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Lecture 11
Bayes估计

LECTURE10

- **随机参量信号检测**
  - 有PD：复合假设检验，计算平均似然函数/似然比
  - 无PDF：估计+检测（广义似然比）或条件似然比
- **随机信号检测（高斯）**

白信号：能量检测器$T(Y)=\sum_{i=1}^{M} y_i^2 \overset{H_1}{\geq} th'$

有色噪声：估计－检测器$T(Y)=Y^T\hat{S}=Y^T C_s\left(C_s+\sigma^2 I\right)^{-1}Y$

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估计背景

- 参量估计
- 理论框架
- 估计性能
- 参量随机/非随机

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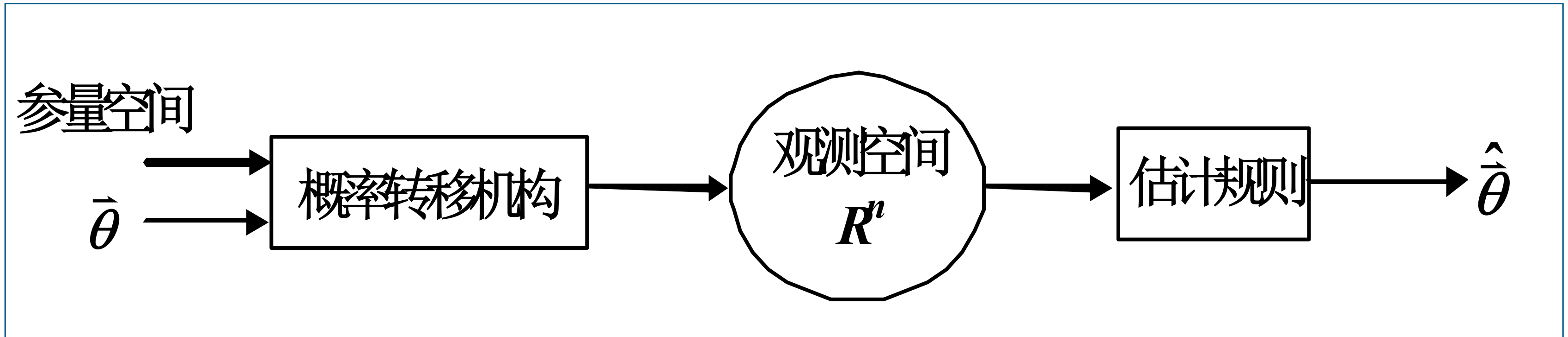
估计模型

- 参量空间
- 概率映射
- 观测空间
- 估计规则



估计性能

- 数学期望（无偏性）

$$E\left\{\hat{\theta}\right\}=E\left\{\theta\right\}$$

- 方差（有效性）
- 均方误差矩阵
- 充分性

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重复上节例题

H_1:$Y=A+N$；H_0:$Y=N$。A未知。高斯白噪声。求检验准则。

$$f\left(\vec{Y}\backslash A\right)=\left(\prod_{k=1}^{N}\frac{1}{\sqrt{2\pi}\sigma_n}\right)exp\left\{\sum_{k=1}^{N}-\frac{1}{2\sigma_n^2}\left(y_k-A\right)^2\right\}$$

ML方程: $\left.\frac{\partial}{\partial A}ln\,f\left(\vec{Y}\backslash A\right)\right|=0$，即 $\left.\sum_{k=1}^{N}\frac{1}{\sigma_n^2}\left(y_k-A\right)\right|=0\Rightarrow\hat{A}=\frac{1}{N}\sum_{k=1}^{N}y_k$

$$E\left(\hat{A}\right)=E\left[\frac{1}{N}\sum_{k=1}^{N}\left(A+n_k\right)\right]=A$$

$$E\left[\left(A-\hat{A}\right)^2\right]=E\left[\left(\frac{1}{N}\sum_{k=1}^{N}n_k\right)^2\right]=\frac{\sigma_n^2}{N}$$

