中国科学院大学

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Lecture 6

多元假设&序贯检测

LECTURE5

- 高斯噪声，确知信号
- 多样本观测：M维高斯分布
- 白噪声：独立同分布，相关检测，检测性能由偏移系数决定
- 非白噪声：不等均值等协方差，检验统计量为观测的线性型
  等均值不等协方差，检验统计量为观测的二次型

检测场景

- 多元假设：假设数量多个
- 累积效应：二元假设检验，样本数量多个且固定
- 序贯检测：样本数量多个且不固定

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Bayes平均风险最小准则

$$平均风险\bar{C}=\sum_{i=0}^{M\text{-}1}\sum_{j=0}^{M\text{-}1}C_{ij}P\left(D_i\backslash H_j\right)P\left(H_j\right)$$

$$=\sum_{i=0}^{M-1}P\left(H_i\right)C_{ii}\int_{Y\in R_i^n}f\left(Y\backslash H_i\right)dY+\sum_{i=0}^{M\text{-}1}\sum_{\substack{j=0\\i\neq j}}^{M\text{-}1}P\left(H_j\right)C_{ij}\int_{Y\in R_i^n}f\left(Y\backslash H_j\right)dY$$

$$=\sum_{i=0}^{M\text{-}1}P\left(H_i\right)C_{ii}-\sum_{i=0}^{M\text{-}1}P\left(H_i\right)C_{ii}\int_{Y\in\bigcup_{\substack{j=0\\j\neq i}}^{M-1}R_j^n}f\left(Y\backslash H_i\right)dY$$

$$+\sum_{i=0}^{M\text{-}1}\sum_{\substack{j=0\\i\neq j}}^{M\text{-}1}P\left(H_j\right)C_{ij}\int_{Y\in R_i^n}f\left(Y\backslash H_j\right)dY$$

Bayes平均风险最小准则

$$\bar{C}=\sum_{i=0}^{M-1}P\left(H_i\right)C_{ii}-\sum_{i=0}^{M-1}\sum_{\substack{j=0\\i\neq j}}^{M-1}P\left(H_j\right)C_{jj}\int_{Y\in R_i^n}f\left(Y\backslash H_j\right)dY$$

$$+\sum_{i=0}^{M-1}\sum_{\substack{j=0\\i\neq j}}^{M-1}P\left(H_j\right)C_{ij}\int_{Y\in R_i^n}f\left(Y\backslash H_j\right)dY$$

$$=\sum_{i=0}^{M-1}P\left(H_i\right)C_{ii}+\sum_{i=0}^{M-1}\sum_{\substack{j=0\\i\neq j}}^{M-1}\int_{Y\in R_i^n}P\left(H_j\right)\left(C_{ij}-C_{jj}\right)f\left(Y\backslash H_j\right)dY$$

$$=\sum_{i=0}^{M-1}P\left(H_i\right)C_{ii}+\sum_{i=0}^{M-1}\int_{Y\in R_i^n}\sum_{\substack{j=0\\i\neq j}}^{M-1}P\left(H_j\right)\left(C_{ij}-C_{jj}\right)f\left(Y\backslash H_j\right)dY$$

Bayes平均风险最小准则

$$判决函数 I_i(Y)=\sum_{\substack{j=0\\ i\neq j}}^{M-1} P(H_j)(C_{ij}-C_{jj})\, f(Y\backslash H_j)$$

R_i的划分：所有使$I_i(Y)$最小的Y

判决规则：$\boxed{I_{i_0}(Y)\leq I_i(Y)判为H_{i_0}}$

最小平均错误概率准则

$$\bar{C} = \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} C_{ij} P\left(D_i \setminus H_j\right) P\left(H_j\right)$$

$$= \sum_{i=0}^{M-1}\sum_{\substack{j=0 \\ i\neq j}}^{M-1} P\left(D_i \setminus H_j\right) P\left(H_j\right)$$

$$= P_e$$

$$I_{i_0}(Y) \le I_i(Y) \text{判为} H_{i_0}$$

$$\Leftrightarrow$$

$$\frac{f(Y \setminus H_{i_0})}{f(Y \setminus H_i)} \ge \frac{P(H_i)}{P(H_{i_0})} \text{判为} H_{i_0}$$

