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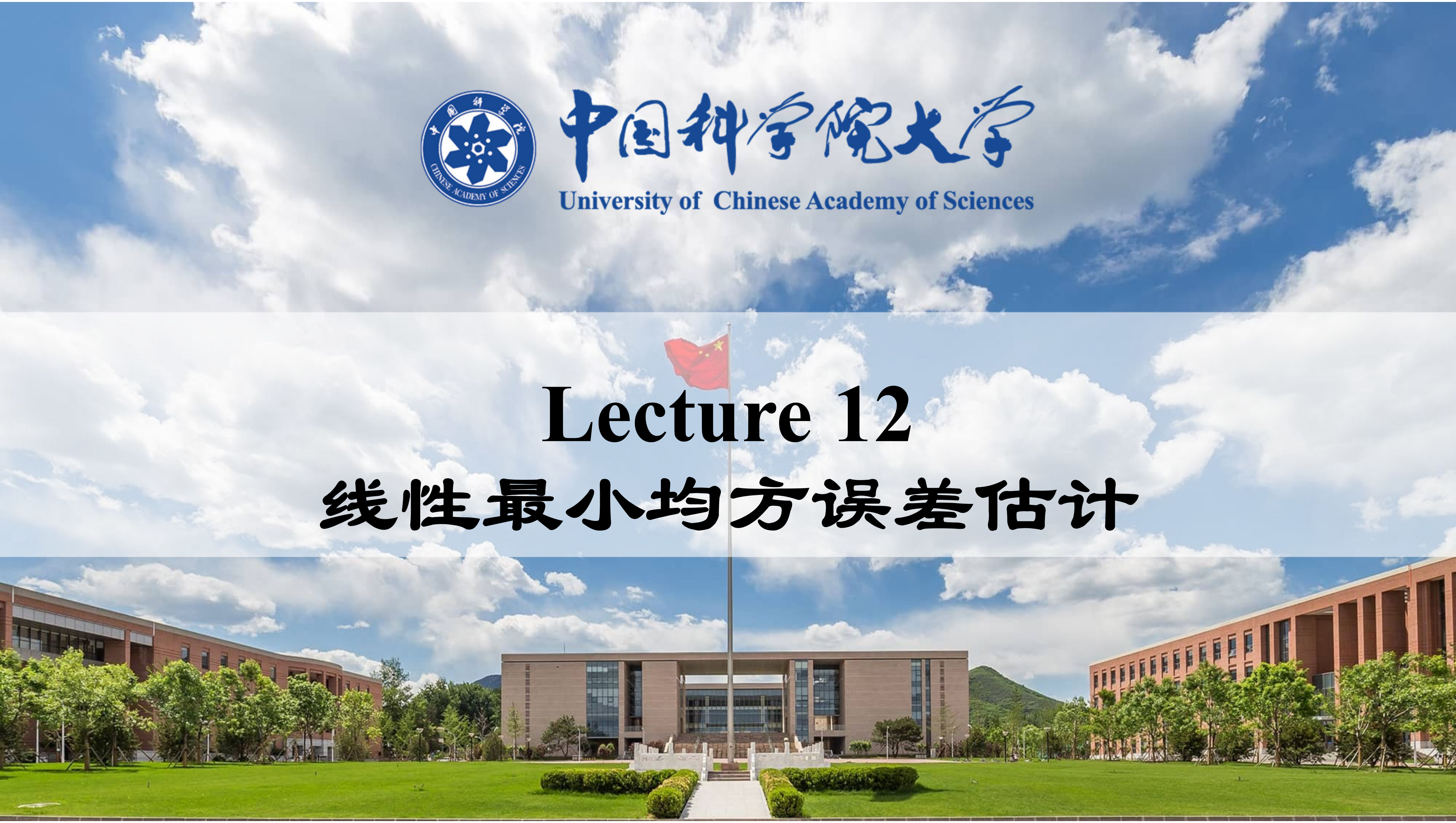
Lecture 12

线性最小均方误差估计



LECTURE11

- **随机参量，已知PDF：BAYES估计**
  - 均匀代价函数：MAP $$\frac{\partial}{\partial\theta}\left[\ln f\left(Y\setminus\theta\right)+\ln f\left(\theta\right)\right]\Big|_{\theta=\hat{\theta}_{\mathrm{MAP}}}=0$$
  - 平方代价函数：MMSE $$\hat{\theta}_{MS}\left(Y\right)=\int\theta f\left(\theta\setminus Y\right)d\theta$$
  - 绝对值代价函数：MED $$\int_{-\infty}^{\hat{\theta}}f\left(\theta\setminus Y\right)d\theta=\int_{\hat{\theta}}^{\infty}f\left(\theta\setminus Y\right)d\theta$$
- **非随机参量：ML估计** $$\frac{\partial}{\partial\theta}\left[lnf\left(Y\setminus\theta\right)\right]\Big|_{\theta=\hat{\theta}_{ML}}=0$$
- **估计性能评价：一阶矩、二阶矩**

估计背景

- 平稳过程
- 线性估计方程
- 待估参数与观测噪声无关
- PDF?

