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Lecture 13

最小二乘估计&性能分析

lecture12

- **线性估计则只需一阶矩、二阶矩**

$$\hat{\theta}_{LMS} = A_L + B_L Y = E\{\theta\} + \mathrm{cov}(\theta, Y)\,\mathrm{cov}(Y, Y)^{-1}\left[Y - E\{Y\}\right]$$

- **均方误差最小→估计误差与观测值正交**

$$E\left\{\left(\theta - \hat{\theta}_{LMS}\right)Y^T\right\} = E\left\{e\left(\theta, \hat{\theta}_{LMS}\right)Y^T\right\} = 0$$

- **均方误差矩阵最小**

$$E\left\{\left[\theta - \hat{\theta}_{LMS}\right]\cdot\left[\theta - \hat{\theta}_{LMS}\right]^T\right\}^T = cov(\theta,\theta) - cov(\theta, Y)\left[cov(Y, Y)\right]^{-1} cov(Y,\theta)$$

- **线性观测下可基于均值、方差和线性系数实现序贯估计**

$$\hat{\theta}_{LMS} = \theta_0 + k\sum_{i=1}^{m} h_i\left(y_i - h_i\theta_0\right)$$

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估计背景

- 无统计先验知识
- 线性观测方程
- 估计性能

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一般形式

- **Least Square**
- **线性观测方程：** $Y=H\theta+N$
- 性能指标

$$T\left(\hat{\theta}\right)=\left[Y-H\hat{\theta}\right]^{T}\left[Y-H\hat{\theta}\right]$$

- **指标最小**

$$\nabla_{\hat{\theta}}T\left(\hat{\theta}\right)=-2H^{T}\left[Y-H\hat{\theta}\right]=0\Rightarrow\hat{\theta}_{LS}=\left[H^{T}H\right]^{-1}H^{T}Y$$

估计性质

- $\hat{\theta}_{LS}$是观测样本的线性估计
- 若E{N}=0，则LS估计是无偏估计：

$$E\left\{\hat{\theta}_{LS}\right\} = E\left\{\left[H^T H\right]^{-1} H^T Y\right\}$$

$$= E\left\{\left[H^T H\right]^{-1} H^T \left(H\theta + N\right)\right\}$$

$$= E\left\{\theta\right\}$$

估计性质

- 设R_N=E{NN^T}，则LS估计的误差矩阵

$$E\left\{\left[\theta-\hat{\theta}_{LS}\right]\left[\theta-\hat{\theta}_{LS}\right]^T\right\}$$

$$=E\left\{\left[\theta-\left[H^TH\right]^{-1}H^TY\right]\cdot\left[\theta-\left[H^TH\right]^{-1}H^TY\right]^T\right\}$$

$$=E\left\{\left[\theta-\left[H^TH\right]^{-1}H^T\left[H\theta+N\right]\right]\cdot\left[\theta-\left[H^TH\right]^{-1}H^T\left[H\theta+N\right]\right]^T\right\}$$

$$=E\left\{\left[-\left[H^TH\right]^{-1}H^TN\right]\cdot\left[-\left[H^TH\right]^{-1}H^TN\right]^T\right\}$$

$$=\left[H^TH\right]^{-1}H^TR_NH\left[H^TH\right]^{-1}$$

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加权形式

- **加权最小二乘（Least Square Weighted）**
- **线性观测方程：** $Y=H\theta+N$
- 性能指标

$$T_W\left(\hat{\theta}\right)=\left[Y-H\hat{\theta}\right]^T W\left[Y-H\hat{\theta}\right]$$