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Bayes估计

- **每次估计的条件风险代价**

$$R(\theta)=\int C\left(\hat{\theta}(Y),\theta\right)f(Y\backslash\theta)\,dY$$

- **平均风险代价**

$$\overline{C}=\int R(\theta)f(\theta)\,d\theta=\int f(Y)\,dY\int C\left(\hat{\theta}(Y),\theta\right)f(\theta\backslash Y)\,d\theta$$

- **Bayes估计**

$$\hat{\theta}\rightarrow min\left\{\overline{C}\left(\hat{\theta},\theta\right)\right\}$$

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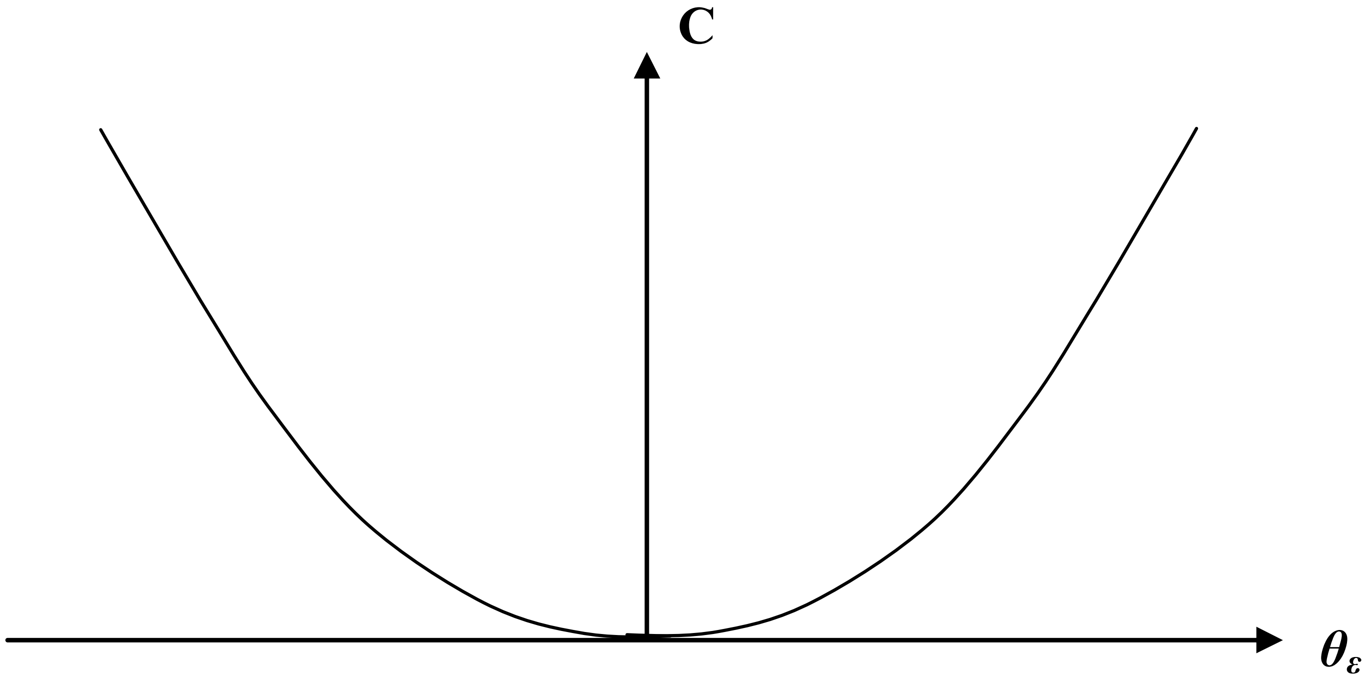
误差平方代价函数

$$C\left(\hat{\theta}\right)=C\left[\theta-\hat{\theta}(y)\right]=\left[\theta-\hat{\theta}(y)\right]^2=\theta_\varepsilon^2$$



