FINA 4130 Empirical Finance

Group Project

Normal-GARCH(1,1) model for stock returns

1. Introduction

Modelling financial time series is a major application in probability theory and statistics. However, the traditional normal assumption $\frac{u_i-\mu}{\sigma} \sim N(0,1)$ has weak prediction power and seriously deviate from reality. Many solutions have been proposed to adjust the normal assumption. One of the challenges particular to this modelling financial time series field is the presence of heteroskedastic effects, meaning that the volatility of the considered process is generally not constant. Our solution is to combine normal assumption and GARCH (1,1) (Generalized Autoregressive Conditional Heteroskedasticity) to model the stock return. GARCH (1,1) model can show great prediction power and model the presence of heteroskedasticity well. We also include some advanced GARCH models in the end of the report to improve our modelling process.

In this part, we will study the following ten stocks: 0001.HK (CK Hutchison), 0002.HK (CLP Group), 0066.HK (MTR), 2333.HK (Great Wall Motor), 0868.HK (Xinyi Glass), 0700.HK (Tencent), 3998.HK (Bosideng), 1368.HK (Xtep), 0135.HK (Kunlun Energy), 0992.HK (Lenovo). And the examination period is from June 2016 to June 2021.

2. Behaviour of the stock return

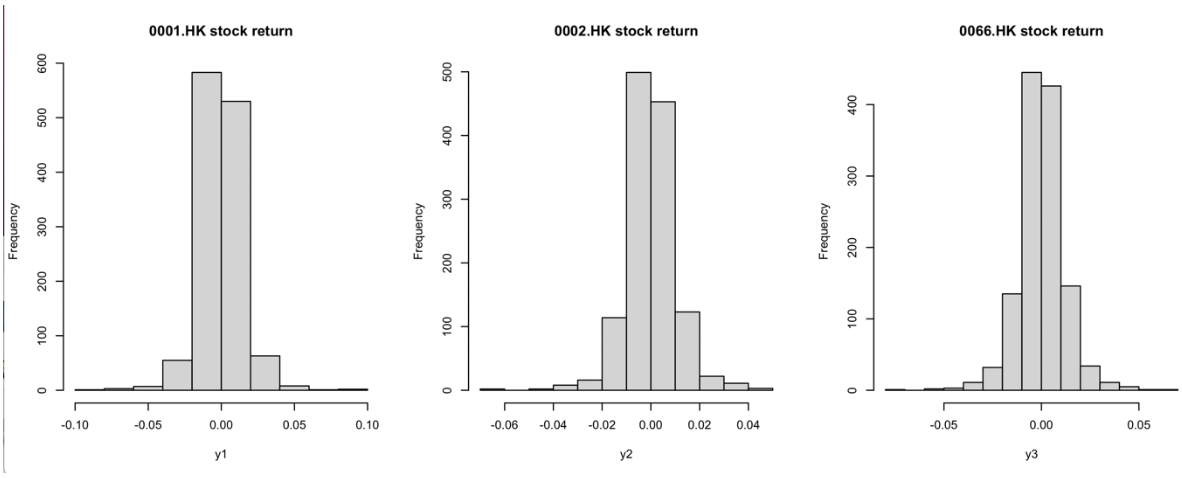
We have tested the behaviour of the stock return using our 10 selected stocks. As the pages are limited, in this part, we demonstrate our progress to test for the behaviour of the stock return of 0001.HK, 0002.HK and 0066.HK using R programming output.

We define that the stock return of 0001.HK as y1, the stock return of 0002.HK as y2 and the stock return of 0066.HK as y3 in this part.

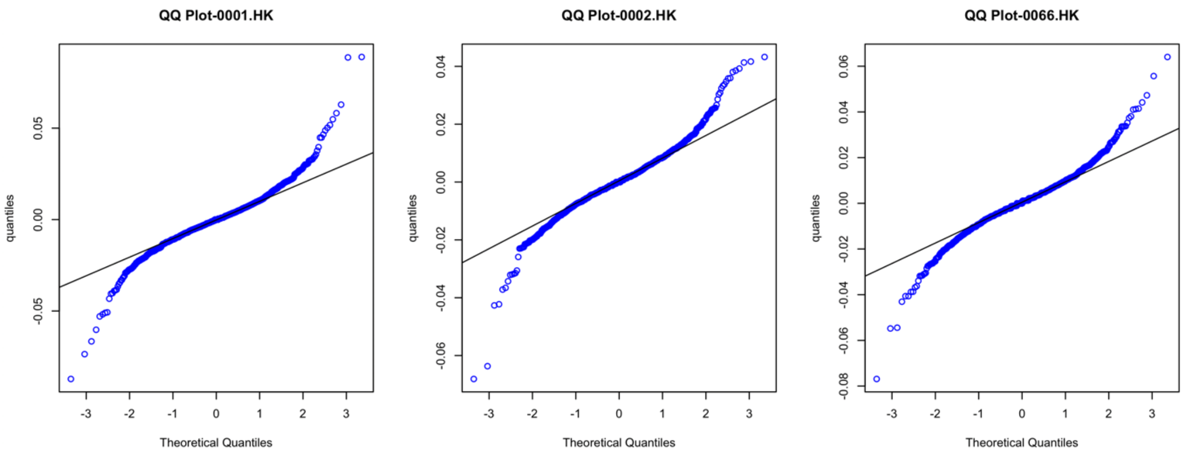
A. Test for normality

In this section, we will test whether stock returns follow the normal distribution.

Firstly, let us look at histogram for stock returns and normal Q-Q plot to visually judge whether stock returns follow normal distribution. As the histograms are bell-shaped and the points are around the solid line in QQ plots approximately, we can conclude that the stock returns follow approximate normal distributions. Also, as we can see that the points in the QQ plots have a steeper slope compared with the solid line, we know that the stock returns are approximate normal distributions with relatively heavy tails.



(Graph 1: Histograms for returns of 0001.HK, 0002.HK and 0066.HK)



(Graph 2: Normal QQ plot for returns of 0001.HK, 0002.HK and 0066.HK)

Next, we conduct Kolmogorov-Smirnov test and Jarque-Bera test to test the normality of stock returns.

