

Anti-Ramsey number of intersecting cliques*

Hongliang Lu, Xinyue Luo and Xinxin Ma
 School of Mathematics and Statistics
 Xi'an Jiaotong University
 Xi'an, Shaanxi 710049, China

Abstract

An edge-colored graph is called a rainbow graph if all its edges have distinct colors. The anti-Ramsey number $ar(n, G)$, for a graph G and a positive integer n , is defined as the minimum number of colors r such that every exact r -edge-coloring of the complete graph K_n contains at least one rainbow copy of G . A (k, r) -fan graph, denoted $F_{k,r}$, is a graph composed of k cliques each of size r , all intersecting at exactly one common vertex. In this paper, we determine $ar(n, F_{k,r})$ for $n \geq 256r^{16}(k+1)^5$, $k \geq 1$, and $r \geq 2$.

Key words: anti-Ramsey number; intersecting cliques; rainbow; edge-coloring

1 Introduction

All graphs considered in this paper are finite and simple, meaning they have no multiple edges or loops. We use $G = (V(G), E(G))$ to denote a graph with vertex set $V(G)$ and edge set $E(G)$. The order of G , denoted by $|G|$, is defined as the cardinality of the vertex set $V(G)$, that is $|G| := |V(G)|$, and the size of G , denoted by $e(G)$, is the cardinality of the edge set $E(G)$, so $e(G) := |E(G)|$. For any subset $X \subseteq V(G)$, we use $G - X$ to represent the subgraph of G obtained by deleting all vertices in X and all edges incident to any vertex in X . When $X = \{x\}$, we simply write it as $G - x$. For $Y \subseteq E(G)$, $G - Y$ represents the subgraph obtained from G by removing all edges in Y . For an edge set Y such that $Y \cap E(G) = \emptyset$, we let $G + Y$ denote the graph whose vertex set is $V(G) \cup V(Y)$ and whose edge set is $E(G) \cup E(Y)$. For two nonempty subsets $S, T \subseteq V(G)$, let $E_G(S, T)$ denote the set of all edges in G with one endpoint in S and one endpoint in T . The neighborhood of vertex x in G is denoted by $N_G(x)$. The *degree* of x in G , denoted by $d_G(x)$, is equal to the size of $N_G(x)$. We use $\delta(G)$ and $\Delta(G)$ to denote the minimum and maximum degrees of the vertices of G . A *matching* in G is a set of edges from $E(G)$ such that no two edges share a common vertex. The *matching number* of G , denoted by $\nu(G)$, is the maximum number of edges in a matching in G .

Given two graphs H and G , if G does not contain H as a subgraph, we say that G is *H -free*. For integers $n \geq p \geq 1$, let $T_{n,p}$ denote the Turán graph, i.e., the complete p -partite graph on n vertices where each partition class has either $\lfloor n/p \rfloor$ or $\lceil n/p \rceil$ vertices, with edges connecting all pairs of vertices from different parts. Let $t_p(n)$ denote the number of edges in

*Research supported by National Key Research and Development Program of China 2023YFA1010200 and National Natural Science Foundation of China grant 12271425.

$T_{n,p}$. Let K_n be the complete graph with n vertices. Given two positive integers r, k , let kK_r denote the union of k vertex-disjoint complete graph K_r . Let $[n] := \{1, 2, \dots, n\}$. For two vertex-disjoint graphs G and H , the *join* of G and H , denoted by $G \vee H$, is the graph obtained from $G \cup H$ by adding edges joining every vertex of G to every vertex of H . The *union* of G and H is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

An *r -edge-coloring* of a graph is an assignment of r colors to the edges of the graph. An *exactly r -edge-coloring* of a graph is an r -edge-coloring that uses all r colors. An edge-colored graph is called *rainbow* if all edges have distinct colors. Given a positive integer n and a graph set \mathcal{G} , the *anti-Ramsey number* $ar(n, \mathcal{G})$ is the minimum number of colors r such that each edge-coloring of K_n with exactly r colors contains a rainbow copy of $G \in \mathcal{G}$. When $\mathcal{G} = \{G\}$, we denote $ar(n, \mathcal{G})$ by $ar(n, G)$ for simplicity. The *Turán number* $ex(n, G)$ is the maximum number of edges of a graph on n vertices containing no subgraph isomorphic to G . A graph H on n vertices with $ex(n, G)$ edges and without a copy of G is called *extremal graph* for G -free graphs. We use $EX(n, G)$ to denote the set of extremal graphs for G , i.e.

$$EX(n, G) := \{H \mid |V(H)| = n, e(H) = ex(n, G), \text{ and } H \text{ is } G\text{-free}\}.$$

The value of $ar(n, G)$ is closely related to the Turán number $ex(n, G)$ as the following inequality

$$2 + ex(n, \mathcal{G}) \leq ar(n, G) \leq 1 + ex(n, G), \quad (1)$$

where $\mathcal{G} = \{G - e \mid e \in E(G)\}$. Erdős, Simonovits and Sós [8] proved there exists a number $n_0(p)$ such that $ar(n, K_p) = t_{p-1}(n) + 2$ for $n > n_0(p)$. Later, Montellano-Ballesteros [16] and, independently, Neumann-Lara [15] extended this result to all n and p satisfying $n > p \geq 3$.

Another interesting problem in anti-Ramsey number theory is determining the anti-Ramsey numbers for cycles. Let $l \geq 3$ be a positive integer and let C_l denote the cycle of length l . Erdős, Simonovits and Sós [8] conjectured that

$$ar(n, C_l) = \left(\frac{l-2}{2} + \frac{1}{l-1} \right) n + O(1)$$

and verified the conjecture for $l = 3$. Alon [2] confirm the case for $l = 4$ by showing that $ar(n, C_4) = \lfloor \frac{4n}{3} \rfloor - 1$. Jiang, Schiermeyer and West [13] extended the proof to all $l \leq 7$. The conjecture was fully resolved by Montellano-Ballesteros and Neumann-Lara [16].