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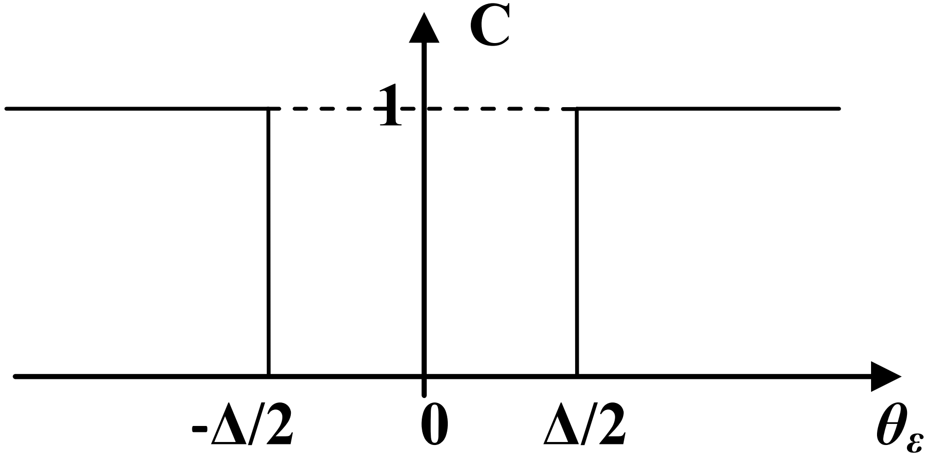
均匀代价函数

$$C\left(\hat{\theta}\right)=C\left[\theta-\hat{\theta}\left(y\right)\right]=\begin{cases}1, \left|\theta-\hat{\theta}\left(y\right)\right|\geq\dfrac{\Delta}{2}\\0, \left|\theta-\hat{\theta}\left(y\right)\right|<\dfrac{\Delta}{2}\end{cases}$$



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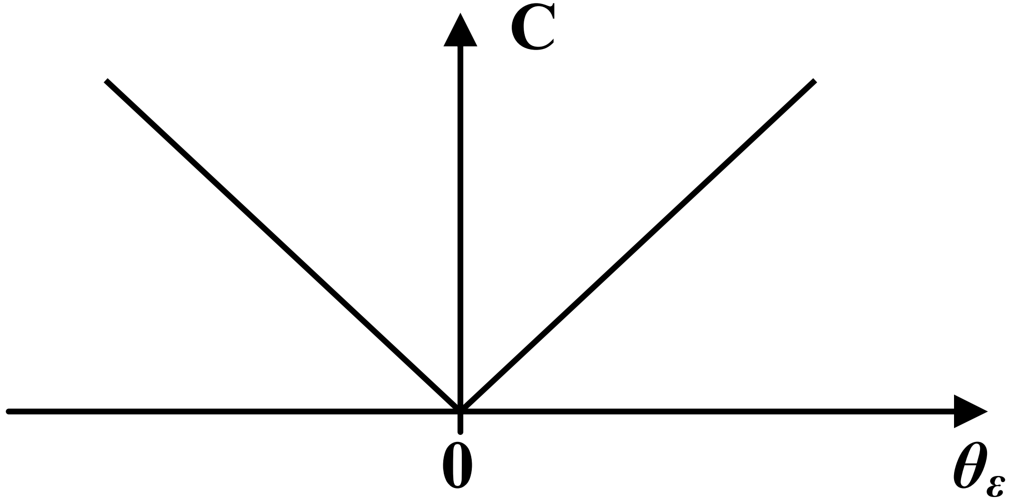
误差绝对值代价函数

$$C\left(\hat{\theta}\right)=C\left[\theta-\hat{\theta}\left(y\right)\right]=\left|\theta-\hat{\theta}\left(y\right)\right|$$



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MAP检测到MAP估计

- 离散待估参量θ
- 多元假设检验H_i：$\theta=\theta_i$
- MAP检测判决：

$P(H_{i_0} \backslash Y) \geq P(H_i \backslash Y)$判$H_{i_0}$为真

- MAP估计：

$$K \to \infty : f\left(\hat{\theta}_{MAP} \backslash Y\right) = \max_{\theta} f\left(\theta \backslash Y\right)$$