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目录
Contents

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目录
Contents

一般形式

- **Linear Mean Square**
- **线性估计方程**

$$\hat{\theta}_{LMS} = A + BY$$

- **均方误差最小**

$$E\left\{e^2\left(\theta,\hat{\theta}\right)\right\} = E\left\{\left[\theta - A - BY\right]^T\left[\theta - A - BY\right]\right\}$$

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial A}E\left\{e^2\left(\theta,\hat{\theta}\right)\right\} = E\left\{-2\left(\theta - A - BY\right)\right\} = 0 \\ \dfrac{\partial}{\partial B}E\left\{e^2\left(\theta,\hat{\theta}\right)\right\} = E\left\{-2\left(\theta - A - BY\right)Y^T\right\} = 0 \end{cases}$$

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一般形式

$$\begin{cases} A_L = E\{\theta\} - B_L E\{Y\} \\ E\{\theta Y^T\} = A_L E\{Y^T\} + B_L E\{Y \cdot Y^T\} \end{cases}$$

$$\Rightarrow E\{\theta Y^T\} = E\{\theta\} E\{Y^T\} - B_L E\{Y\} E\{Y^T\} + B_L E\{Y \cdot Y^T\}$$

$$\Rightarrow B_L E\{Y \cdot Y^T\} - B_L E\{Y\} E\{Y^T\} = E\{\theta Y^T\} - E\{\theta\} E\{Y^T\}$$

$$\Rightarrow B_L \left\{E\left\{\left[Y\text{-}E\left(Y\right)\right] \cdot \left[Y\text{-}E\left(Y\right)\right]^T\right\}\right\} = E\left\{\left[\theta\text{-}E\left(\theta\right)\right] \cdot \left[Y\text{-}E\left(Y\right)\right]^T\right\}$$

$$\Rightarrow \begin{cases} B_L = cov\left(\theta, Y\right) cov\left(Y, Y\right)^{-1} \\ A_L = E\{\theta\} - cov\left(\theta, Y\right) cov\left(Y, Y\right)^{-1} E\{Y\} \end{cases}$$

$$\hat{\theta}_{LMS} = A_L + B_L Y = E\{\theta\} + \mathrm{cov}\left(\theta, Y\right) \mathrm{cov}\left(Y, Y\right)^{-1} \left[Y - E\{Y\}\right]$$

特殊形式

$$\hat{\theta}_{LMS} = BY$$

$$E\left\{e^2\left(\theta,\hat{\theta}\right)\right\} = E\left\{\left[\theta - BY\right]^T\left[\theta - BY\right]\right\}$$

$$\Rightarrow \frac{\partial}{\partial B} E\left\{e^2\left(\theta,\hat{\theta}\right)\right\} = E\left\{-2\left(\theta - BY\right)Y^T\right\} = 0$$

$$B_L = R_{\theta Y} R_Y^{-1}$$

$$LMS: \hat{\theta}_{LMS} = B_L Y = R_{\theta Y} R_Y^{-1} Y$$

正交性

由系数B_L的求取过程：

$$\hat{\theta}_{LMS} = A_L + B_L Y:$$

$$E\left\{\left(\theta - \hat{\theta}_{LMS}\right)Y^T\right\} = E\left\{e\left(\theta, \hat{\theta}_{LMS}\right)Y^T\right\} = E\left\{\left(\theta - A - BY\right)Y^T\right\} = 0$$

$$\hat{\theta}_{LMS} = B_L Y:$$

$$E\left\{\left(\theta - \hat{\theta}_{LMS}\right)Y^T\right\} = E\left\{e\left(\theta, \hat{\theta}_{LMS}\right)Y^T\right\} = E\left\{\left(\theta - BY\right)Y^T\right\} = 0$$

估计误差与观测值之间正交

Q：矢量空间?

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均方误差阵最小

$$\hat{\theta} = A + BY$$

令$C = A - E\{\theta\} + BE\{Y\}$

$$E\left\{\left[\theta - \hat{\theta}\right]\left[\theta - \hat{\theta}\right]^T\right\} = E\left\{[\theta - A - BY][\theta - A - BY]^T\right\}$$

$$= E\left\{\left[\theta - E\{\theta\} - C\text{-}B\left(Y\text{-}E\{Y\}\right)\right]\left[\theta - E\{\theta\} - C\text{-}B\left(Y\text{-}E\{Y\}\right)\right]^T\right\}$$

$$= cov(\theta,\theta) + CC^T + B\left[cov(Y,Y)\right]B^T - \left[cov(\theta,Y)\right]B^T - B\left[cov(Y,\theta)\right]$$

补项$\text{cov}(\theta,Y)\left[\text{cov}(Y,Y)\right]^{-1}\text{cov}(Y,\theta)$+后三项

$$\rightarrow \left[B\,\text{cov}(Y,Y) - \text{cov}(\theta,Y)\right]\left\{B - \text{cov}(\theta,Y)\left[\text{cov}(Y,Y)\right]^{-1}\right\}^T$$

