On Spatial Transformations

Lecture 3

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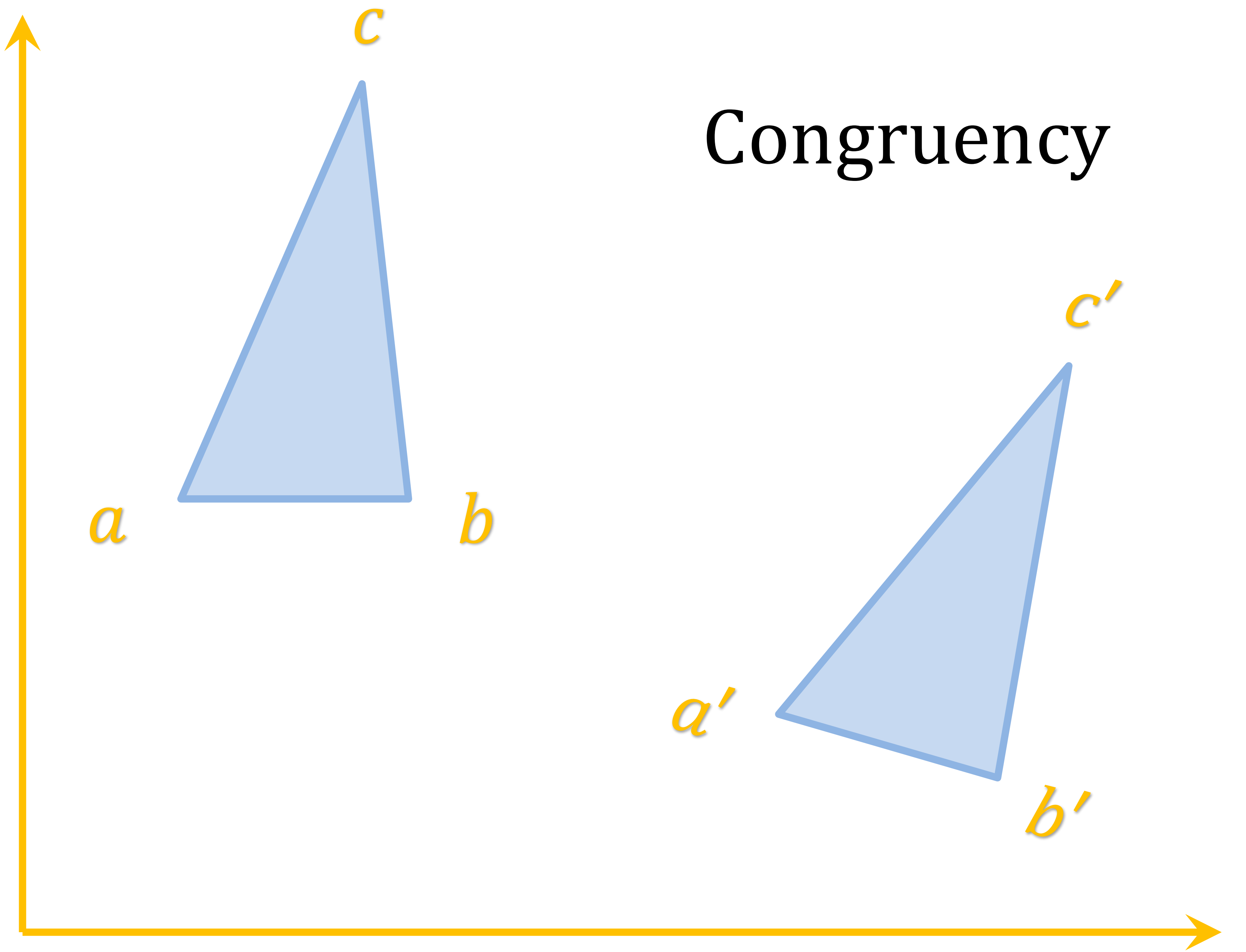
Congruency



Pose and Shape of An Object

- Pose

  The position and orientation of the object with respect to the camera.