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MAP估计(随机待估参量)

- **MAP方程：**

$$\frac{\partial}{\partial\theta} f\left(\theta \setminus Y\right)\Big|_{\theta=\hat{\theta}_{MAP}} = 0$$

$$\Leftrightarrow \frac{\partial}{\partial\theta}\left[\ln f\left(Y \setminus \theta\right) + \ln f\left(\theta\right)\right]\Big|_{\theta=\hat{\theta}_{MAP}} = 0$$

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MAP估计代价

平均代价

$$\overline{C}=\int_{-\infty}^{\infty}f(Y)dY\left[\int_{-\infty}^{\hat{\theta}-\frac{\Delta}{2}}f(\theta\backslash Y)d\theta+\int_{\hat{\theta}+\frac{\Delta}{2}}^{\infty}f(\theta\backslash Y)d\theta\right]$$

$$=\int_{-\infty}^{\infty}f(Y)dY\left[1-\int_{\hat{\theta}-\frac{\Delta}{2}}^{\hat{\theta}+\frac{\Delta}{2}}f(\theta\backslash Y)d\theta\right]$$

$min\overline{C}\Leftrightarrow max\ f(\theta\backslash Y)\Rightarrow MAP$准则

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ML方程（非随机参量）

- 最大似然判决准则：

$$\frac{f(Y \backslash H_{i_0})}{f(Y \backslash H_i)} \geq 1，判为 H_{i_0}$$

- 最大似然估计方程：

$$\frac{\partial}{\partial \theta}\left[lnf\left(Y \backslash \theta\right)\right]\Big|_{\theta=\hat{\theta}_{ML}} = 0$$

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观测数据为$y_k=m+n_k$， k=1,2...N, m为待估参量，均值为零，方差σ_θ^2， 高斯分布；$\{n_k\}$为独立于m的均值零、方差 σ_n^2 的高斯噪声。

解：

m的先验分布 $f(m)=\frac{1}{\sqrt{2\pi}\sigma_\theta}\exp\left\{-\frac{m^2}{2\sigma_\theta^2}\right\}$

观测矢量 $\vec{Y}=\left[y_1,y_2,\cdots y_N\right]^T$

似然函数 $f(\vec{Y}\backslash m)=\left(\frac{1}{\sqrt{2\pi}\sigma_n}\right)^N exp\left\{-\frac{1}{2\sigma_n^2}\sum_{k=1}^{N}(y_k-m)^2\right\}$

后验概率

$$f\left(m \backslash \vec{Y}\right) = \frac{f\left(\vec{Y} \backslash m\right) f\left(m\right)}{f\left(\vec{Y}\right)}$$

$$= \frac{f\left(\vec{Y} \backslash m\right) f\left(m\right)}{\int_{-\infty}^{\infty} f\left(\vec{Y} \backslash m\right) f\left(m\right) dm}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_p} exp\left\{-\frac{1}{2\sigma_p^2}\left(m - \frac{\sigma_\theta^2}{\sigma_\theta^2 + \sigma_n^2 / N}\bar{y}\right)^2\right\}$$

$$\Rightarrow \hat{m}_{MAP} = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \sigma_n^2 / N}\bar{y}$$

其中 $\begin{cases} \sigma_p^2 = \dfrac{\sigma_\theta^2 \cdot \sigma_n^2}{N\sigma_\theta^2 + \sigma_n^2} \\ \bar{y} = \dfrac{1}{N}\sum\limits_{k=1}^{N} y_k \end{cases}$