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均方误差阵最小

$$=CC^T+\operatorname{cov}(\theta,\theta)-\operatorname{cov}(\theta,Y)\left[\operatorname{cov}(Y,Y)\right]^{-1}\operatorname{cov}(Y,\theta)$$

$$+\left[B\operatorname{cov}(Y,Y)-\operatorname{cov}(\theta,Y)\right]\left\{B-\operatorname{cov}(\theta,Y)\left[\operatorname{cov}(Y,Y)\right]^{-1}\right\}^T$$

$$=CC^T$$

$$+\left[B-\operatorname{cov}(\theta,Y)\operatorname{cov}^{-1}(Y,Y)\right]\operatorname{cov}(Y,Y)\cdot\left[B-\operatorname{cov}(\theta,Y)\operatorname{cov}^{-1}(Y,Y)\right]^T$$

$$+\operatorname{cov}(\theta,\theta)-\operatorname{cov}(\theta,Y)\left[\operatorname{cov}(Y,Y)\right]^{-1}\operatorname{cov}(Y,\theta)$$

$$\geq\operatorname{cov}(\theta,\theta)-\operatorname{cov}(\theta,Y)\left[\operatorname{cov}(Y,Y)\right]^{-1}\operatorname{cov}(Y,\theta)$$

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均方误差阵最小

$$E\left\{\left[\theta-\hat{\theta}_{LMS}\right]\cdot\left[\theta-\hat{\theta}_{LMS}\right]^{T}\right\}$$

$$=E\{\left[\theta-E\{\theta\}-cov(\theta,Y)cov(Y,Y)^{-1}\left[Y\text{-}E\{Y\}\right]\right]$$

$$\cdot\left[\theta-E\{\theta\}-cov(\theta,Y)cov(Y,Y)^{-1}\left[Y\text{-}E\{Y\}\right]\right]^{T}\}$$

$$=cov(\theta,\theta)-cov(\theta,Y)\left[cov(Y,Y)\right]^{-1}cov(Y,\theta)$$

$$\Rightarrow E\left\{\left[\theta-\hat{\theta}\right]\cdot\left[\theta-\hat{\theta}\right]^{T}\right\}\geq E\left\{\left[\theta-\hat{\theta}_{LMS}\right]\cdot\left[\theta-\hat{\theta}_{LMS}\right]^{T}\right\}$$

无偏性

$$E\left\{\hat{\theta}_{LMS}\right\}=E\left\{\theta\right\}+\mathrm{cov}\left(\theta,Y\right)\mathrm{cov}\left(Y,Y\right)^{-1}\left[E\left\{Y\right\}-E\left\{Y\right\}\right]$$

$$=E\left\{\theta\right\}$$

$$=\theta_0$$

BLUE：最佳线性无偏估计

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目录
Contents

线性场景

线性观测方程：Y=Hθ+N

$$E\{\theta\}=\theta_0$$

$$cov(\theta,\theta)=E\{[\theta-\theta_0][\theta-\theta_0]^T\}=C_\theta$$

$$E\{N\}=0$$

$$cov\{N,N\}=E\{N\cdot N^T\}=R_N$$

$$E\{\theta\cdot N^T\}=0$$

$$E\{Y\}=HE\{\theta\}=H\theta_0$$

系数计算

$$cov\left(Y,Y\right)=E\left\{\left[Y\text{-}E\left\{Y\right\}\right]\left[Y\text{-}E\left\{Y\right\}\right]^{T}\right\}$$

$$=E\left\{\left[H\left(\theta-\theta_{0}\right)+N\right]\left[H\left(\theta-\theta_{0}\right)+N\right]^{T}\right\}=HC_{\theta}H^{T}+R_{N}$$

$$cov\left(\theta,Y\right)=E\left\{\left[\theta-\theta_{0}\right]\left[Y\text{-}E\left\{Y\right\}\right]^{T}\right\}$$

$$=E\left\{\left[\theta-\theta_{0}\right]\left[H\left(\theta-\theta_{0}\right)+N\right]^{T}\right\}=C_{\theta}H^{T}$$

$$\Rightarrow\begin{cases}A_{L}=E\left\{\theta\right\}\text{-}cov\left(\theta,Y\right)cov\left(Y,Y\right)^{-1}E\left\{Y\right\}\\ \quad =\theta_{0}-C_{\theta}H^{T}\left[HC_{\theta}H^{T}+R_{N}\right]^{-1}H\theta_{0}\\ B_{L}=cov\left(\theta,Y\right)cov\left(Y,Y\right)^{-1}=C_{\theta}H^{T}\left[HC_{\theta}H^{T}+R_{N}\right]^{-1}\end{cases}$$