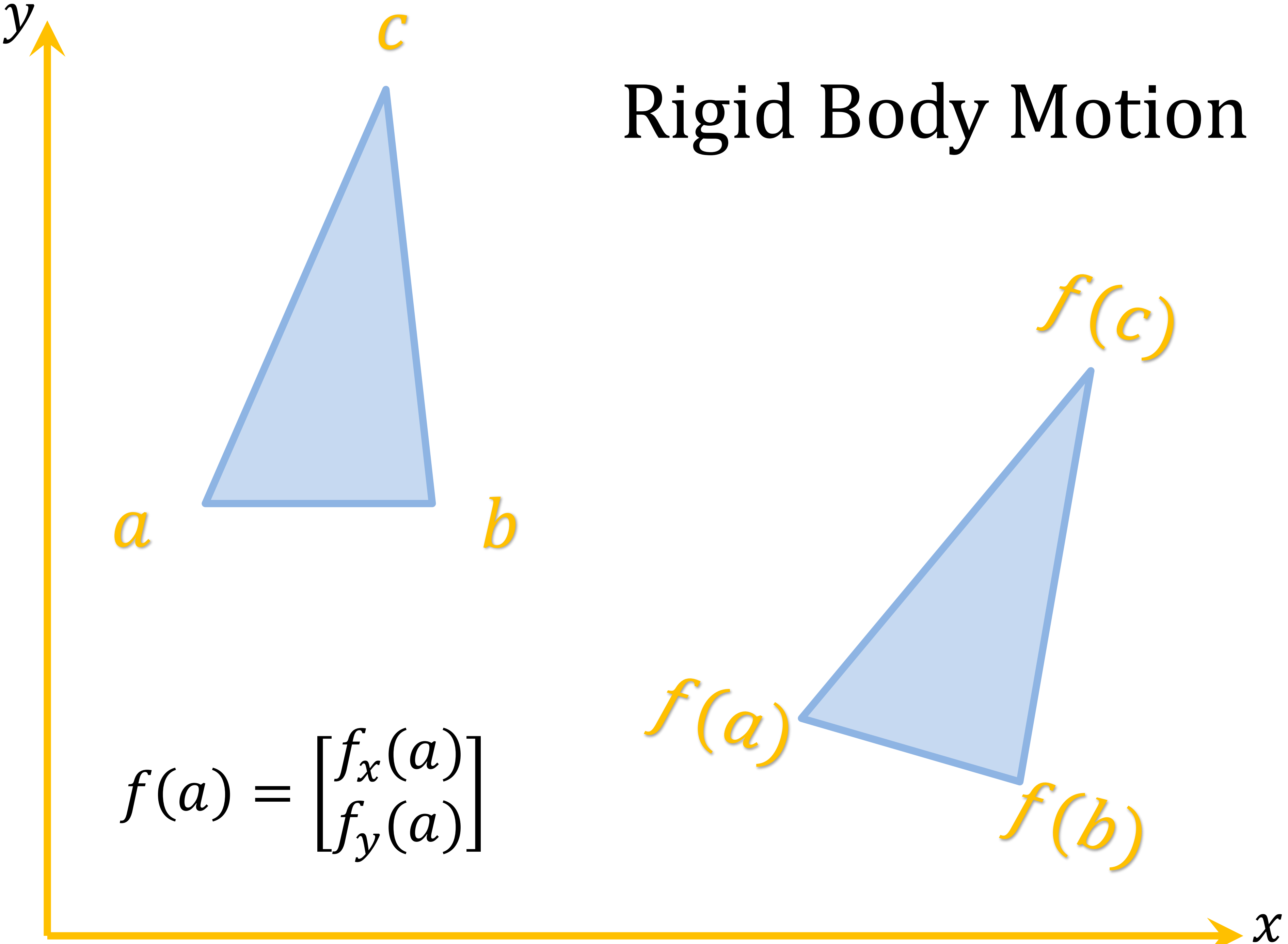
- Shape

  The coordinates of the points of an object relative to a coordinate frame on the object.

Rigid Object

- An object that does not change the shape.
- The distance between any two points on the object does not change over time.
- Euclidean transformation =
  isometry =
  rigid body motion

Rigid Body Motion



$$f(a) = \begin{bmatrix} f_x(a) \\ f_y(a) \end{bmatrix}$$

Euclidean transformations are transformations that preserve distances between pairs of points.

$$|f(a) - f(b)| = |a - b|$$

Translations are isometries since

$$f(a) = a + t$$

$$|f(a) - f(b)| = |(a + t) - (b + t)| = |a - b|.$$

Orthogonal transformations are also isometries.

Linear transformations : For some matrix A,

$$f(a) = A \cdot a.$$

Orthogonal transformations are linear transformations which preserve inner products:

$$a \cdot b = f(a) \cdot f(b).$$

Examples: rotations and reflections.

Theorem:

Any rigid body motion can be expressed as an orthogonal transformation A followed by a translation t.

$$f(a) = A\,a + t$$

Property 1:

Orthogonal transformations preserve norms.

Proof:

$$a \cdot a = f(a) \cdot f(a)$$

$$\Downarrow$$

$$|a| = |f(a)|$$

Property 2:

Orthogonal transformations are isometries.

Proof:

$$\big(f(a) - f(b)\big) \cdot \big(f(a) - f(b)\big)$$

$$= |f(a)|^2 + |f(b)|^2 - 2\big(f(a) \cdot f(b)\big)$$

$$= |a|^2 + |b|^2 - 2(a \cdot b)$$

$$= (a - b) \cdot (a - b)$$

Orthogonal Matrices

Let f be an orthogonal transformation whose action can be represented by

$$f(a) = A\,a$$

Since it preserves inner products,

$$f(a) \cdot f(b) = a \cdot b$$

we have: $(A\,a)'A\,b = a'(A'A)b \;=\; a'b$

$$A'A = I \;\;\Rightarrow\;\; A' = A^{-1}$$

$$\det(A)^2 = 1 \;\;\Rightarrow\;\; \det(A) = \pm 1$$

A has orthonormal rows and columns.

Orthogonal Matrices in 2D

Rotation, $\det = +1$: $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

Reflection, $\det = -1$: $A = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}$

two successive rotations = another rotation:

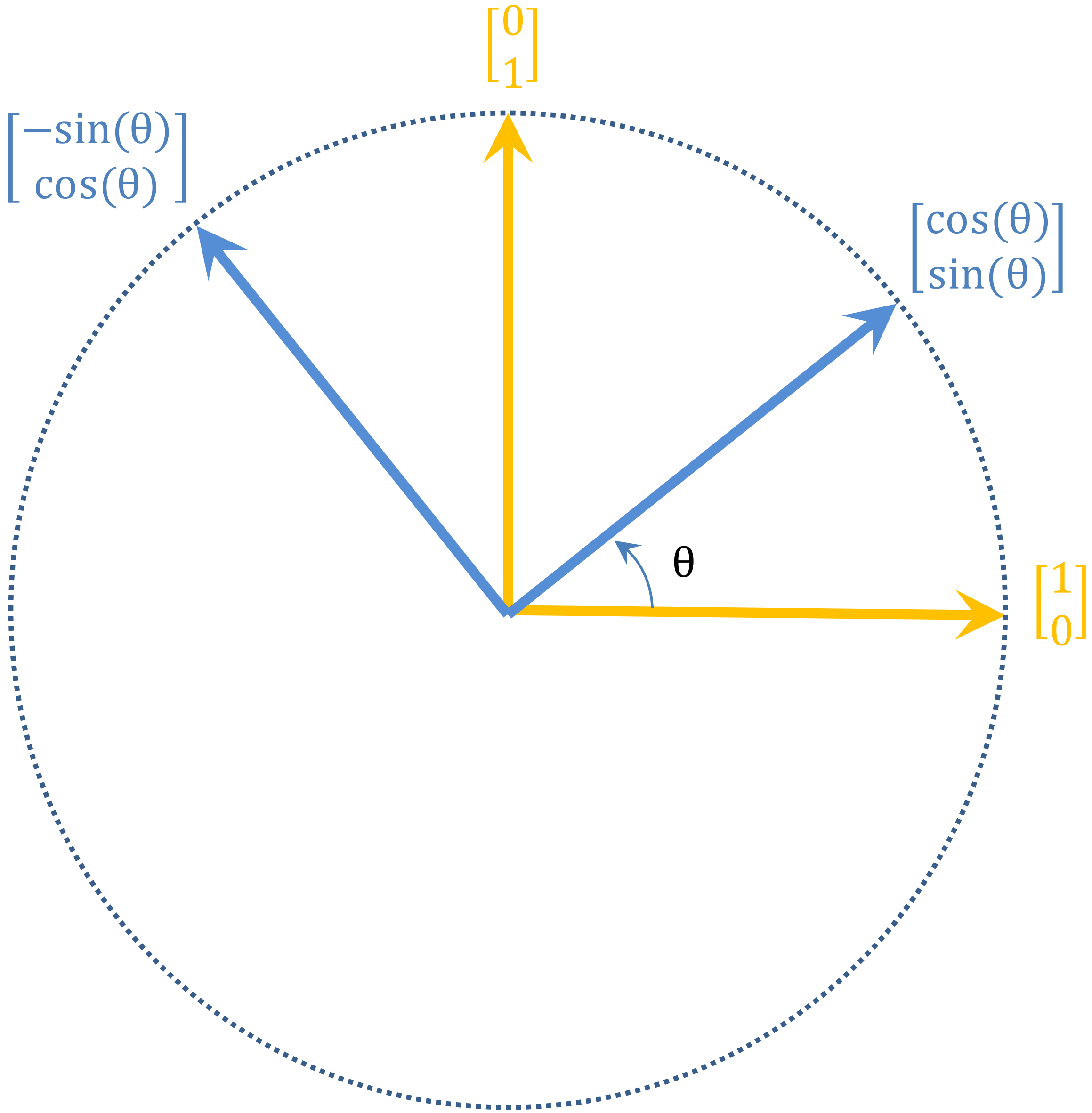
$$Rot(\beta)Rot(\alpha) = Rot(\alpha + \beta)$$