The Turán number for matchings is determined by Erdős and Gallai [7] for $n \geq 2t \geq 2$. For anti-Ramsey numbers of matchings, Schiermeyer [17] first established that $ar(n, tK_2) = ex(n, (t-1)K_2) + 2$ for $n \geq 3t + 3$. This result was later improved by Fujita et al. [10], who extended the range to $n \geq 2s + 1$. A complete resolution for all $s \geq 2$ and $n \geq 2s$ was achieved by Chen, Li, and Tu [5], who determined the exact value of $ar(n, M_s)$. Interestingly, Haas and Young [12] gave a simple proof for the case $n = 2s$. There is a large volume of literature on the anti-Ramsey number of graphs. Interested readers are referred to the survey by Fujita, Magnant, and Ozeki [10].

A graph on $(r-1)k+1$ vertices consisting of k cliques each with r vertices, which intersect in exactly one common vertex, is called a (k, r) -*fan* and denoted by $F_{k,r}$. Liu, Lu and Luo [14] determined the anti-Ramsey number $ar(n, F_{k,3})$ for $n \geq 100k^2$. In this paper, we determined $ar(n, F_{k,r})$ for large n .

Theorem 1 *Let n, k, r be integers such that $k \geq 1$ and $r \geq 2$, and $n \geq 256(k+1)^5 r^{16}$. The following holds*

$$ar(n, F_{k+1, r}) = \text{ex}(n, F_{k, r}) + 2.$$

The rest of the paper is organized as follows. In Section 2, we introduce some technical lemmas. In Section 3, we complete the proof of Lemma 9. In Section 4, we give the proof of Theorem 1.

2 Several Technical Lemmas

For two integers $\nu, \Delta \geq 1$, define $f(\nu, \Delta) = \max\{e(G) \mid \nu(G) \leq \nu, \Delta(G) \leq \Delta\}$. Let $\mathcal{F}_{\nu, \Delta} := \{G \mid \nu(G) \leq \nu, \Delta(G) \leq \Delta, e(G) = f(\nu, \Delta)\}$. In 1972, Abbott, Hanson and Sauer [1] determined $f(k, k)$.

Theorem 2 (Abbott, Hanson and Sauer, [1]) *Let $k \geq 2$ be an integer. Then*

$$f(k, k) = \begin{cases} k^2 + (k-1)/2, & \text{if } k \text{ is odd,} \\ k^2 + k, & \text{if } k \text{ is even.} \end{cases}$$

Later Chvátal and Hanson [3] proved the following result, which generalized Theorem 2.

Theorem 3 (Chvátal and Hanson, [3]) *For every $\nu \geq 1$ and $\Delta \geq 1$,*

$$f(\nu, \Delta) = \nu\Delta + \lfloor \frac{\Delta}{2} \rfloor \lfloor \frac{\nu}{\lceil \Delta/2 \rceil} \rfloor \leq \nu\Delta + \nu.$$

Lemma 4 (Chen et al., [4]) *Let G be a graph of order n . Let k be an integer and c some constant independent from n . If $e(G) \geq \text{ex}(n, K_r) + c$ and $d_G(x) \leq (\frac{r-2}{r-1})n - (k+1)$, then $e(G-x) \geq \text{ex}(n-1, K_r) + c + (k+1)$.*

Erdős et al. [9] determined the $\text{ex}(n, F_{k,3})$ for large n .

Theorem 5 (Erdős, Füredi, Gould and Gunderson, [9]) *For two integers n, k such that $k \geq 2$ and $n \geq 50k^2$,*

$$\text{ex}(n, F_k) = \begin{cases} \lfloor \frac{n^2}{4} \rfloor + k^2 - k, & \text{if } k \text{ is odd,} \\ \lfloor \frac{n^2}{4} \rfloor + k^2 - \frac{3}{2}k, & \text{if } k \text{ is even.} \end{cases}$$

Erdős, Füredi, Gould and Gunderson [9] showed that each member in $EX(n, F_{k,3})$ could be obtained from a Turán graph $T_{n,2}$ with a graph G embedding in one partite set, where G is empty for $k=1$ and $\mathcal{F}_{k-1, k-1}$ for $k \geq 2$. Chen et al. determined the $\text{ex}(n, F_{k,r})$ for large n .

Theorem 6 (Chen et al., [4]) *For every $k \geq 1$ and $r \geq 3$, and for every $n \geq 16k^3 r^8$,*

$$\text{ex}(n, F_{k,r}) = \begin{cases} \text{ex}(n, K_r) + k^2 - k, & \text{if } k \text{ is odd,} \\ \text{ex}(n, K_r) + k^2 - \frac{3}{2}k, & \text{if } k \text{ is even.} \end{cases}$$

Each member in $EX(n, F_{k,r})$ may be obtained from a Turán graph $T_{n, r-1}$ with a graph G embedding in one partite set, where G is empty for $k=1$ and $\mathcal{F}_{k-1, k-1}$ for $k \geq 2$.

Let b be a positive integer and let X and Y be two disjoint vertex subsets of $V(G)$: We say that X *dominates* Y with b -deficiency if $d_Y(x) \geq |Y| - b$ for each $x \in X$. Let V_1, \dots, V_m be disjoint subsets of $V(G)$. We say that $\{V_1, \dots, V_m\}$ is b -deficiency complete if V_i dominates V_j with deficiency b for every pair $i \neq j$ with $i, j = 1, 2, \dots, m$.

Lemma 7 (Chen et al., [4]) *Let b be a positive integer. Let G be a graph and let $\{X_1, \dots, X_m\}$ be a b -deficiency complete partition of $V(G)$ with $|X_i| \geq mb + 2t$ for each i . Suppose that C_1, C_2, \dots, C_t are t cliques of G with the properties:*

- (a) $|C_i \cap X_j| \leq 2$ for each pair i and j ,
 - (b) $|C_i \cap X_j| = 2$ at most one j for each i .
- Then, there exist t cliques D_1, \dots, D_t satisfying:*

- (1) $C_i \subseteq D_i$ for each i ,
- (2) $D_1 - C_1, \dots, D_t - C_t$ are mutually disjoint,
- (3) for each i we have that $|D_i \cap X_j| = 1$ for all j except possibly one at which $|D_i \cap X_j| = |C_i \cap X_j| = 2$.