- **指标最小**

$$\nabla_{\hat{\theta}}T_W\left(\hat{\theta}\right)=-2H^TW\left[Y-H\hat{\theta}\right]=0$$

$$\Rightarrow \hat{\theta}_{LSW}=\left[H^TWH\right]^{-1}H^TWY$$

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估计性质

- $\hat{\theta}_{LSW}$是观测样本的线性估计
- 若E{N}=0，则LSW估计是无偏估计：

$$E\left\{\hat{\theta}_{LSW}\right\} = E\left\{\left[H^T W H\right]^{-1} H^T W Y\right\}$$

$$= E\left\{\left[H^T W H\right]^{-1} H^T W \left(H\theta + N\right)\right\}$$

$$= E\left\{\theta\right\}$$

估计性质

- 设R_N=E{NN^T}，则LSW估计的误差矩阵

$$E\left\{\left[\theta-\hat{\theta}_{LSW}\right]\left[\theta-\hat{\theta}_{LSW}\right]^T\right\}$$

$$=E\left\{\left[\theta-\left[H^TWH\right]^{-1}H^TWY\right]\cdot\left[\theta-\left[H^TWH\right]^{-1}H^TWY\right]^T\right\}$$

$$=E\left\{\left[-\left[H^TWH\right]^{-1}H^TWN\right]\cdot\left[-\left[H^TWH\right]^{-1}H^TWN\right]^T\right\}$$

$$=\left[H^TWH\right]^{-1}H^TWR_NWH\left[H^TWH\right]^{-1}$$

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均方误差阵最小

- 当$W=R_N^{-1}$时，则LSW估计的误差矩阵

$$E\left\{\left[\theta-\hat{\theta}_{LSW}\right]\left[\theta-\hat{\theta}_{LSW}\right]^T\right\}$$

$$=\left[H^TWH\right]^{-1}H^TWR_NWH\left[H^TWH\right]^{-1}$$

$$\overset{R_N=D^TD}{=}\left[H^TWH\right]^{-1}H^TWD^TDWH\left[H^TWH\right]^{-1}=B^TB$$

$$\geq\left[AB\right]^T\left[AA^T\right]^{-1}\left[AB\right]=\left[AA^T\right]^{-1}=\left[H^TR_N^{-1}H\right]^{-1}$$

令：$A=H^TD^{-1}$；$B=DWH\left[H^TWH\right]^{-1}$

均方误差阵最小

当$W = R_N^{-1}$时

$$E\left\{\left[\theta-\hat{\theta}_{LSW}\right]\left[\theta-\hat{\theta}_{LSW}\right]^T\right\}$$

$$=\left[H^T R_N^{-1} H\right]^{-1} H^T R_N^{-1} R_N R_N^{-1} H\left[H^T R_N^{-1} H\right]^{-1}$$

$$=H^{-1} R_N\left[H^T\right]^{-1}=\left[H^T R_N^{-1} H\right]^{-1}$$

$$\Rightarrow \hat{\theta}_{LSW}=\left[H^T R_N^{-1} H\right]^{-1} H^T R_N^{-1} Y$$

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观测某个点的匀速直线运动，设观测数据为 $y_k=\theta_0+\theta_1 t_k+n_k$，$k$=1,2...N, 式中$\theta_0$为$t$=0时的初始距离，$\theta_1$为目标的速度，$t_k$为观测时间（已知），$n_k$为随机测量误差。求对$\theta_0$和$\theta_1$的LS估计。

$$Y = H\theta + N$$

$$\theta = \begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}, \quad N = \begin{bmatrix} n_1 \\ \vdots \\ n_N \end{bmatrix}, \quad H = \begin{bmatrix} 1 & t_1 \\ \vdots & \vdots \\ 1 & t_N \end{bmatrix}$$

$$\hat{\theta}_{LS} = \left[H^T H \right]^{-1} H^T Y$$

$$H^T H = N \begin{bmatrix} 1 & \overline{t} \\ \overline{t} & \overline{t^2} \end{bmatrix} \Rightarrow \left[H^T H \right]^{-1} = \frac{1}{N \Delta t} \begin{bmatrix} \overline{t^2} & -\overline{t} \\ -\overline{t} & 1 \end{bmatrix}$$

$$\text{其中} \begin{cases} \overline{t} = \dfrac{1}{N} \sum\limits_{k=1}^{N} t_k, \\ \overline{t^2} = \dfrac{1}{N} \sum\limits_{k=1}^{N} t_k^2, \\ \Delta t = \dfrac{1}{N} \sum\limits_{k=1}^{N} \left[\overline{t^2} - \left(\overline{t} \right)^2 \right] \end{cases}$$

