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Lecture 4

统计检测理论Ⅰ

理想检验

- 单样本
- 二元假设:是或否
- 简单： 确知信号+噪声概率分布已知

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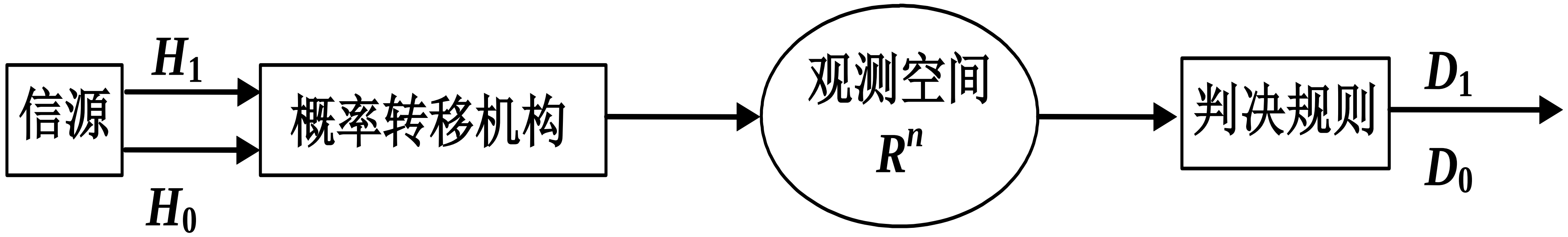
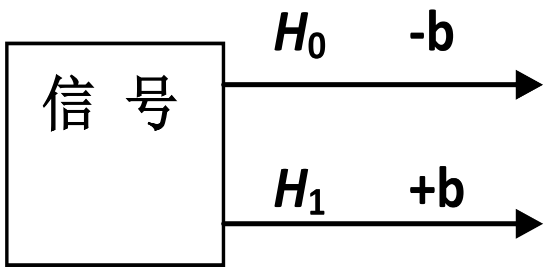
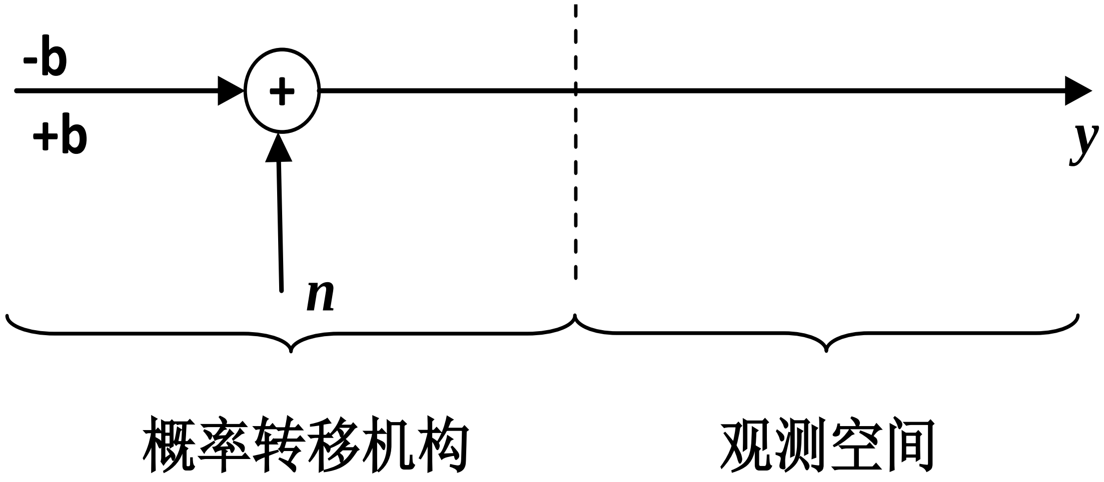
信号

