On Projective Transformations

Lecture 4

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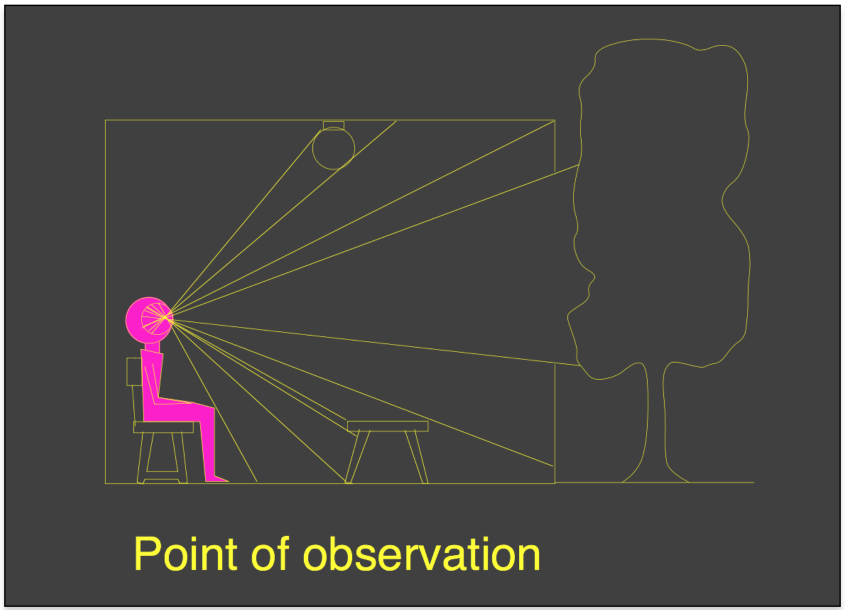
(r,g,b) = (255,255,251) (222,15,7) (0,0,0) (89,120,1)

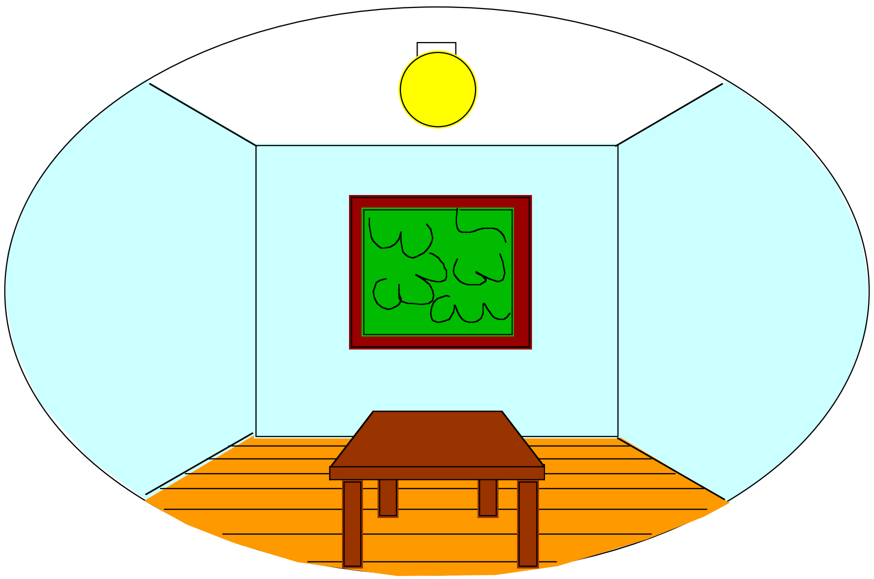
(246,99,0)

(19,37,87) (255,255,115)

