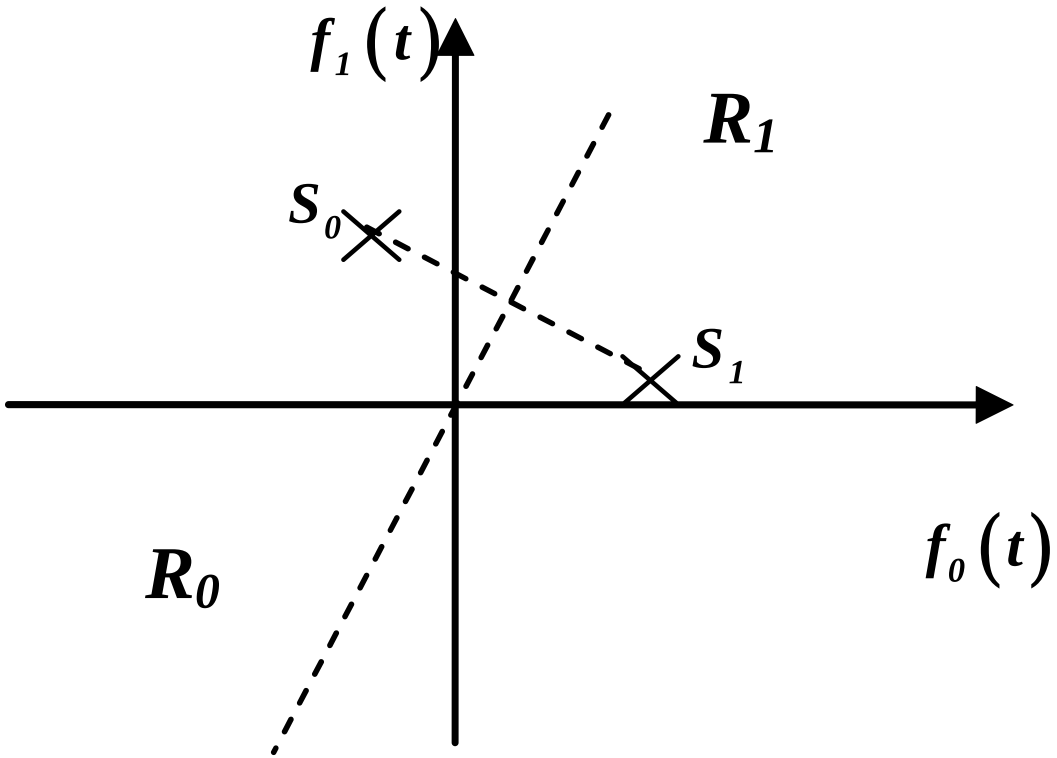
$$\begin{cases} x=1+n; \\ x=n \end{cases}$$

最小欧氏距离判决准则（ML）

$$R_1 = \left\{ Y : \left\| Y - S_1 \right\|^2 < \left\| Y - S_0 \right\|^2 \right\}$$

$$R_0 = \left\{ Y : \left\| Y - S_1 \right\|^2 \geq \left\| Y - S_0 \right\|^2 \right\}$$





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似然函数

$Y=[Y_1, Y_2,\dots ,Y_n]^{T}$

均值：$E(Y\backslash H_i)=\left[E\{Y_1\},E\{Y_2\},\cdots,E\{Y_n\}\right]^T=\left[m_{Y_{i1}},m_{Y_{i2}},\cdots,m_{Y_{in}}\right]^T$

协方差矩阵：$Cov(Y\backslash H_i)=\begin{bmatrix} E\left\{\left(Y_1-m_{Y_{i1}}\right)^2\right\} & \cdots & \cdots & E\left\{\left(Y_1-m_{Y_{i1}}\right)\left(Y_n-m_{Y_{in}}\right)\right\} \\ \vdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \vdots \\ E\left\{\left(Y_n-m_{Y_{in}}\right)\left(Y_1-m_{Y_{i1}}\right)\right\} & \cdots & \cdots & E\left\{\left(Y_n-m_{Y_{in}}\right)^2\right\} \end{bmatrix}$

似然函数：$f(Y\backslash H_i)=\dfrac{1}{(2\pi)^{n/2}\left|C_i\right|^{1/2}}\exp\left\{-\dfrac{1}{2}\left[(Y-m_i)^T C_i^{-1}(Y-m_i)\right]\right\}$

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检验统计量

- 似然比 $L(Y)=\dfrac{f(Y\backslash H_1)}{f(Y\backslash H_0)}\overset{H_1}{\geq} th$
- 对数似然比 $l(Y)=(Y-m_0)^T C_0^{-1}(Y-m_0)-(Y-m_1)^T C_1^{-1}(Y-m_1)$

$$\geq 2\ln th+\ln|C_1|-\ln|C_0|$$

- 等协方差

$$\Rightarrow 2\Delta m^T C^{-1}Y+m_0^T C^{-1}m_0-m_1^T C^{-1}m_1\geq 2\ln th$$

其中$\Delta m=m_1-m_0$

- 判决准则

$$T(Y)=\Delta m^T C^{-1}Y\geq \ln th+\frac{1}{2}\left(m_0^T C^{-1}m_0-m_1^T C^{-1}m_1\right)$$

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检验统计量的分布

$$E\left\{T\left(Y\setminus H_0\right)\right\}=\Delta m^T C^{-1} E\left(Y\setminus H_0\right)=\Delta m^T C^{-1} m_0$$

$$C\left\{T\left(Y\setminus H_0\right)\right\}=E\left\{\left[\Delta m^T C^{-1} Y-\Delta m^T C^{-1} m_0\right]\left[\Delta m^T C^{-1} Y-\Delta m^T C^{-1} m_0\right]^T\right\}$$

$$=\Delta m^T C^{-1} E\left\{\left[Y-m_0\right]\left[Y-m_0\right]^T\right\} C^{-1}\Delta m=\Delta m^T C^{-1}\Delta m$$