- 统计判决
- 假设检验



二元数字通信，双极性不归零码，单样本，加性高斯白噪声均值为0，方差为σ_n^2。求判决准则和判决概率。





① 信源：

H_0： y=-b+n

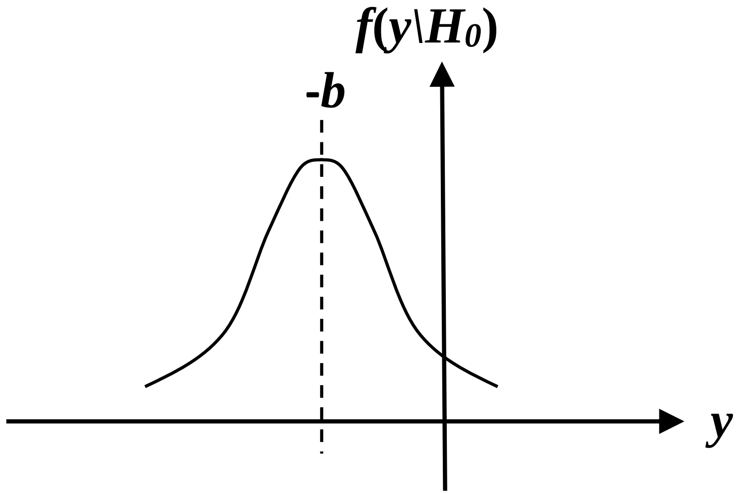
H_1： y=b+n

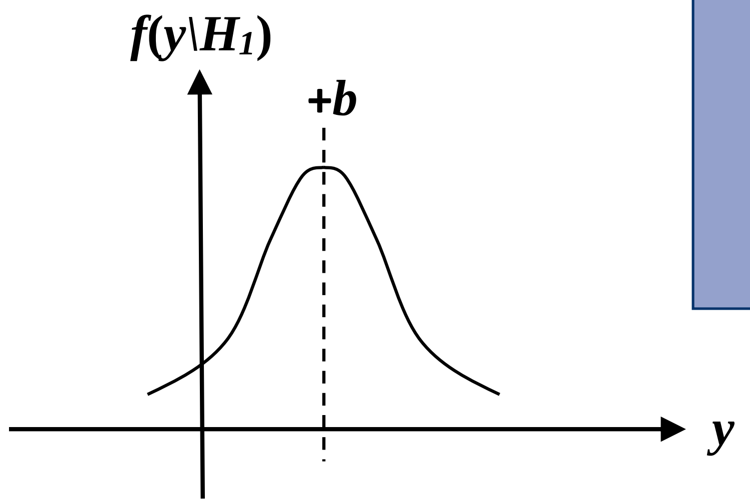
② 概率转移机构

似然函数

$$f\left(y \backslash H_0\right)=\frac{1}{\sqrt{2\pi\sigma_n^2}}exp\left\{-\frac{\left(y+b\right)^2}{2\sigma_n^2}\right\}$$

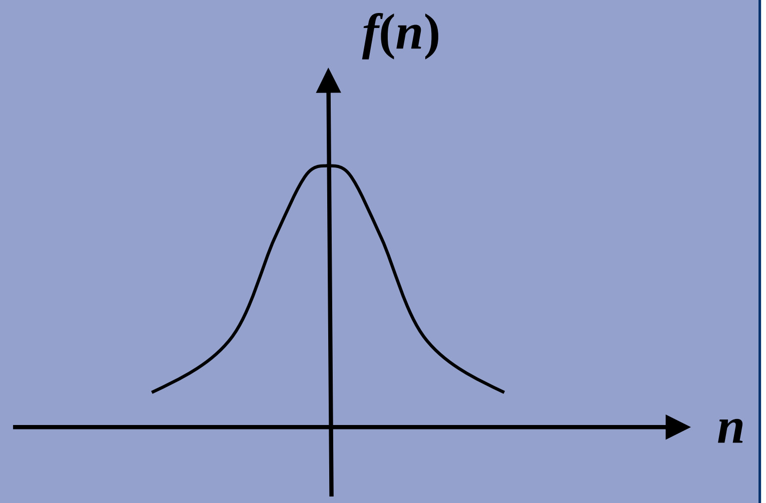
$$f\left(y \backslash H_1\right)=\frac{1}{\sqrt{2\pi\sigma_n^2}}exp\left\{-\frac{\left(y-b\right)^2}{2\sigma_n^2}\right\}$$