估计方程

$$
\begin{aligned}
&\hat{\theta}_{LMS} \\
&= A_L + B_L Y \\
&= E\{\theta\} + cov(\theta, Y)\,cov(Y, Y)^{-1}\left[Y - E\{Y\}\right] \\
&= \theta_0 - C_\theta H^T\left[H C_\theta H^T + R_N\right]^{-1} H\theta_0 \\
&\qquad + C_\theta H^T\left[H C_\theta H^T + R_N\right]^{-1} Y \\
&= \theta_0 + C_\theta H^T\left[H C_\theta H^T + R_N\right]^{-1}\left[Y - H\theta_0\right]
\end{aligned}
$$

误差矩阵

$$
\begin{aligned}
cov(Y,\theta) &= E\left\{\left[Y - E\{Y\}\right]\left[\theta-\theta_0\right]^T\right\} \\
&= E\left\{\left[H(\theta-\theta_0)+N\right](\theta-\theta_0)^T\right\} \\
&= HC_\theta
\end{aligned}
$$

$$
\begin{aligned}
& E\left\{\left[\theta-\hat{\theta}_{LMS}\right]\left[\theta-\hat{\theta}_{LMS}\right]^T\right\} \\
&= cov(\theta,\theta) - cov(\theta,Y)\left[cov(Y,Y)\right]^{-1} cov(Y,\theta) \\
&= C_\theta - C_\theta H^T\left[HC_\theta H^T + R_N\right]^{-1} HC_\theta
\end{aligned}
$$

傅里叶分析。数据模型表示为$y_k=a\cos k\omega_0+b\sin k\omega_0+n_k$，$k=1,2...N$, f_0为$(1/N)$的倍数。a和b**为待估**参量，均值为零，方差σ_θ^2，高斯分布；$\{n_k\}$为独立于信号的均值零、方差σ_n^2的高斯噪声。**N次独立观测。MMSE估计。**

$$Y=H\theta+N$$

$$H=\begin{bmatrix} \cos\omega_0 & \sin\omega_0 \\ \cos 2\omega_0 & \sin 2\omega_0 \\ \vdots & \vdots \\ \cos N\omega_0 & \sin N\omega_0 \end{bmatrix}$$

$$\theta=\begin{bmatrix} a & b \end{bmatrix}^T$$

$$E\{\theta\}=0$$

$$C_\theta=\sigma_\theta^2 I$$

$$C_n=\sigma_n^2 I$$

$$\hat{\theta} = E\{\theta \setminus Y\} = E\{\theta\} + C_{\theta Y} C_{YY}^{-1} \left(Y - E\{Y\} \right)$$

$$C_{YY} = E\left\{ \left[Y\text{-}E\{Y\} \right] \left[Y\text{-}E\{Y\} \right]^T \right\}$$

$$= E\left\{ \left[H\theta + N \right] \left[H\theta + N \right]^T \right\} = HC_\theta H^T + C_n = \sigma_\theta^2 HH^T + C_n$$

$$= \left(\frac{N\sigma_\theta^2}{2} + \sigma_n^2 \right) I$$

$$C_{\theta Y} = E\left\{ \left[\theta \right] \left[Y\text{-}E\{Y\} \right]^T \right\}$$

$$= E\left\{ \left[\theta \right] \left[H\theta + N \right]^T \right\} = C_\theta H^T = \sigma_\theta^2 H^T$$

$$\hat{\theta} = C_{\theta Y} C_{YY}^{-1} Y$$

$$= \begin{bmatrix} \cos\omega_0 & \cos 2\omega_0 & \cdots & \cos N\omega_0 \\ \sin\omega_0 & \sin 2\omega_0 & \cdots & \sin N\omega_0 \end{bmatrix} \cdot$$

$$\begin{bmatrix} \dfrac{\sigma_\theta^2}{\dfrac{N\sigma_\theta^2}{2} + \sigma_n^2} & \cdots & 0 \\ \vdots & \dfrac{\sigma_\theta^2}{\dfrac{N\sigma_\theta^2}{2} + \sigma_n^2} & \vdots \\ 0 & \cdots & \dfrac{\sigma_\theta^2}{\dfrac{N\sigma_\theta^2}{2} + \sigma_n^2} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} = \frac{\dfrac{2}{N}}{1 + \dfrac{2\sigma_n^2 / N}{\sigma_\theta^2}} \begin{bmatrix} \sum\limits_{k=1}^{N} y_k \cos k\omega_0 \\ \sum\limits_{k=1}^{N} y_k \sin k\omega_0 \end{bmatrix}$$

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目录
Contents

理想观测

观测样本 $Y=\left[y_1,y_2,\ldots y_m\right]^T$

观测方程 $y_i=\theta+n_i, i=1,2,\ldots m$

观测噪声 n_i: $E\left\{n_i\right\}=0; E\left\{n_i n_j\right\}=\sigma_n^2\delta_{ij}$

待估参量（随机标量）θ: $E\left\{\theta\right\}=\theta_0; Var\left\{\theta\right\}=\sigma_\theta^2$

$$
\begin{aligned}
&cov(\theta,Y)\\
&=E\{(\theta-\theta_0)[y_1-E\{y_1\},y_2-E\{y_2\},\cdots,y_m-E\{y_m\}]\}\\
&=E\{(\theta-\theta_0)[(\theta-\theta_0)+n_1,(\theta-\theta_0)+n_2,\cdots,(\theta-\theta_0)+n_m]\}\\
&=\left[\sigma_\theta^2,\sigma_\theta^2,\cdots,\sigma_\theta^2\right]=P_\theta
\end{aligned}
$$

