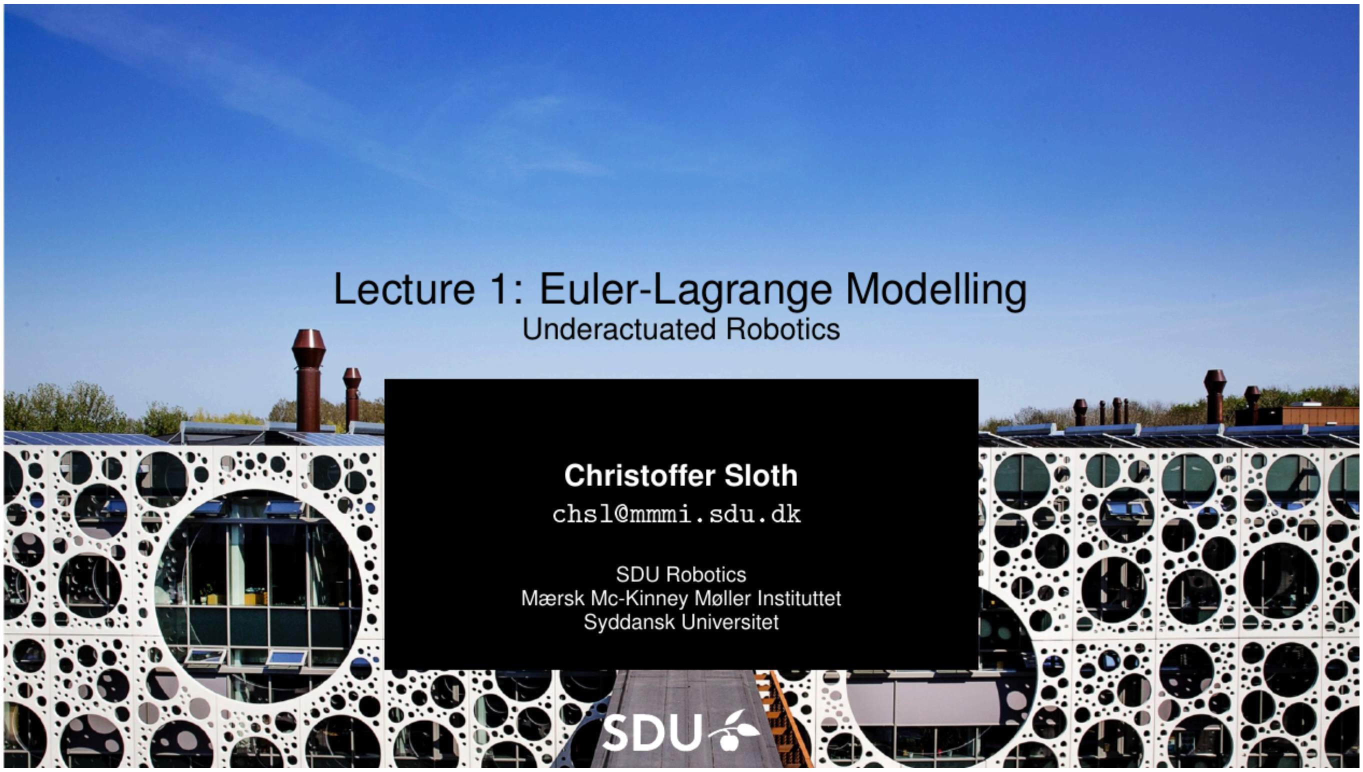


Agenda

- Introduction
- Euler-Lagrange Modelling
  - Modelling of Conservative Systems
  - Modelling of Non-Conservative Systems
  - Properties of Dynamical Robot Models
- Robot with Two Joints
  - Kinematics
  - Potential Energy
  - Kinetic Energy
  - Dynamics
- Summary

Euler-Langrange: for many "connected" bodies

Acrobot: 2 revolute dof

Definition of Underactuated Robot

Introduction
What is an Underactuated Robot?

An n-degrees of freedom robot is said to be ***underactuated*** if the number of input signals m (controls) is lower then m, i.e. $m < n$.

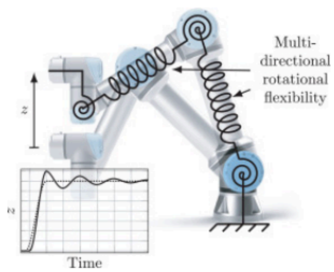
Introduction
Examples of Underactuated Robots

Spherical robot

There is only control of the inside system --> which makes the ball roll
But no control of the ball directly
--> Underactuated Robot

Introduction
Flexible-Link Robots

Robots with flexible links.