加性高斯噪声n

$$f\left(n\right)=\frac{1}{\sqrt{2\pi\sigma_n^2}}exp\left\{-\frac{n^2}{2\sigma_n^2}\right\}$$



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③ 按照某种“最佳”准则？？

$y \geq \tau_{th}$，判定H_1为真（D_1）$\Rightarrow y \in R_1$

$y < \tau_{th}$，判定H_0为真（D_0）$\Rightarrow y \in R_0$

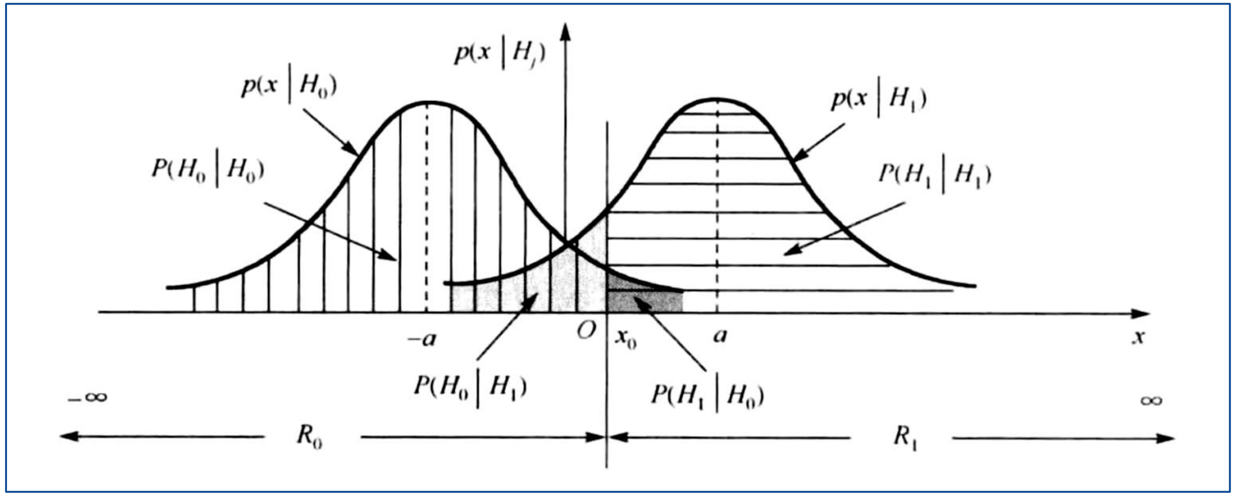
④ 检测性能（判决概率） $P_{ij} = \int_{R_i} f\left(y \backslash H_j\right) dy$

虚警概率 $\alpha = P(D_1 \backslash H_0) = P_{fa}$

漏警概率 $\beta = P(D_0 \backslash H_1) = P_m$

发现概率 $P(D_1 \backslash H_1) = P_d$

误码率 $P_e = P(H_0)\alpha + P(H_1)\beta$



统计处理方法

- 统计描述信号随机特性
  - 概率密度函数
  - 高阶矩（如相关函数、协方差函数）
  - 功率谱密度
- 准则为统计意义下“最佳”
- 性能评价为统计平均量

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系统模型

- **接收信号（观测值）**：$y=s_i+n \quad i=0,1$

用假设H_i表示发送相应信号s_i的场景

- **代价因子**

$C_{ij}>C_{jj}$

判 决	假 设(H_0)	假 设(H_1)
D_0	C_{00}	C_{01}
D_1	C_{10}	C_{11}

- **似然函数**：H_i情景下获得y观测的概率

平均风险

$$\overline{C} = P(H_0)\left[C_{00}P(D_0 \backslash H_0) + C_{10}P(D_1 \backslash H_0)\right] + P(H_1)\left[C_{01}P(D_0 \backslash H_1) + C_{11}P(D_1 \backslash H_1)\right]$$

$$P_{ij} = \int_{R_i} f\left(y \backslash H_j\right) dy$$

$$\overline{C} = P(H_0)\left[C_{00}\int_{R_0} f(y \backslash H_0)dy + C_{10}\int_{R_1} f(y \backslash H_0)dy\right]$$

$$+P(H_1)\left[C_{01}\int_{R_0} f(y \backslash H_1)dy + C_{11}\int_{R_1} f(y \backslash H_1)dy\right]$$