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雷达测距系统，目标真实距离为m，由于噪声的干扰，每次测量的结果为$y_k=m+n_k$，$k=1,2...N$, n_k为均值零、方差σ_n^2的高斯干扰或噪声。N次独立观测。

$$f\left(\vec{Y}\backslash m\right)=\left(\prod_{k=1}^{N}\frac{1}{\sqrt{2\pi}\sigma_k}\right)\exp\left\{\sum_{k=1}^{N}-\frac{1}{2\sigma_k^2}\left(y_k-m\right)^2\right\}$$

$$ML\text{方程}:\frac{\partial}{\partial m}\ln f\left(\vec{Y}\backslash m\right)\Big|_{m=\hat{m}_{ML}}=0,\quad \text{即}\sum_{k=1}^{N}\frac{1}{2\sigma_k^2}\left(y_k-m\right)\Big|_{m=\hat{m}_{ML}}=0$$

$$\Rightarrow \hat{m}_{ML}=\frac{\sum_{k=1}^{N}\frac{y_k}{\sigma_k^2}}{\sum_{k=1}^{N}\frac{1}{\sigma_k^2}}\overset{\text{方差相等}}{=\!=\!=\!=}\frac{1}{N}\sum_{k=1}^{N}y_k$$

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对噪声中正弦序列信号的相位进行估计。设观测为$y_k=A\cos(k\omega_0+\varphi)+n_k$，$k=1,2...N$, 幅度$A$和频率$f_0$为已知的，噪声是方差为$\sigma_n^2$的高斯白噪声。

$$f(Y\setminus\varphi)=\frac{1}{\left(2\pi\sigma_n^2\right)^{\frac{N}{2}}}exp\left\{\sum_{k=1}^{N}-\frac{1}{2\sigma_n^2}\left[y_k-A\cos\left(2\pi kf_0+\varphi\right)\right]^2\right\}$$

$$\frac{\partial}{\partial\varphi}ln\,f\left(Y\setminus\varphi\right)=0,$$

即$$\left[2\sum_{k=1}^{N}\left[y_k-A\cos\left(2\pi kf_0+\varphi\right)\right]A\sin\left(2\pi kf_0+\varphi\right)\right]=0$$

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$$\Rightarrow \sum_{k=1}^{N} y_k \sin\left(2\pi k f_0 + \hat{\varphi}\right) = A\sum_{k=1}^{N} \sin\left(4\pi k f_0 + 2\hat{\varphi}\right) \approx 0$$

$$\Rightarrow \sum_{k=1}^{N} y_k \sin\left(2\pi k f_0\right)\cos\hat{\varphi} = \sum_{k=1}^{N} y_k \cos\left(2\pi k f_0\right)\sin\hat{\varphi}$$

$$\Rightarrow \hat{\varphi} = \arctan\frac{\sum_{k=1}^{N} y_k \sin\left(2\pi k f_0\right)}{\sum_{k=1}^{N} y_k \cos\left(2\pi k f_0\right)}$$

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Minimum Mean Square Error(MMSE)

- **估计误差**

$$\theta_{\varepsilon}=\hat{\theta}-\theta$$

- **代价函数**

$$C\left(\hat{\theta},\theta\right)=\theta_{\varepsilon}^{T}\theta_{\varepsilon}$$

- **条件平均风险**

$$R(\theta)=\int C\left(\hat{\theta}(Y),\theta\right)f\left(Y\backslash\theta\right)dY$$

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Minimum Mean Square Error(MMSE)

- **平均代价**

$$\bar{C}_{MS} = \int R(\theta) f(\theta) d\theta$$

$$= \int f(Y) dY \int C\left(\hat{\theta}(Y), \theta\right) f(\theta \setminus Y) d\theta$$

$$= \int f(Y) dY \int \left[\hat{\theta}(Y) - \theta\right]^T \left[\hat{\theta}(Y) - \theta\right] f(\theta \setminus Y) d\theta$$

- **MMSE估计方程**

$$\frac{\partial}{\partial \hat{\theta}} \bar{C}_{MS} = 0 \Rightarrow \hat{\theta}_{MS}(Y) = \int \theta f(\theta \setminus Y) d\theta$$