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$$\hat{\theta}=\left[\hat{\theta}_0,\hat{\theta}_1\right]^T=\frac{1}{N\Delta t}\begin{bmatrix}\overline{t^2} & -\bar{t}\\ -\bar{t} & 1\end{bmatrix}\begin{bmatrix}1 & \cdots & 1\\ t_1 & \cdots & t_N\end{bmatrix}\begin{bmatrix}y_1\\ \vdots\\ y_N\end{bmatrix}$$

$$=\frac{1}{\Delta t}\begin{bmatrix}\bar{Y}\,\overline{t^2}-\bar{t}\,\overline{Yt}\\ \overline{Yt}-\bar{Y}\bar{t}\end{bmatrix}$$

其中 $\begin{cases}\bar{y}=\dfrac{1}{N}\displaystyle\sum_{k=1}^{N}y_k\\ \overline{Yt}=\dfrac{1}{N}\displaystyle\sum_{k=1}^{N}y_k t_k\end{cases}\Rightarrow\begin{cases}\hat{\theta}_0=\dfrac{\bar{Y}\,\overline{t^2}-\bar{t}\,\overline{Yt}}{\Delta t}\\ \hat{\theta}_1=\dfrac{\overline{Yt}-\bar{Y}\bar{t}}{\Delta t}\end{cases}$

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线性场景

线性观测方程：Y=Hθ+N

$$y_i = \sum_{l=1}^{L} h_{il}\theta_l + n_i, i = 1,\ldots,k$$

$$H = \begin{bmatrix} h_{11} & \cdots & h_{1L} \\ \vdots & \ddots & \vdots \\ h_{k1} & \cdots & h_{kL} \end{bmatrix}$$

$$Y = \left[y_1,\ldots,y_k\right]^T$$

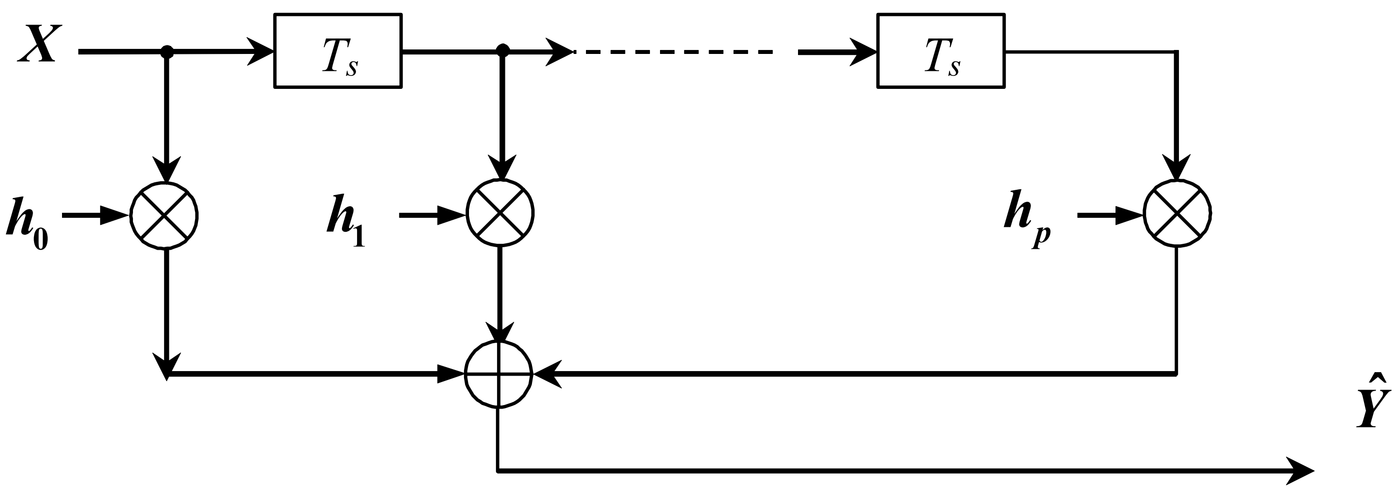
$$\theta = \left[\theta_1,\ldots,\theta_L\right]^T$$