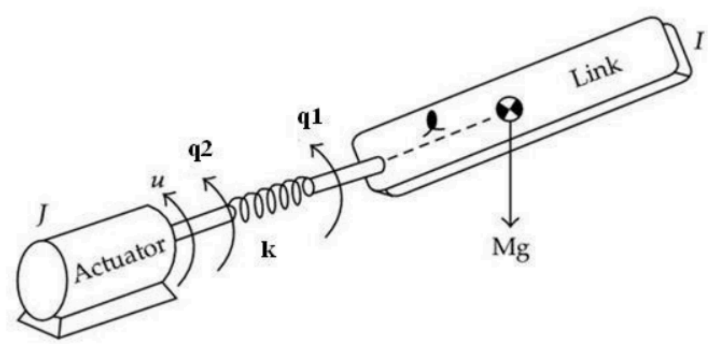


Flexiibility of the links (included in the model) --> means extra DOFs
Only really intereting for modelling vibrations?
A gear box could also be an extra dof?

Underactuated joint model

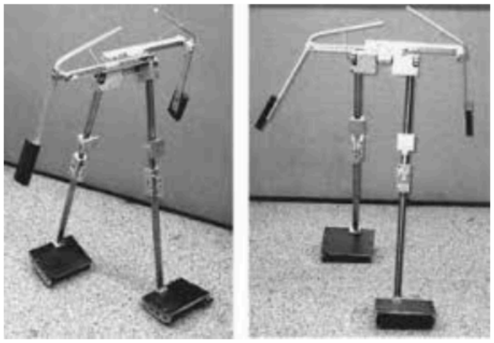
Introduction
Flexible-Joint Robots

Robots with flexible joints.



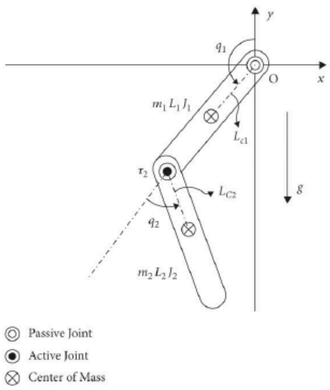
Introduction
Passive-Dynamic Walking Robot

Three-Dimensional Passive-Dynamic Walking Robot with Two Legs and Knees.



Introduction
Acrobot

Robots with passive joints.



Active joint = joint with actuator

Robots with compliant coupling between joints.

the object has DOFs on top of the hands DOFs --> so if we want to control object in hand --> think of it has underactuated System

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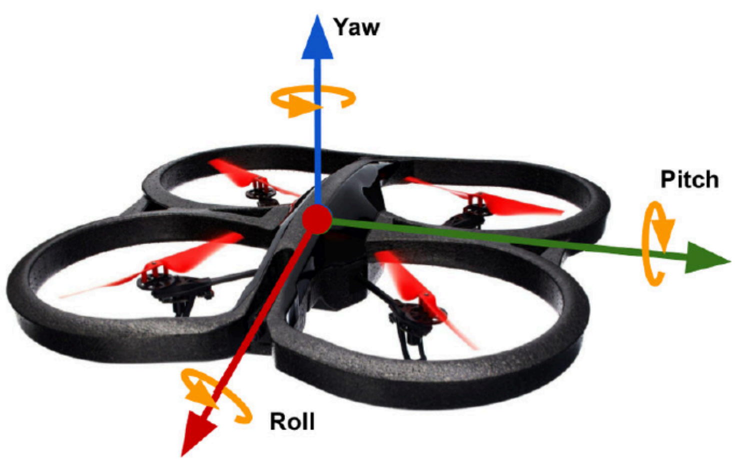
Systems with fewer actuators than degrees of freedom.



Introduction
Underactuated Robotics

Knowledge:

- Derive dynamical state-space models of robots as control systems
- Analyze the stability of low dimensional linear and nonlinear systems
- Analyze the observability and controllability of linear control systems
- Use a variety of controllers for underactuated robots

Skills:

- Implement simulations of control systems in software
- Create concise technical reports presenting solutions to proposed problems

Competencies:

- Choose appropriate modern control techniques to solve control problems in robotics
- Apply modern control techniques to control simulated underactuated robots

We work in time domain

Course Structure

- **Lesson 1**: Euler-Lagrange Modelling
- **Lesson 2**: Simulation of Dynamical Systems
- **Lesson 3**: Stability Analysis
- **Lesson 4**: Help with hand-in (Modelling and Simulation)
- **Lesson 5**: Feedback Linearisation
- **Lesson 6**: Optimal Control
- **Lesson 7**: Energy Shaping Control
- **Lesson 8**: Help with hand-in (Model-Based Control)
- **Lesson 9**: Example
- **Lesson 10**: Sliding Mode Control
- **Lesson 11**: Data-Driven Control
- **Lesson 12**: Help with hand-in (Data-Driven Control)

Conservative System: total mechanical energy is constant --> no energy dissipation
This is "ideal" - a system with no friction, the basis of modelling

Non-conservative systems models friction using Euler-Lagrange modelling

---

Hamiltons Principle

The motion of a mechanical system from time a to b is such that the integral

$$I(t, q, \dot{q}) = \int_a^b \mathcal{L}(t, q, \dot{q}) dt,$$

where $\mathcal{L} = E_{\text{kin}} - E_{\text{pot}}$ has a stationary value. The function $\mathcal{L}$ is called the **Lagrangian**.