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雷达测距系统，目标真实距离为m，由于噪声的干扰， 观测数据为$y_k=m+n_k$，$k=1,2...N$, m为待估参量，均值为零，方差σ_θ^2，高斯分布；$\{n_k\}$为独立于m的均值零、方差 σ_n^2 的高斯噪声。**N次独立观测。MMSE估计。**

解：

$$f\left(m \mid \vec{Y}\right)=\frac{1}{\sqrt{2\pi}\sigma_p}exp\left\{-\frac{1}{2\sigma_p^2}\left(m-\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_n^2/N}\bar{y}\right)^2\right\}$$

$$\Rightarrow \hat{m}_{MMSE}=\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_n^2/N}\bar{y}=\hat{m}_{MAP}$$

无偏性

$$
\begin{aligned}
E\left\{\hat{\theta}_{MS}(Y)\right\} &= \int_{(Y)}\left[\int_{(\theta)}\theta f(\theta \setminus Y)\,d\theta\right]f(Y)\,dY \\
&= \int_{(\theta)}\theta\int_{(Y)}f(\theta, Y)\,dY\,d\theta \\
&= \int_{(\theta)}\theta f(\theta)\,d\theta \\
&= E\{\theta\}
\end{aligned}
$$

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傅里叶分析。数据模型表示为$y_k=a\cos k\omega_0+b\sin k\omega_0+n_k$， $k=1...N, f_0$为$(1/N)$的倍数。a和b**为待估**参量，均值为零，方差σ_θ^2， 高斯分布；$\{n_k\}$为独立于m的均值零、方差 σ_n^2 的高斯噪声。**N次独立观测。MMSE估计。**

$$Y=H\theta+N$$

$$H=\begin{bmatrix} \cos\omega_0 & \sin\omega_0 \\ \cos 2\omega_0 & \sin 2\omega_0 \\ \vdots & \vdots \\ \cos N\omega_0 & \sin N\omega_0 \end{bmatrix}$$

$$\theta=\begin{bmatrix} a & b \end{bmatrix}^T$$

$$E\{\theta\}=0$$

$$C_\theta=\sigma_\theta^2 I$$

$$C_n=\sigma_n^2 I$$

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$$f\left(Y \backslash \theta\right) \sim N\left(H\theta, C_n\right)$$

$$f\left(\theta\right) \sim N\left(0, C_\theta\right)$$

由 $\dfrac{\partial}{\partial\theta}\left[\ln f\left(Y \backslash \theta\right) + \ln f\left(\theta\right)\right]\Big|_{\theta=\hat{\theta}_{MAP}} = 0$

针对 $\dfrac{1}{\left(2\pi\right)^{N}\left|C_n\right|^{1/2}} exp\left\{-\dfrac{1}{2}\left[(Y-H\theta)^T C_n^{-1}(Y-H\theta)\right]\right\} \dfrac{1}{\left|C_\theta\right|^{1/2}} exp\left\{-\dfrac{1}{2}\left[\theta^T C_\theta^{-1}\theta\right]\right\}$

取对数后求导得

$$H^T C_n^{-1}\left[Y - H\hat{\theta}\right] = C_\theta^{-1}\hat{\theta} \quad \Rightarrow H^T C_n^{-1} Y = \left[H^T C_n^{-1} H + C_\theta^{-1}\right]\hat{\theta}$$

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$$\begin{bmatrix} \cos\omega_0 & \cos 2\omega_0 & \cdots & \cos N\omega_0 \\ \sin\omega_0 & \sin 2\omega_0 & \cdots & \sin N\omega_0 \end{bmatrix} \begin{bmatrix} 1/\sigma_n^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1/\sigma_n^2 \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}$$

$$= \left\{ \begin{bmatrix} \cos\omega_0 & \cos 2\omega_0 & \cdots & \cos N\omega_0 \\ \sin\omega_0 & \sin 2\omega_0 & \cdots & \sin N\omega_0 \end{bmatrix} \begin{bmatrix} 1/\sigma_n^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1/\sigma_n^2 \end{bmatrix} \begin{bmatrix} \cos\omega_0 & \sin\omega_0 \\ \cos 2\omega_0 & \sin 2\omega_0 \\ \vdots & \vdots \\ \cos N\omega_0 & \sin N\omega_0 \end{bmatrix} + \begin{bmatrix} 1/\sigma_\theta^2 & 0 \\ 0 & 1/\sigma_\theta^2 \end{bmatrix} \right\} \begin{bmatrix} a \\ b \end{bmatrix}$$