- 完备集 $R = R_0 \cup R_1; \quad R_0 \cap R_1 = \varnothing$

$$P(D_0 \backslash H_0) + P(D_1 \backslash H_0) = 1$$

$$P(D_0 \backslash H_1) + P(D_1 \backslash H_1) = 1$$

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平均风险

$$\bar{C} = P(H_0)C_{10} + P(H_1)C_{11}$$

$$+P(H_0)\left[C_{00}P(D_0 \backslash H_0) - C_{10}P(D_0 \backslash H_0)\right] + P(H_1)\left[C_{01}P(D_0 \backslash H_1) - C_{11}P(D_0 \backslash H_1)\right]$$

$$\Rightarrow$$

$$\bar{C} = P(H_0)C_{10} + P(H_1)C_{11}$$

$$+\int_{R_0} [P(H_1)(C_{01} - C_{11}) f(y \backslash H_1) - P(H_0)(C_{10} - C_{00}) f(y \backslash H_0)] dy$$

$$\Rightarrow$$

$$R_0 = \{y : P(H_1)(C_{01} - C_{11}) f(y \backslash H_1) - P(H_0)(C_{10} - C_{00}) f(y \backslash H_0) < 0\}$$

Bayes平均风险最小准则

判决准则：

$y \in R_0$，判定H_0；$y \in R_1$，判定H_1。

$$\frac{f\left(y \backslash H_1\right)}{f\left(y \backslash H_0\right)} \geq \frac{P\left(H_0\right)\left(C_{10}-C_{00}\right)}{P\left(H_1\right)\left(C_{01}-C_{11}\right)}，\text{判}H_1;$$

$$\frac{f\left(y \backslash H_1\right)}{f\left(y \backslash H_0\right)} < \frac{P\left(H_0\right)\left(C_{10}-C_{00}\right)}{P\left(H_1\right)\left(C_{01}-C_{11}\right)}，\text{判}H_0\text{。}$$

最小平均错误概率准则

- **Minimum Mean Probability of Error**
- $C_{00}=C_{11}=0$；$C_{01}=C_{10}=1$
  这时平均风险退化成了什么？？
- **由Bayes，判决准则等效为：**

$$\frac{f(y\backslash H_1)}{f(y\backslash H_0)} \geq \frac{P(H_0)}{P(H_1)} = \frac{P(H_0)}{1\text{-}P(H_0)} = \frac{1\text{-}P(H_1)}{P(H_1)}，\text{判为}H_1;$$

$$\frac{f(y\backslash H_1)}{f(y\backslash H_0)} < \frac{P(H_0)}{P(H_1)} = \frac{P(H_0)}{1\text{-}P(H_0)} = \frac{1\text{-}P(H_1)}{P(H_1)}，\text{判为}H_0。$$

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最大后验准则 (MAP)

- **Maximum A Posteriori**
- **后验概率**：获得y条件下假设为真的概率

$$P\left(H_i \backslash y\right)=\frac{P\left(H_i\right) f\left(y \backslash H_i\right)}{f\left(y\right)}$$

- **由最小平均错误，判决准则等效为：**

$$\frac{P\left(H_1 \backslash y\right)}{P\left(H_0 \backslash y\right)} \geq 1，\text{判为}H_1；\frac{P\left(H_1 \backslash y\right)}{P\left(H_0 \backslash y\right)} < 1，\text{判为}H_0。$$

最大似然准则(ML)

- **Maximum Likelihood**
- **先验概率：**

$$P(H_1)=P(H_0)=\frac{1}{2}$$

- **判决准则为：**

$$\frac{f(y\backslash H_1)}{f(y\backslash H_0)}\ge\frac{P(H_0)}{P(H_1)}=1,\text{ 判为}H_1\text{；否则，判为}H_0\text{。}$$