Stock	**KS test statistic**	**p-value**	**H_0: normal distribution**
0001.HK	0.07119	6.108e-06	Reject H0
0002.HK	0.07149	5.478e-06	Reject H0
0066.HK	0.07556	1.224e-06	Reject H0

(Table 1: KS test for 0001.HK, 0002.HK and 0066.HK)

Stock	**JB test statistic**	**p-value**	**H_0: normal distribution**
0001.HK	2091.1	2.2e-16	Reject H0
0002.HK	1229.1	2.2e-16	Reject H0
0066.HK	925.8	2.2e-16	Reject H0

(Table 2: JB test for 0001.HK, 0002.HK and 0066.HK)

Therefore, from both the KS test and JB test results, we should reject H_0 of normal distributions in a more precise quantitative perspective.

The above tests are tests on univariate normal distribution. The following two tests are aiming at testing multivariate normal distribution. The first test we would like to use is Mahanalobis distance (Mahanalobis, 1936).

For a multivariate normal distribution $N_p(\mu, \Sigma)$, when sample size is large, we have

$$d_i^2 = (u_i - \overline{u})^T S^{-1} (u_i - \overline{u}) \sim \chi_{\mathrm{p}}^2 \tag{1}$$

where d_i^2 stands for Mahanalobis distance and S is the sample variance matrix. The test results from R programming are listed below. As we have over 1,000 entries in total, we only choose the first 200 lines to be displayed in our report.

```
> y<-cbind(y1,y2,y3)
> mahalanobis(y, colMeans(y), cov(y))
  [1]  3.977504079  1.052708529  0.605473092  1.702443503  3.788728587  0.447074402  4.203492045  2.399288132
  [9]  4.692543542  1.744390029  2.934350872  3.328632339  0.235751069  0.141952580  1.439598025 19.445067910
 [17] 18.871382968  4.538358381 13.684852875  4.570668283  1.212642909  2.214408327  2.821713069  1.210476500
 [25]  2.120196165  3.917126193  4.067089501  3.644466554  2.026293171  0.469570584  1.005596084  0.793395667
 [33]  4.012728183  4.545497311  3.193979731  2.235744174  2.307455808  8.782623661  2.828377170  1.609973400
 [41]  4.350510708  0.001980947 17.239616361  3.952768800  2.056506247  2.595466098  0.989485066 13.641026753
 [49]  1.345541512  5.486503602  4.323476006  2.450005481  0.888878795  2.156932838  2.986718905  2.073986205
 [57]  0.998725342  0.859935503  3.844887318  0.743977976  4.943242860  0.510292642  2.620292368  0.361000470
 [65]  6.098434987  0.102759622  0.818703673  1.086053452  0.811633109  1.909531286  7.097972957  1.611503389
 [73]  0.592895614  0.088901502  1.537937687  3.416956218  0.328697839  1.046524508  0.564876904  1.212691449
 [81]  3.255038937  0.242697030  2.182184296  1.224396615  0.435186525  1.390356397  0.612007835  1.838964888
 [89]  1.869743196  4.040617098  1.962002633  2.939918817  1.894068844  0.135992077  1.234443722  0.488352106
 [97]  0.394012574  0.001980947  3.658856199  1.840638783  1.320931860  0.848454564  0.888330144  1.723762213
[105]  2.473570610  2.486985232  0.646875956  0.084254144  2.224195014  0.148030476  2.909928746  0.348104962
[113] 10.198029051  4.191496992  1.621057792  5.518868528  3.661653325  2.441273466  3.289216517  1.050419163
[121]  1.303050074  0.553317856  0.471266897  1.721367255  4.313500031  0.474436773  3.340667074  2.316453649
[129]  1.334333637  1.581615493  1.178370839  0.580287889  0.499675640  3.265783131  4.532688852  0.105904282
[137]  1.401529154  0.573278211  1.314821443  1.291372392  0.132079899  1.528084021  0.678561998  0.657354841
[145]  0.217342478  0.256361875  6.169086674  1.211151094  0.726825478  0.381538055  0.215638260  5.539091417
[153]  1.663072034  1.086527494  0.784095845  2.097952711  0.238474044  1.193408253  2.015988162  0.583998112
```

(Graph 3: Mahanalobis distance for the returns of 0001.HK, 0002.HK and 0066.HK)

We then compare the above Mahanalobis distances to 95% percentile of Chi-square distribution with degree of freedom 3, and there are 91 points we find that should be categorized as outliers. However, we should assume that there are around 63 points $(1252 \times 5\%)$ categorized as outliers. Therefore, based on Mahanalobis distance, we should reject the null hypothesis that these stock returns are joint normal.

```
 [1] 19.445068 18.871383 13.684853  8.782624 17.239616 13.641027 10.198029 32.064461 31.592279 10.323805 12.592457 19.062381  8.062537 15.196559 11.556632 10.370130 15.716129 25.176453 12.671440 12.037992
[21] 11.838094 12.551914 10.952844 10.391176  8.053154 12.078457 10.106788 12.782514 10.603673 12.370314 10.567113  7.957449 28.392732 10.842935 10.457416  9.902902 20.689245 11.220231 16.576591 25.069112
[41]  9.827495 31.472980  8.887176  7.928208 11.760820  8.107539 16.784963 15.907471 14.853265 22.139948  9.882212 16.108282 14.493604 23.481584 20.436874 21.855606 77.715629 57.971812 51.538704 15.371096
[61] 20.190405 25.068008 25.054490 34.985522 13.358670 46.621780 11.321750 13.602563  9.937579  9.128188  8.422660 11.945685 10.317066 14.873455  7.965583 54.179611  7.897572 20.454888 18.691259 10.210744
[81]  9.251719  8.682790  7.981708 24.113339 15.230717 13.936432 10.030137 19.449268 12.509478 13.291001 11.639708
```

(Graph 4: Outliers detected based on Mahanalobis distance)

The other test for multivariate normality is Bowman-Shenton test.