由DFT正交性 $\begin{cases} \sum_{n=1}^{N} \cos\left(\frac{2\pi in}{N}\right)\cos\left(\frac{2\pi jn}{N}\right) = \frac{N}{2}\delta_{ij} \\ \sum_{n=1}^{N} \sin\left(\frac{2\pi in}{N}\right)\sin\left(\frac{2\pi jn}{N}\right) = \frac{N}{2}\delta_{ij} \\ \sum_{n=1}^{N} \cos\left(\frac{2\pi in}{N}\right)\sin\left(\frac{2\pi jn}{N}\right) = 0 \end{cases}$

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$$\frac{1}{\sigma_n^2}\begin{bmatrix}\sum_{k=1}^{N} y_k \cos k\omega_0 \\ \sum_{k=1}^{N} y_k \sin k\omega_0\end{bmatrix} = \begin{bmatrix}\frac{N}{2\sigma_n^2}+\frac{1}{\sigma_\theta^2} & 0 \\ 0 & \frac{N}{2\sigma_n^2}+\frac{1}{\sigma_\theta^2}\end{bmatrix}\begin{bmatrix}\hat{a} \\ \hat{b}\end{bmatrix}$$

其中 $$\frac{\frac{1}{\sigma_n^2}}{\frac{N}{2\sigma_n^2}+\frac{1}{\sigma_\theta^2}} = \frac{1}{\frac{N}{2}+\frac{\sigma_n^2}{\sigma_\theta^2}}$$

$$\Rightarrow \hat{\theta} = \begin{bmatrix}\hat{a} \\ \hat{b}\end{bmatrix} = \frac{\frac{2}{N}}{1+\frac{2\sigma_n^2/N}{\sigma_\theta^2}}\begin{bmatrix}\sum_{k=1}^{N} y_k \cos k\omega_0 \\ \sum_{k=1}^{N} y_k \sin k\omega_0\end{bmatrix}$$

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条件中值估计

误差绝对值代价函数

$$\bar{C}=\int R(\theta)f(\theta)d\theta=\int_{-\infty}^{\infty}f(Y)dY\int\left|\hat{\theta}(Y)-\theta\right|f(\theta\backslash Y)d\theta$$

$$=\int_{-\infty}^{\hat{\theta}}\left(\widehat{\theta(Y)}-\theta\right)f(\theta\backslash Y)d\theta+\int_{\hat{\theta}}^{\infty}\left(\theta-\widehat{\theta(Y)}\right)f(\theta\backslash Y)d\theta$$

求导等于零，可得

$$\int_{-\infty}^{\hat{\theta}}f(\theta\backslash Y)d\theta=\int_{\hat{\theta}}^{\infty}f(\theta\backslash Y)d\theta$$

线性观测：$x=m+n$，m为待估参量，在$[-A, A]$区间均匀分布；n为独立于m的均值零、方差 σ_n^2 的高斯噪声。MAP和MMSE**估计。**

$$f(x \backslash m) = \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left\{-\frac{(x-m)^2}{2\sigma_n^2}\right\}$$

$$f(m) = \begin{cases} \dfrac{1}{2A} & -A \le m \le A \\ 0 & \text{其它} \end{cases}$$