判决准则

- 似然比 $$L(y)=\frac{f\left(y\backslash H_1\right)}{f\left(y\backslash H_0\right)}$$

- 判决区域 $$\begin{cases} L(y)\geq\lambda_{th}\rightarrow y\in R_1 \\ L(y)<\lambda_{th}\rightarrow y\in R_0 \end{cases} \Leftrightarrow \begin{cases} y\geq th\rightarrow y\in R_1 \\ y<th\rightarrow y\in R_0 \end{cases}$$

- 判决概率：

$$P_{ij}=\int_{R_i} f\left(y\backslash H_j\right)dy$$

二元数字通信，双极性不归零码，单样本，加性高斯白噪声均值为0，方差为σ_n^2。求判决准则和判决概率。

解：

H_0：$y=-b+n$
$$f\left(y\backslash H_0\right)=\frac{1}{\sqrt{2\pi\sigma_n^2}}exp\left\{-\frac{\left(y+b\right)^2}{2\sigma_n^2}\right\}$$

H_1：$y=b+n$
$$f\left(y\backslash H_1\right)=\frac{1}{\sqrt{2\pi\sigma_n^2}}exp\left\{-\frac{\left(y-b\right)^2}{2\sigma_n^2}\right\}$$

$$L\left(y\right)=\frac{f\left(y\backslash H_1\right)}{f\left(y\backslash H_0\right)}=exp\left\{\frac{\left(y+b\right)^2}{2\sigma_n^2}-\frac{\left(y-b\right)^2}{2\sigma_n^2}\right\}=exp\left\{\frac{2yb}{\sigma_n^2}\right\}$$

若：$P(H_1)=0.2; P(H_0)=0.8;\quad \sigma_n^2=1; b=1$

C_{00} =C_{11} =0； C_{01}=C_{10}=1

由最小平均错误概率准则

$$L(y)=exp\left\{\frac{2yb}{\sigma_n^2}\right\}\geq 4\text{判}H_1 \Leftrightarrow y\geq\frac{\sigma_n^2}{2b}ln4(\approx 0.69)\text{判}H_1$$

各判决概率？？？

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极大极小准则

设$P(H_0)=Q$

$$
\begin{aligned}
\overline{C} &= P(H_0)\left[C_{00}P(D_0 \backslash H_0)+C_{10}P(D_1 \backslash H_0)\right] \\
&\quad +P(H_1)\left[C_{01}P(D_0 \backslash H_1)+C_{11}P(D_1 \backslash H_1)\right] \\
&= Q\left\{C_{00}\left[1-P(D_1 \backslash H_0)\right]+C_{10}P(D_1 \backslash H_0)\right\} \\
&\quad +(1-Q)\left\{C_{01}P(D_0 \backslash H_1)+C_{11}\left[1-P(D_0 \backslash H_1)\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
\overline{C}_{min} &= Q\left\{C_{00}\left[1-\alpha(Q)\right]+C_{10}\alpha(Q)\right\} \\
&\quad +(1-Q)\left\{C_{01}\beta(Q)+C_{11}\left[1-\beta(Q)\right]\right\}
\end{aligned}
$$