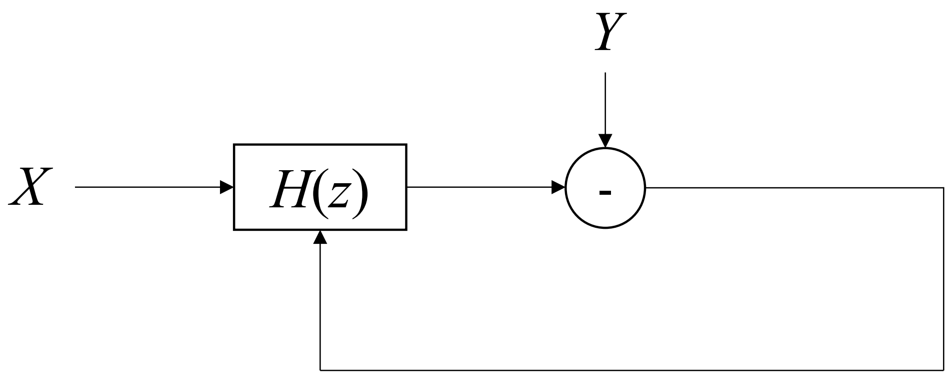
最优滤波器设计

$$y(n)=\left[x(n)x(n-1)\cdots x(n-p)\right]\begin{bmatrix} h(0) \\ h(1) \\ \vdots \\ h(p) \end{bmatrix}$$

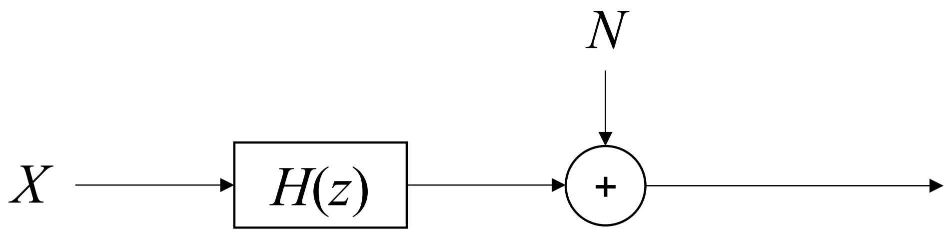
$$\Rightarrow E=D-\begin{bmatrix} x(1) & x(0) & \cdots & x(1-p) \\ x(2) & x(1) & \cdots & x(2-p) \\ \vdots & \vdots & \cdots & \vdots \\ x(N) & x(N-1) & \cdots & x(n-p) \end{bmatrix}\cdot\begin{bmatrix} h(0) \\ h(1) \\ \vdots \\ h(p) \end{bmatrix}$$

最优滤波器设计





线性系统辨识



$$\hat{y}_i = \left[x(i)x(i-1)\cdots x(i-q)\right]\begin{bmatrix}\hat{h}_0\\ \hat{h}_1\\ \vdots\\ \hat{h}_q\end{bmatrix}$$

$$\Rightarrow E = Y - \begin{bmatrix} x(1) & x(0) & \cdots & x(1-q)\\ x(2) & x(1) & \cdots & x(2-q)\\ \vdots & \vdots & \cdots & \vdots\\ x(N) & x(N-1) & \cdots & x(n-q)\end{bmatrix}\cdot\begin{bmatrix}\hat{h}_0\\ \hat{h}_1\\ \vdots\\ \hat{h}_q\end{bmatrix}$$