Statistic	p-value	Result
5379.89	0	NO

(Table 3: Bowman-Shenton test for 0001.HK, 0002.HK and 0066.HK)

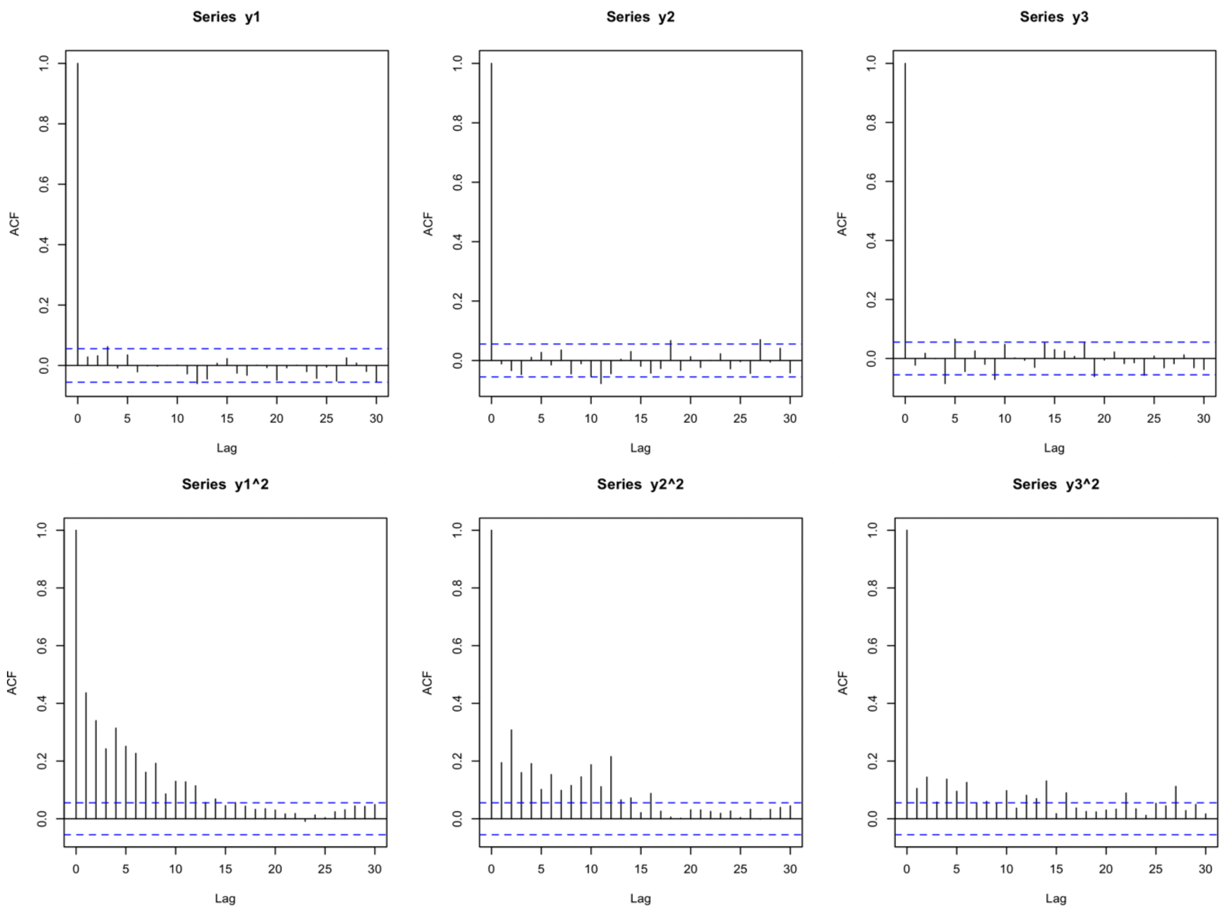
Therefore, the statistics are not multivariate normal.

From the previous normality test, we know that traditional normal assumption on stock returns does not work well.

B. Test for independence and autocorrelation

We would also like to test whether stock returns are independent and autocorrelated over time.

Firstly, let us look at the autocorrelation functions of the aforementioned three stocks.



(Graph 5: ACF plots for returns and squared returns of 0001.HK, 0002.HK and 0066.HK)

From the ACF test results, we can conclude stock returns are not serially correlated. However, we can find that the squared returns are serially correlated.

We further conduct Ljung-Box test to test whether these two series are autocorrelated or not.

Ljung-Box test of stock return with lag 12			
Stock	**X-squared**	**P-value**	**H0: no autocorrelation**
0001.HK	14.853	0.2496	Not reject H0
0002.HK	23.738	0.02207	Reject H0
0066.HK	28.607	0.004504	Reject H0

Ljung-Box test of squared stock return with lag 12			
Stock	**X-squared**	**p-value**	**H_0: no autocorrelation**
0001.HK	874.62	< 2.2e-16	Reject H0
0002.HK	462.19	< 2.2e-16	Reject H0
0066.HK	133.27	< 2.2e-16	Reject H0

(Table 4: Ljung-Box test for returns and squared returns of 0001.HK, 0002.HK and 0066.HK)

Therefore, from the test results, we know that both log stock return and squared log stock return have autocorrelation, which should not exist based on traditional normal assumption. In the other hand, to test for independence, we will conduct BDS test.

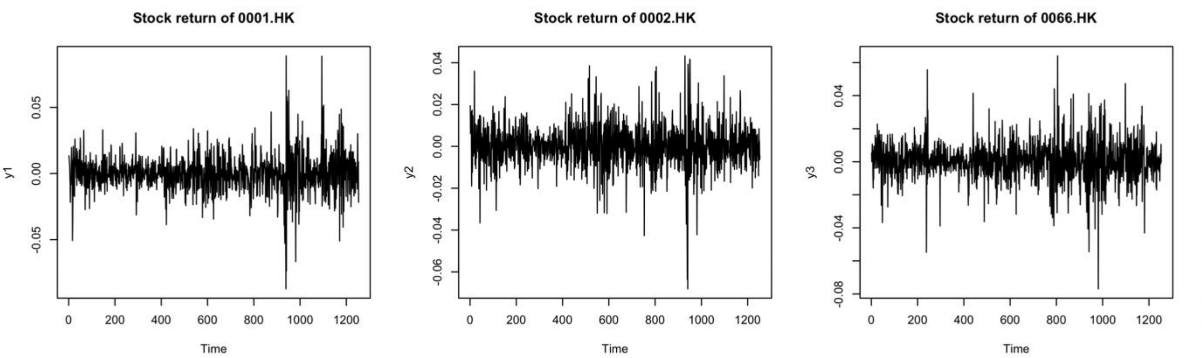
Stock	p-value	H_0: i.i.d. random variables
0001.HK	< 2.2e-16	Reject H0
0002.HK	6.425e-07	Reject H0
0066.HK	< 2.2e-16	Reject H0

(Table 5: BDS test for returns of 0001.HK, 0002.HK and 0066.HK)

Therefore, from the test results, we know that all the stock returns are not independent overtime. Actually, they are time series with time dependence.

