One-step worst-case optimal bivariate algorithm for bi-objective optimization

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Abstract

Bivariate one-step worst-case optimal algorithm for bi-objective Lipschitz optimization problems is presented. The feasible region is partitioned by simplices. The bivariate algorithm is used to compute discreate Pareto front representation with predefined accuracy. The proposed algorithm is constructed by generalizing one-step worst-case optimal univariate algorithm for bivariate problems. Lipschitz condition with respect to Euclidean distance is used.

The problem

The problem of bivariate bi-objective non-convex optimization

$$\min_{\mathbf{x}\in A} f(\mathbf{x}), \quad f(\mathbf{x}) = (f^1(\mathbf{x}), f^2(\mathbf{x}))^T, \quad A \subset \mathbb{R}^2 \quad (1)$$

is considered, where the feasible region is bounded

$$A = \{\mathbf{x} : a_k \le x_k \le b_k, k = 1, 2\} \quad (2)$$

Lipschitz condition

Practical objective functions normally have bounded rate of change. Which allows the detection of the global minimum with a prescribed accuracy [1].

If a function $f(x)$ is in Lipschitz objective functions class $\Phi(L)$ then it has a constant bound L on the first derivative. Lets say

$$\mathbf{x} \in A \subset \mathbb{R}^d \quad (3)$$

$$L = (L^1, ..., L^n), \quad L^k > 0, \quad k = 1, ..., n \quad (4)$$

$$f(\mathbf{x}) = (f^1(\mathbf{x}), ..., f^n(\mathbf{x})), \quad f(\mathbf{x}) \in \Phi(L) \quad (5)$$

and A is bounded, then the Lipschitz condition holds:

$$|f^k(\mathbf{x}) - f^k(\mathbf{t})| \le L^k ||\mathbf{x} - \mathbf{t}||, \quad k = 1, ..., n, \quad (6)$$

for $\mathbf{x} \in A$, $\mathbf{t} \in A$. Here $||\cdot||$ is usually *Euclidean distance*, but other distance norms can also be used, e.g. *city-block distance* is used in [3]. Lipschitz condition with respect to Euclidean distance is used in this paper.

Univariate one-step worst-case optimal algorithm

The one-step worst-case optimal algorithm for bi-objective univariate optimization introduced in [2] uses **tolerance** (also called **tightness** [3]) to bound the error of Pareto front approximation. The tolerance of the lower Lipschitz bound is defined as the maximum distance between local lower Lipschitz bound $V(f(x_1), f(x_2), x_1, x_2)$ and $\{f(x_1), f(x_2)\}$, $x_1 < x_2$ and it is denoted by

$$\Delta(f(x_1), f(x_2), x_1, x_2) = \max\left(\min_{\xi\in V(f(x_1),f(x_2),x_1,x_2)} ||\xi - f(x_1)||, \min_{\xi\in V(f(x_1),f(x_2),x_1,x_2)} ||\xi - f(x_2)||\right) \quad (7)$$

The algorithm iteratively decreases the error of Pareto front approximation by decreasing the tolerance Δ. The algorithm stops when the maximum number of iterations *max_iters* is exceeded or the maximum of the tolerance is lower than predefined accuracy ϵ.

Univariate algorithm adaptation for two variable problems

To use the univariate tolerance definition (7) for the bi-variate optimization problems we have to reduce two dimensional space, e.g. simplex, tolerance calculation to one dimensional tolerance calculation. The local lower Lipschitz bound for 2-simplex can be approximated by its longest edge local lower Lipschitz bound. This can be achieved by multiplying the actual Lipschitz constant L^k by a constant $\hat{\alpha}^k$ before calculating the longest 2-simplex edge local lower Lipschitz bound, where $\hat{\alpha}^k$ is

$$\hat{\alpha}^k = \frac{b_3 + a_3 - 2\min_{\mathbf{x}\in ABC} \mathcal{L}^k(\mathbf{x})}{L^k ||A - B||} \quad (8)$$

$$\min_{\mathbf{x}\in ABC} \mathcal{L}^k(\mathbf{x}) = \max(AC' \cap Cone_B, BC' \cap Cone_A) \quad (9)$$

, here $Cone_A$, $Cone_B$ are cones constructed by Lipschitz lower bound drawn from vertices A, B. Here AC', BC' are generatrixes of $Cone_A$, $Cone_B$, which pass through C point's projection C' on to the cone's surface.

Pseudocode

1. Initialization: start by partitioning the feasible region by simplices. Compute the function values at the simplex vertices and the lower Lipschitz bound, compose the first approximation of the Pareto front.
2. Choose the candidate simplex with the largest tolerance Δ. If the tolerance is smaller than ϵ, stop.
3. Divide the chosen simplex into two smaller simplex by dividing the longest edge.
4. Compute the function value at new simplex vertex and the Lower Lipschitz bounds, update the current approximation of the Pareto front.
5. If the predefined number of function evaluations is not exceeded, return to Step 2.

Combinatorial triangulation of feasable region

Combinatorial triangulation was chosen for the proposed algorithm. This method is deterministic and based on enumeration of permutations of $1, \ldots, d$. The number of simplices is known in advance and is equal to $d!$. For the higher dimension than 2 other triangulation methods should be chosen.

Experimental results

The experiments were made with bi-objective problem which has feasible region $[0,1]^2$ and Lipschitz constants $2, 1$:

$$\begin{aligned} f^1(x_1, x_2) &= (x_1 - 1)x_2^2 + 1 \\ f^2(x_1, x_2) &= x_2 \end{aligned} \quad (10)$$

, but experiments with greater objective function diversity should be made to get objective assesment.

Conclusions

One-step worst-case optimal bivariate algorithm for bi-objective optimization was proposed in this paper. It was shown that it is possible to generalise the one-step worst-case optimal univariate algorithm for the higher dimensional problems using Lipschitz condition with respect to Euclidean distance. In addition, it was shown that partitioning by simplices can be used with the proposed algorithm. Thurthermore, the proposed algorithm has relatively smaller part of additional computations than the alternative bivariate algorithm [3].

Future work

In the future work, the proposed algorithm should be generalized for a higher dimension and other feasable region triangulation methods should be analysed. The benefits of hyper-simplex approximation by a hyper-tetrahedron should be investigated. Also the proposed algorithms should be compared with other global optimization algorithms in more details.

References

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