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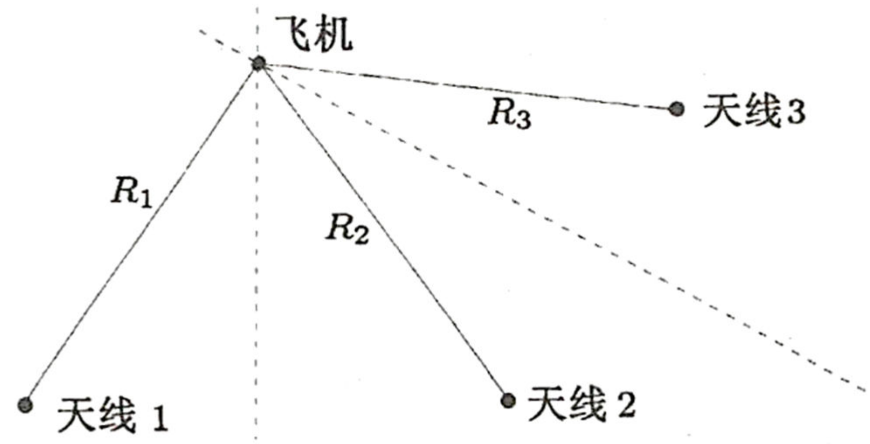
GPS

到达时间=发送时间+传输时延+噪声

多天线

测量值:到达时间
待估参量:位置坐标



GPS

到达时间

$$l_i = \frac{1}{c}\sqrt{\left(x_i - x_0 - \Delta x\right)^2 + \left(y_i - y_0 - \Delta y\right)^2 + \left(z_i - z_0 - \Delta z\right)^2} + t_i + \Delta t + n_i$$

$$\left(Taylor\right)$$

$$\approx \frac{d_i}{c} - \frac{x_i - x_0}{d_i c}\Delta x - \frac{y_i - y_0}{d_i c}\Delta y - \frac{z_i - z_0}{d_i c}\Delta z + t_i + \Delta t + n_i$$

其中$d_i = \frac{1}{c}\sqrt{\left(x_i - x_0\right)^2 + \left(y_i - y_0\right)^2 + \left(z_i - z_0\right)^2}$

$$m_i = l_i - \frac{d_i}{c} - t_i = -\frac{x_i - x_0}{d_i c}\Delta x - \frac{y_i - y_0}{d_i c}\Delta y - \frac{z_i - z_0}{d_i c}\Delta z + \Delta t + n_i$$

$i = 1,2,3,4$（四天线）

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GPS

$$H = \begin{bmatrix} -\dfrac{x_1 - x_0}{d_1 c} & -\dfrac{y_1 - y_0}{d_1 c} & -\dfrac{z_1 - z_0}{d_1 c} & 1 \\ -\dfrac{x_2 - x_0}{d_2 c} & -\dfrac{y_2 - y_0}{d_2 c} & -\dfrac{z_2 - z_0}{d_2 c} & 1 \\ -\dfrac{x_3 - x_0}{d_3 c} & -\dfrac{y_3 - y_0}{d_3 c} & -\dfrac{z_3 - z_0}{d_3 c} & 1 \\ -\dfrac{x_4 - x_0}{d_4 c} & -\dfrac{y_4 - y_0}{d_4 c} & -\dfrac{z_4 - z_0}{d_4 c} & 1 \end{bmatrix}$$

$$M = \left[m_1, \ldots, m_4\right]^T$$

$$\theta = \left[\Delta x, \Delta y, \Delta z, \Delta t\right]^T$$

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无偏性

- **非随机量的无偏性**

$$E\left\{\hat{\theta}\right\}=\int_{-\infty}^{\infty}\hat{\theta}f\left(y\setminus\theta\right)dy=\theta+b\left(\hat{\theta}\right)\underset{b\left(\hat{\theta}\right)=0}{=}\theta$$

- **随机量的无偏性**

$$E\left\{\hat{\theta}\right\}=E\left\{\theta\right\}$$

- **渐近无偏性能指标**

$$\lim_{m\to\infty}E\left\{\hat{\theta}_m\right\}=\theta$$

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有效性

- **任意两个无偏估计量**

$$E\left\{\left(\theta-\hat{\theta}_1\right)^2\right\} < E\left\{\left(\theta-\hat{\theta}_2\right)^2\right\}, \text{则}\hat{\theta}_1\text{比}\hat{\theta}_2\text{有效}$$