C. Time-changing Volatility

We use R programming to have the time series plots as follows.



(Graph 6: Return plots for 0001.HK, 0002.HK and 0066.HK)

From the time series plots, we find that the volatility of stock returns is actually changing with time. Besides, we find that the large volatility usually follows large volatility and small volatility usually follows small volatility, which we call "volatility clustering" effect.

From the previous normality, autocorrelation and volatility clustering effect tests, we can have our conclusion that we can use CARCH model to do our stock return simulation.

3. Normal-GARCH(1,1) model

A. An introduction to GARCH(1,1) model

Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model is commonly used in statistics for estimating the volatilities of the time series. GARCH(p,q) takes the following form

$$\sigma_n^2 = \gamma V_L + \alpha_1 u_{n-1}^2 + \alpha_2 u_{n-2}^2 + \cdots + \alpha_p u_{n-p}^2 + \beta_1 \sigma_{n-1}^2 + \beta_2 \sigma_{n-2}^2 + \cdots + \beta_q \sigma_{n-q}^2 \qquad (2)$$

In the above model, u_i stands for the security return on day i and σ_i is the standard deviation of stock return on day i. As for the coefficient, $\gamma + \Sigma_{i=1}^{p} \alpha_{n-i} + \Sigma_{j=1}^{q} \beta_{n-j} = 1$ and V_L stands for long-term variance rate (We shall further elaborate this point in the following sections).

We further assume following conditional independence of u_i's:

$$\frac{u_i}{\sigma_i}\bigg| u_{i-1}, u_{i-2}, \cdots, u_1 \overset{d}{\Leftrightarrow} \frac{u_i}{\sigma_i}\bigg| \sigma_i \sim N(0,1) \qquad (3)$$

Since daily return for stocks is typically very small, it is reasonable to assume that the mean of daily stock return is 0. We shall use this assumption in all the remaining sections.

It is easy for us to find that this model can capture time-changing volatility of stock returns. Further, this model also can be used to illustrate volatility clustering. If previous p days have large volatilities, then the volatility on day n will also be large, and vice versa.

What's more, if u_t follows normal-GARCH(p,q) model, then u_t^2 actually follows ARMA(max(p,q),p) model. Since u_t^2 follows ARMA model, the correlation between u_i^2 and u_j^2 is well captured.

In this part, we will only use GARCH(1,1) model, i.e.,

$$\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2 \qquad (4)$$

This can simplify our model a bit without loss of much modelling power.

B. Datasets

As mentioned before, the securities that we are going to study are 0001.HK, 0002.HK, 0066.HK, 2333.HK, 0868.HK, 0700.HK, 3998.HK, 1368.HK, 0135.HK and 0992.HK. We will use the adjusted close prices of these stocks from 2016/06/01 to 2021/06/01 to build our model.

C. Model fitting with training set

To fit GARCH(1,1) model, we need to first estimate the parameters in the model. Maximum likelihood estimation (MLE) is a commonly used method to estimate parameters. Recall that $u_i \sim N(0, \sigma_i^2)$. Therefore, the likelihood function of u_i is

$$\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{u_i^2}{2\sigma_i^2}} \tag{5}$$

The log-likelihood function therefore is

$$-\frac{n}{2}\ln(2\pi) + \frac{1}{2}\Sigma_{i=1}^{n}(-\ln\left(\sigma_i^2 - \frac{u_i^2}{\sigma_i^2}\right)) \tag{6}$$

Thus, we need to choose suitable γ, V_L, α and β so that the log-likelihood function is maximized. However, this is a very complicated maximization problem and is beyond the scope of this part. We shall rely on the garch() in tseries library in R to get the MLE of GARCH(1,1).

The following table summarizes the MLE after fitting GARCH(1,1) model with training set (Please refer to the attached R code). In the table, we use w to represent γV_L.

Security	$\boldsymbol{w}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$
0001.HK	1.0056×10^{-5}	0.1581	0.7906
0002.HK	1.9479×10^{-6}	0.0647	0.9162
0066.HK	5.4961×10^{-6}	0.1529	0.8200
2333.HK	5.1478×10^{-5}	0.0856	0.8596
0868.HK	3.7655×10^{-6}	0.0393	0.9546
0700.HK	7.6017×10^{-6}	0.0726	0.9091
3998.HK	9.3432×10^{-6}	0.0738	0.9216
1368.HK	1.0564×10^{-6}	0.0183	0.9830
0135.HK	7.6122×10^{-6}	0.0615	0.9238
0992.HK	1.1365×10^{-5}	0.1644	0.8418

(Table 7: GARCH estimation for the 10 stocks)