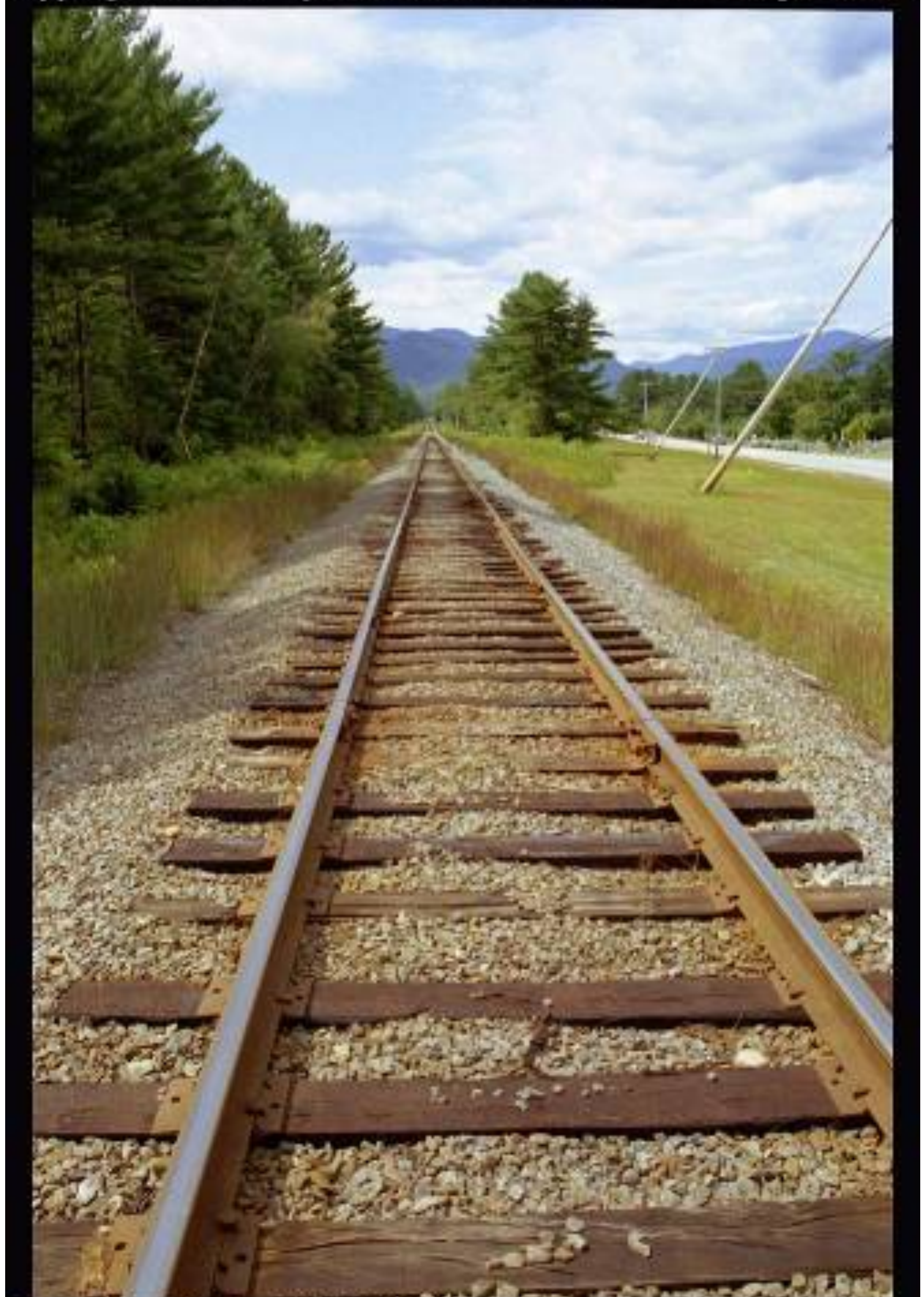
The Lost 3-rd Dimension



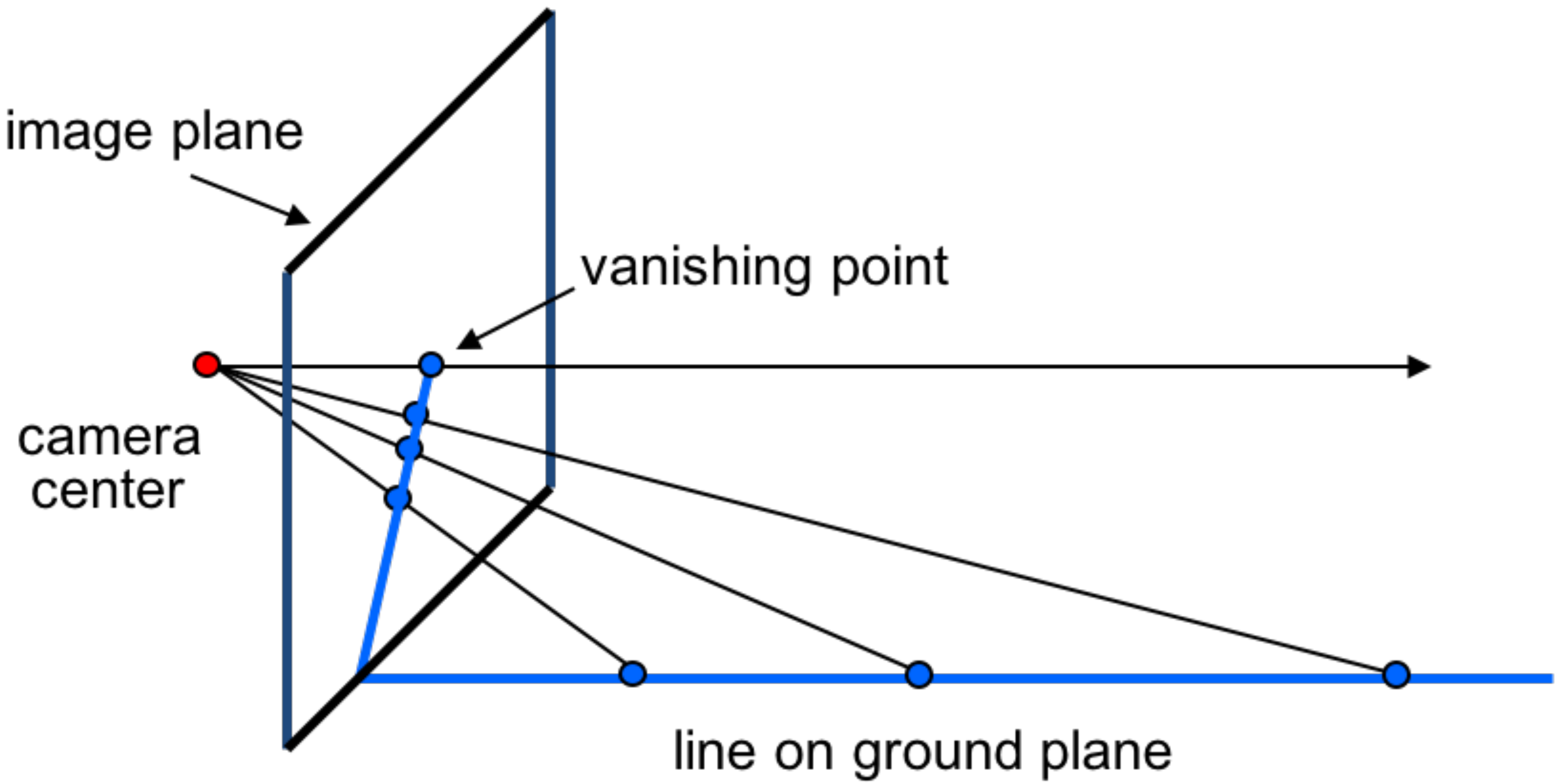
3D world



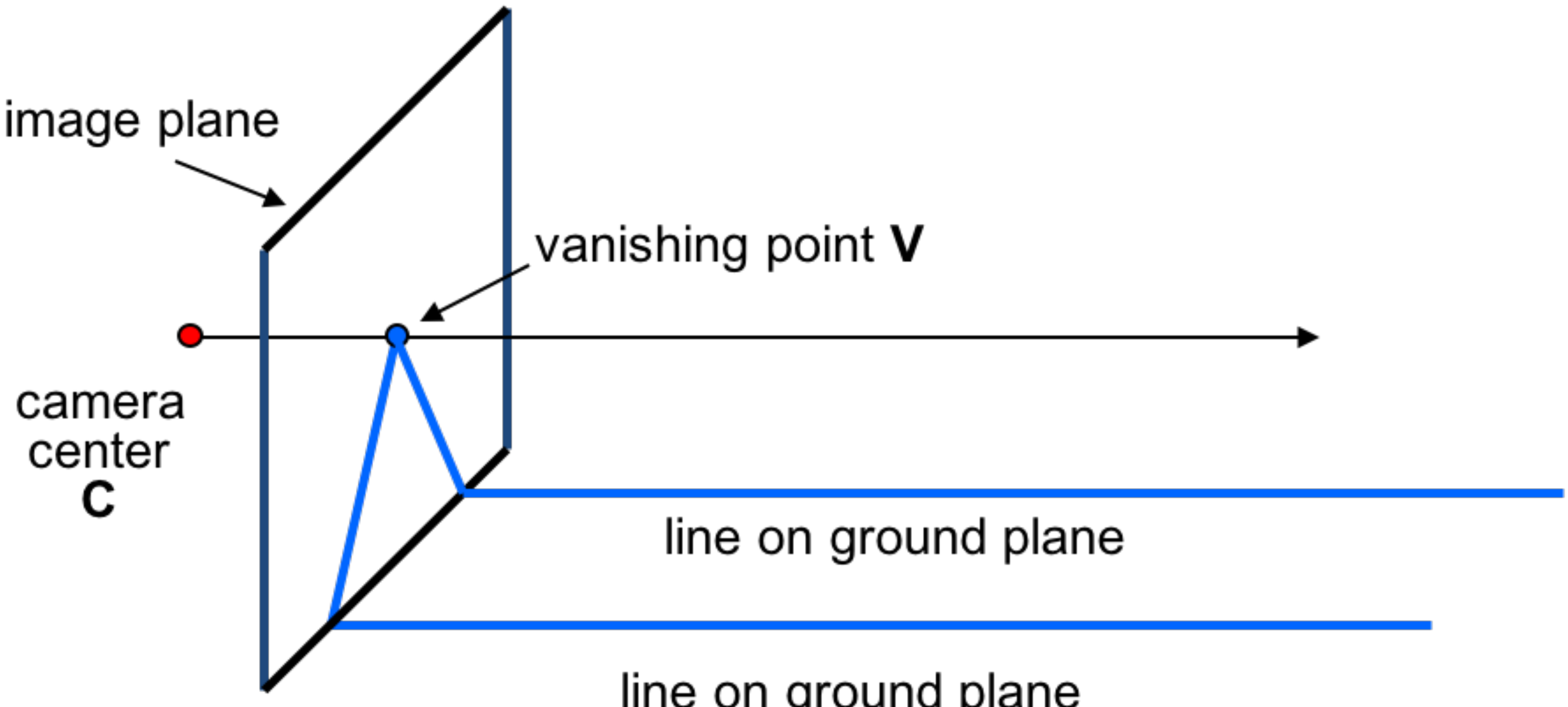
