Exercise 1 - Euler-Lagrange Modelling

Consider the cart-pole system shown in Figure 1. Derive the equations of motions of the cart-pole system using the Euler-Lagrange modelling symbolically. Parameters can be selected as $m = 1$ kg, $M = 10$ kg, $l = 0.5$ m for simulation purposes.

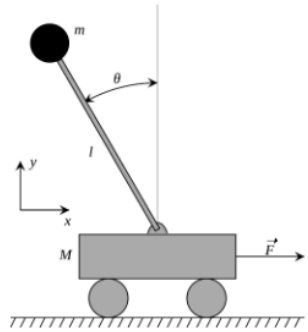


Figure 1: Sketch of cart-pole system.

(1) Coordinates/Kinematics

1) Choose generalized coordinates

Use

$$q = \begin{bmatrix} x \\ \theta \end{bmatrix}, \qquad \dot{q} = \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix}$$

- x = cart position (horizontal)
- θ = pole angle (I'll assume **measured from vertical**, like the figure)

2) Kinematics (position of each COM)

Cart COM (just moves in x, y=0):

$$p_c = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

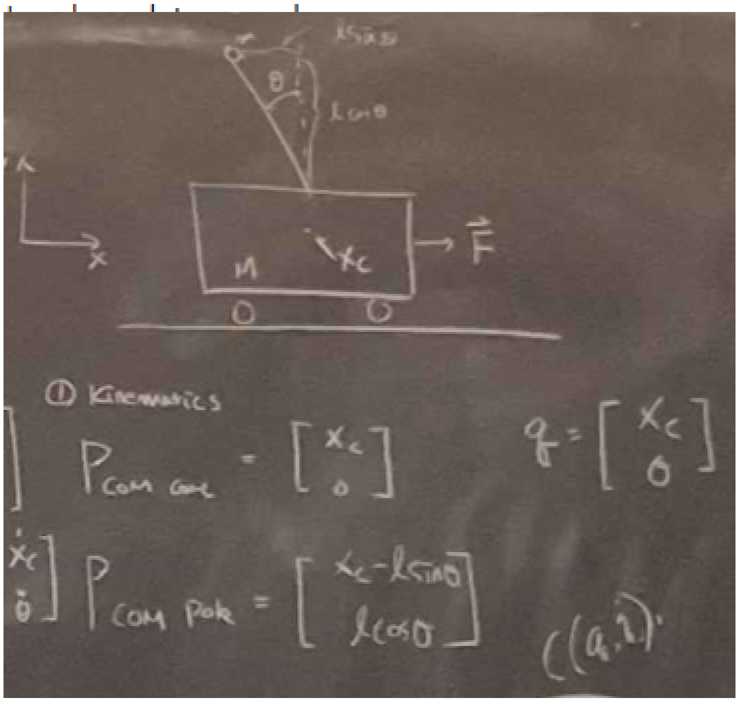
Pole point-mass COM at distance l from pivot:
A clean common choice (pivot at cart top/center) is:

$$p_p = \begin{bmatrix} x + l \sin\theta \\ l \cos\theta \end{bmatrix}$$

(Your sign on the $l \sin\theta$ term was opposite; that's not "wrong" if your θ sign convention matches it, but you must be consistent everywhere.)

Velocity via Jacobian:

$$\dot{p} = J(q)\dot{q}, \quad J = \frac{\partial p}{\partial q}$$



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(2) Jacobian/Energies

3) Energies (this is where your code mainly broke)

Potential energy

Gravity in 2D: easiest is **just use** $V = mgy$.

- Cart: $V_c = Mg \cdot 0 = 0$
- Pole: $V_p = mg(l\cos\theta)$

So:

$$V = mgl\cos\theta$$

What went wrong in your code: you did something like `1/2*M*(g0')*P_com_cart` which is not the formula for gravitational potential energy. There's no `1/2`, and you want a **dot product** with the gravity direction, or simply mgy.

Kinetic energy

- Cart: $T_c = \frac{1}{2}M\dot{x}^2$
- Pole (point mass at COM): $T_p = \frac{1}{2}m\|\dot{p}_p\|^2$

Compute $\dot{p}_p$:

$$\dot{p}_p = \begin{bmatrix} \dot{x} + l\cos\theta\,\dot{\theta} \\ -l\sin\theta\,\dot{\theta} \end{bmatrix}$$

Then

$$\|\dot{p}_p\|^2 = (\dot{x} + l\cos\theta\,\dot{\theta})^2 + (l\sin\theta\,\dot{\theta})^2 = \dot{x}^2 + 2l\cos\theta\,\dot{x}\dot{\theta} + l^2\dot{\theta}^2$$

So:

$$T = \frac{1}{2}(M+m)\dot{x}^2 + ml\cos\theta\,\dot{x}\dot{\theta} + \frac{1}{2}ml^2\dot{\theta}^2$$

What went wrong in your code: you were multiplyin[illegible]ctors elementwise and never did $v^T v$. Kinetic energy must be scalar.