- **均方误差下界**

  Cramer-Rao不等式

- **若无偏估计量的均方误差达到Cramer-Rao界则为优效估计量**

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渐进无偏估计量的有效性

$$M^2\left(\hat{\theta}\right)=E\left\{\left(\hat{\theta}-\theta\right)^2\right\}$$

$$=E\left\{\left[\hat{\theta}-E\left(\hat{\theta}\right)+E\left(\hat{\theta}\right)-\theta\right]^2\right\}$$

$$=E\left\{\left[\hat{\theta}-E\left(\hat{\theta}\right)\right]^2\right\}+E\left\{\left[E\left(\hat{\theta}\right)-\theta\right]^2\right\}+2E\left\{\left[\hat{\theta}-E\left(\hat{\theta}\right)\right]\left[E\left(\hat{\theta}\right)-\theta\right]\right\}$$

$$=var\left(\hat{\theta}\right)+b^2\left(\hat{\theta}\right)$$

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一致性

- 一致估计量

$$\lim_{m\to\infty} P\left[\theta-\varepsilon<\hat{\theta}_m<\theta+\varepsilon\right]=1$$

$$\lim_{m\to\infty} P\left[\left|\theta-\hat{\theta}_m\right|>\varepsilon\right]=0$$

- 均方一致估计量

$$\lim_{m\to\infty} E\left\{\left(\theta-\hat{\theta}_m\right)^2\right\}=0$$

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充分性

- Fisher分解：对于t=$T(Y)$，可分解为

  $f(Y;\theta)=g(T(Y),\theta)h(Y)$，$h(Y)\geq 0$

  则t是θ的充分统计量。

- 有效统计量必然是充分统计量。

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Fisher信息

- 品质函数

$$V(Y) = \frac{\partial}{\partial\theta} ln\, f(Y \setminus \theta) = \frac{\frac{\partial}{\partial\theta} f(Y \setminus \theta)}{f(Y \setminus \theta)}$$

- Fisher信息函数

$$J\{\theta\} = E\left\{\left[\frac{\partial}{\partial\theta} ln\, f(Y \setminus \theta)\right]^2\right\}$$

$$= -E\left\{\frac{\partial^2}{\partial\theta^2} ln\, f(Y \setminus \theta)\right\}$$

非随机标量的Cramer-Rao不等式

$$var\left\{\hat{\theta}\right\} \geq \frac{1}{-E\left\{\frac{\partial^2}{\partial\theta^2} ln\, f\left(Y \backslash \theta\right)\right\}}$$

$$var\left\{\hat{\theta}\right\} \geq \frac{1}{E\left\{\left[\frac{\partial}{\partial\theta} ln\, f\left(Y \backslash \theta\right)\right]^2\right\}}$$

$$\frac{\partial}{\partial\theta} ln\, f\left(Y \backslash \theta\right) = \left(\theta - \hat{\theta}\right) k\left(\theta\right)$$时等号成立

非随机标量的Cramer-Rao不等式

$$\frac{\partial}{\partial\theta}E\left(\hat{\theta}-\theta\right)=\frac{\partial}{\partial\theta}\int_{-\infty}^{\infty}\left(\hat{\theta}-\theta\right)f\left(Y\setminus\theta\right)dY$$

$$=\int_{-\infty}^{\infty}\frac{\partial}{\partial\theta}\left[\left(\hat{\theta}-\theta\right)f\left(Y\setminus\theta\right)\right]dY=0$$

$$\Rightarrow -\int_{-\infty}^{\infty}f\left(Y\setminus\theta\right)dY+\left(\hat{\theta}-\theta\right)\int_{-\infty}^{\infty}\frac{\partial}{\partial\theta}f\left(Y\setminus\theta\right)dY=0$$

$$\frac{\partial}{\partial\theta}f\left(Y\setminus\theta\right)=\left[\frac{\partial}{\partial\theta}\ln f\left(Y\setminus\theta\right)\right]f\left(Y\setminus\theta\right)$$

$$\Rightarrow$$

$$\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial\theta}\ln f\left(Y\setminus\theta\right)\right]f\left(Y\setminus\theta\right)\left(\hat{\theta}-\theta\right)dY=1$$