Lemma 8 (Chen et al., [4]) *Let G be a graph and V_1, V_2, \dots, V_m be m vertex disjoint subsets of $V(G)$ and $Y_0 \subseteq V(G) - \cup_{i=1}^m V_i$ such that $|Y_i| \geq (i-1)b + k$ for each $i = 1, \dots, m$. If Y_j dominates V_i with b -deficiency for each $i = 1, \dots, m, j = 0, 1, \dots, m$ and $i \neq j$, then, there are k vertex disjoint cliques C_1, C_2, \dots, C_k satisfying $|C_i| = m$ and $|C_i \cap V_j| = 1$ for each i and $j \geq 1$. Furthermore, if $|Y_0| \geq mb + k$, then there are k vertex disjoint cliques D_1, D_2, \dots, D_k with the property that $|D_i| = m + 1$ and $|D_i \cap V_j| = 1$ for each $i = 1, \dots, k$ and $j = 0, 1, \dots, m$.*

For complete the proof of Theorem 1, we need the following lemma, whose proof employs some idea presented in [4].

Lemma 9 *For every $k \geq 1$ and $r \geq 4$, suppose that G is an $F_{k+1,r}$ -free graph on n vertices with $n \geq 4(k+1)^2 r^4$ and with minimum degree $\delta(G) > \binom{r-2}{r-1}n - (k+1)$, and $e(G) > ex(n, F_{k,r})$. Then there exists a partition $V(G) = V_0 \cup V_1 \cup \dots \cup V_{r-2}$, such that for all $i = 0, \dots, r-2$, the following three statements hold:*

- (i) $|V_i| \geq \frac{n}{r-1} - (k+1)(2r-3)$;
- (ii) $\sum_{j \neq i} \nu(G[V_j]) \leq k$ and $\Delta(G[V_i]) \leq k$;
- (iii) $d_{G[V_i]}(x) + \sum_{j \neq i} \nu(G[N(x) \cap V_j]) \leq k$ for any $x \in V_i$.

Let G be a graph with a partition of the vertices into $r-1$ non-empty parts $V(G) = V_0 \cup V_1 \cup \dots \cup V_{r-2}$. Define $G_i = G[V_i]$ for each $i = 0, 1, \dots, r-2$, and define

$$G_{cr} = G - (\cup_{i=0}^{r-2} E(G[V_i])).$$

Lemma 10 (Chen et al., [4]) *Suppose G is partitioned as above so that (??) and (??) are satisfied. If G is $F_{k+1,r}$ -free, then*

$$\sum_{i=0}^{r-2} e(G_i) - \left(\sum_{0 \leq i < j \leq r-2} |V_i||V_j| - e(G_{cr}) \right) \leq f(k, k).$$

3 Proof of Lemma 9

Proof of Lemma 9. Since $e(G) > ex(n, F_{k,r})$, then G contains k edge disjoint cliques D_1, \dots, D_k sharing one vertex v_0 with $|D_i| = r$ for $i = 1, \dots, k$. Let K be a clique of maximum size in $G - (\cup_{i=1}^k V(D_i) \setminus \{v_0\})$ containing the vertex v_0 . Denote $V(K) = \{v_0, v_1, \dots, v_s\}$. From the inequality $\delta(G) > (\frac{r-2}{r-1})n - (k+1)$, it follows that $|K| \geq 2$. So we have $s \geq 1$ and $|\cup_{i=1}^k D_i \cup K| = k(r-1) + s + 1$. For each $i = 0, \dots, s$, let $X_i = \cap_{j \neq i} N_G(v_j) - V(\cup_{j=1}^k D_j \cup K)$. Since K is a clique that contains v_0 and has the maximum order and is vertex-disjoint with D_1, \dots, D_k , it follows

$$X_i \cap X_j = \emptyset \text{ for } i \neq j.$$

Note that $\delta(G) > (\frac{r-2}{r-1})n - (k+1)$. So we may infer that

$$|X_i \cup V(\cup_{z=1}^k D_z \cup K)| \geq n - \left(\frac{n}{r-1} + (k+1) \right) s,$$

which implies that

$$|X_i| \geq \frac{r-1-s}{r-1}n - (k+1)s - k(r-1) - s - 1. \quad (2)$$

Claim 1. X_i is an independent set for $i = 1, \dots, s$.

Otherwise, suppose that there is an edge $uv \in E(G[X_i])$. Replacing v_i by the edge uv in D , we obtain a copy of clique whose size greater than K , a contradiction. This completes the proof of Claim 1.

Claim 2. $s = r - 2$.

By contradiction, suppose that $s \neq r - 2$. Since G is $F_{k+1,r}$ -free, we may assume that $s \leq r - 3$. For $i \in [s]$, by Claim 1 and (2),

$$\begin{aligned} d_G(x_i) &\leq n - |X_i| \\ &\leq n - \left(\frac{2}{r-1}n - (k+1)(r-3) - k(r-1) - (r-2) \right) \\ &= \frac{r-3}{r-1}n + (k+1)(r-3) + k(r-1) + r - 2. \end{aligned}$$

contradicts with the fact that $\delta(G) > \frac{r-2}{r-1}n - (k+1)$. This completes the proof of Claim 2.

By Claim 2 and (2), we may infer that for each $i = 0, \dots, r - 2$,

$$|X_i| \geq \frac{1}{r-1}n - (k+1)(2r-3). \quad (3)$$

Claim 3. For $\{i, j\} \in \binom{[s] \cup \{0\}}{2}$ with $i \neq 0$, X_i dominates X_j with $2(k+1)(r-1)$ -deficiency.