最大后验概率准则

判决函数

$$I_i(Y)=\sum_{\substack{j=0\\ i\neq j}}^{M-1} P(H_j)\,f(Y\setminus H_j)$$

$$=\sum_{\substack{j=0\\ i\neq j}}^{M-1} f(Y)\,P(H_j\setminus Y)$$

$$=[1-P(H_i\setminus Y)]\,f(Y)$$

$$\Leftrightarrow P_{i_0}(Y)\geq P_i(Y)\text{判为}H_{i_0}$$

n维观测样本Y=(y_1,y_2,…y_n)，在4种不同的假设下H_k: y_i=k+n_i， n_i i=1,…,n是均值为0、方差是σ²，彼此统计独立的高斯噪声。且已知各种假设出现的概率彼此相等，若采用平均错误概率最小准则，求相应的判决规则。

$$f(Y \backslash H_k)=\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{y_i\text{-}k}{\sigma}\right)^2\right\},k=1,2,3,4$$

$$\frac{f(Y \backslash H_{i_0})}{f(Y \backslash H_i)}\geq 1,\text{判为}H_{i_0},\text{最大似然}$$

即对特定Y，寻求使$2\sum_{i=1}^{n} y_i k - nk^2$最大的$H_k$

$$
\begin{cases}
H_1: \frac{2}{n}\sum_{i=1}^{n} y_i - 1 \\
H_2: \frac{4}{n}\sum_{i=1}^{n} y_i - 4 \\
H_3: \frac{6}{n}\sum_{i=1}^{n} y_i - 9 \\
H_4: \frac{8}{n}\sum_{i=1}^{n} y_i - 16
\end{cases}
\overset{\bar{Y}=\frac{1}{n}\sum_{i=1}^{n} y_i}{\Rightarrow}
\begin{cases}
\bar{Y} \le 1.5 \\
1.5 < \bar{Y} \le 2.5 \\
2.5 < \bar{Y} \le 3.5 \\
\bar{Y} > 3.5
\end{cases}
$$

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单样本DC电平检测

- **二元假设检验：** $H_1: y = A + n$

  $H_0: y = n$

- **N-P准则**

$$L = \frac{f(y \backslash H_1)}{f(y \backslash H_0)} = exp\left\{\frac{1}{\sigma^2}\left(Ay - \frac{1}{2}A^2\right)\right\} \geq th \Leftrightarrow y \geq th'$$

$$P_{fa} = \int_{th'/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} exp\left\{-\frac{y^2}{2}\right\} dy = \alpha$$

$$P_d = \int_{th'/\sigma - A/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} exp\left\{-\frac{y^2}{2}\right\} dy$$

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多样本DC电平检测

- **二元假设检验：** $H_1: Y=A+N$

  $H_0: Y=N$

- $Y=[y_1,y_2,\ldots y_M]^T$； $N=[n_1,n_2,\ldots n_M]^T$
- **白噪声，IID，均值为零：**

$$E\left(n_i n_j\right)=\begin{cases}\sigma^2, i=j\\ 0, i\neq j\end{cases}$$

检测统计量

$$L(Y)=\frac{\prod_{i=1}^{M} f\left(y_i \backslash H_1\right)}{\prod_{i=1}^{M} f\left(y_i \backslash H_0\right)}=\prod_{i=1}^{M} exp\left\{\frac{1}{\sigma^2}\left(Ay_i-\frac{1}{2}A^2\right)\right\}$$

$$=exp\left\{\frac{1}{\sigma^2}\left(A\sum_{i=1}^{M} y_i-\frac{M}{2}A^2\right)\right\}\geq th$$

$$\Leftrightarrow L'=\sum_{i=1}^{M} y_i \geq th'$$

令 $Z=\frac{1}{\sqrt{M}}\sum_{i=1}^{M} y_i, th=\frac{1}{\sqrt{M}}th'$

$$\Leftrightarrow Z \overset{H_1}{\geq} th$$

NP准则

- H_0假设

$$E\{Z \backslash H_0\} = \frac{1}{\sqrt{M}} E\left\{\sum_{i=1}^{M} y_i \backslash H_0\right\} = 0$$

$$V\{Z \backslash H_0\} = E\left\{\frac{1}{M}\left(\sum_{i=1}^{M} n_i\right)^2\right\} = \sigma^2$$

$$P_{fa} = \int_{th}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{-\frac{Z^2}{2\sigma^2}\right\} dZ = \int_{th/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} dx$$