同理 $E\left\{T\left(Y\setminus H_1\right)\right\}=\Delta m^T C^{-1} m_1$

$$C\left\{T\left(Y\setminus H_0\right)\right\}=\Delta m^T C^{-1}\Delta m$$

$$d^2=\frac{\left[E\left(T\setminus H_1\right)-E\left(T\setminus H_0\right)\right]^2}{C\left(T\setminus H_0\right)}=\Delta m^T C^{-1}\Delta m$$

Q：白？有色？

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检验统计量

$$
\begin{aligned}
l(Y) &= (Y-m)^T C_0^{-1}(Y-m) - (Y-m)^T C_1^{-1}(Y-m) \\
&= (Y-m)^T \left(C_0^{-1} - C_1^{-1}\right)(Y-m) \\
&\geq 2\ln th + \ln|C_1| - \ln|C_0|
\end{aligned}
$$

令$m_0 = m_1 = m = 0$

$\Delta C^{-1} = C_1^{-1} - C_0^{-1}$

二次型函数 $T(Y) = Y^T \Delta C^{-1} Y \geq 2\ln th + \ln|C_1| - \ln|C_0|$

Q：白？有色？

$$\begin{aligned}H_0: y \sim N\left(m,\sigma_0^2\right)\\ H_1: y \sim N\left(m,\sigma_1^2\right)\end{aligned}$$，若$\sigma_1^2 > \sigma_0^2$且先验概率相等，求检测准则。

似然函数

$$f\left(y \backslash H_0\right)=\frac{1}{\sqrt{2\pi\sigma_0^2}}exp\left\{-\frac{\left(y-m\right)^2}{2\sigma_0^2}\right\}$$

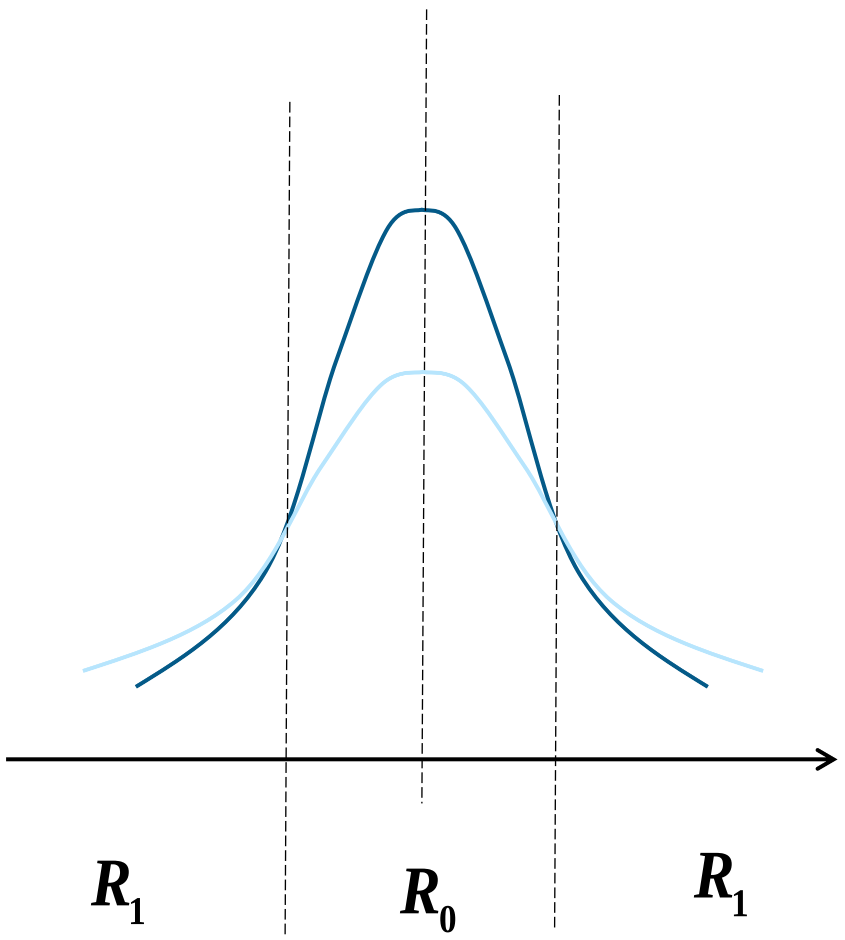
$$f\left(y \backslash H_1\right)=\frac{1}{\sqrt{2\pi\sigma_1^2}}exp\left\{-\frac{\left(y-m\right)^2}{2\sigma_1^2}\right\}$$

$$\Rightarrow \frac{\sigma_1^2-\sigma_0^2}{2\sigma_1^2\sigma_0^2}\left(y-m\right)^2 \overset{H_1}{\geq} ln\frac{\sigma_1}{\sigma_0}$$

当$y \geq th' + m$或$y \leq -th' + m$时，判决H_1为真

当$-th' + m < y < th' + m$时，判决H_0为真

其中$th' = \frac{2\sigma_1^2\sigma_0^2}{\sigma_1^2 - \sigma_0^2} ln \frac{\sigma_1}{\sigma_0}$



Q:此时的虚警、漏警概率？

均值为0时有区别吗？

噪声 IID时多样本？

summary

- ✓ 高斯噪声、参量检测
- ✓ 最佳检测等效于相关接收
- ✓ 检测性能由偏移系数确定

Ref：§3.1-§3.5(赵版)or §3.1-§3.7、 §4.5(KAY版)

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Q

多样本的必要性？

样本数如何影响检测性能？

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FIN