Euler-Lagrange modelling can be used for finding the equations of motion of e.g. mechanical systems using the system's potential energy E_{pot} and kinetic energy E_{kin}.

instead of free-body diagrams we use Ekin and Epot as base for our model

Euler-Lagrange modelling
Generalized Coordinates

Consider a mechanical system with n degrees of freedom. The system is modelled with n ***generalized coordinates*** $q_1, \ldots, q_n$.

Euler-Lagrange modelling
Generalized Coordinates

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Consider a mechanical system with n degrees of freedom. The system is modelled with n ***generalized coordinates*** $q_1, \ldots, q_n$.

The generalized coordinates should be

- **Minimal**
- **Independent**
  If all but one coordinate is fixed then the last coordinate should take values in a continuous domain.
- **Complete**
  Should describe all configurations to any time.

Euler-Lagrange modelling
Generalized Coordinates

Consider a mechanical system with n degrees of freedom. The system is modelled with n ***generalized coordinates*** $q_1, \ldots, q_n$.

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- **Minimal**
- **Independent**
  If all but one coordinate is fixed then the last coordinate should take values in a continuous domain.
- **Complete**
  Should describe all configurations to any time.

Generalized coordinates will most often be positions and/or angles of a mechanical system.

•

Euler-Lagrange modelling
Euler-Lagrange Equation

If q is a trajectory of a conservative mechanical system then

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} = 0$$

where q is an n-dimensional vector of generalized coordinates and $\mathcal{L}$ is the *Lagrangian* given by

$$\mathcal{L} = E_{\text{kin}} - E_{\text{pot}} \quad \text{[J]}$$

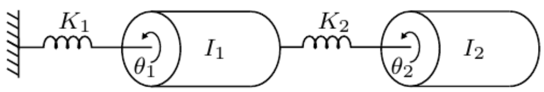
where E_{pot} is the system's potential energy and E_{kin} is the system's kinetic energy.

used instead of N2
first term is the time derivative of partial velocity
the second term is the partial derivative os position (generalized coodinates)

$q = [\theta_1 \theta_2]$
OG ligningen ovenfor indgår --> derfor et system af to ligninger

ma= f,
mx'' = f <-- two derivitions means 2 dofs

Euler-Lagrange modelling
Example: Rotational Mass-Spring System



The rotational mass-spring system has dynamics given by

2 dofs --> 2 second order diff-equations
hooks law: $= k\theta$

$Itheta'' = \tau_n et$

Euler-Lagrange model for rotational mass-spring system

Euler-Lagrange modelling
Example: Rotational Mass-Spring System

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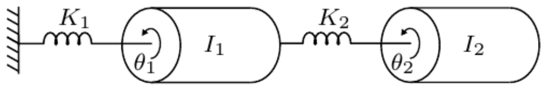

The rotational mass-spring system has dynamics given by

$$I_1\ddot{\theta}_1 = -K_1\theta_1 - K_2(\theta_1 - \theta_2) \qquad \text{[Nm]}$$
$$I_2\ddot{\theta}_2 = -K_2(\theta_2 - \theta_1) \qquad \text{[Nm]}$$

where I_1, I_2 are moments of inertia [kgm^2] and K_1, K_2 are stiffnesses [N/rad].

The potential and kinetic energies are

$$E_{\text{pot}} = \frac{1}{2}K_1\theta_1^2 + \frac{1}{2}K_2(\theta_1 - \theta_2)^2$$
$$E_{\text{kin}} = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2$$

From Euler-Lagrange Equation with generalized coordinates $\boldsymbol{q} = (q_1, q_2) = (\theta_1, \theta_2)$ we obtain

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} = 0$$

where

$$\mathcal{L} = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 - \left(\frac{1}{2}K_1\theta_1^2 + \frac{1}{2}K_2(\theta_1 - \theta_2)^2\right)$$

no translation --> no gravitational potential/kinetic energy

Euler-Lagrange modelling
Example: Rotational Mass-Spring System

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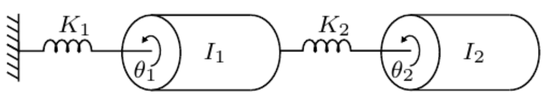

The rotational mass-spring system has dynamics given by

$$I_1\ddot{\theta}_1 = -K_1\theta_1 - K_2(\theta_1 - \theta_2) \qquad \text{[Nm]}$$
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where I_1, I_2 are moments of inertia [kgm^2] and K_1, K_2 are stiffnesses [N/rad].