2D image



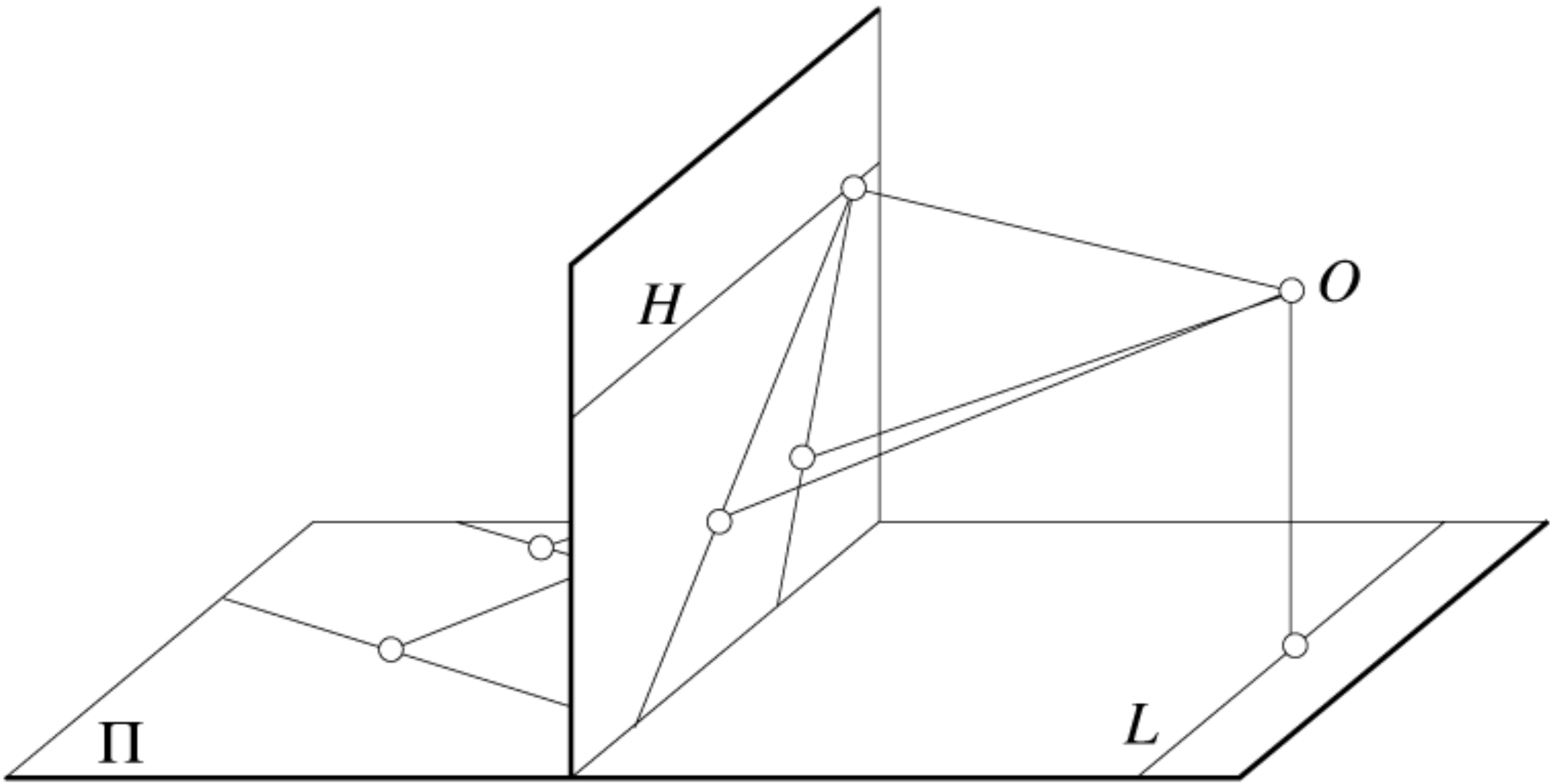
Imaging A Line On the Ground



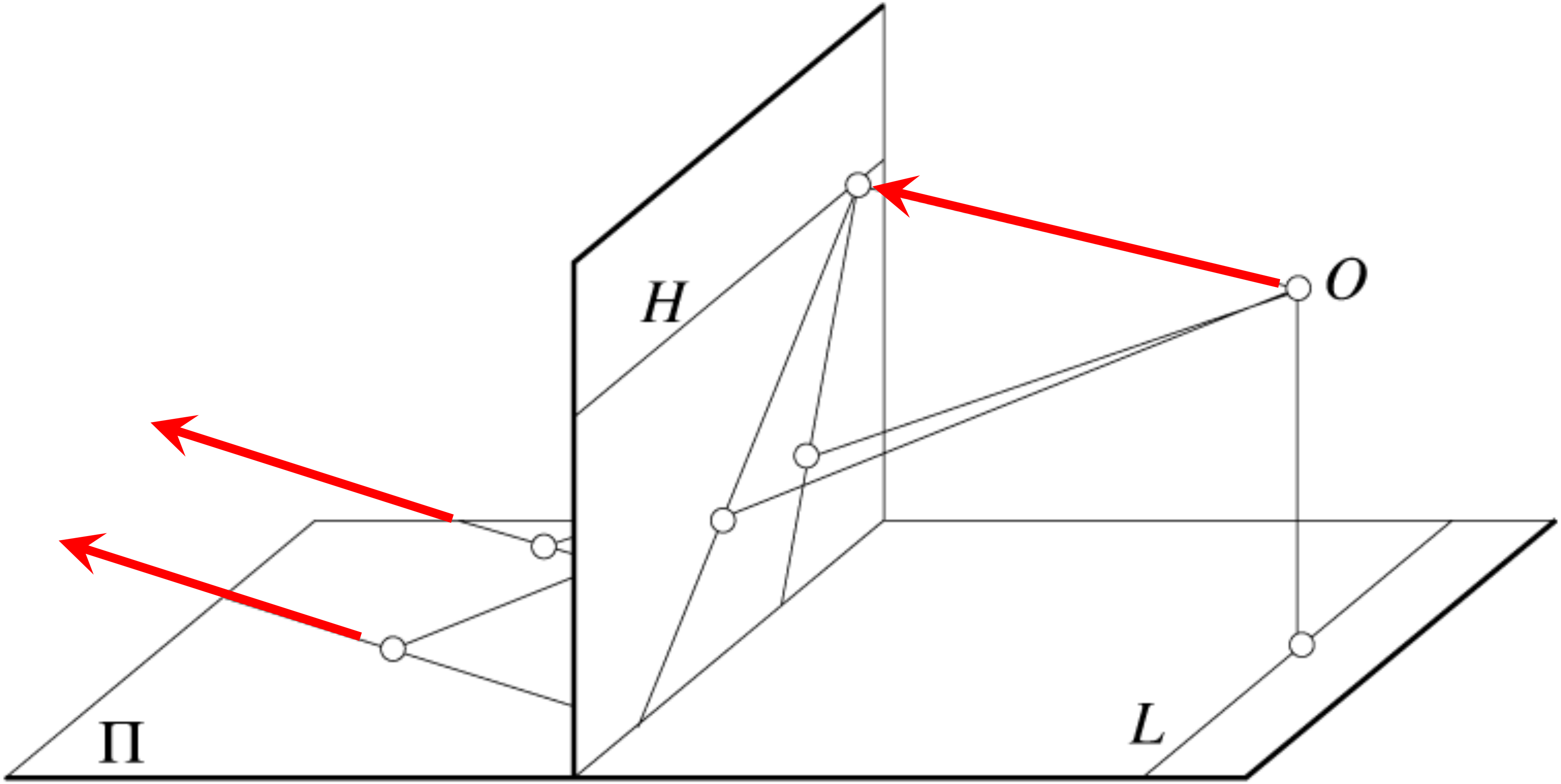
Imaging Parallel Lines