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$$\boxed{\hat{\theta}_{MS}(Y)=\int \theta f(\theta \backslash Y)\,d\theta}$$

$$\hat{m}_{MS}=\frac{\int m f(x\backslash m)f(m)\,dm}{f(x)}=\frac{\int_{-\infty}^{\infty} m f(x\backslash m)f(m)\,dm}{\int_{-\infty}^{\infty} f(x\backslash m)f(m)\,dm}$$

$$=\frac{\int_{-\infty}^{\infty} m\frac{1}{\sqrt{2\pi}\sigma_n}exp\left[-\frac{(x-m)^2}{2\sigma_n^2}\right]\frac{1}{2A}dm}{\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma_n}exp\left[-\frac{(x-m)^2}{2\sigma_n^2}\right]\frac{1}{2A}dm}$$

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$$\hat{m}_{MS}=\frac{\int_{-\infty}^{\infty} m\exp\left[-\frac{(x-m)^2}{2\sigma_n^2}\right]dm}{\int_{-\infty}^{\infty}\exp\left[-\frac{(x-m)^2}{2\sigma_n^2}\right]dm}=x-\frac{\sigma_n\int_{(x/\sigma_n-A/\sigma_n)^2/2}^{(x/\sigma_n+A/\sigma_n)^2/2}\exp(-v)\,dv}{\sqrt{2\pi}\int_{x/\sigma_n-A/\sigma_n}^{x/\sigma_n+A/\sigma_n}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{u^2}{2}\right)du}$$

$$=x-\frac{\sigma_n\left\{\exp\left[-{(x/\sigma_n-A/\sigma_n)^2}/{2}\right]-\exp\left[-{(x/\sigma_n+A/\sigma_n)^2}/{2}\right]\right\}}{\sqrt{2\pi}\left[Q(x/\sigma_n+A/\sigma_n)-Q(x/\sigma_n-A/\sigma_n)\right]}$$

$$\boxed{令u=\frac{x-m}{\sigma_n},\quad v=\frac{u^2}{2}}$$

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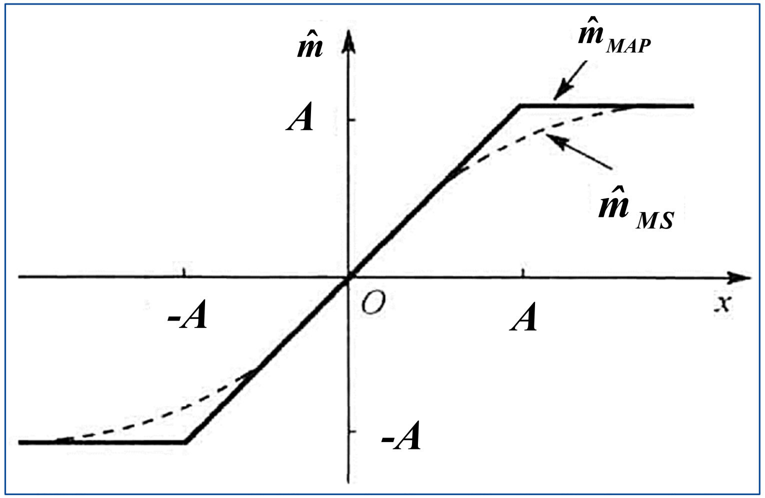
$$\frac{\partial}{\partial \theta}\left[\ln f\left(Y \backslash \theta\right)+\ln f\left(\theta\right)\right]\Big|_{\theta=\hat{\theta}_{\mathrm{MAP}}}=0$$

$$\Rightarrow \frac{x}{\sigma_n^2}-\frac{\hat{m}}{\sigma_n^2}=0, -A \le m \le A$$

$$\Rightarrow \hat{m}_{MAP}=\begin{cases} x & -A \le m \le A \\ -A & m < -A \\ A & m > A \end{cases}$$



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summary

- **随机参量，已知PDF：BAYES估计**
  - 均匀代价函数：MAP
  - 平方代价函数：MMSE
  - 绝对值代价函数：MED
- **非随机参量：ML估计**
- **估计性能评价：一阶矩、二阶矩**

Ref：§5.1-5.4(赵版)、第7章、第10章 、第11章 (KAY版)


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