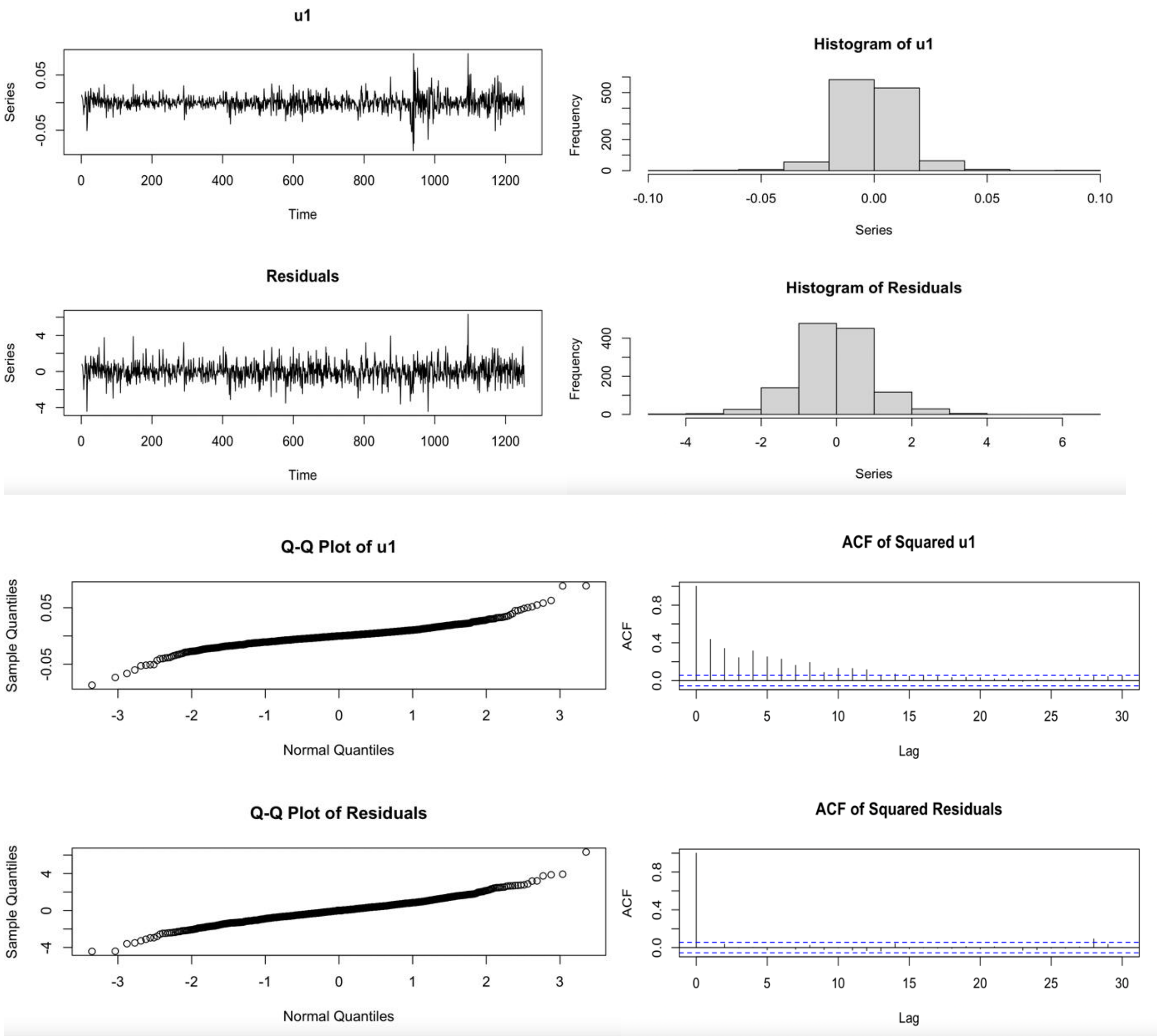
D. Diagnostic Checks

Next, we would like to conduct diagnostic checks on the model to check whether the assumptions hold well. If the model holds well, the residuals should perform like Gaussian white noise.

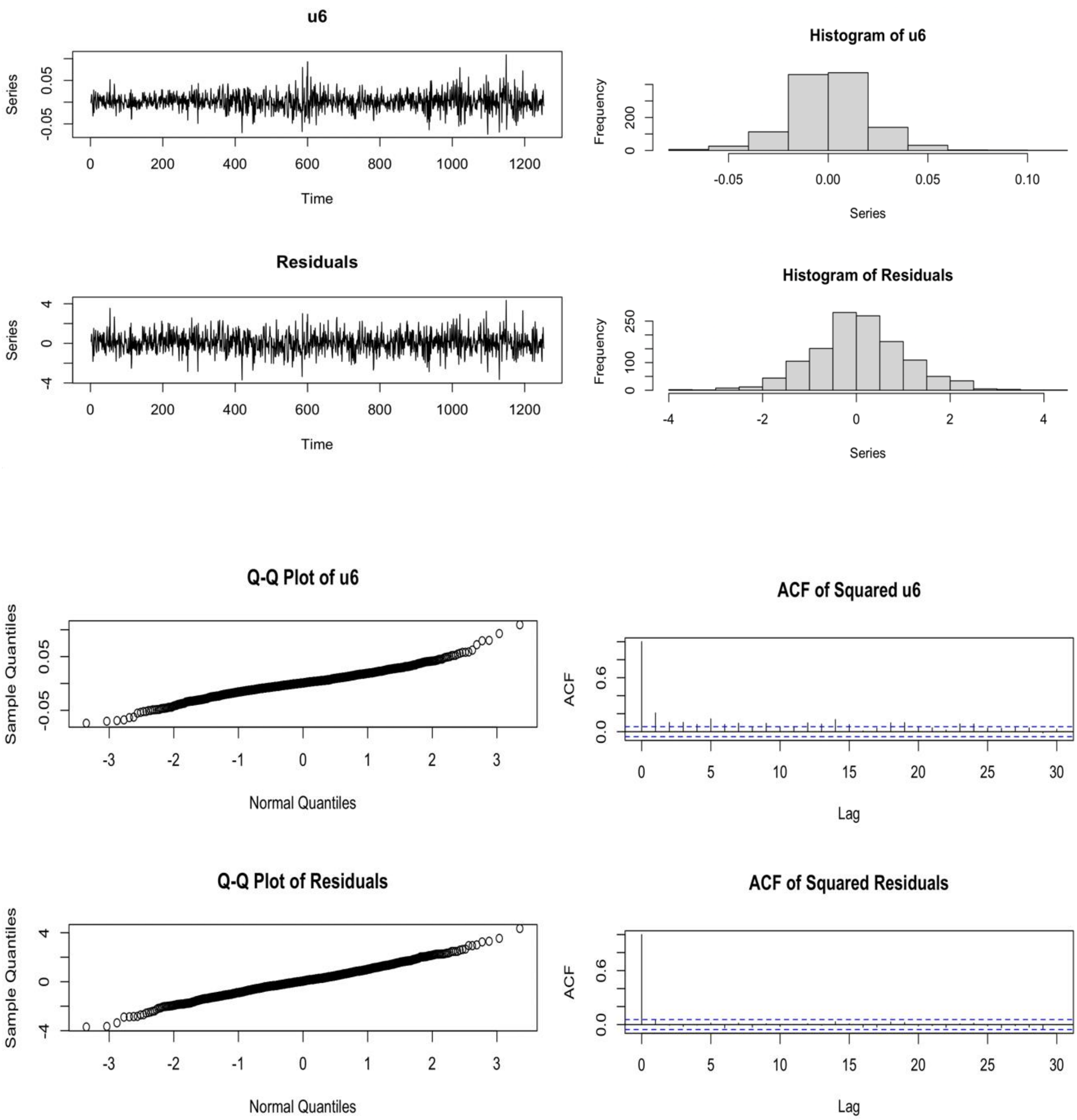
Firstly, we draw several plots to visually judge whether the model is good. Due to page limit, we will only put the graphs of 0001.HK and 0700.HK here. For the remaining graphs, please refer to the attached R code.