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系数计算

$$cov(Y,Y) = E\{[Y - E\{Y\}][Y - E\{Y\}]^T\}$$

$$= E\left\{\begin{bmatrix} y_1 - \theta_0 \\ \vdots \\ y_m - \theta_0 \end{bmatrix}[y_1 - \theta_0, \cdots y_m - \theta_0]\right\} = \begin{bmatrix} \sigma_\theta^2 + \sigma_n^2 & \sigma_\theta^2 & \cdots & \sigma_\theta^2 \\ \sigma_\theta^2 & \sigma_\theta^2 + \sigma_n^2 & \cdots & \sigma_\theta^2 \\ \vdots & \cdots & \cdots & \vdots \\ \sigma_\theta^2 & \sigma_\theta^2 & \cdots & \sigma_\theta^2 + \sigma_n^2 \end{bmatrix}$$

$$= G$$

令$cov(\theta, Y)[cov(Y,Y)]^{-1} = P_\theta G^{-1} = K = [k_1, k_2, \cdots, k_m]$

$$\Rightarrow \begin{bmatrix} \sigma_\theta^2 + \sigma_n^2 & \sigma_\theta^2 & \cdots & \sigma_\theta^2 \\ \sigma_\theta^2 & \sigma_\theta^2 + \sigma_n^2 & \cdots & \sigma_\theta^2 \\ \vdots & \cdots & \cdots & \vdots \\ \sigma_\theta^2 & \sigma_\theta^2 & \cdots & \sigma_\theta^2 + \sigma_n^2 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{bmatrix} = \begin{bmatrix} \sigma_\theta^2 \\ \sigma_\theta^2 \\ \vdots \\ \sigma_\theta^2 \end{bmatrix}$$

估计方程

$$\Rightarrow k_i = \frac{\sigma_\theta^2}{m\sigma_\theta^2 + \sigma_n^2} = \frac{1}{m+b}$$

其中$b = \sigma_n^2 / \sigma_\theta^2$

$$\hat{\theta}_{LMS} = E\{\theta\} + \mathrm{cov}(\theta, Y)\,\mathrm{cov}(Y,Y)^{-1}\left[Y - E\{Y\}\right]$$

$$= \theta_0 + K\begin{bmatrix} y_1 - \theta_0 \\ \vdots \\ y_m - \theta_0 \end{bmatrix} = \theta_0 + \frac{1}{m+b}\sum_{i=1}^{m}(y_i - \theta_0)$$

$$E\left\{e^2\left(\theta, \hat{\theta}_{LMS}\right)\right\} = \mathrm{cov}(\theta,\theta) - \mathrm{cov}(\theta, Y)\left[\mathrm{cov}(Y,Y)\right]^{-1}\mathrm{cov}(Y,\theta)$$

$$= \frac{b}{m+b}\sigma_\theta^2$$

线性观测

$$y_i = h_i\theta + n_i$$

令$K = \left[k_1, k_2, \cdots, k_m\right] = cov\left(\theta, Y\right)\left[cov\left(Y, Y\right)\right]^{-1}$

$$cov\left(\theta, Y\right)$$

$$= E\left\{\left(\theta - \theta_0\right)\left[Y - E\left\{Y\right\}\right]^T\right\}$$

$$= E\left\{\left(\theta - \theta_0\right)\left[h_1\left(\theta - \theta_0\right) + n_1, h_2\left(\theta - \theta_0\right) + n_2, \cdots, h_m\left(\theta - \theta_0\right) + n_m\right]\right\}$$

$$= \left[h_1\sigma_\theta^2, h_2\sigma_\theta^2, \cdots, h_m\sigma_\theta^2\right]$$

$$= P_\theta$$

线性观测

$$cov(Y,Y)=E\left\{[Y-E\{Y\}][Y-E\{Y\}]^T\right\}$$

$$=E\left\{\begin{bmatrix} h_1(\theta-\theta_0)+n_1 \\ \vdots \\ h_m(\theta-\theta_0)+n_m \end{bmatrix}[h_1(\theta-\theta_0)+n_1,\cdots,h_m(\theta-\theta_0)+n_m]\right\}$$

$$=\begin{bmatrix} h_1^2\sigma_\theta^2+\sigma_n^2 & h_1h_2\sigma_\theta^2 & \cdots & h_1h_m\sigma_\theta^2 \\ h_1h_2\sigma_\theta^2 & h_2^2\sigma_\theta^2+\sigma_n^2 & \cdots & h_2h_2\sigma_\theta^2 \\ \vdots & \cdots & \cdots & \vdots \\ h_1h_m\sigma_\theta^2 & h_2h_m\sigma_\theta^2 & \cdots & h_m^2\sigma_\theta^2+\sigma_n^2 \end{bmatrix}=G$$