Vanishing Points



Vanishing Points



Beuchet Chair Illusion

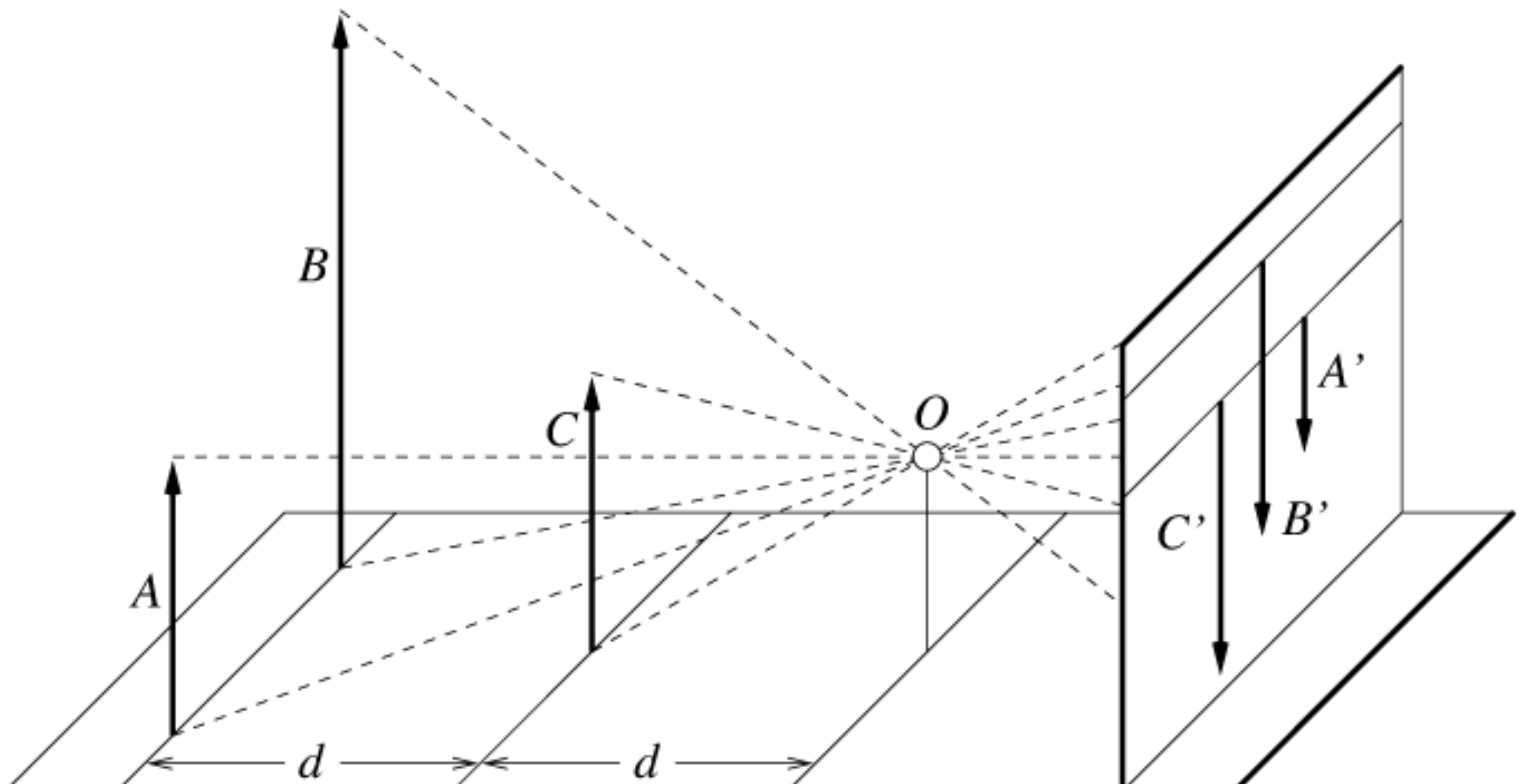
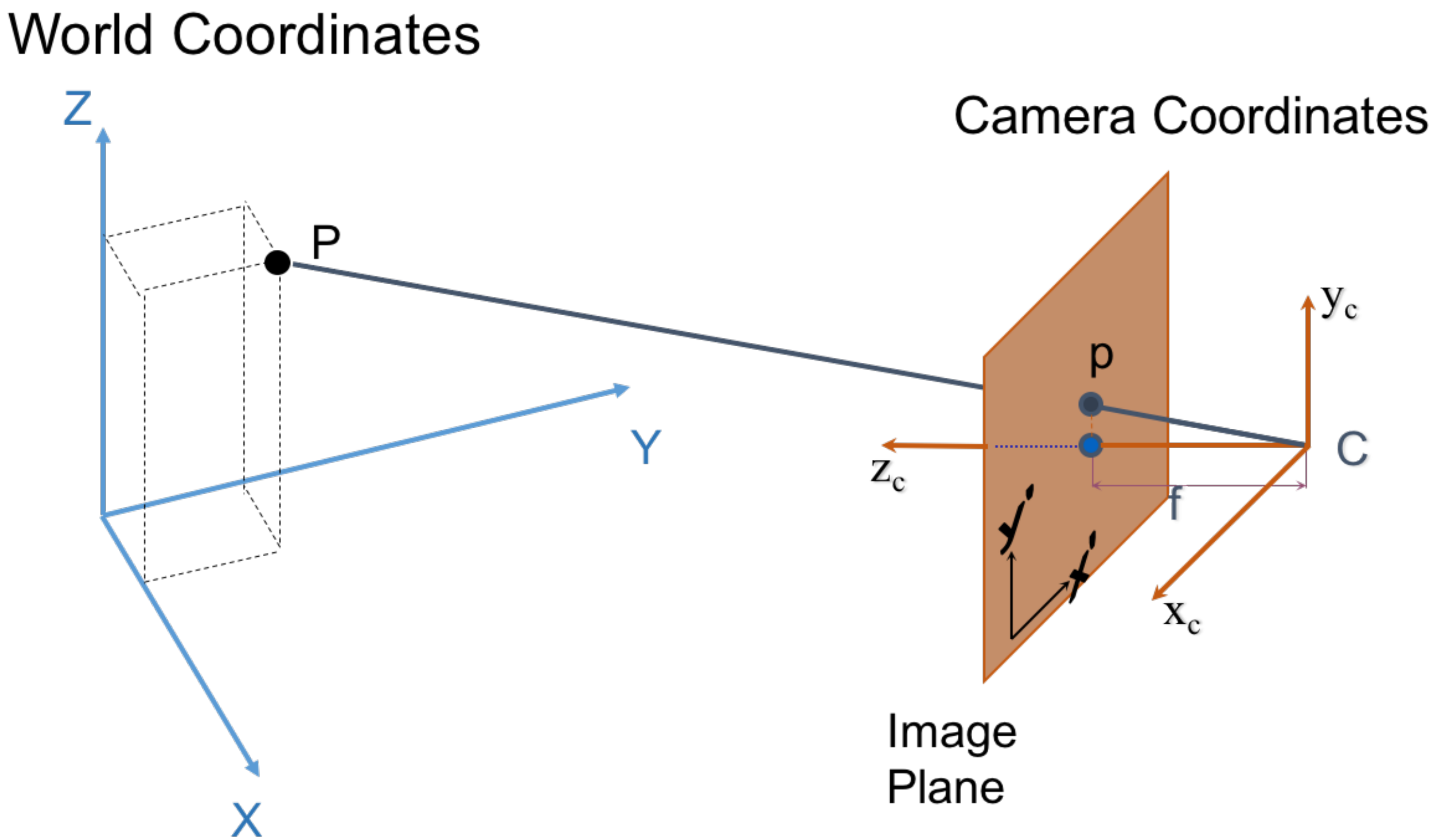