NP准则

- H_1假设

$$E\left\{Z \setminus H_1\right\} = \frac{1}{\sqrt{M}} E\left\{\sum_{i=1}^{M} y_i \setminus H_1\right\} = \sqrt{M}A$$

$$V\left\{Z \setminus H_1\right\} = \sigma^2$$

$$P_d = \int_{th}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{\left(Z - \sqrt{M}A\right)^2}{2\sigma^2}\right\} dZ$$

$$= \int_{th/\sigma - \sqrt{M}A/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} dx$$

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检测性能

- H_1假设时检验统计量

$$Z=\frac{1}{\sqrt{M}}\sum_{i=1}^{M}y_i=\sum_{i=1}^{M}\left(\frac{A}{\sqrt{M}}+\frac{n_i}{\sqrt{M}}\right)$$

- 检验信噪比

$$\frac{S}{N}=\frac{(MA)^2}{E\left(\sum_{i=1}^{M}n_i\right)^2}=\frac{(MA)^2}{M\sigma^2}=M\frac{A^2}{\sigma^2}$$

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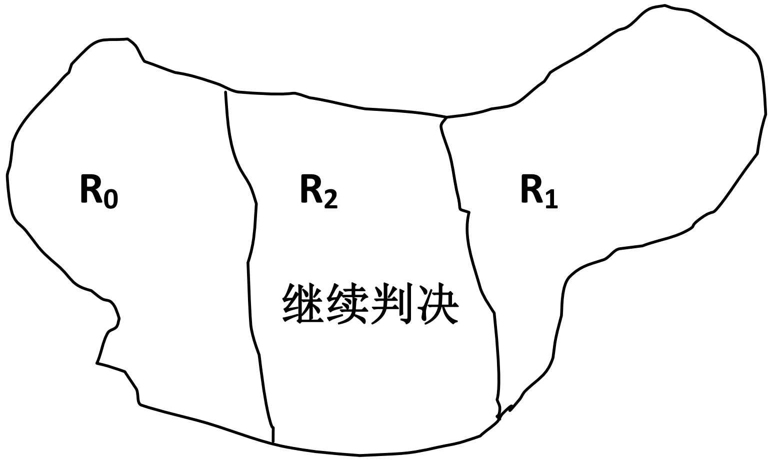
SPRT

- 二元假设检验：H_1:$Y=S_1+N$

  H_0:$Y=S_0+N$
- $Y=[y_1,y_2,\ldots y_i]^T$；　$S_j=[s_{j1},s_{j2},\ldots s_{jM}]^T$
- $N=[n_1,n_2,\ldots n_i]^T$



$$L(Y_i)=\frac{f(Y_i\backslash H_1)}{f(Y_i\backslash H_0)}\geq th_1,\ \text{判为}H_1$$

$$L(Y_i)=\frac{f(Y_i\backslash H_1)}{f(Y_i\backslash H_0)}\leq th_2,\ \text{判为}H_0$$

$$th_2<L(Y_i)<th_1,\text{观测样本变为}Y_{i+1}=[y_1,y_2,\cdots y_i,y_{i+1}]^T$$

修正的NP准则

似然比（独立分布）：

$$L(Y_N)=\frac{f(Y_N \backslash H_1)}{f(Y_N \backslash H_0)}=\prod_{i=1}^{N}\frac{f(y_i \backslash H_1)}{f(y_i \backslash H_0)}=\frac{f(y_N \backslash H_1)}{f(y_N \backslash H_0)}\prod_{i=1}^{N-1}\frac{f(y_i \backslash H_1)}{f(y_i \backslash H_0)}$$

$$\Rightarrow L(Y_N)=L(y_N)L(Y_{N-1})$$

$$\ln L(Y_N)=\sum_{i=1}^{N-1}\ln L(y_i)+\ln L(y_N)$$

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虚警概率和漏警概率

$$\alpha = P_{fa} = \int_{R_1} f\left(Y_N \backslash H_0\right) dY_N$$

$$P_d = 1 - \beta = \int_{R_1} f\left(Y_N \backslash H_1\right) dY_N$$

$$= \int_{R_1} f\left(Y_N \backslash H_0\right) L\left(Y_N\right) dY_N$$

$$1 - \beta \geq th_1 \cdot \int_{R_1} f\left(Y_N \backslash H_0\right) dY_N = th_1 \cdot \alpha$$