two successive reflections = a rotation:

$$Ref(\beta)Ref(\alpha) = Rot(\alpha + \beta)$$

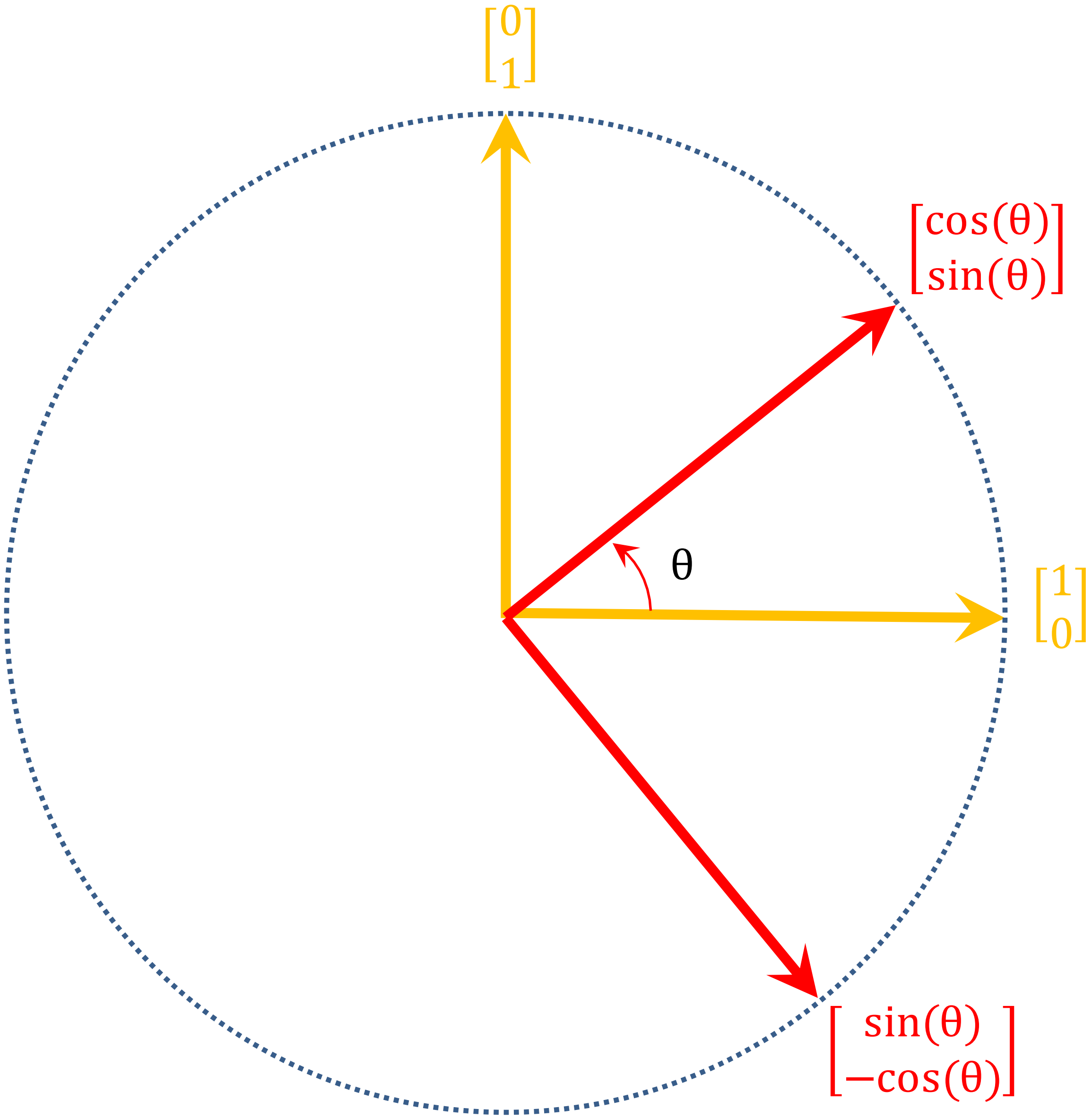
Rotation in 2D

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$



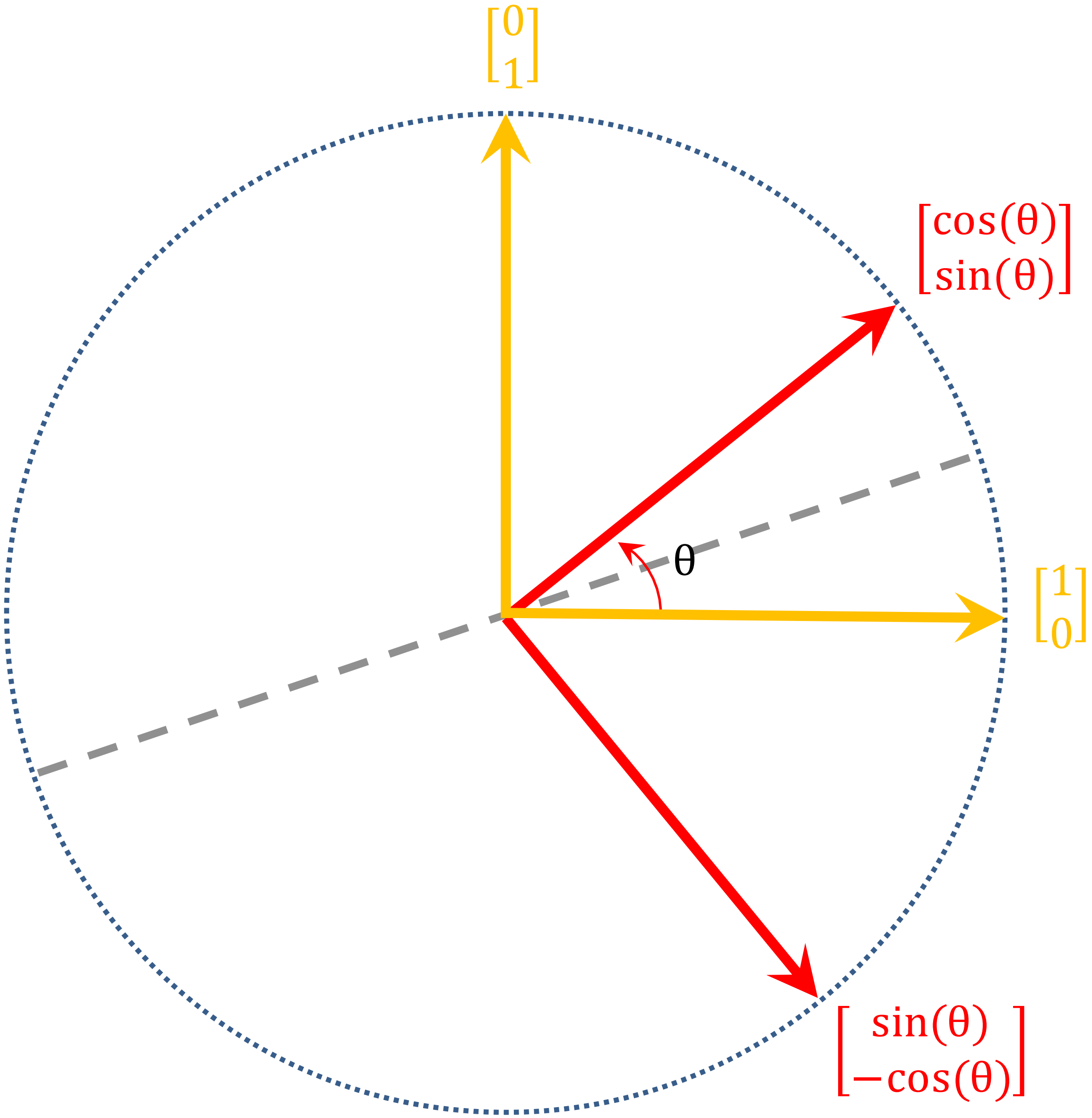
Reflection in 2D

$$A = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}$$



Reflection in 2D

$$A = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}$$



Orthogonal Matrices in 3D

- 9 parameters

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

- 6 constraints

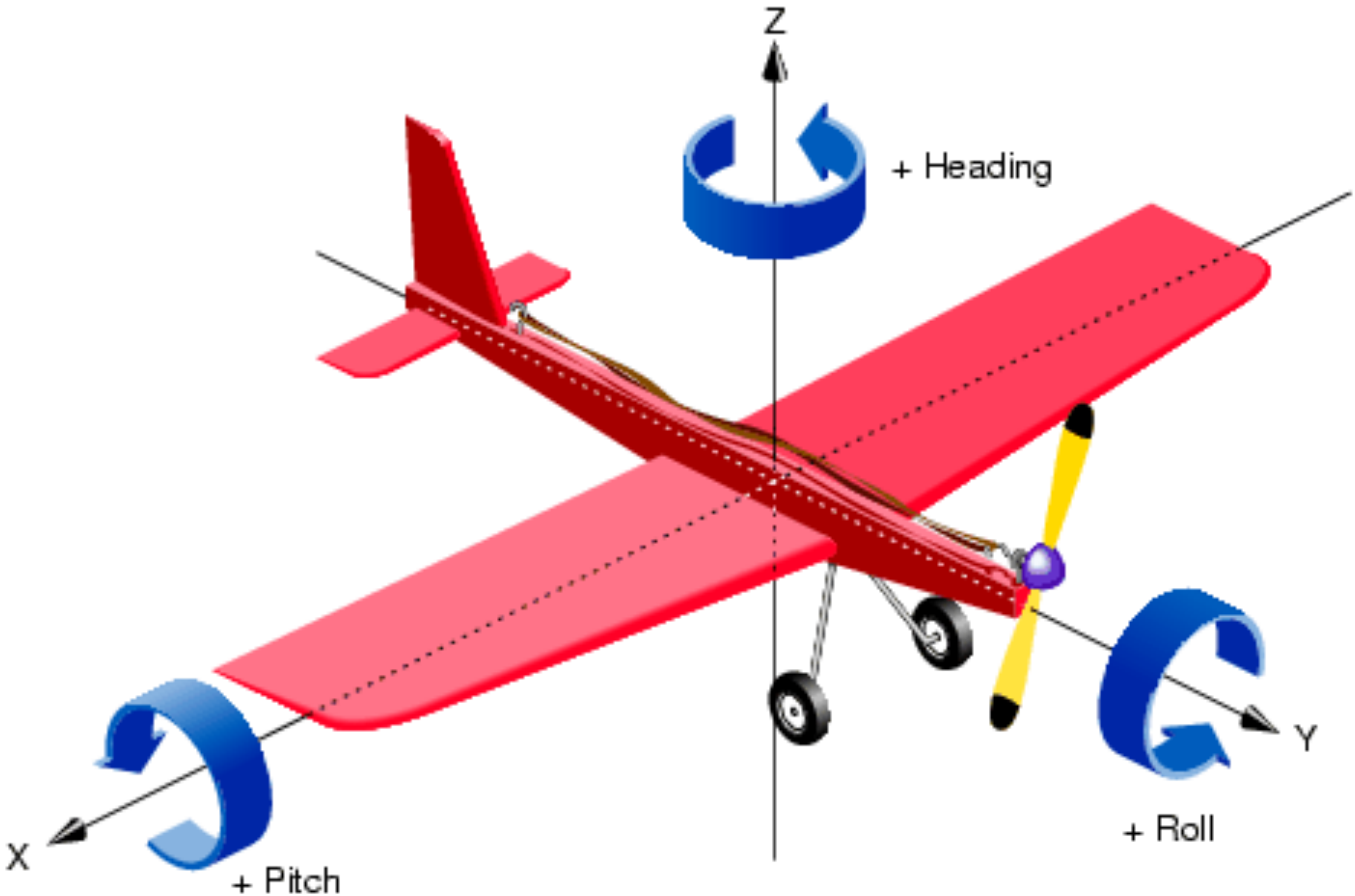
$$A'A = I$$

- 3 degrees of freedom

Rotations in 3D

$$R_z = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$



Z
+ Heading
X
+ Pitch
Y
+ Roll

Parameterizing Rotations in 3D

1. Euler angles for rotations about 3 axes
2. Quaternions generalizing complex numbers from 2D to 3D. Note that a complex number can represent a rotation in 2D.
3. Axis plus amount of rotation. We prefer this representation of an orthogonal matrix. Given rotation matrix R, we want to find (s, θ) – the unit vector of the axis of rotation and the amount of rotation.

Skew Symmetric Matrix

Cross (or vector) product of two vectors:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \wedge \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

Define the skew symmetric matrix of vector a:

$$\hat{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

we turn cross products into matrix multiplications:

$$a \wedge b = \hat{a} b$$

Motion of A Rotating Point

Let vector $q(t)$ denote the position of a point q on a rotating body over time t.

Let the direction of vector ω specify the axis of rotation, and the magnitude the angular speed.

The equation of motion is:

$$\dot{q}(t) = \omega \wedge q(t) = \widehat{\omega} q(t).$$

Solution

The solution to this differential equation is:

$$q(t) = q(0)e^{\hat{\omega}t}$$

The matrix exponential is defined as:

$$e^{\hat{\omega}t} = I + \hat{\omega}t + \frac{(\hat{\omega}t)^2}{2!} + \frac{(\hat{\omega}t)^3}{3!} + \cdots$$

Collecting odd and even terms separately, we get to Roderigues' Formula for a rotation matrix R:

$$R = e^{\theta\hat{s}} = I + \sin(\theta)\ \hat{s}\ + \big(1 - \cos(\theta)\big)\,\hat{s}^{\,2}$$

$$\theta = |\omega|t, \qquad s = \frac{\omega}{|\omega|}$$

Compositions of two isometries is an isometry.

