On one-step worst-case optimal trisection in univariate bi-objective Lipschitz optimization

Antanas Žilinskas   Gražina Gimbutienė

Institute of Mathematics and Informatics, University of Vilnius

Abstract

The bi-objective Lipschitz optimization with univariate objectives is considered. The concept of the tolerance of the lower Lipschitz bound over an interval is generalized to arbitrary subintervals of the search region. The one-step worst-case optimality of trisecting an interval with respect to the resulting tolerance is established. The theoretical investigation supports the previous usage of trisection in other algorithms. The trisection-based algorithm is introduced. Some numerical examples illustrating the performance of the algorithm are provided. [1]

The problem and definitions

A one-dimensional bi-objective Lipschitz function $f(t) = (f_1(t), f_2(t)), t \in [x_j, x_{j+1}]$, is considered, where

$$|f_k(u) - f_k(t)| \le L_k |u - t|, k = 1, 2, \quad (1)$$

for $\forall u, t \in [x_j, x_{j+1}]$, $L = (L_1, L_2)^T, L_k > 0, k = 1, 2$.

We will use the following notation:

$$f_1(x_i) = y_i, f_2(x_i) = z_i, i = j, j+1, \quad (2)$$

$$\delta y = |y_j - y_{j+1}|, \delta z = |z_j - z_{j+1}|, \quad (3)$$

$$C = \frac{1}{2}\sqrt{L_1^2 + L_2^2}, \quad (4)$$

$$\Psi = \{f(\cdot) : \text{function } f(\cdot) \text{ satisfies (1) and } f(x_i) = (y_i, z_i), i = j, j+1\}. \quad (5)$$

Definition 1. For any interval $[x_j, x_{j+1}]$ with known $f(x_j), f(x_{j+1})$

$$\Delta((x_j, x_{j+1}), (f(x_j), f(x_{j+1}))) = C \max\left((x_{j+1} - x_j) - \frac{\delta y}{L_1}, (x_{j+1} - x_j) - \frac{\delta z}{L_2}\right). \quad (6)$$

Definition 2. For an interval $[r_1, r_2], r_1 \le r_2$, such that $[r_1, r_2] \subseteq [x_j, x_{j+1}]$, and known $f(x_j), f(x_{j+1})$, the tolerance is defined as:

$$\bar{\Delta}((r_1, r_2, x_j, x_{j+1}), (f(x_j), f(x_{j+1}))) = \max_{f(\cdot) \in \Psi} \Delta((r_1, r_2), (f(r_1), f(r_2))). \quad (7)$$

Worst-case optimal trisection

We are interested in trisecting the interval $[r_1, r_2]$, $x_j \le r_1 < r_2 \le x_{j+1}$, using points $\tilde{a}, \tilde{b} \in (r_1, r_2), \tilde{a} < \tilde{b}$. The worst-case optimality criterion for the choice of the points is defined as follows:

$$(\tilde{a}, \tilde{b}) = \arg\min_{a,b} \bar{\Delta}^*((a, b, r_1, r_2, x_j, x_{j+1}), (f(x_j), f(x_{j+1}))), \quad (8)$$

$$\bar{\Delta}^*(\cdot) = \max_{f(\cdot) \in \Psi} \max[\bar{\Delta}((r_1, a, x_j, b), (f(x_j), f(b))), \bar{\Delta}((a, b, x_j, x_{j+1}), (f(x_j), f(x_{j+1}))), \bar{\Delta}((b, r_2, a, x_{j+1}), (f(a), f(x_{j+1})))]. \quad (9)$$

Experimental results

Pseudocode

Theorems

Lemma 1. Denote

$$\beta = \beta((x_j, x_{j+1}), (f(x_j), f(x_{j+1}))) = (x_{j+1} - x_j) - \min\left(\frac{\delta y}{L_1}, \frac{\delta z}{L_2}\right). \quad (10)$$

Then

$$\bar{\Delta}((r_1, r_2, x_j, x_{j+1}), (f(x_j), f(x_{j+1}))) = C \times \begin{cases} r_2 - r_1, & \text{if } r_2 - r_1 \le \beta, \\ \beta & \text{otherwise.} \end{cases} \quad (11)$$

Theorem 1. Let $x_j \le r_1 < r_2 \le x_{j+1}$ and β defined by (10). The worst-case optimal division points $\tilde{a}$ and $\tilde{b}$ of (r_1, r_2), solving (8), are defined as follows:

1) $\tilde{a} = r_1 + \frac{1}{3}(r_2 - r_1), \tilde{b} = r_1 + \frac{2}{3}(r_2 - r_1)$, if $\frac{1}{3}(r_2 - r_1) < \beta$;
2) an arbitrary choice $\tilde{a} \in (r_1, r_2), \tilde{b} \in (r_1, r_2)$, otherwise.

Then $\bar{\Delta}^*(\cdot)$ equals $\frac{C}{3}(r_2 - r_1)$ in the first case, and $C\beta$ in the second.

Conclusions

References

[1] A. Žilinskas and G. Gimbutienė, *On one-step worst-case optimal trisection in univariate bi-objective Lipschitz optimization*, Communications in Nonlinear Science and Numerical Simulation, 35C (2016), pp. 123–136.