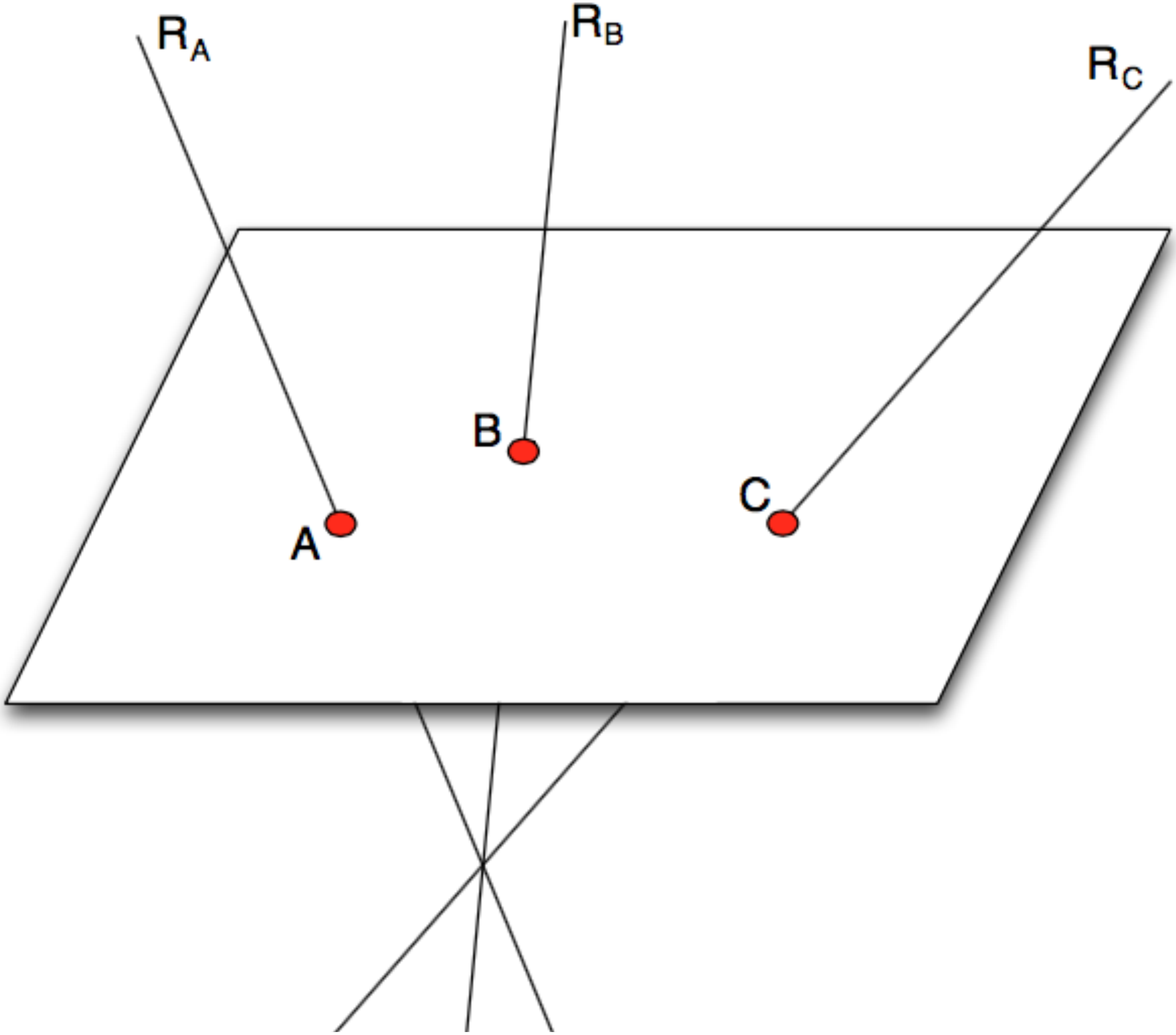
Figure by David Forsyth

3D to 2D Projection



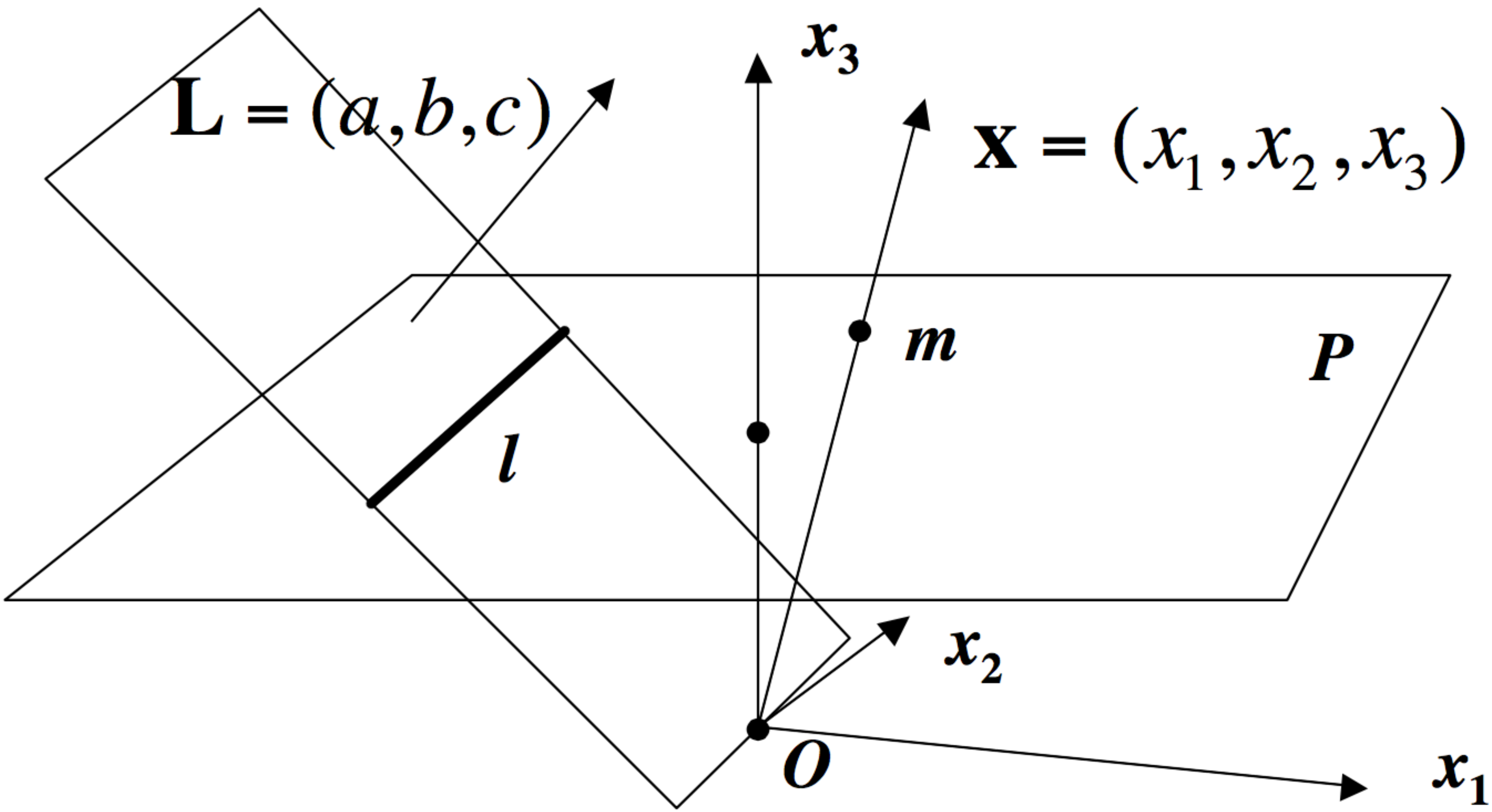
Projective Geometry

- Euclidean geometry describes shapes as they are. Properties of objects that are unchanged by rigid motions: lengths, angles, parallelism.
- Projective geometry describes objects as they appear. Lengths, angles, parallelism become distorted when we look at objects. It is a mathematical model for how images of the 3D world are formed.