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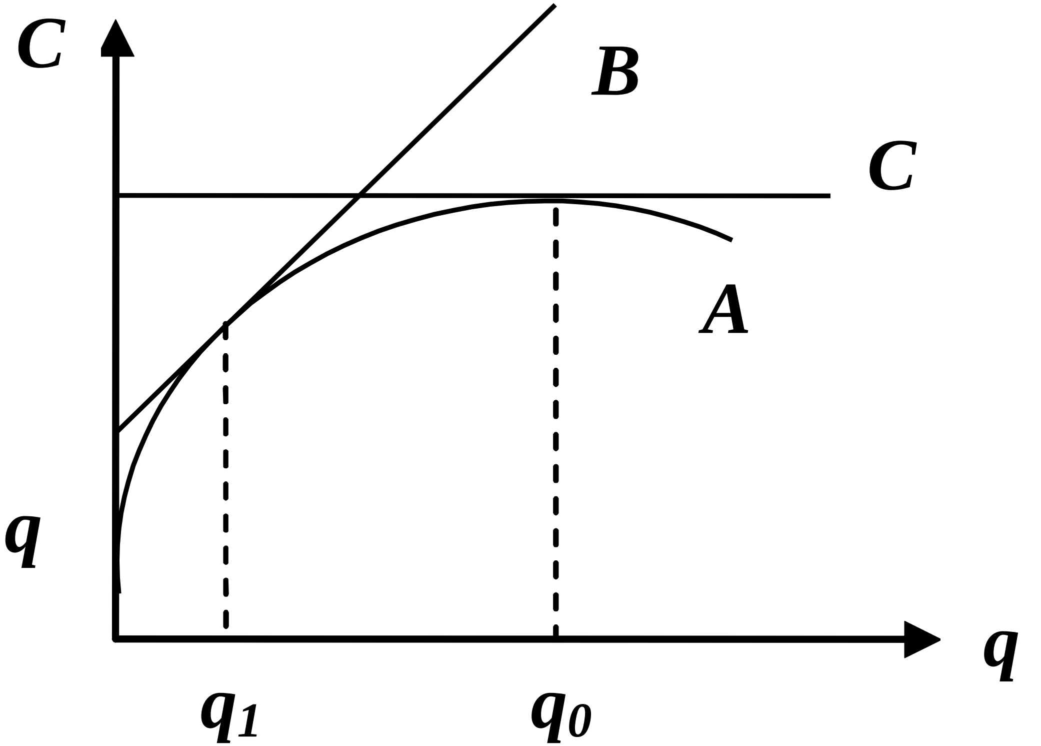
极大极小准则

- C_{ij}已知，Q未知，平均代价是Q的上凸曲线
- 预估$Q=q_1$,实际为q



$$\begin{aligned}\overline{C}(q,q_1) &= q\left\{C_{00}\left[1-\alpha(q_1)\right]+C_{10}\alpha(q_1)\right\} \\ &\quad +(1-q)\left\{C_{01}\beta(q_1)+C_{11}\left[1-\beta(q_1)\right]\right\} \\ &= C_{00}q+C_{11}(1-q)+(C_{10}-C_{00})\alpha(q_1)q \\ &\quad +(C_{01}-C_{11})\beta(q_1)(1-q)\end{aligned}$$

$$\overline{C}_{\min} = C_{00}q+C_{11}(1-q)+(C_{10}-C_{00})\alpha(q)q+(C_{01}-C_{11})\beta(q)(1-q)$$

极大极小准则

B直线可能出现最大平均风险($q=1$时)

C直线，切点q_0

$$\left.\frac{\partial \overline{C}(q,q_1)}{\partial q}\right|_{q_1=q_0}$$

$$=C_{00}-C_{11}+(C_{10}-C_{00})\alpha(q)-(C_{01}-C_{11})\beta(q)(1-q)=0$$

$$\Leftrightarrow C_{10}\alpha(q_0)+C_{00}\left[1-\alpha(q_0)\right]=C_{01}\beta(q_0)+C_{11}\left[1-\beta(q_0)\right]$$

二元通信系统，使用归一化的单极性不归零码，AGWN信道（方差为1、均值为0），正确检测无代价，错误检测代价为1，求判决准则。

解：

由$C_{00}=C_{11}=0$；$C_{01}=C_{10}=1$

极大极小方程

$$C_{10}\alpha(q_0)+C_{00}\left[1-\alpha(q_0)\right]=C_{01}\beta(q_0)+C_{11}\left[1-\beta(q_0)\right]$$