Fix i and j such that $i \neq 0$. For every $x \in X_i$, by Claim 1,

$$\begin{aligned}
d_{X_j}(x) &\geq d_G(x_i) - \left(\bigcup_{l \in ([s] \setminus \{i,j\}) \cup \{0\}} |X_l| \right) - |V(\cup_{j=1}^k D_j \cup K)| \\
&= d_G(x_i) - (n - |X_i| - |X_j|) \quad (\text{by (3)}) \\
&\geq |X_j| - (k+1) - (k+1)(2r-3) \\
&= |X_j| - 2(k+1)(r-1).
\end{aligned}$$

This completes the proof of Claim 3.

Claim 4. Let $\{x_1, x_2, \dots, x_{r-2}\} \subseteq \cup_{i=1}^{r-2} X_i$ such that $x_i \in X_i$ for $i \in [r-2]$. For any $Y_0 \subseteq X_0$ with $|Y_0| \geq 2(k+1)(r-1)^2$, $|\cap_{i=1}^{r-2} N_G(x_i) \cap Y_0| \geq k+1$.

By (3), we may infer that

$$|\cap_{i=1}^{r-2} N_G(x_i) \cap X_0| \geq |X_0| - 2(k+1)(r-1)(r-2).$$

Hence

$$\begin{aligned}
|\cap_{i=1}^{r-2} N_G(x_i) \cap Y_0| &\geq |Y_0| - (|X_0| - |\cap_{i=1}^{r-2} N_G(x_i) \cap X_0|) \\
&\geq 2(k+1)(r-1)^2 - 2(k+1)(r-1)(r-2) \\
&= 2(k+1)(r-1) > (k+1).
\end{aligned}$$

This completes the proof of Claim 4.

Let X_0^* denote the set of all vertices of X_0 of degree at least $2(k+1)(r-1)^2$ in X_0 , i.e.,

$$X_0^* := \{x \in X_0 \mid d_{G[X_0]}(x) \geq 2(k+1)(r-1)^2\}.$$

Claim 5. $|X_0^*| \leq 2(k+1)(r-1)(r-2)$.

By contradiction, suppose that $|X_0^*| > 2(k+1)(r-1)(r-2)$. For each i , let

$$X_{0i}^* = \{x \in X_0^* \mid d_{X_i}(x) \geq |X_i|/(2(k+1)(r-1)+1)\}. \quad (4)$$

By (3), $d_{X_0}(x_i) \geq |X_0| - 2(k+1)(r-1)$ for every $x_i \in X_i$. Hence for any $S \subseteq X_0$ with $|S| = 2(k+1)(r-1)+1$, every vertex in X_i has at least one neighbor in S , which implies that

$$X_i \subseteq N(S). \quad (5)$$

By taking the average value, X_0 contains at least $|X_0| - 2(k+1)(r-1)$ vertices x such that $d_{G[X_i]}(x) \geq 2(k+1)(r-1)^2$. It follows that

$$|X_{0i}^*| \geq |X_0^*| - 2(k+1)(r-1). \quad (6)$$

By (6), we may infer that

$$|\cap_{i=1}^{r-2} X_{0i}^*| \geq |X_0^*| - 2(k+1)(r-1)(r-2) \geq 1. \quad (7)$$

Let $\mathcal{X}_{0i}^* = X_0^* - X_{0i}^*$. For $T \in \binom{\mathcal{X}_{0i}^*}{2(k+1)(r-1)+1}$, we have $\sum_{v \in T} d_{X_i}(v) \geq |X_i|$ by (5), which means there exists $v \in T$ such that $d_{X_i}(v) \geq |X_i|/(2(k+1)(r-1)+1)$ contradicting the definition of \mathcal{X}_{0i}^* by (4).

By (7), there is an $x_0 \in X_0^*$ such that $|N(x_0) \cap X_i| \geq |X_i|/(2(k+1)(r-1)+1)$ for each $i = 1, \dots, r-2$. Since $n \geq 4(k+1)^2 r^4$, by (3) we have

$$|N_{X_i}(x_0)| \geq |X_i|/(2(k+1)(r-1)+1) \geq 2(k+1)(r-1)(r-2) + (k+1).$$

Applying Lemma 8 with $Y_0 = N_{X_0}(x_0), \dots, Y_{r-2} = N_{X_{r-2}}(x_0)$ and $b = 2(k+1)(r-1)$, we obtain $k+1$ vertex disjoint cliques of size $r-1$ in $N(x_0)$. Then G contains a copy of $F_{k+1,r}$ contradicting the fact that G is $F_{k+1,r}$ -free. This completes the proof of Claim 5.

Let $Z_0 := X_0 - X_0^*$ and $Z_i := X_i$ for each $i = 1, \dots, r-2$. By Claim 5 and (3), we have that

$$\begin{aligned} |V(G) - \cup_{i=0}^{r-2} Z_i| &= |V(G) - \cup_{i=0}^{r-2} X_i| + |X_0^*| \\ &\leq (k+1)(2r-3)(r-1) + 2(k+1)(r-1)(r-2) \\ &< 4(k+1)(r-1)^2. \end{aligned}$$

Moreover,

$$\begin{aligned} |Z_0| &= |X_0| - |X_0^*| \\ &\geq \frac{n}{r-1} - (k+1)(2r-3) - 2(k+1)(r-1)(r-2) \\ &= \frac{n}{r-1} - 2(k+1)(r-1)^2. \end{aligned}$$

Claim 6. For every $v \in V - \cup_{i=0}^{r-2} Z_i$, there exists unique $j \in \{0, 1, \dots, r-2\}$ such that $d_{Z_j}(v) < 2(k+1)(r-1)^2 + (k+1)$.

Firstly, suppose that there is $v \in V(G) - \cup_{i=0}^{r-2} Z_i$ such that $d_{Z_j}(v) \geq 2(k+1)(r-1)^2 + (k+1)$ for $j = 0, \dots, r-2$. Let $b := 2(k+1)(r-1)$ and $m := r-1$. Then for $j = 0, \dots, r-2$, $d_{Z_j}(v) \geq mb + (k+1)$ and $d_{Z_j}(z_i) \geq |Z_j| - b$ for all $z_i \in Z_i$, where $i > 0$ and $i \neq j$. Applying Lemma 8, there are $k+1$ vertex disjoint cliques of size $r-1$ in $N(x_0)$. Then G contains a copy of $F_{k+1,r}$ contradicting the fact that G is $F_{k+1,r}$ -free.