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非随机标量的Cramer-Rao不等式

$$\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial\theta}\ln f\left(Y\backslash\theta\right)\sqrt{f\left(Y\backslash\theta\right)}\right]\left(\hat{\theta}-\theta\right)\sqrt{f\left(Y\backslash\theta\right)}dY=1$$

由柯西-许瓦兹不等式

$$\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial\theta}\ln f\left(Y\backslash\theta\right)\right]^{2}f\left(Y\backslash\theta\right)dY\int_{-\infty}^{\infty}\left(\hat{\theta}-\theta\right)^{2}f\left(Y\backslash\theta\right)dY\geq 1$$

$$\int_{-\infty}^{\infty}\left(\hat{\theta}-\theta\right)^{2}f\left(Y\backslash\theta\right)dY\geq\frac{1}{\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial\theta}\ln f\left(Y\backslash\theta\right)\right]^{2}f\left(Y\backslash\theta\right)dY}$$

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随机标量

$$var\left\{\hat{\theta}\right\} \geq \frac{1}{-E\left\{\frac{\partial^2}{\partial\theta^2} ln\, f\left(Y,\theta\right)\right\}}$$

$$var\left\{\hat{\theta}\right\} \geq \frac{1}{E\left\{\left[\frac{\partial}{\partial\theta} ln\, f\left(Y,\theta\right)\right]^2\right\}}$$

$\frac{\partial}{\partial\theta} ln\, f\left(Y,\ \theta\right) = \left(\theta - \hat{\theta}\right)k$时等号成立

非随机矢量

$$var\left\{\hat{\theta}_m\right\} \geq \varphi_{mm};\psi = J^{-1}\left(Fisher\right)$$

$$j_{mn} = -E\left\{\frac{\partial\, ln\, f\left(Y \backslash \vec{\theta}\right)}{\partial\theta_m}\frac{\partial\, ln\, f\left(Y \backslash \vec{\theta}\right)}{\partial\theta_n}\right\}$$

$$= --E\left\{\frac{\partial^2\, ln\, f\left(Y \backslash \vec{\theta}\right)}{\partial\theta_m\partial\theta_n}\right\}, m, n = 1, 2, \cdots, N$$

$$\frac{\partial}{\partial\vec{\theta}} ln\, f\left(Y \backslash \vec{\theta}\right) = -J\left(\vec{\theta} - \hat{\vec{\theta}}\right)$$时等号成立

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随机矢量

$$E\left\{\vec{\theta}_\varepsilon \vec{\theta}_\varepsilon^T\right\} \geq J^{-1}$$

$$j_{mn} = -E\left\{\frac{\partial^2 \ln f\left(Y \backslash \vec{\theta}\right)}{\partial \theta_m \partial \theta_n}\right\} - E\left\{\frac{\partial^2 \ln f\left(\vec{\theta}\right)}{\partial \theta_m \partial \theta_n}\right\}$$

$$m, n = 1, 2, \cdots, N$$

$$\frac{\partial}{\partial \vec{\theta}} \ln f\left(Y, \vec{\theta}\right) = -J\left(\vec{\theta} - \hat{\vec{\theta}}\right)$$时等号成立

ML估计例

$$f\left(\vec{Y}\backslash m\right)=\left(\prod_{k=1}^{N}\frac{1}{\sqrt{2\pi}\sigma_k}\right)exp\left\{\sum_{k=1}^{N}-\frac{1}{2\sigma_k^2}\left(y_k-m\right)^2\right\}$$

$$ML\text{方程}:\frac{\partial}{\partial m}ln\, f\left(\vec{Y}\backslash m\right)\Big|_{m=\hat{m}_{ML}}=0,\ \text{即}\sum_{k=1}^{N}\frac{1}{\sigma_k^2}\left(y_k-m\right)\Big|_{m=\hat{m}_{ML}}=0$$

$$\Rightarrow \hat{m}_{ML}=\frac{\sum_{k=1}^{N}\frac{y_k}{\sigma_k^2}}{\sum_{k=1}^{N}\frac{1}{\sigma_k^2}}\xlongequal{\text{方差相等}}\frac{1}{N}\sum_{k=1}^{N}y_k$$