From Euler-Lagrange Equation with generalized coordinates $\boldsymbol{q} = (q_1, q_2) = (\theta_1, \theta_2)$ we obtain

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} = 0$$

where

$$\mathcal{L} = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 - \left(\frac{1}{2}K_1\theta_1^2 + \frac{1}{2}K_2(\theta_1 - \theta_2)^2\right)$$

This can be written as

$$\begin{bmatrix} I_1\ddot{\theta}_1 + K_1\theta_1 + K_2(\theta_1 - \theta_2) \\ I_2\ddot{\theta}_2 - K_2(\theta_1 - \theta_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Øverst: partial derivative of langrangian with respect to qdot

Nederst: partial derivative of langrangian with respect to q

This can be written as

$$\begin{bmatrix} I_1\ddot{\theta}_1 + K_1\theta_1 + K_2(\theta_1 - \theta_2) \\ I_2\ddot{\theta}_2 - K_2(\theta_1 - \theta_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Disse er udgangspunktet for nedenstående

The rotational mass-spring system has dynamics given by

$$I_1\ddot{\theta}_1 = -K_1\theta_1 - K_2(\theta_1 - \theta_2) \qquad [\text{Nm}]$$
$$I_2\ddot{\theta}_2 = -K_2(\theta_2 - \theta_1) \qquad [\text{Nm}]$$

hvor theta 1 og theta 2 er byttet om

Euler-Lagrange modelling
Exercise: Spring Pendulum

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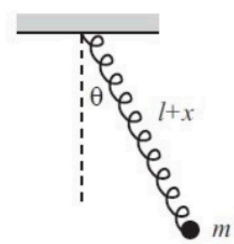
Derive a model of the following spring pendulum system using Euler-Lagrange modelling.



generalized coordinates: $q = [\theta, x]$
angle contributes 1 dof
there is also variation in x (due to spring)
Epot = mgh

$$Epot = 1/2\ k\ x^2 + m\ g \cos(\theta)\ (l + x)$$

$$Ekin = 1/2\ m\ x_{dot}^2 + 1/2\ m\ (l + x)\theta_{dot}^2$$

- $E_{pot} = \frac{1}{2}kx^2 - mg(l + x)\cos(\theta)$

- $E_{kin} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m(l+x)^2\dot{\theta}^2$

Modelling of Non-Conservative Systems

Euler-Lagrange Modelling
Generalized Forces

Physical systems are often affected by external controllable forces and dissipative forces such as friction. Therefore, Euler-Lagrange Equation is extended with generalized forces $\boldsymbol{Q}$, which are not necessarily conservative.

-

Euler-Lagrange Modelling
Generalized Forces

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Physical systems are often affected by external controllable forces and dissipative forces such as friction. Therefore, Euler-Lagrange Equation is extended with generalized forces $\boldsymbol{Q}$, which are not necessarily conservative.

This extension is called ***Lagrange–D'Alembert's Principle***.

•

Euler-Lagrange Modelling
Lagrange–D'Alembert's Principle

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If q is a trajectory of a mechanical system that is affected by a generalized force Q then

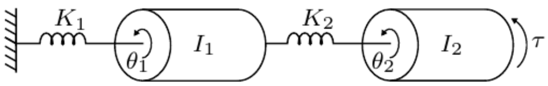
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} = \boldsymbol{Q}$$

where Q is an n-dimensional vector of generalized forces. ***Lagrange–D'Alembert's Principle*** can be written as (for $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)$)

$$\begin{aligned} \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} &- \frac{\partial \mathcal{L}}{\partial q_1} &= Q_1 \\ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2} &- \frac{\partial \mathcal{L}}{\partial q_2} &= Q_2 \\ \vdots \quad & \quad \vdots & \vdots \\ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_n} &- \frac{\partial \mathcal{L}}{\partial q_n} &= Q_n \end{aligned}$$

Only dissipitative forces go into capital Q
otherwise they are modelled twice

Euler-Lagrange Modelling
Example: Rotational Mass-Spring System with External Force



The above rotational mass-spring system has dynamics

$$I_1\ddot{\theta}_1 = -K_1\theta_1 - K_2(\theta_1 - \theta_2) \qquad \text{[Nm]}$$
$$I_2\ddot{\theta}_2 = -K_2(\theta_2 - \theta_1) + \tau \qquad \text{[Nm]}$$

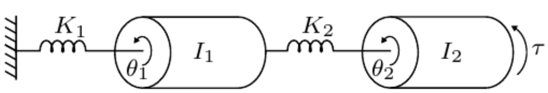
where I_1, I_2 are moments of inertia [kgm^2] and K_1, K_2 are stiffnesses [N/rad].