线性观测

$$G^T K^T = P_\theta^T \Rightarrow \begin{cases} \left(h_1^2\sigma_\theta^2 + \sigma_n^2\right)k_1 + h_1 h_2 \sigma_\theta^2 k_2 + \cdots h_1 h_m \sigma_\theta^2 k_m = h_1 \sigma_\theta^2 \\ h_1 h_2 \sigma_\theta^2 k_1 + \left(h_2^2\sigma_\theta^2 + \sigma_n^2\right)k_2 + \cdots + h_2 h_2 \sigma_\theta^2 k_m = h_2 \sigma_\theta^2 \\ \vdots \\ h_1 h_m \sigma_\theta^2 k_1 + h_2 h_m \sigma_\theta^2 k_2 + \cdots + \left(h_m^2\sigma_\theta^2 + \sigma_n^2\right)k_m = h_m \sigma_\theta^2 \end{cases}$$

$$\Rightarrow \left[\sum_{i=1}^m h_i^2 + b\right] h_1 k_1 + \left[\sum_{i=1}^m h_i^2 + b\right] h_2 k_2 + \cdots + \left[\sum_{i=1}^m h_i^2 + b\right] h_m k_m = \sum_{i=1}^m h_i^2$$

$$\Rightarrow k_i = k h_i, \left(k = {1} \Big/ {\sum_{i=1}^m h_i^2 + b}, b = {\sigma_n^2} \Big/ {\sigma_\theta^2} \right)$$

线性观测

$$\hat{\theta}_{LMS} = E\{\theta\} + \mathrm{cov}(\theta, Y)\,\mathrm{cov}(Y, Y)^{-1}\left[Y - E\{Y\}\right]$$

$$= \theta_0 + \left[kh_1, \cdots kh_m\right]\begin{bmatrix} y_1 - h_1\theta_0 \\ \vdots \\ y_m - h_m\theta_0 \end{bmatrix} = \theta_0 + k\sum_{i=1}^{m} h_i\left(y_i - h_i\theta_0\right)$$

$$E\left\{e^2\left(\theta, \hat{\theta}_{LMS}\right)\right\} = \mathrm{cov}(\theta, \theta) - \mathrm{cov}(\theta, Y)\left[\mathrm{cov}(Y, Y)\right]^{-1}\mathrm{cov}(Y, \theta)$$

$$= \sigma_\theta^2 - \left[kh_1, \cdots kh_m\right]\begin{bmatrix} h_1\sigma_\theta^2 \\ \vdots \\ h_m\sigma_\theta^2 \end{bmatrix} = \sigma_\theta^2 - k\sum_{i=1}^{m} h_i^2\sigma_\theta^2 = \frac{\sigma_n^2}{\sum_{i=1}^{m} h_i^2 + b}$$

观测某个点的匀速直线运动。已知观测时间间隔△t=1min，$\mathrm{var}(v)=0.3(\mathrm{km/min})^2$，$E(v)=10\mathrm{km/min}$；观测噪声$n_k$，$E(n_k)=0$，$E(n_j n_k)=0.6\delta_{jk}(\mathrm{km/min})^2$，且$E(vn_k)=0$。观测方程$z_k=kv+n_k$，$k=1,2...5$。在获得观测值$z_1=9.8\mathrm{km}$，$z_2=20.4\mathrm{km}$，$z_3=30.6\mathrm{km}$，$z_4=40.2\mathrm{km}$，$z_5=49.7\mathrm{km}$的情况下，求速度$v$的LMS估计。

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解：$\hat{v}_{LMS}(z) = E(v) + \dfrac{1}{\sum\limits_{k=1}^{N} h_k^2 + b} \sum\limits_{k=1}^{N} h_k \left[z_k - h_k E(v) \right]$

其中$E(v) = 10 km/min$；$b = \dfrac{\sigma_n^2}{\sigma_v^2} = \dfrac{0.6}{0.3} = 2$；$h_k = k, k = 1, 2, \ldots 5$

$$\hat{v}_{LMS}(z) = 10 + \frac{1}{55 + 2} \sum_{k=1}^{N} k \left[z_k - 10k \right]$$

$$= 10 \frac{17}{570} km/min \approx 10.03 km/min$$

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目录
Contents

time step:m=0→m=1

$$\hat{\theta}_0 = E\{\theta\} = \theta_0$$

$$\xi_0 = E\left\{(\theta - \theta_0)^2\right\} = \sigma_\theta^2$$

$$\hat{\theta}_1 = \hat{\theta}_0 + k_1 h_1\left(y_1 - h_1\hat{\theta}_0\right) = \theta_0 + k_1 h_1\left(y_1 - h_1\theta_0\right)$$