② Jacobian

$$\dot{P}_{com,cart} = \frac{\partial P_{com,cart}}{\partial q}\dot{q} = \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}^{\text{Jacobian}} \begin{bmatrix} \dot{x}_c \\ \dot{\theta} \end{bmatrix} \quad ①$$

$$\dot{P}_{com,pole} = \frac{\partial P_{com,pole}}{\partial q}\dot{q} = \underbrace{\begin{bmatrix} 1 & -l\cos\theta \\ 0 & -l\sin\theta \end{bmatrix}}_{\text{Jacobian}} \begin{bmatrix} \dot{x}_c \\ \dot{\theta} \end{bmatrix}$$

$$E_{pot} = -\sum_{i=1}^{n} m \, g_0^T P_{l_i}(q)$$

$$E_{pot} = -\Big(\underbrace{M\begin{bmatrix} 0 & -g \end{bmatrix}\overbrace{\begin{bmatrix} x_c \\ 0 \end{bmatrix}}^{\text{cart}}}_{0} + m\begin{bmatrix} 0 & -g \end{bmatrix}\overbrace{\begin{bmatrix} x_c - l\sin\theta \\ l\cos\theta \end{bmatrix}}^{\text{pole}}\Big)$$

$$= mgl\cos\theta \qquad v^2 \quad \boxed{v^T v}$$

$$\boxed{\dot{P} = J\dot{q}}$$

$$E_{kin} = \sum_{i=1}^{n} \frac{1}{2} m \, \dot{q}^T J_P^{l_i T} J_P^{l_i} \dot{q}$$

$$v = Hw$$

$$\boxed{\frac{1}{2} m v^2}$$

$$\boxed{B(q)}$$

$$L = E_{kin} - E_{pot}$$

4) Lagrangian and Euler–Lagrange

$$L = T - V$$

Euler–Lagrange with generalized forces $Q = [F,\ 0]^T$:

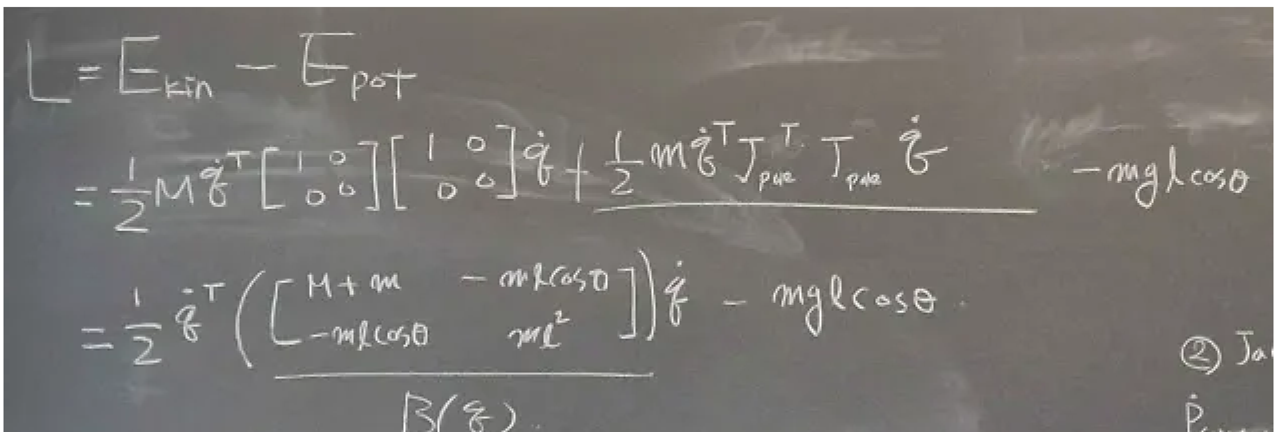
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q$$

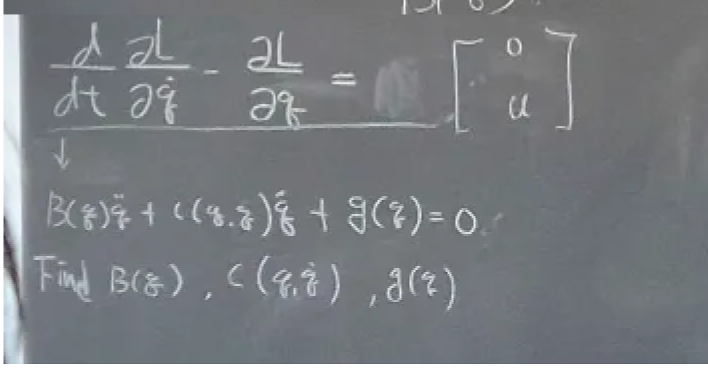
This gives the standard coupled equations:

$$(M+m)\ddot{x} + ml\cos\theta\,\ddot{\theta} - ml\sin\theta\,\dot{\theta}^2 = F$$

$$ml\cos\theta\,\ddot{x} + ml^2\ddot{\theta} - mgl\sin\theta = 0$$

That's already a valid "symbolic derivation" result.





$$B(q)\ddot{q} + \dot{B}(q)\dot{q} - \frac{\partial L}{\partial q}$$

$$= B(q)\ddot{q} + \begin{bmatrix} 0 & ml\sin\theta\dot{\theta} \\ ml\sin\theta\dot{\theta} & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_c \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ -ml\sin\theta\dot{\theta}\dot{x}_c - mgl\sin\theta \end{bmatrix} = B(q)\ddot{q} + \begin{bmatrix} ml\sin\theta\dot{\theta}^2 \\ \cancel{ml\sin\theta\dot{\theta}\dot{x}_c} \cancel{- ml\sin\theta\dot{\theta}\dot{x}_c} - mgl\sin\theta \end{bmatrix}$$

$$= B(q)\ddot{q} + \begin{bmatrix} 0 & ml\sin\theta\dot{\theta} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_c \\ \dot{\theta} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -mgl\sin\theta \end{bmatrix}}_{g(q)} = \begin{bmatrix} 0 \\ u \end{bmatrix}$$