考察估计量的性质。

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- 无偏估计量

$$E\left(\hat{m}_{ML}\right)=E\left(\frac{1}{N}\sum_{k=1}^{N}y_k\right)=\frac{1}{N}\sum_{k=1}^{N}E\left(m+n_k\right)=m$$

- 充分统计量

$$\sum_{i=1}^{N}\left(y_i-m\right)^2=N\left[m^2-2\overline{y}m+\frac{1}{N}\sum_{i=1}^{N}y_i^2\right]$$

$$=N\left[m^2-2\overline{y}m+\overline{y}^2+\frac{1}{N}\sum_{i=1}^{N}y_i^2-\overline{y}^2\right]$$

$$f\left(\vec{Y}\backslash m\right)=\left(\frac{N}{\sqrt{2\pi\sigma_n^2}}\right)^{1/2}\exp\left\{-\frac{N}{2\sigma_n^2}\sum_{k=1}^{N}\left(\hat{m}_{ML}-m\right)^2\right\}$$

$$\cdot\left(\frac{1}{\sqrt{2\pi\sigma_n^2}}\right)^{\frac{N-1}{2}}\frac{1}{N^{\frac{1}{2}}}\exp\left\{-\frac{N}{2\sigma_n^2}\left[\frac{1}{N}\sum_{i=1}^{N}y_i^2-\left(\frac{1}{N}\sum_{i=1}^{N}y_i\right)^2\right]\right\}$$

- **Cramer-Rao界**

$$f\left(\vec{Y}\backslash m\right)=\left(\frac{1}{\sqrt{2\pi}\sigma_n}\right)^N exp\left\{\sum_{k=1}^{N}-\frac{1}{2\sigma_n^2}\left(y_k-m\right)^2\right\}$$

$$var\left\{\widehat{m}_{ML}\right\}=E\left\{\left(m-\widehat{m}_{ML}\right)^2\right\}=\frac{1}{-E\left\{\dfrac{\partial^2 \ln f\left(\vec{Y}\backslash m\right)}{\partial m^2}\right\}}$$

$$=\frac{1}{-E\left(-\dfrac{N}{\sigma_n^2}\right)}=\frac{\sigma_n^2}{N}$$

- 有效估计量

$$\frac{\partial \ln f\left(\vec{Y}\backslash m\right)}{\partial m}=\frac{1}{\sigma_n^2}\sum_{k=1}^{N}\left(y_k - m\right)=\left(m-\frac{1}{N}\sum_{k=1}^{N}y_k\right)\left(-\frac{N}{\sigma_n^2}\right)$$

$$=\left(m-\hat{m}_{ML}\right)k\left(m\right)$$

- 一致性

$$\lim_{N\to\infty} P\left[\left|m - \widehat{m}_N\right| > \varepsilon\right] = \lim_{N\to\infty} P\left[\left|m - \frac{1}{N}\sum_{k=1}^{N} y_k\right| > \varepsilon\right]$$

$$= \lim_{N\to\infty} P\left[\left|m - \frac{1}{N}\sum_{k=1}^{N}\left(m+n_k\right)\right| > \varepsilon\right]$$

$$= \lim_{N\to\infty} P\left[\left|\frac{1}{N}\sum_{k=1}^{N} n_k\right| > \varepsilon\right] = 0$$

$$\lim_{N\to\infty} E\left\{\left(m-\hat{m}_N\right)^2\right\} = \lim_{N\to\infty}\frac{\sigma_n^2}{N} = 0$$

summary

- 最小二乘法无需统计先验知识
- 以噪声二阶矩作为权重因子的LSW误差矩阵最小
- 性能评价：无偏、有效、一致、充分
- CRLB

Ref： §5.5& §5.9(赵版)、第3章&第8章 (KAY版)

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