Secondly, to show the uniqueness, suppose there are two distinct j_1 and j_2 such that $d_{Z_{j_i}}(v) < 2(k+1)(r-1)^2 + (k+1)$ for both $i = 1, 2$. Since $n \geq 6(k+1)^2 r^4$, we have

$$\begin{aligned} d_G(v) &\leq n - |Z_{j_1} \cup Z_{j_2}| + 4(k+1)(r-1)^2 + 2(k+1) \\ &\leq n - \left[\left(\frac{n}{r-1} - 2(k+1)(r-1)^2 \right) + \left(\frac{n}{r-1} - (k+1)(2r-3) \right) \right] + 4(k+1)(r-1)^2 + 2(k+1) \\ &= \frac{r-2}{r-1}n - \frac{n}{r-1} + 6(k+1)(r-1)^2 + (k+1)(2r-1) \\ &< \frac{r-2}{r-1}n - (k+1). \end{aligned}$$

which contradicts with the fact that $d_G(x_i) > (\frac{r-2}{r-1})n - (k+1)$. This completes the proof of Claim 6.

Adding each $v \in V(G) - (\cup_{i=0}^{r-2} Z_i)$ to $Z_{j(v)}$, we obtain a partition of $V(G) = V_0 \cup \dots \cup V_{r-2}$. Clearly, for $i \in \{0, \dots, r-2\}$,

$$|V_i| \geq |Z_i| \geq \frac{n}{r-1} - 2(k+1)(r-1)^2. \quad (8)$$

For $v_i \in V_i$, by Claims 1 and 6,

$$\Delta(G[V_i]) \leq \Delta(G[Z_i]) + |V(G) - \cup_{i=0}^{r-2} Z_i| \leq 6(k+1)(r-1)^2. \quad (9)$$

Now for $\{i, j\} \in \binom{[r-2] \cup \{0\}}{2}$, we have

$$\begin{aligned} d_{V_j}(v_i) &= d_G(v_i) - d_{G-V_j}(v_i) \\ &\geq \delta(G) - (n - |V_j| - (|V_i| - \Delta(G[V_i]))) \\ &\geq |V_j| - \left(\frac{r-2}{r-1}\right)n - (k+1) - n + \left(\frac{n}{r-1} - 2(k+1)(r-1)^2\right) - 6(k+1)(r-1)^2 \quad (\text{by (8) and (9)}) \\ &> |V_j| - 8(k+1)r^2, \end{aligned}$$

i.e.,

$$d_{V_j}(v_i) \geq |V_j| - 8(k+1)r^2. \quad (10)$$

We will show that V_0, \dots, V_{r-2} be a partition of $V(G)$ satisfying (i), (ii) and (iii). Let $b' := 8(k+1)r^2$. Since $n \geq 4(k+1)^2r^4$, for any j , by (8), we have

$$|V_j| \geq \frac{n}{r-1} - 2(k+1)(r-1)^2 \geq (r-1)b' + 2(k+1).$$

Now we show $d_{V_i}(y) \leq k$ for all $y \in V_i$. Suppose for some $y \in V_i$, $d_{V_i}(y) \geq k+1$. Let $y_1, \dots, y_{k+1} \in N(y) \cap V_i$ in V_i . By Lemma 7, there are $k+1$ cliques D_1, \dots, D_{k+1} such that $y, y_j \in D_j$ and $|D_j| = r$ for each j . Further, $D_j \cap D_l = \{y\}$ for any $\{j, l\} \in \binom{[k+1]}{2}$. $\cup_{i=1}^{k+1} D_i$ is a copy of $F_{k+1, r}$, a contradiction.

Next we show that for all $i \in [r-2] \cup \{0\}$,

$$\sum_{j \in (\{0\} \cup [r-2]) \setminus \{i\}} \nu(G[V_j]) \leq k.$$

Suppose that there exists $l \in \{0\} \cup [r-2]$, say $l = 0$ such that

$$\sum_{j \in [r-2]} \nu(G[V_j]) \geq k+1.$$

Let $\{y_1 z_1, \dots, y_{k+1} z_{k+1}\}$ be a $k+1$ -matching of size $k+1$ in $\cup_{i=1}^{r-2} G[V_i]$. Now, since $n \geq 4(k+1)^2r^4$,

$$\begin{aligned} |\cap_{i=1}^{k+1} (N_{V_i}(y_j) \cap N_{V_i}(z_j))| &> |V_i| - 2(k+1)8(k+1)r^2 \\ &\geq \frac{n}{r-1} - 2(k+1)(r-1)^2 - 16(k+1)^2r^2 \geq 1. \end{aligned}$$

Therefore, there exists a vertex $y \in V_0$ such that $\cup_{j=1}^{k+1} \{y_j, z_j\} \subseteq N(y)$. By Lemma 7, there are $k+1$ cliques D_1, \dots, D_{k+1} such that $y, y_j, z_j \in D_j$ and $|D_j| = r$ for $j \in [k+1]$. Furthermore, $V(D_j) \cap V(D_i) = \{y\}$ for all $\{j, i\} \in \binom{[k+1]}{2}$. It follows that G contains a copy of $F_{k+1, r}$, a contradiction.