$$f_1(a) = R_1 a + t_1$$

$$f_2(a) = R_2 a + t_2$$

$$f_2(f_1(a)) = R_2(R_1 a + t_1) + t_2$$

$$= (R_2 R_1)a + (R_2 t_1 + t_2)$$

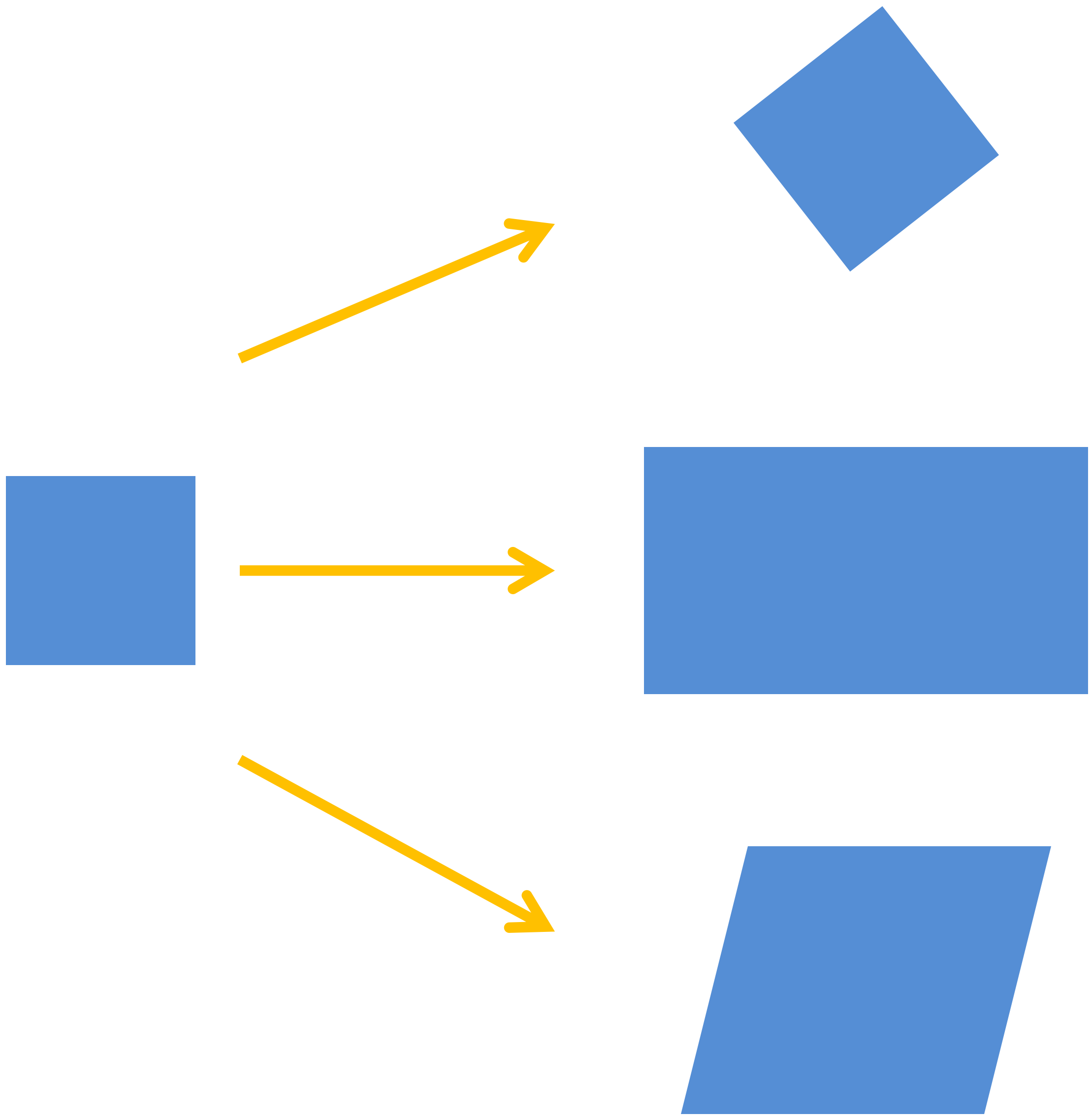
$$= R_3 \, a + t_3$$

Affine Transformations

An affine transformation is a nonsingular linear transformation followed by a translation.