The real projective space P^n, of dimension n, associated to $R^{\mathrm{n+1}}$, is the set of rays of $R^{\mathrm{n+1}}$.

Each 2D Pixel Is A 3D Ray



Homogeneous Coordinates

Instead of using n coordinates for n-dimensional space, we use $n+1$ coordinates.

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in P^1$ the projective line $= R^1 \cup \{\text{points at } \infty\}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in P^2$ the projective plane $= R^2 \cup \{\text{lines at } \infty\}$

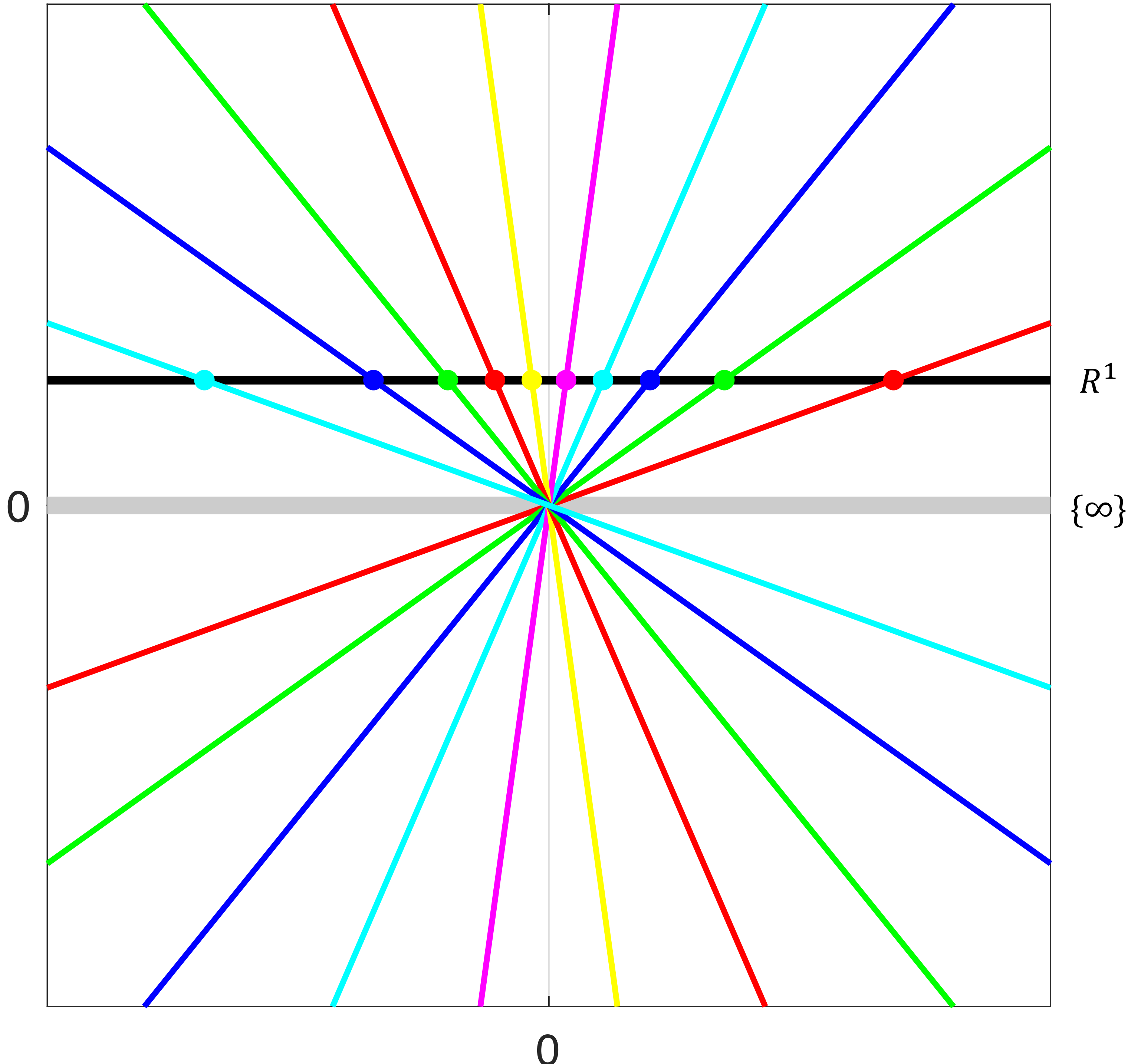
$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in P^3$ the projective space $= R^3 \cup \{\text{planes at } \infty\}$

Key Rules

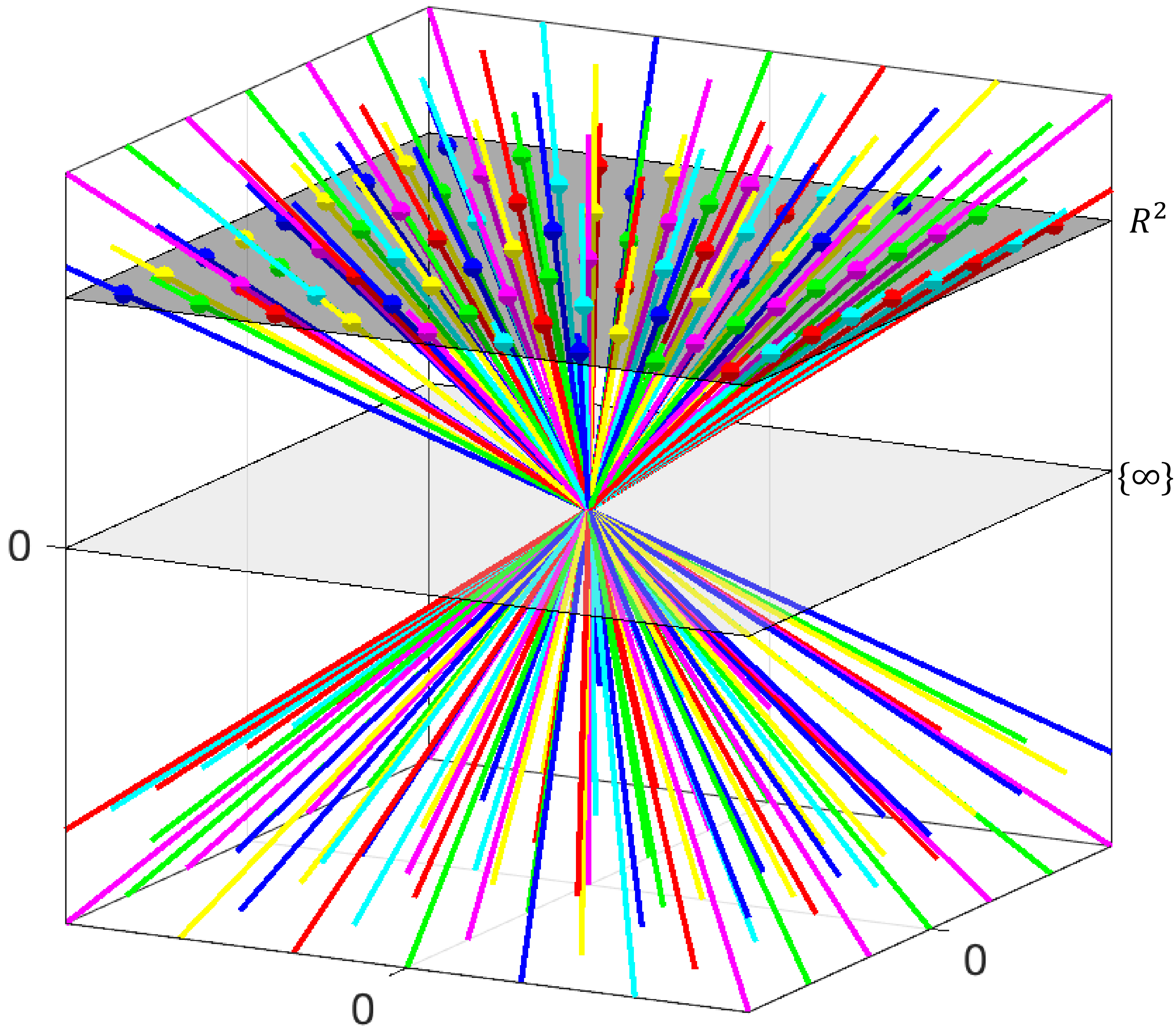
- $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{bmatrix}$ are the same point in P^{n-1}.