Finally, we show that (iii) holds by contradiction. Suppose that there exist i, v , say $i = k+1$ such that $v \in V_{k+1}$ and

$$d_{G[V_{k+1}]}(x) + \sum_{j=1}^k \nu(G[N(x) \cap V_j]) \geq k+1. \quad (11)$$

Write $h_1 := d_{G[V_{k+1}]}(x)$ and $h_2 := \sum_{j=1}^k \nu(G[N(x) \cap V_j])$. Let $v_1, \dots, v_{h_1} \in N_{G[V_{k+1}]}(x)$ and let $\{y_1 z_1, \dots, y_{h_2} z_{h_2}\}$ be a matching of size h_2 in $(\cup_{i=1}^k G[N(x) \cap V_i])$. By (??), both h_1 and h_2 are less than $k+1$. Consider $k+1$ of the cliques $\{v, x_1\}, \dots, \{v, x_s\}, \{v, y_1, z_1\}, \dots, \{v, y_t, z_t\}$. Applying Lemma 7 again, there are $k+1$ cliques D_1, \dots, D_{k+1} in G such that $\{v, x_i\} \subseteq D_i$ for $1 \leq i \leq h_1$ and $\{v, y_j, z_j\} \subseteq D_{j+h_1}$ for $1 \leq j \leq h_2$. Hence G contains $F_{k+1, r}$ as a subgraph, a contradiction. This completes the proof of Lemma 9. \square

4 Proof of Theorem 1

Proof of Theorem 1. By (1), we have $ar(n, F_{k+1, r}) \geq \text{ex}(n, F_{k, r}) + 2$. So the lower bound is followed. Write $c(n, k) = \text{ex}(n, F_{k, r}) + 2$.

For the upper bound, we prove it by contradiction. Suppose that the result does not hold. Then there exists an edge-coloring $c : E(K_n) \rightarrow [c(n, k)]$ such that c is surjective and the colored graph denoted by H contains no rainbow $F_{k+1, r}$. Let G be a rainbow subgraph with exactly $c(n, k)$ colors. Obviously, G is $F_{k+1, r}$ -free and $\Delta(G) > (\frac{r-2}{r-1})n$. We discuss two cases. For an edge subset $Q \subseteq E(H)$, let $c(Q) := \{c(e) \mid e \in Q\}$.

Case 1. $\delta(G) > (\frac{r-2}{r-1})n - (k+1)$.

By Lemma 9, there exists a partition $V(G) = V_0 \cup V_1 \cup \dots \cup V_{r-2}$, such that (i), (ii) and (iii) hold. By Lemma 10, we may get that

$$\begin{aligned} \sum_{0 \leq i < j \leq r-2} |V_i||V_j| &\geq e(G) - f(k, k) \\ &= \text{ex}(n, F_{k, r}) + 2 - f(k, k) \\ &= \begin{cases} \text{ex}(n, K_r) - \frac{3}{2}k + \frac{5}{2}, & \text{if } k \text{ is odd,} \\ \text{ex}(n, K_r) - \frac{5}{2}k + 2, & \text{if } k \text{ is even.} \end{cases} \end{aligned}$$

i.e.,

$$\sum_{0 \leq i < j \leq r-2} |V_i||V_j| \geq \begin{cases} \text{ex}(n, K_r) - \frac{3}{2}k + \frac{5}{2}, & \text{if } k \text{ is odd,} \\ \text{ex}(n, K_r) - \frac{5}{2}k + 2, & \text{if } k \text{ is even.} \end{cases} \quad (12)$$

Let $t = \max\{||V_j| - \frac{n}{r-1}|, j \in \{0, 1, \dots, r-2\}\}$. Without loss of generality, we may assume $t = ||V_0| - \frac{n}{r-1}|$. Then

$$\begin{aligned}
\sum_{0 \leq i < j \leq r-2} |V_i||V_j| &\leq |V_0|(n - V_0) + \sum_{1 \leq i < j \leq r-2} |V_i||V_j| \\
&= |V_0|(n - V_0) + \frac{1}{2} \left(\left(\sum_{j=1}^{r-2} |V_j| \right)^2 - \sum_{j=1}^{r-2} |V_j|^2 \right) \\
&\leq |V_0|(n - V_0) + \frac{1}{2} (n - |V_0|)^2 - \frac{1}{2(r-2)} (n - |V_0|)^2 \\
&< -\frac{r-1}{2(r-2)} t^2 + \frac{r-2}{2(r-1)} n^2.
\end{aligned}$$

where the last second inequality holds by Hölder's inequality, and the last inequality holds since $||V_0| - \frac{n}{r-1}| = t$. That is,

$$\sum_{0 \leq i < j \leq r-2} |V_i||V_j| < -\frac{r-1}{2(r-2)} t^2 + \frac{r-2}{2(r-1)} n^2. \quad (13)$$

By (12) and (13), we obtain that

$$\begin{aligned}
t^2 &< \begin{cases} \frac{2(r-2)}{r-1} \left(\frac{r-2}{2(r-1)} n^2 - \text{ex}(n, K_r) + \frac{3}{2}k - \frac{5}{2} \right), & \text{if } k \text{ is odd,} \\ \frac{2(r-2)}{r-1} \left(\frac{r-2}{2(r-1)} n^2 - \text{ex}(n, K_r) + \frac{5}{2}k - 2 \right), & \text{if } k \text{ is even.} \end{cases} \\
&< \begin{cases} 2 \left(\frac{r-2}{2(r-1)} n^2 - \text{ex}(n, K_r) + \frac{3}{2}k - \frac{5}{2} \right), & \text{if } k \text{ is odd,} \\ 2 \left(\frac{r-2}{2(r-1)} n^2 - \text{ex}(n, K_r) + \frac{5}{2}k - 2 \right), & \text{if } k \text{ is even.} \end{cases}
\end{aligned}$$

By Turán Theorem, we have $\text{ex}(n, K_r) = \frac{r-2}{2(r-1)} n^2 - \epsilon$ where $\epsilon = \frac{l(r-1-l)}{2(r-1)}$ for $l \in \{0, 1, \dots, r-2\}$ with $l \equiv n \pmod{r-1}$. Note that $0 \leq \epsilon \leq \frac{r-1}{8}$, and $\epsilon = 0$ if and only if n is divisible by $r-1$, which implies that

$$\begin{aligned}
t^2 &< \begin{cases} 2 \left(\frac{r-1}{8} + \frac{3}{2}k - \frac{5}{2} \right), & \text{if } k \text{ is odd,} \\ 2 \left(\frac{r-1}{8} + \frac{5}{2}k - 2 \right), & \text{if } k \text{ is even.} \end{cases} \\
&= \begin{cases} \frac{r-1}{4} + 3k - 5, & \text{if } k \text{ is odd,} \\ \frac{r-1}{4} + 5k - 4, & \text{if } k \text{ is even.} \end{cases}
\end{aligned}$$