From the plots, it is easy for us to find that the normal Q-Q plots for residuals are very good except for few outliers, which mean that by incorporating GARCH(1,1). Moreover, from the ACF plots of the residuals and the squared residuals, we find that residuals and squared residuals have no autocorrelations. These suggest that the residuals are possibly Gaussian white noise.

In addition to plots, we will also conduct Jarque-Bera test and Ljung-Box test to check the



(Graph 7: Some plots for 0001.HK)



(Graph 8: Some plots for 0700.HK)

residuals. We summarize related statistics in the following table.

Security	Ljung-Box test statistic for squared residuals	p-value
0001.HK	7.2128×10^{-7}	0.9993
0002.HK	0.002302	0.9617
0066.HK	0.392830	0.5308
2333.HK	0.064441	0.7996
0868.HK	0.015789	0.9000
0700.HK	3.203800	0.0735
3998.HK	1.965100	0.1610
1368.HK	1.868700	0.1716
0135.HK	0.762180	0.3826
0992.HK	0.209300	0.6473

(Table 8: Ljung-Box test for squared residuals of the ten stocks)

The p-values of Ljung-Box tests are all bigger than 5%. We can say that there are no autocorrelations between residuals.

E. Interpretation of GARCH(1,1) with Heston and Nandi model

Actually, after mathematical deduction, GARCH(1,1) is equivalent to the following stochastic process, known as Heston and Nandi model (2011):

$$dV_t = (1 - \alpha - \beta)(V_L - V_t)dt + \alpha\sqrt{2}V_t dW_t \tag{7}$$

where W_t is a Winer process (or standard Brownian motion). This stochastic process shows similar mean-reverting property to the celebrated Vasicek Model and the reverting level is V_L. If the volatility is above V_L, it will be pushed down to V_L. If the volatility is below V_L, it will be pushed up to V_L. Therefore, in the long run, we can expect that the volatility of the stock return will converge to V_L. This explains why V_L is called long-term variance rate.

4. Prediction with GARCH(1,1) model

A. Volatility prediction

In the previous sections, we estimate the MLEs of the parameters in GARCH(1,1) model with R. Once the estimations are got, we can forecast future volatilities with GARCH(1,1) model.
Recall that

$$\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta\sigma_{n-1}^2 = (1 - \alpha - \beta)V_L + \alpha u_{n-1}^2 + \beta\sigma_{n-1}^2 \tag{8}$$

Hence,

$$\sigma_n^2 - V_L = \alpha(u_{n-1}^2 - V_L) + \beta(\sigma_{n-1}^2 - V_L) \tag{9}$$

On the day $n + k$ in the future,

$$\sigma_{n+k}^2 - V_L = \alpha(u_{n+k-1}^2 - V_L) + \beta(\sigma_{n+k-1}^2 - V_L) \tag{10}$$

To forecast the future volatilities, we need to calculate the expectation $E(\sigma_{n+k}^2)$, and this expected value can be regarded as a reasonable prediction for future volatilities.
According to (3),

$$Var(u_i|\sigma_i^2) = E(u_i^2|\sigma_i^2) = \sigma_i^2 \tag{11}$$

Therefore,

$$E(u_{n+k-1}^2 - V_L) = E[E(u_{n+k-1}^2|\sigma_{n+k-1}^2) - V_L) = E(\sigma_{n+k-1}^2 - V_L) \tag{12}$$

The derivation of the above equation utilizes tower property, which states that

$$E(X) = E(E(X|Y)) \tag{13}$$

Based on above equations, we can get

$$E(\sigma_{n+k}^2 - V_L) = \alpha E(\sigma_{n+k-1}^2 - V_L) + \beta E(\sigma_{n+k-1}^2 - V_L) = (\alpha+\beta)E(\sigma_{n+k-1}^2 - V_L) \quad (14)$$

We also know that

$$E(\sigma_{n+k-1}^2 - V_L) = (\alpha+\beta)E(\sigma_{n+k-2}^2 - V_L) \quad (15)$$

Thus,

$$E(\sigma_{n+k}^2 - V_L) = (\alpha+\beta)^2 E(\sigma_{n+k-2}^2 - V_L) \quad (16)$$

We apply this k times and then get

$$E(\sigma_{n+k}^2) = V_L + (\alpha+\beta)^k E(\sigma_n^2 - V_L) \quad (17)$$

From this equation, we can find that as $k \to \infty$, since $0 < \alpha + \beta < 1$, $(\alpha+\beta)^k E(\sigma_n^2 - V_L) \to 0$. Thus, $E(\sigma_{n+k}^2) \to V_L$. This explains why V_L is called long-term variance rate.

Based on the equation, we can predict the future volatility of a stock, and this is crucial to option pricing since when applying Black-Scholes-Merton Model to price an option, we need to have the volatility of the underlying stock.

In the previous section, we divide the whole dataset into training set and testing set. However, in this section, we shall use the whole dataset to estimate the parameters in the model and predict future 10-day volatilities of the ten securities.

Let us take 0001.HK as an example.

The MLE estimations based on whole data set is $\gamma V_L = 1.006 \times 10^{-5}$, $\alpha = 0.1581$ and $\beta = 0.7906$. Thus, $V_L = 0.0001962$.

Let $\theta = -\ln(\alpha+\beta)$ and V_0 stands for current volatility estimated from GARCH(1,1), which is 0.0001626514. Then, we have $E(V_t) = V_L + e^{-\theta t}(V_0 - V_L)$, and V_t stands for the variance at day t in the future.

Then, the average of the estimated variance from n to n+T days is

$$\frac{1}{T}\int_0^T E(V_t)dt = V_L + \frac{1-e^{-\theta T}}{\theta T}(V_0 - V_L) \quad (18)$$

Plug in the data, we get

$$\frac{1}{10}\int_0^{10} E(V_t)dt = V_L + \frac{1-e^{-10\theta}}{10\theta}(V_0 - V_L) = 0.00019262$$

Note that this is the estimated average daily variance. And we can convert it to the annual variance rate by multiplying by $\sqrt{252}$.

The following table summarizes our estimation for the future 10-day volatility.