The potential and kinetic energies are

$$E_{\text{pot}} = \frac{1}{2}K_1\theta_1^2 + \frac{1}{2}K_2(\theta_1 - \theta_2)^2$$
$$E_{\text{kin}} = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2$$

External force tau added

Euler-Lagrange Modelling
Example: Rotational Mass-Spring System with External Force



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$$E_{\text{kin}} = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2$$

From Euler-Lagrange Equation with generalized coordinates $\boldsymbol{q} = (q_1, q_2) = (\theta_1, \theta_2)$ and generalized force $\boldsymbol{Q} = (0, \tau)$ we obtain

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} = \boldsymbol{Q}$$

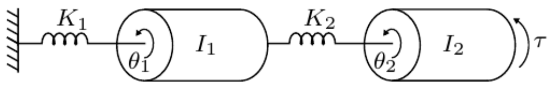
where

$$\mathcal{L} = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 - \left(\frac{1}{2}K_1\theta_1^2 + \frac{1}{2}K_2(\theta_1 - \theta_2)^2\right)$$

- tau only applied to second body

$\boldsymbol{q} = (q_1, q_2) = (\theta_1, \theta_2)$
$\boldsymbol{Q} = (0, \tau)$ we obtain

Euler-Lagrange Modelling
Example: Rotational Mass-Spring System with External Force



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where I_1, I_2 are moments of inertia [kgm^2] and K_1, K_2 are stiffnesses [N/rad].

From Euler-Lagrange Equation with generalized coordinates $\boldsymbol{q} = (q_1, q_2) = (\theta_1, \theta_2)$ and generalized force $\boldsymbol{Q} = (0, \tau)$ we obtain

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This can be written as

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Properties of Dynamical Robot Models

If q is a trajectory of a mechanical system that is affected by a generalized force Q then

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = Q$$

where Q is an n-dimensional vector of generalized forces and $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)$ is the generalized coordinate and the Lagrangian is given by

$$\mathcal{L}(q, \dot{q}) = E_{\text{kin}}(q, \dot{q}) - E_{\text{pot}}(q).$$

Kinetic and Potential Energy calculation

The gravitational potential energy is given by

$$E_{\text{pot}}(\boldsymbol{q}) = -\sum_{i=1}^{n} m_{l_i} \boldsymbol{g}_0^T \boldsymbol{p}_{l_i}(\boldsymbol{q}) \qquad \text{[J]}$$

where m_{l_i} is the mass of Link i [kg], $\boldsymbol{g}_0$ is the gravitational acceleration in Base Frame [m/s^2] and $\boldsymbol{p}_{l_i}(\boldsymbol{q})$ is the position of the center of mass of Link i in Base Frame [m]; and

$$E_{\text{kin}}(q, \dot{q}) = \frac{1}{2}\dot{q}^T B(q)\dot{q} \qquad \text{[J]}$$

where $B(q)$ is the inertia tensor in Base Frame.

Makes it possible to use kinematics --> this is a vector [x, y, z]

, $\boldsymbol{g}_0$ is the gravitational acceleration in Base Frame

systematic approach

first write down kinematics

then we have center of mass
compute potential and kinetic energies

$B(q)$

this is invertable (?) so the kinetic energy is positive(?)

Properties of Dynamical Robot Models
Gravity Torque

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From Lagrange–D'Alembert's Principle, it is seen that

$$\frac{d}{dt}\frac{\partial E_{\text{kin}}}{\partial \dot{q}} - \frac{\partial E_{\text{kin}}}{\partial q} + \frac{\partial E_{\text{pot}}}{\partial q} = Q$$

where

$$\frac{\partial E_{\text{pot}}}{\partial q_i} = -\sum_{i=1}^{n} m_{l_i} \boldsymbol{g}_0^T \underbrace{\frac{\partial \boldsymbol{p}_{l_i}(\boldsymbol{q})}{\partial q_i}}_{=J_{P_i}^{l_i}}$$

positional jacobian

$$\underbrace{\frac{\partial \boldsymbol{p}_{l_i}(\boldsymbol{q})}{\partial q_i}}_{=J_{P_i}^{l_i}}$$

From Lagrange–D'Alembert's Principle, it is seen that

$$\frac{d}{dt}\frac{\partial E_{\text{kin}}}{\partial \dot{q}} - \frac{\partial E_{\text{kin}}}{\partial q} + \frac{\partial E_{\text{pot}}}{\partial q} = Q$$

where

$$\frac{\partial E_{\text{pot}}}{\partial q_i} = -\sum_{i=1}^{n} m_{l_i} \boldsymbol{g}_0^T \underbrace{\frac{\partial \boldsymbol{p}_{l_i}(\boldsymbol{q})}{\partial q_i}}_{=J_{P_i}^{l_i}}$$