$$E\left\{\left(\theta - \hat{\theta}_1\right) y_1\right\}$$

$$= E\left\{\left[\theta - \theta_0 - k_1 h_1\left(y_1 - h_1\theta_0\right)\right]\left(h_1\theta + n_1\right)\right\}$$

$$= E\left\{\left[\overline{\theta - \theta_0} - k_1 h_1\left(h_1\overline{\theta - \theta_0} + n_1\right)\right]\left(h_1\overline{\theta - \theta_0} + h_1\theta_0 + n_1\right)\right\}$$

$$= h_1\sigma_\theta^2 - k_1 h_1\left(h_1^2\sigma_\theta^2 + \sigma_n^2\right) = 0$$

$$\Rightarrow k_1 = \frac{\sigma_\theta^2}{h_1^2\sigma_\theta^2 + \sigma_n^2}$$

time step: m=1

$$\boxed{\hat{\theta}_1 = \theta_0 + \frac{1}{h_1^2 + \sigma_n^2 / \sigma_\theta^2} h_1 \left(y_1 - h_1 \theta_0 \right)}$$

$$\xi_1 = E\left\{\left(\theta - \hat{\theta}_1\right)^2\right\}$$

$$= E\left\{\left(\theta - \hat{\theta}_1\right)\left[\theta - \theta_0 - k_1 h_1 \left(y_1 - h_1\theta_0\right)\right]\right\}$$

$$= E\left\{\left(\theta - \hat{\theta}_1\right)\left[\theta - \theta_0 - k_1 h_1^2 \theta_0\right]\right\}$$

$$= E\left\{\left(\theta - \hat{\theta}_1\right)\left(\theta - \theta_0\right)\right\} = E\left\{\left[\theta - \theta_0 - k_1 h_1 \left(y_1 - h_1\theta_0\right)\right]\left(\theta - \theta_0\right)\right\}$$

$$= \sigma_\theta^2 - k_1 h_1^2 \sigma_\theta^2 = k_1 \sigma_n^2 = \sigma_n^2 \frac{1}{h_1^2 + \sigma_n^2 / \sigma_\theta^2}$$

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time step: m=2

$$\boxed{\hat{\theta}_2 = \hat{\theta}_1 + k_2 h_2 \left(y_2 - h_2 \hat{\theta}_1 \right)}$$

$$E\left\{ \left(\theta - \hat{\theta}_2 \right) y_2 \right\}$$

$$= E\left\{ \left[\theta - \hat{\theta}_1 - k_2 h_2 \left(y_2 - h_2 \hat{\theta}_1 \right) \right] \left(h_2 \theta + n_2 \right) \right\}$$

$$= E\{ [\overline{\theta - \theta_0} - k_1 h_1 \left(h_1 \overline{\theta - \theta_0} + n_1 \right) - k_2 h_2 [h_2 \theta + n_2$$

$$- h_2 \left(\theta_0 + k_1 h_1 \overline{h_1 \theta - h_1 \theta_0} + k_1 h_1 n_1 \right)] \cdot \left(h_2 \theta + n_2 \right) \}$$

$$= E\{ [\overline{\theta - \theta_0} - k_1 h_1 \left(h_1 \overline{\theta - \theta_0} + n_1 \right) - k_2 h_2 \left(h_2 \overline{\theta - \theta_0} + n_2 \right)$$

$$+ k_1 k_2 h_1^2 h_2^2 \overline{\theta - \theta_0} - k_1 k_2 h_1 h_2^2 n_1] \cdot \left(h_2 \overline{\theta - \theta_0} + h_2 \theta_0 + n_2 \right) \}$$

time step: m=2

$$= E\{[\overline{\theta-\theta_0} - k_1 h_1^2 \overline{\theta-\theta_0} - k_1 h_1 n_1 - k_2 h_2^2 \overline{\theta-\theta_0} - k_2 h_2 n_2$$

$$+ k_1 k_2 h_1^2 h_2^2 \overline{\theta-\theta_0} - k_1 k_2 h_1 h_2^2 n_1] \cdot \left(h_2 \overline{\theta-\theta_0} + h_2 \theta_0 + n_2 \right)\}$$

$$= h_2 \left[\sigma_\theta^2 - k_1 h_1^2 \sigma_\theta^2 - k_2 h_2^2 \sigma_\theta^2 + k_1 k_2 h_1^2 h_2^2 \sigma_\theta^2 - k_2 \sigma_n^2 \right] = 0$$

$$\Rightarrow \sigma_\theta^2 - k_1 h_1^2 \sigma_\theta^2 - k_2 h_2^2 \sigma_\theta^2 + k_1 k_2 h_1^2 h_2^2 \sigma_\theta^2 - k_2 \sigma_n^2 = 0$$

$$\Rightarrow k_2 = \frac{\sigma_\theta^2 - k_1 h_1^2 \sigma_\theta^2}{h_2^2 \sigma_\theta^2 - k_1 h_1^2 h_2^2 \sigma_\theta^2 + \sigma_n^2} = \frac{k_1}{1 + k_1 h_2^2}$$

$$\xi_2 = \frac{\sigma_n^2}{h_1^2 + h_2^2 + \sigma_n^2 / \sigma_\theta^2} = k_2 \sigma_n^2 = \frac{k_1}{1 + k_1 h_2^2} \sigma_n^2 = \frac{\xi_1}{1 + k_1 h_2^2}$$