$$f(a) = A\,a + t$$

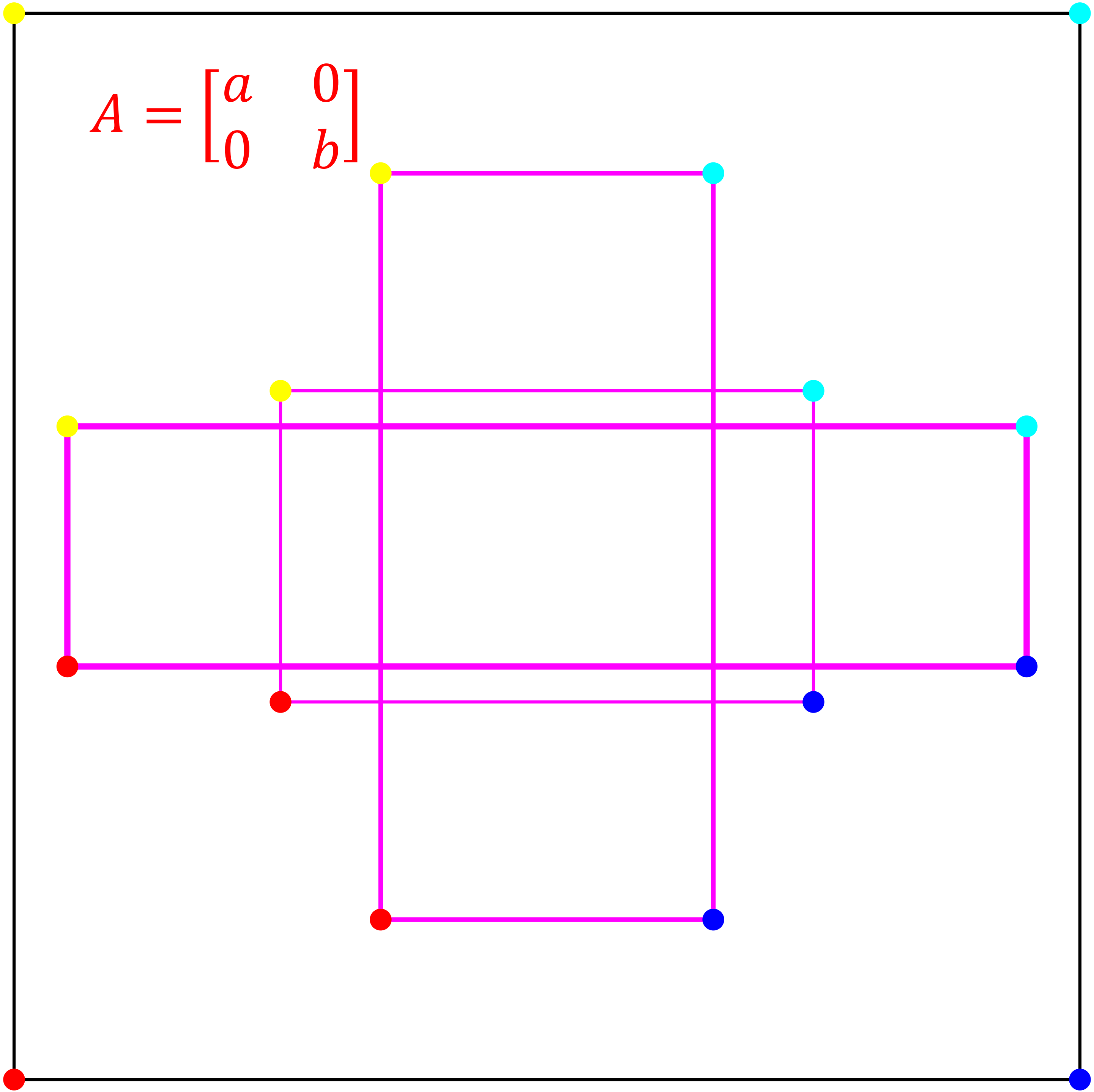
Affine Transform Examples

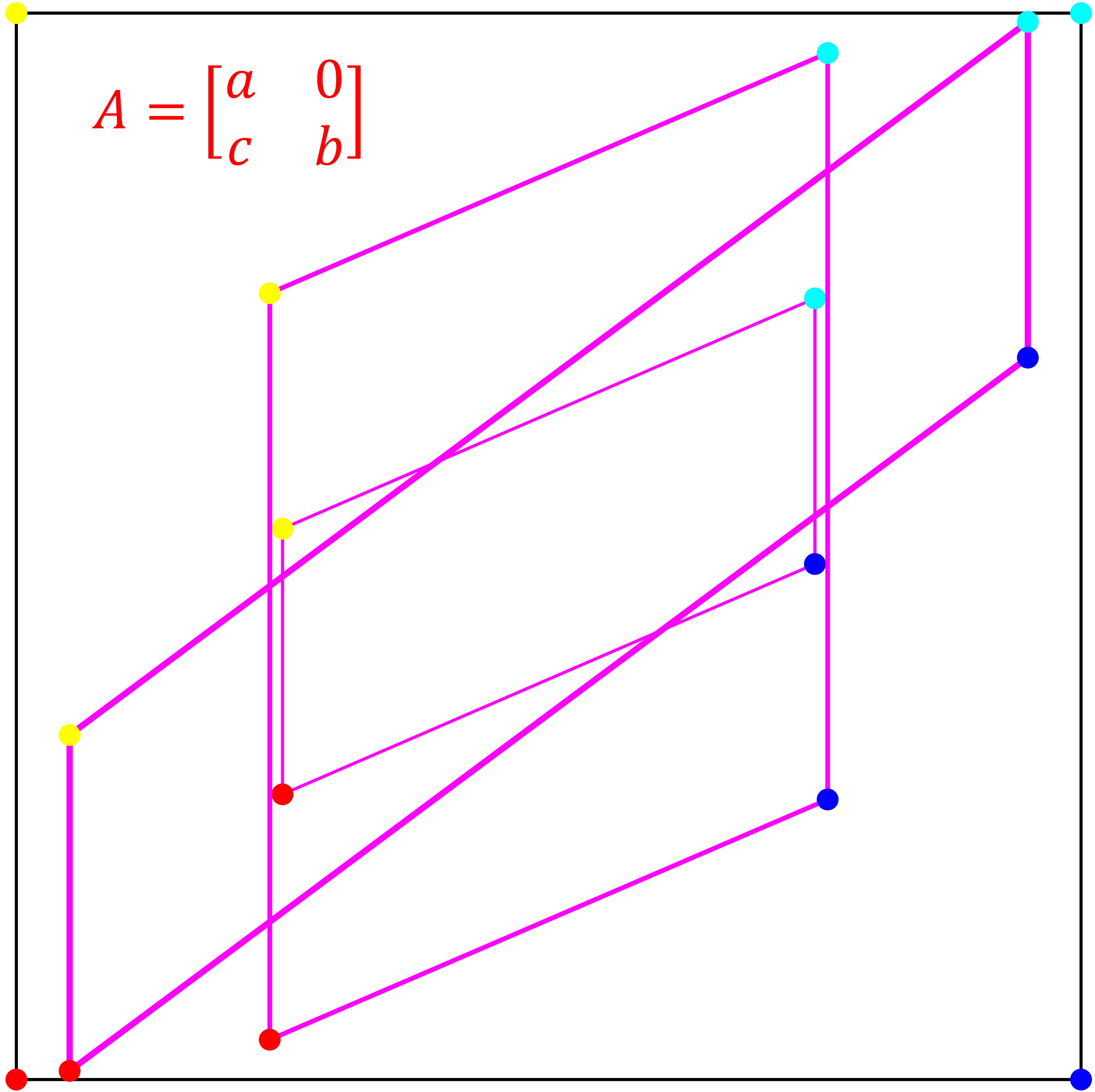


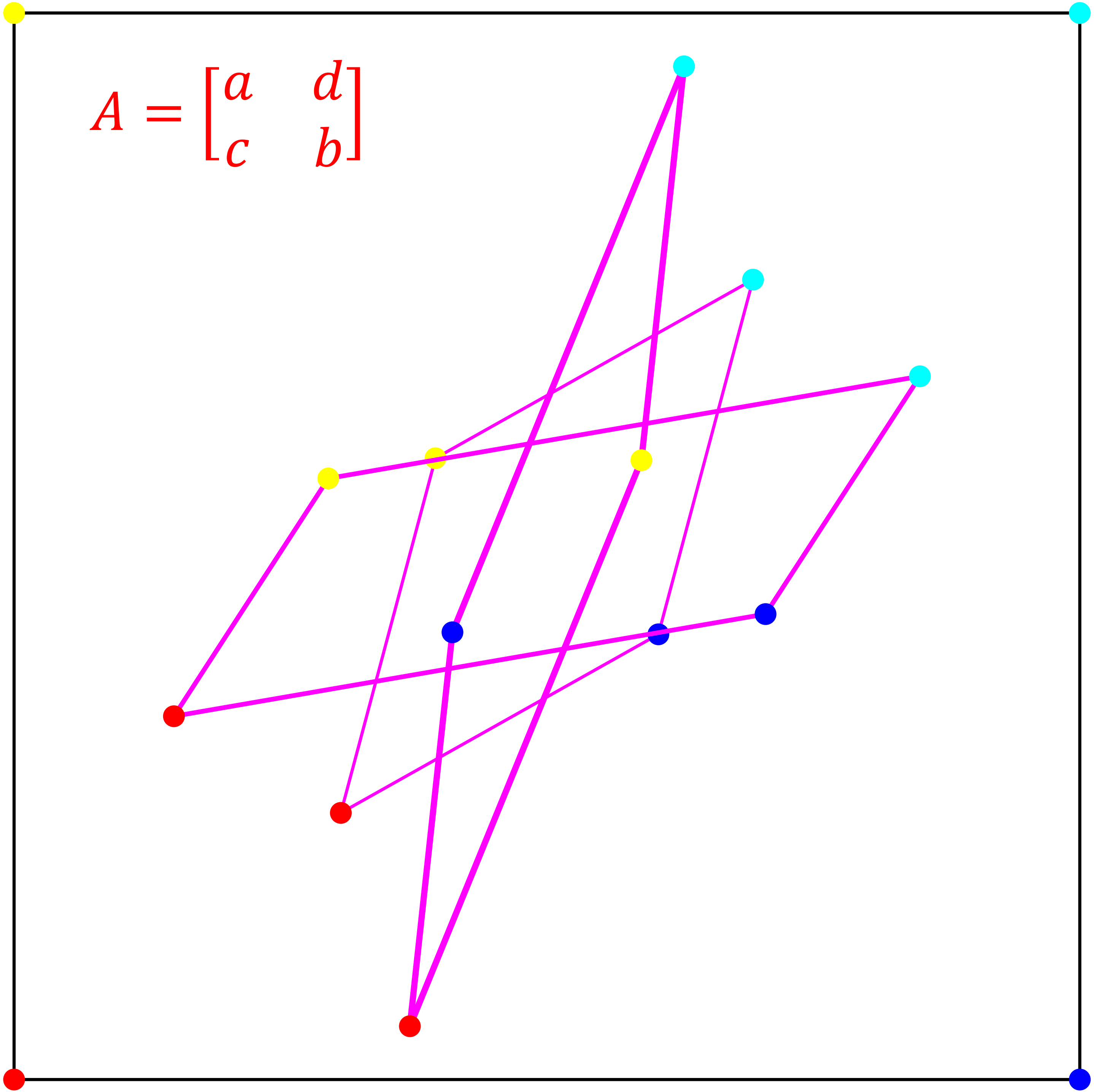
rotation

anisotropic scaling

shearing


$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$


$A = \begin{bmatrix} a & 0 \\ c & b \end{bmatrix}$