We define

$$g(q) = \begin{bmatrix} \frac{\partial E_{\text{pot}}}{\partial q_1} & \frac{\partial E_{\text{pot}}}{\partial q_2} & \cdots & \frac{\partial E_{\text{pot}}}{\partial q_n} \end{bmatrix}^T$$

The dynamical equation

$$\frac{d}{dt}\frac{\partial E_{\text{kin}}}{\partial \dot{q}} - \frac{\partial E_{\text{kin}}}{\partial q} + g(q) = Q$$

can be rewritten by exploiting that

$$\frac{\partial E_{\text{kin}}}{\partial \dot{q}} = B(q)\dot{q}$$

Properties of Dynamical Robot Models
Moment of Inertia Term

The dynamical equation

$$\frac{d}{dt}\frac{\partial E_{\text{kin}}}{\partial \dot{q}} - \frac{\partial E_{\text{kin}}}{\partial q} + g(q) = Q$$

can be rewritten by exploiting that

$$\frac{\partial E_{\text{kin}}}{\partial \dot{q}} = B(q)\dot{q}$$

This implies that

$$\frac{d}{dt}\frac{\partial E_{\text{kin}}}{\partial \dot{q}} = B(q)\ddot{q} + \dot{B}(q)\dot{q}$$

Properties of Dynamical Robot Models
Moment of Inertia Term

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The dynamical equation

$$\frac{d}{dt}\frac{\partial E_{\text{kin}}}{\partial \dot{q}} - \frac{\partial E_{\text{kin}}}{\partial q} + g(q) = Q$$

can be rewritten by exploiting that

$$\frac{\partial E_{\text{kin}}}{\partial \dot{q}} = B(q)\dot{q}$$

This implies that

$$\frac{d}{dt}\frac{\partial E_{\text{kin}}}{\partial \dot{q}} = B(q)\ddot{q} + \dot{B}(q)\dot{q}$$

This leads to

$$B(q)\ddot{q} + \dot{B}(q)\dot{q} - \frac{\partial E_{\text{kin}}}{\partial q} + g(q) = Q$$

The final two terms of

$$B(q)\ddot{q} + \dot{B}(q)\dot{q} - \frac{\partial E_{\text{kin}}}{\partial q} + g(q) = Q$$

can be written as (the chain rule has been applied)

$$(\dot{B}(q)\dot{q})_i = \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial b_{ij}}{\partial q_k}\dot{q}_k\dot{q}_j$$

and

$$\frac{\partial E_{\text{kin}}}{\partial q_i} = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial b_{jk}}{\partial q_i}\dot{q}_k\dot{q}_j$$

It is seen that

$$\dot{B}(q)\dot{q} - \frac{\partial E_{\text{kin}}}{\partial q} = C(q,\dot{q})\dot{q}$$

where

$$c_{ij} = \sum_{k=1}^{n} \frac{1}{2}\left(\frac{\partial b_{ij}}{\partial q_k} + \frac{\partial b_{ik}}{\partial q_j} - \frac{\partial b_{jk}}{\partial q_i}\right)\dot{q}_k$$

This is centrifugal and coriolis forces

Full Euler-Lagrange model

Properties of Euler-Lagrange Systems

Euler-Lagrange Equation on Matrix Form

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The robot model given by the Euler-Lagrange equation can be formulated as

$$B(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau$$

where $B(q)$ is the inertia tensor, $C(q, \dot{q})$ is a matrix containing Coriolis and centrifugal terms, $g(q)$ is the gravity vector, and τ is the actuator torque.

We need to isolate acceleration

- $\ddot{q} = B^-1(q)(\tau - C(q, \dot{q})\dot{q} - g(q))$

Robot with Two Joints

Kinematics

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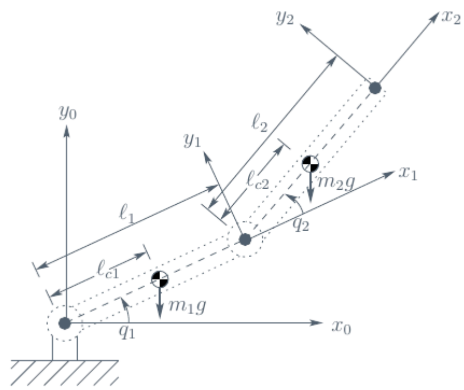
Kinematics of two-joint robot in DH-notation

Robot with Two Joints
DH Parameters

The DH parameters for the robot are given in the following table.