Let $G_i = G[V_i]$ for each $i = 0, \dots, r-2$. Let G_{cr} denote the $(r-1)$ -partite graph with partitions (V_0, \dots, V_{r-2}) and edge set $\cup_{\{i,j\} \in \binom{[r-2] \cup \{0\}}{2}} E_G(V_i, V_j)$. By Lemma 9, we have

$$e(G_i) \leq f(k, k) \quad \text{for } i \in \{0, \dots, r-2\}. \quad (14)$$

So we may infer that

$$\begin{aligned}
e(G_{cr}) &\geq e(G) - (r-1)f(k, k) \\
&\geq \begin{cases} \text{ex}(n, K_r) + k^2 - k + 2 - (r-1)(k^2 + (k-1)/2), & \text{if } k \text{ is odd,} \\ \text{ex}(n, K_r) + k^2 - \frac{3}{2}k + 2 - (r-1)(k^2 + k), & \text{if } k \text{ is even.} \end{cases} \\
&= \begin{cases} \text{ex}(n, K_r) - (r-2)k^2 - \frac{1}{2}k(r+1) + \frac{1}{2}(r+3), & \text{if } k \text{ is odd,} \\ \text{ex}(n, K_r) - (r-2)k^2 - \frac{1}{2}k(2r+1) + 2, & \text{if } k \text{ is even.} \end{cases}
\end{aligned}$$

For $i \in \{0, \dots, r-2\}$, let $V'_i := \{v \in V_i \mid N_G(v) = V(G) - V_i\}$. Then for $i \in \{0, \dots, r-2\}$, one can see that the following statements hold:

$$(i) \quad |V'_i| \geq |V_i| - (r-2)k^2 - \frac{1}{2}k(2r+1) + 2;$$

$$(ii) \quad \text{for any } v \in V_i, d_{G_{cr}}(v) \geq n - |V_i| - (r-2)k^2 - \frac{1}{2}k(2r+1) + 2.$$

Claim 5. $H[V_i]$ contains no rainbow $K_{1,k+1}$ or a rainbow $(k+1)K_2$.

By contradiction. Suppose that $H[V_i]$ contains a rainbow $K_{1,k+1}$ or a rainbow $(k+1)K_2$. Without loss generality, we may assume that $G[V_0]$ contains $K_{1,k+1}$ or $(k+1)K_2$. Let $R \in \{K_{1,k+1}, (k+1)K_2\}$ be a subgraph of G . For any $v \in V_j$ with $j > 0$, by (ii),

$$d_{G_j}(v_i) \geq |V_j| - (r-2)k^2 - \frac{1}{2}k(2r+1) + 2 - (k+1) \geq |V_j| - 2rk^2.$$

Let $b := 2rk^2$. So $\{V_0, \dots, V_{r-2}\}$ is b -deficiency complete in G . Since $n \geq 16(k+1)^3r^8$, then for any j , $|V_j| > (r-1)b + 2(k+1)$. So by Lemma 7, R can be greedily extended into a rainbow $F_{k+1,r}$ in H , a contradiction. This completes the proof of Claim 5.

By Claim 5, we may assume that for $i \in \{0, \dots, r-2\}$, $H[V_i]$ contains no rainbow $K_{1,k+1}$ or rainbow $(k+1)K_2$. Since $e(G) > t_{r-1}(n)$, there exists $i \in \{0, 1, \dots, r-2\}$ such that $\nu(G_i) > 0$. For $i \in \{0, 1, \dots, r-2\}$, define $S_i = \{x \in V_i \mid N_G(x) = V(G) - V_i\}$ and $s_i = |S_i|$. Then, by (14), we have

$$\begin{aligned}
|S_i| &\geq |V'_i| - 2(k^2 - k + 1) \\
&\geq |V_i| - (r-2)k^2 - \frac{1}{2}k(2r+1) + 2 - 2(k^2 - k + 1) \\
&\geq |V_i| \geq \frac{n}{r-1} - (k+1)(2r-3) - rk^2 - \frac{1}{2}k(2r+1) + 2k \\
&> 10k^3r^7 \quad (\text{since } n \geq 16(k+1)^3r^8).
\end{aligned}$$

For $i \in \{0, 1, \dots, r-2\}$, since $H[S_i]$ contains no rainbow matching of size $k+1$ and

$$\frac{|S_i|}{2k} \geq \frac{10k^3r^7}{2k} > 5k^2r^7.$$

$H[S_i]$ has a monochromatic matching \mathcal{M}_i on color c_i with at least $5k^2r^7$ edges.

Let G' be the rainbow subgraph obtained by deleting the edges colored by c_0 and c_1 from G . Note that

$$e(G') \geq \text{ex}(n, F_{k+1,r}). \quad (15)$$

Claim 6. G' contains no $F_{k,r}$.

Otherwise, suppose that G' contains a copy R of $F_{k,r}$ with center vertex u in V_z . We may choose $j_0 \in \{0,1\}$ such that $j_0 \neq z$. Since \mathcal{M}_{j_0} is a monochromatic matching of size at least $5k^2r^7$ and $V(\mathcal{M}_{j_0}) \subseteq S_{j_0}$. By the definition of S_{j_0} , there exists an edge $v_{j_0}w_{j_0} \in \mathcal{M}_{j_0}$ such that $uv_{j_0}, uw_{j_0} \in E(G')$ and $\{v_{j_0}, w_{j_0}\} \cap V(R) = \emptyset$. By the definition of G' , $c(v_{j_0}w_{j_0}) \notin \{c(e) \mid e \in E(G')\}$. By Lemma 7, we can greedily find a complete graph K in $G' + v_{j_0}w_{j_0}$ with size of r such that $V(K) \cap V(R) = \{u\}$. It follows that $K \cup R$ is a rainbow copy of $F_{k+1,r}$ in H , a contradiction. This completes the proof of Claim 6.