转换为

$$\alpha(q_0)=\beta(q_0)$$

$$L(y) \geq \frac{P(H_0)}{P(H_1)} = \frac{P(H_0)}{1\text{-}P(H_0)}$$判为H_1；otherwise判为H_0。

$$f(y \backslash H_1) = \frac{1}{\sqrt{2\pi}} exp\left\{-\frac{(y-1)^2}{2}\right\}$$

$$f(y \backslash H_0) = \frac{1}{\sqrt{2\pi}} exp\left\{-\frac{y^2}{2}\right\}$$

判决规则

$$L(y) = exp\left\{y - \frac{1}{2}\right\} \geq \frac{P(H_0)}{1\text{-}P(H_0)}$$判为H_1；

otherwise判为H_0。

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等效判决规则

$$y \geq \frac{1}{2} + ln\left[\frac{P(H_0)}{1\text{-}P(H_0)}\right]$$判为H_1；otherwise判为H_0。

$$\int_{th'}^{\infty} \frac{1}{\sqrt{2\pi}} exp\left\{-\frac{y^2}{2}\right\} dy = \int_{-\infty}^{th'} \frac{1}{\sqrt{2\pi}} exp\left\{-\frac{(y-1)^2}{2}\right\} dy$$

$$= \int_{-\infty}^{th'-1} \frac{1}{\sqrt{2\pi}} exp\left\{-\frac{y^2}{2}\right\} dy$$

$$th' = \frac{1}{2}\left(\Rightarrow P(H_0) = \frac{1}{2}\right)$$

判决：$y \geq \frac{1}{2}$则H_1为真，否则H_0为真

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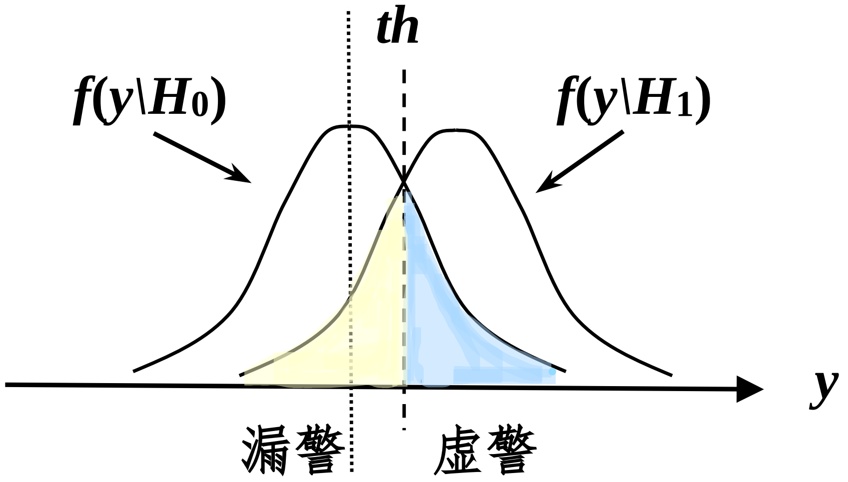
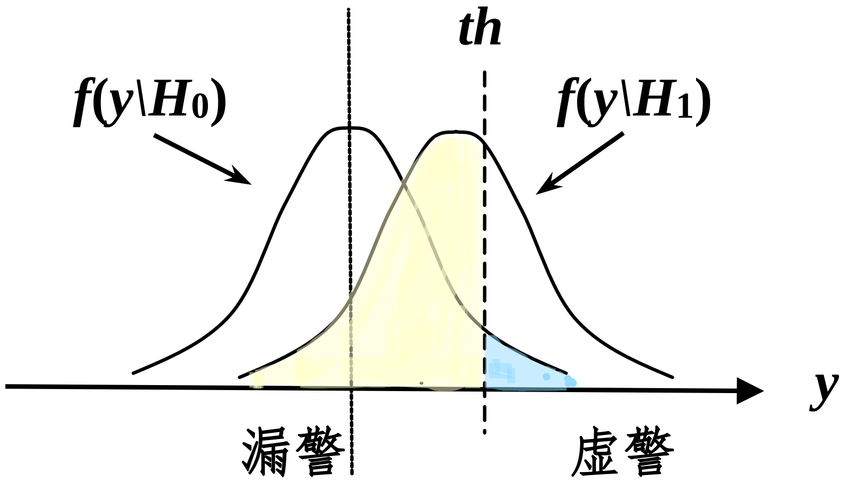
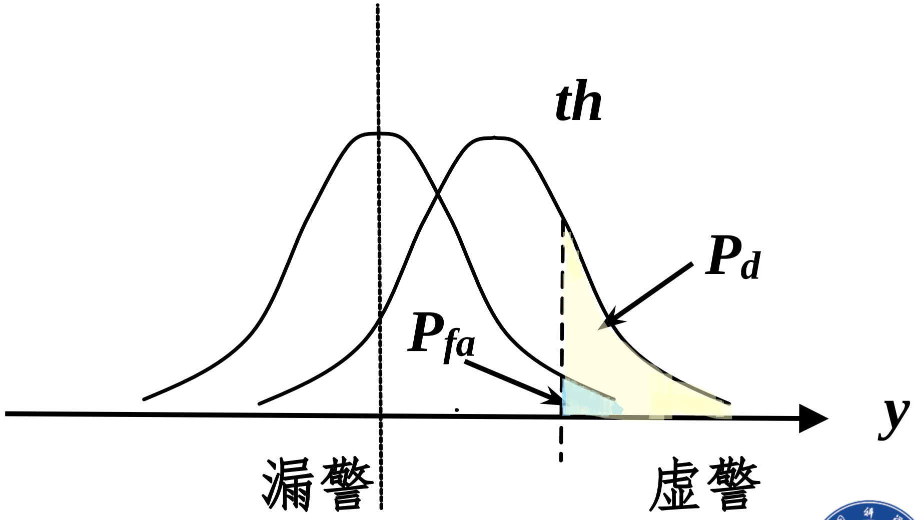
概率耦合