- $n - 1$ degrees of freedom only.

- $x_1, x_2, \ldots, x_n$ may not all be 0.



R^1
0
$\{\infty\}$
0

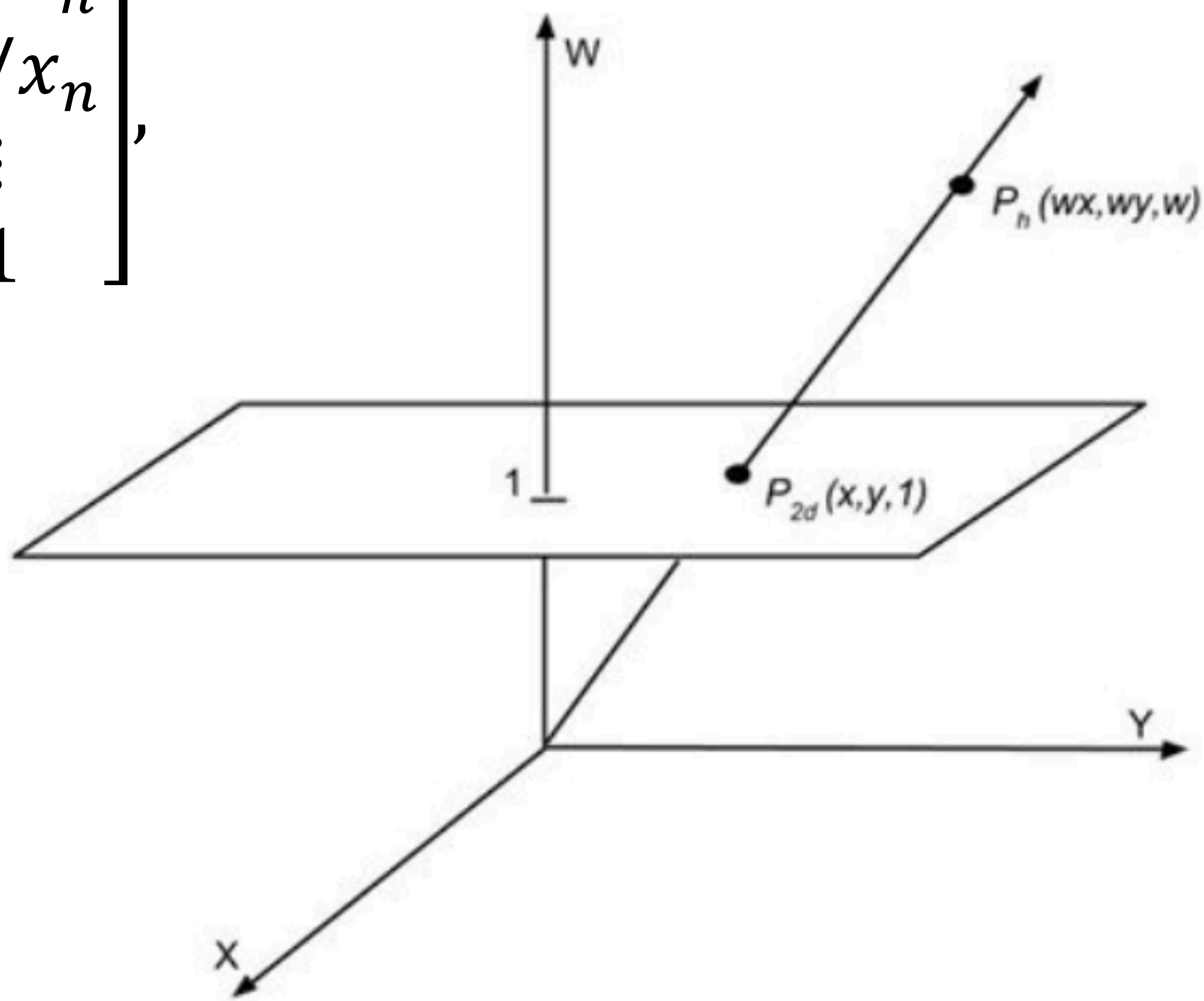


R^2
$\{\infty\}$
0
0
0

Picking a Canonical Representative

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1/x_n \\ x_2/x_n \\ \vdots \\ 1 \end{bmatrix},$$

if $x_n \neq 0$



The Projective Line

- Any finite point x can be represented as

$$\begin{bmatrix} x \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 2x \\ 2 \end{bmatrix} \text{ or } \begin{bmatrix} 6.8\,x \\ 6.8 \end{bmatrix} \text{ or } \ldots$$

- Any infinite point can be expressed as:

$$\begin{bmatrix} x \\ 0 \end{bmatrix}$$

- Note there is only one such point.

The Projective Plane

- Any finite point x can be represented as
  $$\begin{bmatrix} \lambda x \\ \lambda y \\ \lambda \end{bmatrix}, \forall \lambda \neq 0$$
- Any infinite point can be expressed as:
  $$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$
- Note there is a line at infinity. Different ratios of x and y give different points.

Lines in Homogeneous Coordinates

$a_1 x + a_2\, y + a_3 = 0$ is a line.