Note that $e(G') \geq \text{ex}(n, F_{k,r})$. By Claim 6, we have $e(G') = \text{ex}(n, F_{k,r})$ and $G' \in EX(n, F_{k,r})$, which also implies that $c_0 \neq c_1$. By Theorem 6, $G_{cr} = G'_{cr} = T(n, r-1)$ and there exists one part set V_z such that $G'[V_z] \in EX(|V_z|, \{K_{1,k}, kK_2\})$ and $e(G'[V_i]) = 0$ for $i \neq z$. One can see that $\Delta(G'[V_z]) \leq k-1$. By Theorem 3, $G'[V_z]$ contains a matching M_z of size $k-1$. Recall $|\mathcal{M}_i| \geq 5k^2r^7$ for $i \in \{0,1\}$. Then there exist $e_0 \in \mathcal{M}_0$ and $e_1 \in \mathcal{M}_1$ such that $\{e_0, e_1\} \cup M_z$ is a rainbow matching of size $k+1$ in H . Furthermore, we may assume that $e_0, e_1 \in E(G)$. Then $\{e_0, e_1\} \cup M_z$ is a matching of size $k+1$ in G . By combining $G_{cr} = T(n, r-1)$, $\{e_0, e_1\} \cup M_z$ can greedily be extended into a copy of $F_{k+1,r}$ in G , a contradiction.

Case 2. $\delta(G) \leq (\frac{r-2}{r-1})n - (k+1)$.

Let $x_0 \in V(G)$ such that $d_G(x_0) = \delta(G) \leq (\frac{r-2}{r-1})n - (k+1)$. Let $G^0 = G$ and $G^1 = G^0 - x_0$. Then by Lemma 4,

$$e(G^1) = e(G^0 - x_0) \geq \text{ex}(n-1, K_r) + f(k-1, k-1) + 2 + (k+1).$$

If there exists a vertex $x_1 \in G^1$ with degree $d_G(x_1) = \delta(G^1) \leq (\frac{r-2}{r-1})(n-1) - (k+1)$, then delete it to obtain $G^2 = G^1 - x_1$. Continue this process as long as $\delta(G^t) \leq (\frac{r-2}{r-1})(n-t) - (k+1)$, and after l steps, we get we get a subgraph G^l with $\delta(G^l) \geq (\frac{r-2}{r-1})l - (k+1)$. By induction, one can see that

$$\text{ex}(n-l, F_{k+1,r}) \geq e(G_l) \geq \text{ex}(n-l, K_r) + (k+1)l + f(k-1, k-1) + 2,$$

which implies that

$$f(k, k) \geq (k+1)l + f(k-1, k-1) + 2. \quad (16)$$

By (16) and Theorem 2, we may get that $l \leq k$. So we have $n-l > \sqrt{(k+1)n} \geq 16(k+1)^3r^8$. Then with the same discussion as Case 1, $H[V[G^l]]$ contains a rainbow $F_{k+1,r}$, a contradiction. This completes the proof of Theorem 1. \square

References

- [1] H. Abbott, D. Hanson and N. Sauer, Intersection theorems for systems of sets, *J. Combin. Theory Ser. A*, **12** (1972), 381–389.
- [2] N. Alon, On a conjecture of Erdős, Simonovits and Sós concerning anti-Ramsey theorems, *J. Graph Theory*, **7** (1983), 91–94.
- [3] V. Chvátal and D. Hanson, Degrees and matchings, *J. Combin. Theory Ser. B*, **20** (1976), 128–138.
- [4] G. Chen, R. Gould, F. Pfender, and B. Wei, Extremal graphs for intersecting cliques, *J. Combin. Theory Ser. B*, **89**(2003), 159–171.
- [5] H. Chen, X. Li, and J. Tu, Complete solution for the rainbow number of matchings, *Discrete Math.*, **309** (2009), 3370–3380.
- [6] P. Erdős and A.H. Stone, On the structure of linear graphs, *Bull. Amer. Math.*, **52** (1946), 1089–1091.
- [7] P. Erdős, and T. Gallai, On maximal paths and circuits of graphs, *Acta Math. Acad. Sci. Hung.*, **10** (1959), 337–356.
- [8] P. Erdős, M. Simonovits and V.T. Sós, Anti-Ramsey theorems, *Infinite and Finite sets (Colloq. Keszthely 1973)*, *Colloq. Math. Soc. János Bolyai*, **10** (1975), 633–643.
- [9] P. Erdős, Z. Füredi, R.J. Gould and D.S. Gunderson, Extremal graphs for intersecting triangles, *J. Combin. Theory Ser. B*, **64** (1995), 89–100.
- [10] S. Fujita, C. Magnant and K. Ozeki, Rainbow generalizations of Ramsey theory: a survey, *Graphs Combin.*, **26** (2010), 1–30.
- [11] S. Gilboa, and Y. Roditty, Anti-Ramsey numbers of graphs with small connected components, *Graphs Combin.*, **32** (2016), 649–662.
- [12] R. Haas, and M. Young, The anti-Ramsey number of perfect matching, *Discrete Math.*, **312** (2012), 933–937.
- [13] T. Jiang, I. Schiermeyer and D.B. West, The Erdős-Simonovits-Sós conjecture for $k \leq 7$, *unpublished manuscript*.
- [14] W. Liu, H. Lu and X. Luo, Anti-Ramsey Number of Friendship Graphs, preprint.
- [15] J. Montellano-Ballesteros and V. Neumann-Lara, An anti-Ramsey theorem, *Combinatorica*, **22** (2002), 445–449.
- [16] J. Montellano-Ballesteros and V. Neumann-Lara, An anti-Ramsey theorem on cycles, *Graphs Combin.*, **21** (2005), 343–354.
- [17] I. Schiermeyer, Rainbow numbers for matchings and complete graphs, *Discrete Math.*, **286** (2004), 157–162.

- [18] F. Wu, S. Zhang, B. Li and J. Xiao, Anti-Ramsey numbers for vertex-disjoint triangles, *Discrete Math.*, **346** (2023), 113–123.
- [19] L. Yuan and X. Zhang, Anti-Ramsey numbers of graphs with some decomposition family sequences, *arXiv:1903.10319*, 2019.