$A = \begin{bmatrix} a & d \\ c & b \end{bmatrix}$

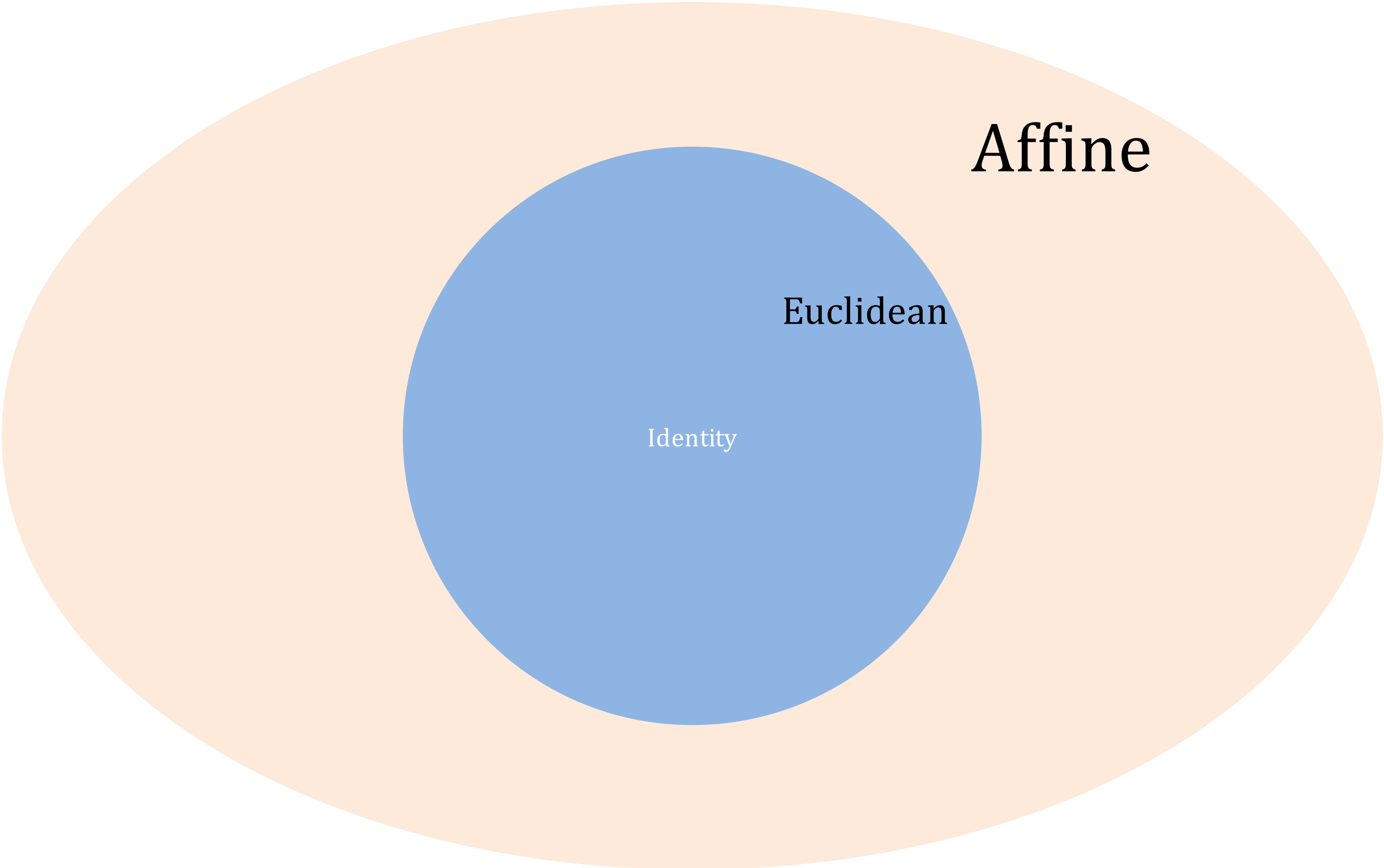
Invariants under Transformation

Euclidean	Affine
length	parallelism
angle	midpoints
area	⋮
⋮	

Number of Parameters Required

	Isometry	Affine
in 2D	1 + 2 = 3	4 + 2 = 6
In 3D	3 + 3 = 6	9 + 3 = 12

The Big Picture



Projective Transformations

- Under perspective projection, parallel lines can map to lines that intersect. Therefore, this cannot be modeled by an affine transform!
- Projective transformations are a more general family which includes affine transforms and perspective projections.
- Projective transformations are linear transformations using homogeneous coordinates.