

Law of Iterated Expectations

$$E(E(X|Y)) = E(X) \quad \text{증명}$$

X	x_1	x_2	\dots	x_m
$P(X)$	$P(X=x_1)$	$P(X=x_2)$	\dots	$P(X=x_m)$

Y	y_1	y_2	\dots	y_n
$P(Y)$	$P(Y=y_1)$	$P(Y=y_2)$	\dots	$P(Y=y_n)$

$$E(E(X|Y)) = E(X|Y=y_1)P(Y=y_1) + \dots + E(X|Y=y_n)P(Y=y_n)$$

$$E(X|Y=y_1) = \sum_{i=1}^m x_i E(X=x_i|Y=y_1)$$

$$= \sum_{i=1}^m x_i E(X=x_i|Y=y_1)P(Y=y_1) + \dots + \sum_{i=1}^m x_i E(X=x_i|Y=y_1)P(Y=y_1)$$

$$= \sum_{i=1}^m x_i P(X=x_i \cap Y=y_1) + \sum_{i=1}^m x_i P(X=x_i \cap Y=y_2) + \dots + \sum_{i=1}^m x_i P(X=x_i \cap Y=y_n)$$

$$= \begin{array}{|l} x_1 P(X=x_1 \cap Y=y_1) \\ + x_1 P(X=x_1 \cap Y=y_2) \\ + x_1 P(X=x_1 \cap Y=y_n) \end{array} + \begin{array}{|l} x_2 P(X=x_2 \cap Y=y_1) \\ + x_2 P(X=x_2 \cap Y=y_2) \\ + x_2 P(X=x_2 \cap Y=y_n) \end{array} + \dots + \begin{array}{|l} x_m P(X=x_m \cap Y=y_1) \\ + x_m P(X=x_m \cap Y=y_2) \\ + x_m P(X=x_m \cap Y=y_n) \end{array}$$

$$= x_1 P(X=x_1) + x_2 P(X=x_2) + \dots + x_m P(X=x_m)$$

$$= E(X)$$