门限

$$th_1 \le \frac{1-\beta}{\alpha}; th_2 \ge \frac{\beta}{1-\alpha}$$

$$\ln L(Y_N) = \sum_{i=1}^{N-1} \ln L(y_i) + \ln L(y_N)$$

$$\ln th_1 \approx \ln\left(\frac{1-\beta}{\alpha}\right)$$

$$\ln th_2 \approx \ln\left(\frac{\beta}{1-\alpha}\right)$$

判决完成

- H_0假设为真时:

$$P\left[\left\{\ln L\left(Y_N\right) \le \ln th_2\right\} \backslash H_0\right] = 1-\alpha$$

$$P\left[\left\{\ln L\left(Y_N\right) \ge \ln th_1\right\} \backslash H_0\right] = \alpha$$

- H_1假设为真时:

$$P\left[\left\{\ln L\left(Y_N\right) \le \ln th_2\right\} \backslash H_1\right] = \beta$$

$$P\left[\left\{\ln L\left(Y_N\right) \ge \ln th_1\right\} \backslash H_1\right] = 1-\beta$$

终结样本数目

- 终结样本的似然比近似为两门限，则

$$E\left[\ln L(Y_N)\backslash H_1\right]=(1-\beta)\ln th_1+\beta\ln th_2$$

$$E\left[\ln L(Y_N)\backslash H_0\right]=\alpha\ln th_1+(1-\alpha)\ln th_2$$

- 观测量IID

$$\ln L(Y_N)=\ln\prod_{i=1}^{N}L(y_i)=\sum_{i=1}^{N}\ln L(y_i)=N\ln L(y)$$

终结样本数目

$$E\{\ln L(Y_N)\backslash H_1\} = E\{N \ln L(y)\backslash H_1\}$$
$$= E\{\ln L(y)\backslash H_1\} E\{N\backslash H_1\}$$

$$E\{N\backslash H_1\} = \frac{E\{\ln L(Y_N)\backslash H_1\}}{E\{\ln L(y)\backslash H_1\}} = \frac{(1-\beta)\ln th_1 + \beta \ln th_2}{E\{\ln L(y)\backslash H_1\}}$$

$$E\{N\backslash H_0\} = \frac{E\{\ln L(Y_N)\backslash H_0\}}{E\{\ln L(y)\backslash H_0\}} = \frac{\alpha \ln th_1 + (1-\alpha)\ln th_2}{E\{\ln L(y)\backslash H_0\}}$$

二元数字通信，两个假设下的观测信号为H_0：$y_i=n_i$

H_1：$y_i=1+n_i$

加性高斯白噪声均值为0，方差为1，各次观测统计独立且顺序进行。若虚警概率和漏警概率都为0.1，试求判决规则和观测次数的期望值。

$$L(Y_N) = \frac{\prod_{i=1}^{N} f(y_i \backslash H_1)}{\prod_{i=1}^{N} f(y_i \backslash H_0)} = exp\left\{\sum_{i=1}^{N} y_i - \frac{N}{2}\right\}$$

$$l(Y_N) = lnL(Y_N) = \sum_{i=1}^{N} y_i - \frac{N}{2}$$

$$lnth_1 = ln\left(\frac{1-\beta}{\alpha}\right) = 2.197$$

$$lnth_2 = ln\left(\frac{\beta}{1-\alpha}\right) = -2.197$$

$$\sum_{i=1}^{N} y_i - \frac{N}{2} \geq 2.197$$，则判H_1成立

$$\sum_{i=1}^{N} y_i - \frac{N}{2} \leq -2.197$$，则判H_0成立

$$2.197 < \sum_{i=1}^{N} y_i - \frac{N}{2} < 2.197$$，则增加一次观测再检验

$$E\{N \setminus H_1\} = \frac{(1-\beta)\ln th_1 + \beta \ln th_2}{E\{\ln L(y) \setminus H_1\}} = \frac{(1-\beta)\ln th_1 + \beta \ln th_2}{\frac{1}{2}} = 3.515$$

同理$E\{N \setminus H_0\} = 3.515$

summary

- 多元假设比较M-1次
- 多样本带来检测性能提升的可能性
- 样本质量决定样本序列的数量

Ref：§3.6、§3.11(赵版)or §3.8(KAY版)

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Q

- 多元假设时的检测性能?
- 噪声分布非高斯?
- 噪声分布未知?


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