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1
2	a_2	0	0	θ_2



theta is the angle we actuate through q1 and q2

Robot with Two Joints
DH Parameters

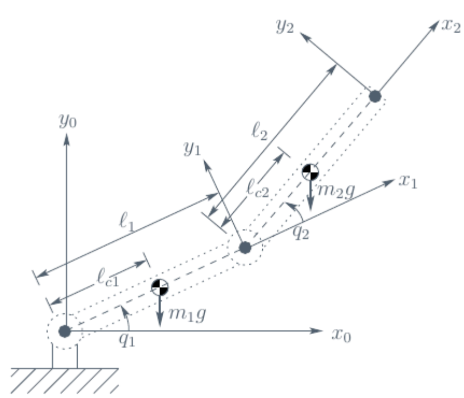
The DH parameters for the robot are given in the following table.

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1
2	a_2	0	0	θ_2

Each coordinate transformation is given by

$$A_i^{i-1}(\theta_i) = \begin{bmatrix} c_i & -s_i & 0 & a_i c_i \\ s_i & c_i & 0 & a_i s_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where c_i (s_i) denotes $\cos(\theta_i)$ $(\sin(\theta_i))$ and c_{ij} (s_{ij}) denotes $\cos(\theta_i + \theta_j)$ $(\sin(\theta_i + \theta_j))$.



Homogeneous transformation between the coordinate frames

Translations

a1 is l1
a2 is l2

Model of Acrobot - center of mass

Robot with Two Joints
Center of Mass

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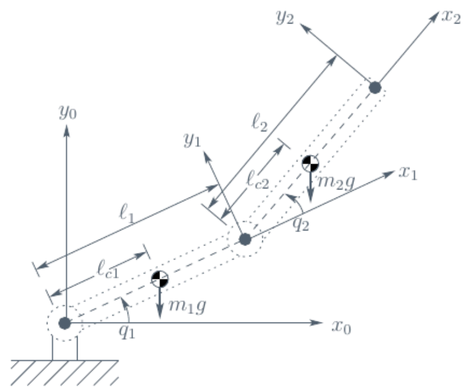
The center of mass for Link 1 in Frame 0 is

$$\begin{bmatrix} \boldsymbol{p}_{l_1} \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} c_1 & -s_1 & 0 & l_1 c_1 \\ s_1 & c_1 & 0 & l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{=A_1^0} \begin{bmatrix} -l_1 + l_{c1} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{c1} c_1 \\ l_{c1} s_1 \\ 0 \\ 1 \end{bmatrix}$$



notation
c: cosine
s: sine
see above slide
start with pos of COM, everything is in relation to base frame

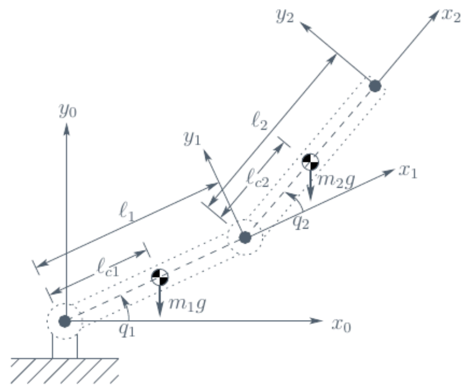
---

Robot with Two Joints
Center of Mass

The center of mass for Link 2 in Frame 0 is

$$\begin{bmatrix} \boldsymbol{p}_{l_2} \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} c_{12} & -s_{12} & 0 & l_1 c_1 + l_2 c_{12} \\ s_{12} & c_{12} & 0 & l_1 s_1 + l_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{=A_1^0 A_2^1} \begin{bmatrix} -l_2 + l_{c2} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_1 c_1 + l_{c2} c_{12} \\ l_1 s_1 + l_{c2} s_{12} \\ 0 \\ 1 \end{bmatrix}$$



Joint 2 is located at frame 1

Joint 1 is located at frame 0 (base frame)

A01 --> translation from 0 to 1

A12 --> translation from 1 to 2

Positional and Orientational Jacobians

Dynamics of Robot
Jacobian

The Jacobian can be used for expressing the velocities of the center of mass of Link i as

$$\dot{\boldsymbol{p}}_{l_i} = J_P^{l_i} \dot{\boldsymbol{q}}$$
$$\boldsymbol{\omega}_i = J_O^{l_i} \dot{\boldsymbol{q}}$$

where

$$J_P^{l_i} = \begin{bmatrix} \boldsymbol{J}_{P1}^{l_i} & \boldsymbol{J}_{P2}^{l_i} & \dots & \boldsymbol{J}_{Pi}^{l_i} & \boldsymbol{0} & \dots & \boldsymbol{0} \end{bmatrix}$$
$$J_O^{l_i} = \begin{bmatrix} \boldsymbol{J}_{O1}^{l_i} & \boldsymbol{J}_{O2}^{l_i} & \dots & \boldsymbol{J}_{Oi}^{l_i} & \boldsymbol{0} & \dots & \boldsymbol{0} \end{bmatrix}$$

we need one jacobian per body, as each has a COM (not just the end-effector)