Security	**Daily Volatility**	**Annual Volatility**
0001.HK	0.000193	0.00306

0002.HK	0.000097	0.00154
0066.HK	0.000189	0.00300
2333.HK	0.000978	0.01550
0868.HK	0.000636	0.01010
0700.HK	0.000449	0.00713
3998.HK	0.001950	0.03090
1368.HK	-0.000382	-0.00607
0135.HK	0.000519	0.00823
0992.HK	-0.001620	-0.00257

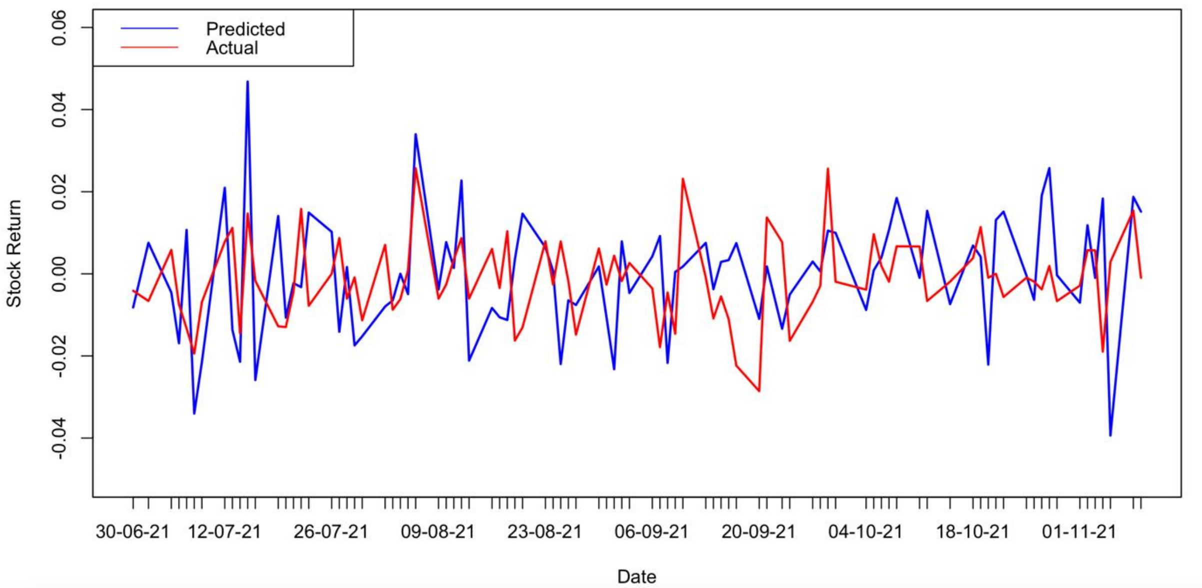
(Table 9: Predicted 10-day volatility for ten stocks)

Similar methodology can also be applied to predict any future volatilities. For example, if we are going to price a call option that matures in a year, we can first use this model to get the volatility of future one year and then apply Black-Scholes-Merton model to get this option price.

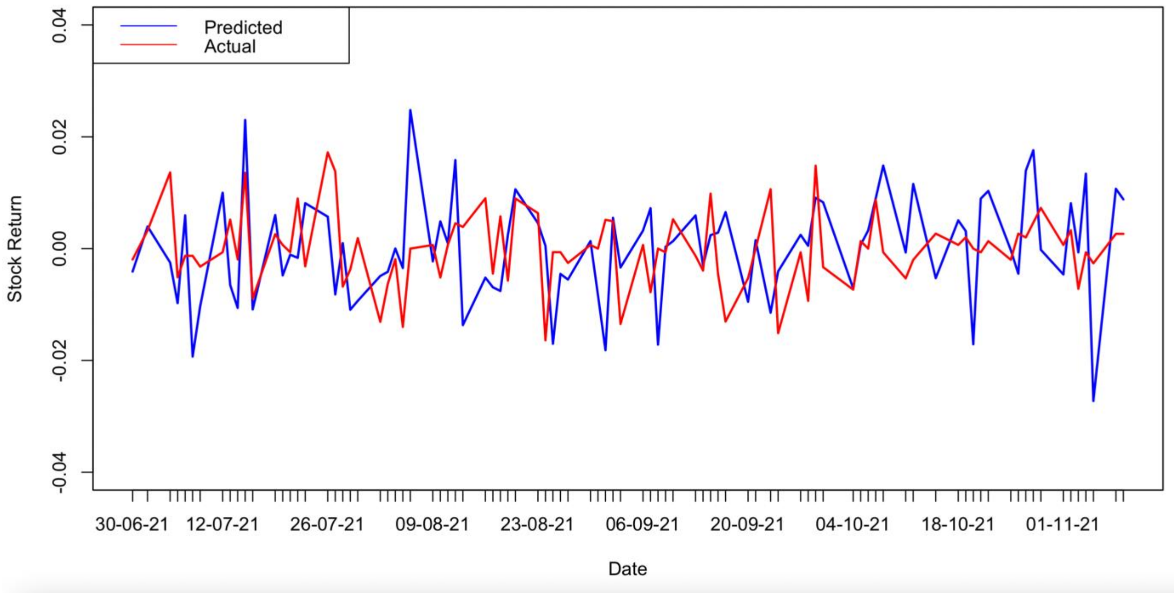
B. Simulation on future stock returns

In this section, we will use Monte Carlo technique to simulate the 90-day (from 06/30/2021 to 11/09/2021) future stock returns based on Normal-GARCH(1,1) model. Then we will compare the predicted stock returns to the actual stock returns.