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time step: m=i+1

标量形式

$$\hat{\theta}_{i+1} = \hat{\theta}_i + k_{i+1} h_{i+1} \left(y_{i+1} - h_{i+1} \hat{\theta}_i \right)$$

$$k_{i+1} = \frac{k_i}{1 + h_{i+1}^2 k_i}$$

$$\xi_{i+1} = k_{i+1} \sigma_n^2 = \frac{\xi_i}{1 + h_{i+1}^2 k_i} < \xi_i$$

矢量形式

$$\widehat{\theta_{i+1}} = \widehat{\theta}_i + K_{i+1} \left(Y_{i+1} - H_{i+1} \widehat{\theta}_i \right)$$

$$K_{i+1} = P_i H_{i+1}^T \left[H_{i+1} P_i H_{i+1}^T + R_N \right]^{-1}$$

$$P_{i+1} = E\left\{ e\left(\theta, \widehat{\theta_{i+1}}\right) e\left(\theta, \widehat{\theta_{i+1}}\right)^T \right\} = \left[P_i^{-1} + H_{i+1}^T R_N H_{i+1} \right]^{-1}$$

观测某个点的匀速直线运动。已知观测时间间隔△t=1min，var(v)=0.3(km/min)^2，E(v)=10km/min；观测噪声n_k，E(n_k)=0，E($n_j n_k$)=0.6δ_{jk}(km/min)^2，且E($v n_k$)=0。观测方程$z_k=kv+n_k$，k=1,2...5。在获得观测值z_1=9.8km，z_2=20.4km，z_3=30.6km，z_4=40.2km，z_5=49.7km的情况下，求速度v的序贯LMS估计。

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起始估计值$v_0 = E(v) = 10km/min$； $k_0 = \frac{\sigma_v^2}{\sigma_n^2} = \frac{0.3}{0.6} = 0.5$

$$
第一次：\begin{cases}
\hat{v}_1(z) = v_0 + k_1 h_1 (y_1 - h_1 v_0) \\
k_1 = \frac{k_0}{h_1^2 k_0 + 1} = \frac{0.5}{1 + 0.5} = \frac{1}{3} \\
\hat{v}_1(z) = 10 + \frac{1}{3}(9.8 - 10) = 9\frac{14}{15} km/min
\end{cases}
$$

第二次:$\begin{cases} \hat{v}_2(z) = \hat{v}_1 + k_2 h_2 \left(y_2 - h_2 \hat{v}_1 \right) \\ k_2 = \dfrac{k_1}{h_2^2 k_1 + 1} = \dfrac{1/3}{1 + 4 \cdot 1/3} = \dfrac{1}{7} \\ \hat{v}_2(z) = 9\dfrac{14}{15} + \dfrac{1}{7} \times 2 \times \left(20.4 - 2 \times 9\dfrac{14}{15} \right) = 10.087 km/min \end{cases}$

$$
\text{第三次：}\begin{cases}
k_3 = \dfrac{k_2}{h_3^2 k_2 + 1} = \dfrac{1/7}{1 + 9 \cdot 1/7} = \dfrac{1}{16} \\
\hat{v}_3(z) = \hat{v}_2 + k_3 h_3 \left(y_3 - h_3 \hat{v}_2 \right) \\
= 10.087 + \dfrac{1}{16} \times 3 \times \left(30.6 - 3 \times 10.087 \right) = 10.151 km/min
\end{cases}
$$

$$
第四次:\begin{cases}
k_4 = \dfrac{k_3}{h_4^2 k_3 + 1} = \dfrac{1/16}{1+16\cdot 1/16} = \dfrac{1}{32} \\
\hat{v}_4(z) = \hat{v}_3 + k_4 h_4 \left(y_4 - h_4 \hat{v}_3 \right) \\
= 10.151 + \dfrac{1}{32} \times 4 \times \left(40.2 - 4 \times 10.151 \right) = 10.101 km/min
\end{cases}
$$

第五次: $\begin{cases} k_5 = \dfrac{k_4}{h_5^2 k_4 + 1} = \dfrac{1/32}{1 + 25 \cdot 1/32} = \dfrac{1}{57} \\ \hat{v}_5(z) = \hat{v}_4 + k_5 h_5 \left(y_5 - h_5 \hat{v}_4 \right) \\ = 10.101 + \dfrac{1}{57} \times 5 \times (49.7 - 5 \times 10.101) = 10.08 km/min \end{cases}$

summary

- 线性估计则只需一阶矩、二阶矩
- 均方误差最小→估计误差与观测值正交
- 均方误差矩阵最小
- 线性观测下可基于均值、方差和线性系数实现序贯估计

Ref：§5.8(赵版)、第12章 (KAY版)


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