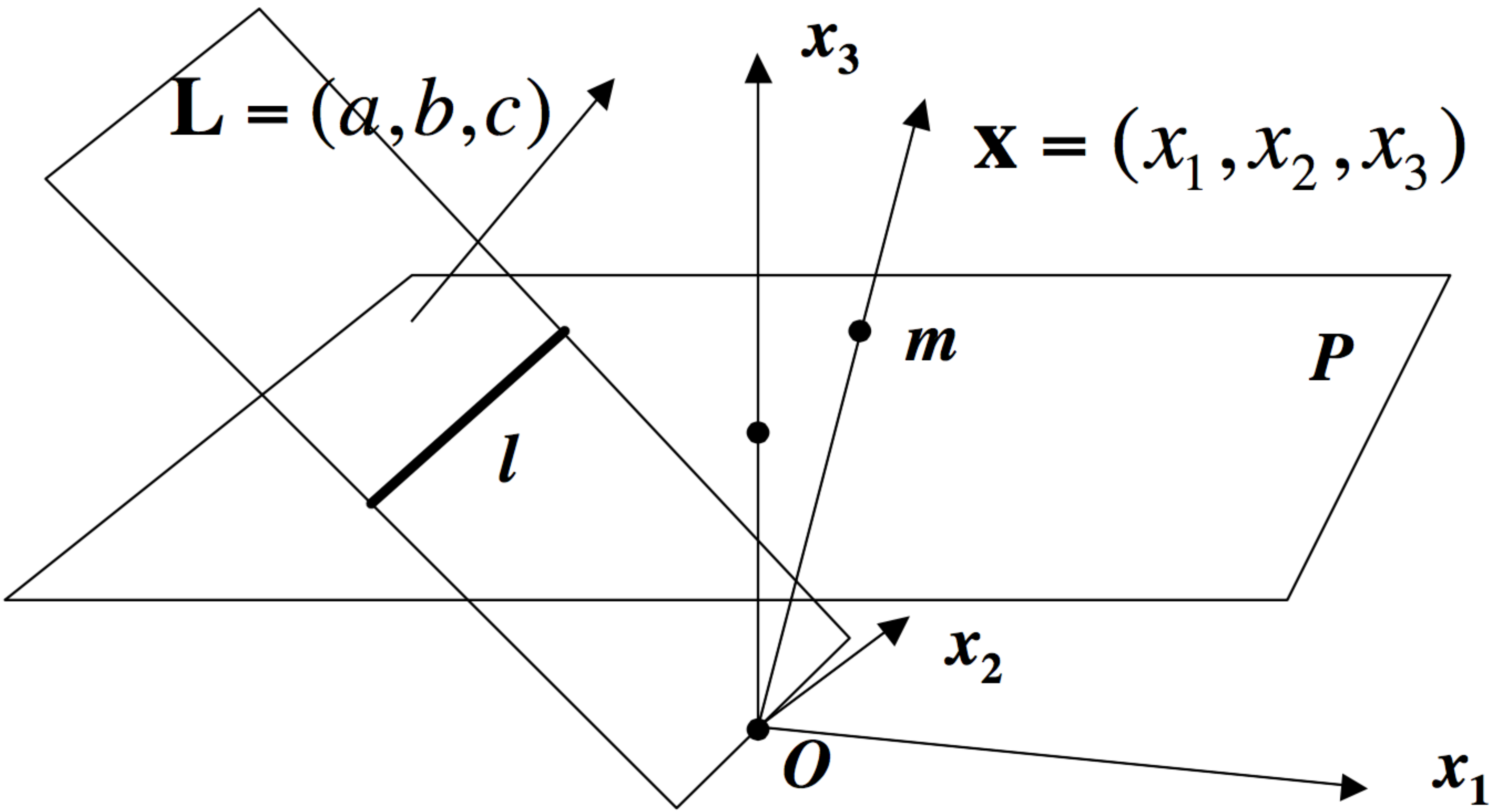
$\lambda a_1 x + \lambda\, a_2\, y + \lambda\, a_3 = 0$ is the same line.

$\begin{bmatrix} x \\ y \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ with $x = \frac{x_1}{x_3}, y = \frac{x_2}{x_3}$

$$a_1 \frac{x_1}{x_3} + a_2 \frac{x_2}{x_3} + a_3 = 0$$

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$$

Duality of Points and Lines



Incidence of Points on Lines

When does a point $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ lie on a line $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$?

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Incidence of Points on Lines

Where do two lines $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ intersect?

At the point $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \wedge \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Incidence of Points on Lines

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$x = 1$ $\quad y = 1$ $\quad (1,1)$

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$x = 1$ $\quad x = 2$ $\quad (0,1)$

Incidence of Points on Lines

What is the line given by points $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$?

$$\begin{bmatrix} x \\ x_2 \\ x_3 \end{bmatrix} \wedge \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

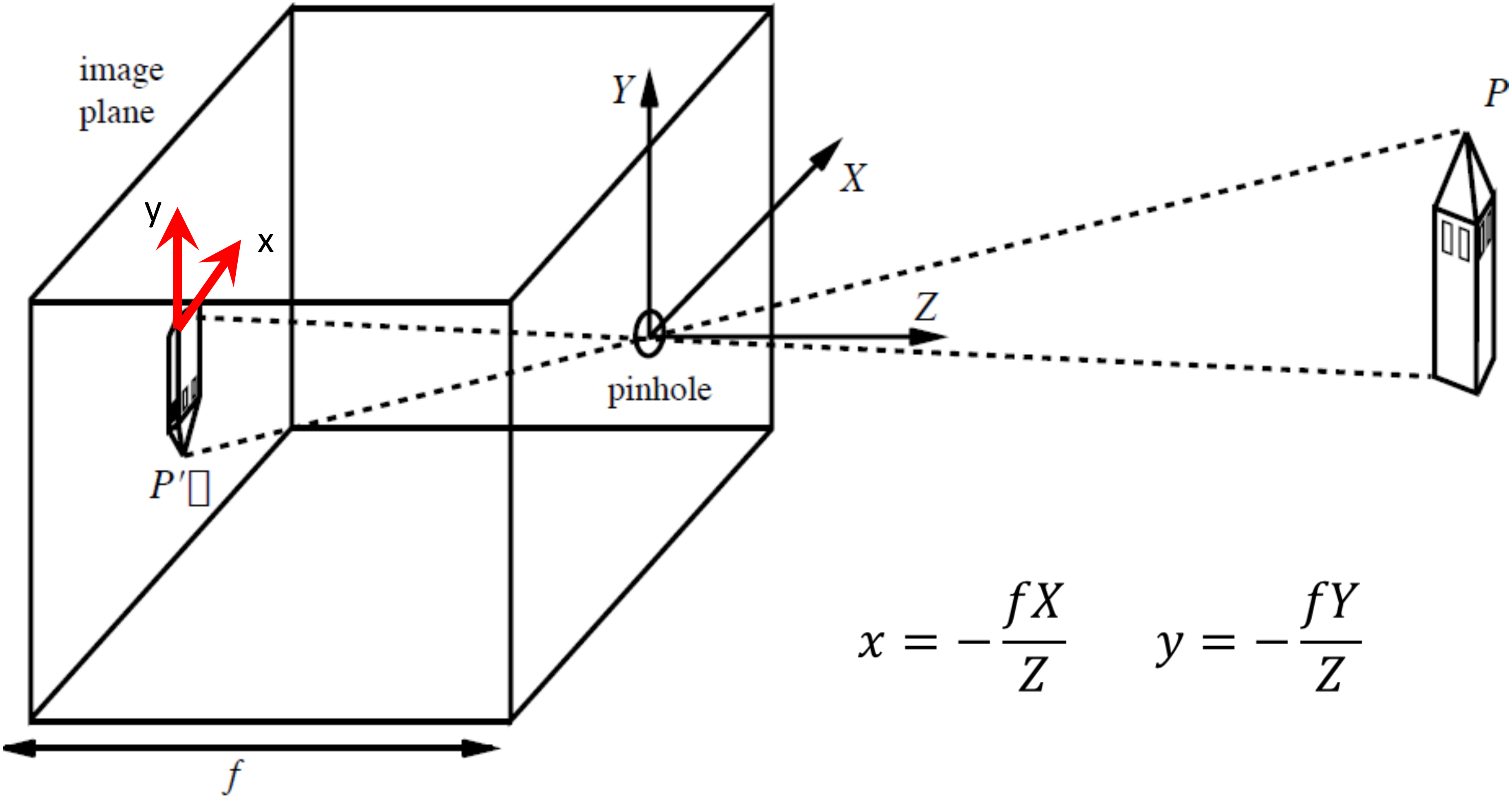
Representing Affine Transformations

$$f(a) = A\,a + t = \begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & t_x \\ A_{21} & A_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$x' = A_{11}x + A_{12}y + t_x$$
$$y' = A_{21}x + A_{22}y + t_y$$
$$w' = 1$$

The Pinhole Camera



$$x = -\frac{fX}{Z} \qquad y = -\frac{fY}{Z}$$

Perspective Projection from P^3 to P^2

$$\begin{bmatrix} x \\ y \\ f \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/\lambda \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/f & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ f \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/f & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z/f \end{bmatrix} = \begin{bmatrix} fX/Z \\ fY/Z \\ 1 \end{bmatrix}$$

Projective Transformations

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

8 independent parameters in P^2.

3 independent parameters in P^1.
15 independent parameters in P^3.

The matrix is required to be non-singular.

The Big Picture