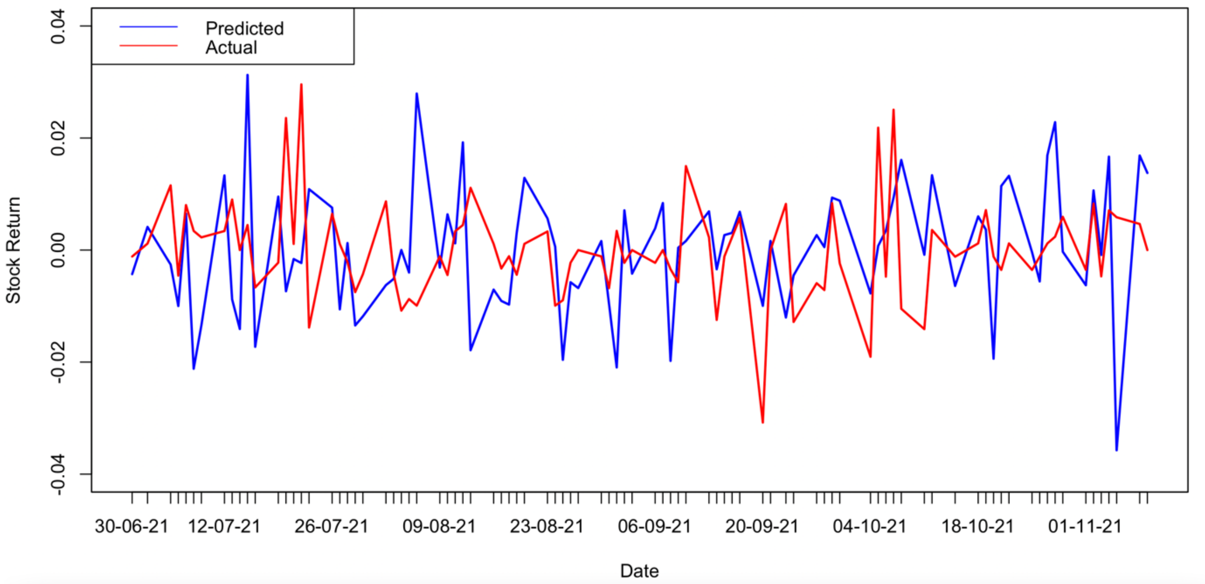
We will generate one sample path for each stock as the representative of the future. Due to the page limits, we only put the graph for 0001.HK, 0002.HK and 0066.HK here. For the rest of the graphs, please refer to the R file.



(Graph 9: Simulated future 90-day stock return and actual return for 0001.HK)



(Graph 10: Simulated future 90-day stock return and actual return for 0002.HK)



(Graph 11: Simulated future 90-day stock return and actual return for 0066.HK)

From the above three plots, it can be found that in terms of tendency, normal-GARCH(1,1) model well captures the future tendency of stock returns although it may not be able to very exactly predict the magnitude of stock returns. However, the good prediction power on tendency of stock returns can already help investors a lot in terms of making their investments.

5. Some possible improvements on the model

According to QQ plot in previous section, we can see that there are still some outliers in our model, especially at two tails, which means that GARCH(1,1) model still fails to explain some outliers.

A. Improvements on GARCH

A.1 Threshold GARCH, also known as leverage GARCH, GJR-GARCH

TGARCH model can handle leverage effects well. A TGARCH (m, s) model assumes the form

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} (\alpha_i + \gamma_i N_{t-i})\, a_{t-i}^2 + \sum_{i=1}^{s} \beta_j \sigma_{t-j}^2 \tag{19}$$

where N_{t-i} is an indicator for negative α_{t-i}, that is,

$$N_{t-i} = \begin{cases} 1, & if\ a_{t-i} < 0 \\ 0, & if\ a_{t-i} \geq 0 \end{cases} \tag{20}$$

And $\alpha_i, \gamma_i\ and\ \beta_j$ are nonnegative parameters satisfying conditions similar to those of GARCH models.

TGARCH can also be written as

$$\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2 + \theta I_n u_{n-1}^2 \tag{21}$$

I_n is an indicator function and takes 1 when $u_{n-1} < 0$. If $\theta > 0$, bad news will increase the volatility while good news has lower effect.

A.2 Exponential GARCH

To overcome the weakness of the GARCH model in handling financial time series that it can only simulate symmetric relationship between conditional volatility and conditional mean value, exponential GARCH can be applied further as it presents asymmetric volatility on positive and negative returns. Exponential GARCH model can utilize multiple volatility measures for the modeling of a return series. The model can specify the dynamic returns and measures, and is characterized by a model of dependence between returns and volatility.

$$n\sigma_n^2 = \gamma V_L + \beta \sigma_{n-1}^2 + \alpha \frac{|u_{n-1}| + \theta u_{n-1}}{\sigma_{n-1}} \tag{22}$$

B. Using t-distribution instead of normal one

Actually, from Q-Q plots for stock returns, heavier tail can be found. Therefore, t-distribution or other heavy tail distributions may be more appropriate compared with normal distribution. We can use the following t-distribution to model our stock returns better:

$$f(T,v) = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\sqrt{v\pi\sigma^2}}\left(1+\frac{T^2}{v\sigma^2}\right)^{-\frac{v+1}{2}}, for\ v > 2 \tag{23}$$

When v is small, it shows heavier tail. When v is large, it converges to normal distribution. Markowitz (1996) points out typical v ranges from 3 to 6. Therefore, using this t-distribution, we can model our stock returns better.

Some moments of the above distribution is summarized in the following table.

Mean	Variance	Skewness	Excess Kurtosis
0	$\frac{\sigma^2 v}{v-2}$	0	$\frac{6}{v-4}$

(Table 10: Moments of the t-distribution)

C. Extreme Value Theory

Actually, outliers of stock returns at two tails mainly come from some extreme conditions like heat market or financial crisis. Instead of integrating these outliers into our current model, we can model them separately by using Extreme Value Theory.

One famous model for is generalized Pareto distribution (Gnedenko, 1943) , which takes the form

$$F_u(y) = P(u < X < u + y | X > u) = \frac{F(u+y) - F(u)}{1 - F(u)} \rightarrow 1 - \left(1 + \frac{\xi y}{\beta}\right)^{-\frac{1}{\xi}} \tag{24}$$

as n goes to infinity. ξ and β are the shape parameter and the scale parameter respectively and need to be estimated.

Typically, ξ for stock returns are positive. Therefore, we have

$$G_{\xi,\beta}(x) = 1 - \left(1 + \frac{\xi}{\beta}(x-u)\right)^{-\frac{1}{\xi}} \tag{25}$$

This model can be used to model stock returns under extreme conditions like financial crisis and heat market.

6. Summary

In this report, we study the behaviour of several stock returns and find that traditional assumption that stock returns follow normal distribution does not hold well. Therefore, we propose normal-GARCH(1,1) model to better explain the stylised facts we discover like volatility clustering, and the model does a very good job. We then turn to use this model to predict future volatilities and future stock returns. In the end, we also propose some possible improvements on the model.

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