Neyman-Pearson准则

- 在虚警概率$P_{fa}=\alpha$条件下，发现概率P_d最大
- Langrange乘子$\lambda(\geq 0)$构建目标函数

$$L(R_0)=\int_{R_0} f(y\backslash H_1)dy+\lambda\left[\int_{R_1} f(y\backslash H_0)dY-\alpha\right]$$

$$=\lambda(1-\alpha)+\int_{R_0}\left[f(y\backslash H_1)-\lambda f(y\backslash H_0)\right]dy$$

$\dfrac{f(y\backslash H_1)}{f(y\backslash H_0)}\geq\lambda$，判为$H_1$；否则，判为$H_0$。

满足$P_{fa}=\int_{R_1} f(y\backslash H_0)dy=\int_{\lambda}^{\infty} f(L\backslash H_0)dL=\alpha$

二元通信系统，使用归一化的单极性不归零码，AGWN信道（方差为1、均值为0），若虚警概率保证0.1，构造接收机。

判决规则

$$L(y)=exp\left\{y-\frac{1}{2}\right\}\geq\lambda$$,判为H_1；otherwise判为H_0。

等效判决规则

$$y\geq\frac{1}{2}+ln\lambda$$ 判为H_1；otherwise判为H_0。

$$P_{fa}=\int_{\lambda'}^{\infty}\frac{1}{\sqrt{2\pi}}exp\left\{-\frac{y^2}{2}\right\}dy=0.1\Rightarrow\lambda'=1.29$$

$$\lambda=exp\left(\lambda'-\frac{1}{2}\right)=2.2$$

判决：$y\geq 1.29$则H_1为真，否则H_0为真

发现概率$P_d=\int_{th'}^{\infty}\frac{1}{\sqrt{2\pi}}exp\left\{-\frac{(y-1)^2}{2}\right\}dy=0.614$

summary

- ✓ 假设检验方法：假设→概率映射→测量样本：统计判决
- ✓ Bayes平均风险最小准则： “最佳”--代价统计平均值最小
  - 最小平均错误概率准则：代价因子归一化
  - 最大后验概率准则：等效于最小平均错误概率准则
  - 最大似然准则：归一化代价+等概率先验
- ✓ 极大极小准则： “最佳”—代价恒定
- ✓ 最小平均错误概率准则： “最佳”—漏警概率最小

Ref：§3.1-§3.4(赵版)or §3.1-§3.7(KAY版)

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Q

多样本时，似然函数怎么构成？

对于NP准则和极大极小准则，离散分布数据如何解方程？

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