The Jacobian can be used for expressing the velocities of the center of mass of Link i as

$$\dot{\boldsymbol{p}}_{l_i} = J_P^{l_i}\dot{\boldsymbol{q}}$$
$$\boldsymbol{\omega}_i = J_O^{l_i}\dot{\boldsymbol{q}}$$

where

$$J_P^{l_i} = \begin{bmatrix}\boldsymbol{J}_{P1}^{l_i} & \boldsymbol{J}_{P2}^{l_i} & \dots & \boldsymbol{J}_{Pi}^{l_i} & \boldsymbol{0} & \dots & \boldsymbol{0}\end{bmatrix}$$
$$J_O^{l_i} = \begin{bmatrix}\boldsymbol{J}_{O1}^{l_i} & \boldsymbol{J}_{O2}^{l_i} & \dots & \boldsymbol{J}_{Oi}^{l_i} & \boldsymbol{0} & \dots & \boldsymbol{0}\end{bmatrix}$$

For a revolute joint it is

$$\boldsymbol{J}_{Pj}^{l_i} = \boldsymbol{z}_{j-1} \times (\boldsymbol{p}_{l_i} - \boldsymbol{p}_{j-1}) \qquad \text{and} \qquad \boldsymbol{J}_{Oj}^{l_i} = \boldsymbol{z}_{j-1}$$

where $\boldsymbol{p}_{j-1}$ is the position vector to the origin of Frame $j-1$ and $\boldsymbol{z}_{j-1}$ is a unit vector in the direction of the z-axis of Frame $j-1$.

PER DH NOTATION: translation along z-axis, and rotation about z-axis
so the jacobian is written in terms of z-axis
"The point we are studying" is pi
Below is translation along z-axis

$$\boldsymbol{J}_{Pj}^{l_i} = \boldsymbol{z}_{j-1} \times (\boldsymbol{p}_{l_i} - \boldsymbol{p}_{j-1})$$

X: cross product (from origin of base frame to com of l1)

Below is rotation about z-axis

$$\boldsymbol{J}_{Oj}^{l_i} = \boldsymbol{z}_{j-1}$$

For Link 1 we obtain

$$\dot{\boldsymbol{p}}_{l_1} = J_P^{l_1}\dot{\boldsymbol{q}} \qquad \text{and} \qquad \boldsymbol{\omega}_1 = J_O^{l_1}\dot{\boldsymbol{q}}$$

where

$$J_P^{l_1} = \begin{bmatrix}\boldsymbol{J}_{P1}^{l_1} & \boldsymbol{0}\end{bmatrix} = \begin{bmatrix}\boldsymbol{z}_0 \times (\boldsymbol{p}_{l_1} - \boldsymbol{p}_0) & \boldsymbol{0}\end{bmatrix}$$
$$J_O^{l_1} = \begin{bmatrix}\boldsymbol{J}_{O1}^{l_1} & \boldsymbol{0}\end{bmatrix} = \begin{bmatrix}\boldsymbol{z}_0 & \boldsymbol{0}\end{bmatrix}$$

Dynamics of Robot
Example: Jacobian (I)
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For Link 1 we obtain

$$\dot{\boldsymbol{p}}_{l_1} = J_P^{l_1}\dot{\boldsymbol{q}} \qquad \text{and} \qquad \boldsymbol{\omega}_1 = J_O^{l_1}\dot{\boldsymbol{q}}$$

where

$$J_P^{l_1} = \begin{bmatrix}\boldsymbol{J}_{P1}^{l_1} & \boldsymbol{0}\end{bmatrix} = \begin{bmatrix}\boldsymbol{z}_0 \times (\boldsymbol{p}_{l_1} - \boldsymbol{p}_0) & \boldsymbol{0}\end{bmatrix}$$
$$J_O^{l_1} = \begin{bmatrix}\boldsymbol{J}_{O1}^{l_1} & \boldsymbol{0}\end{bmatrix} = \begin{bmatrix}\boldsymbol{z}_0 & \boldsymbol{0}\end{bmatrix}$$

This implies that

$$J_P^{l_1} = \begin{bmatrix}\boldsymbol{J}_{P1}^{l_1} & \boldsymbol{0}\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix}l_{c1}c_1\\l_{c1}s_1\\0\end{bmatrix} & \boldsymbol{0}\end{bmatrix} = \begin{bmatrix}-l_{c1}s_1 & 0\\ l_{c1}c_1 & 0\\ 0 & 0\end{bmatrix}$$
$$J_O^{l_1} = \begin{bmatrix}\boldsymbol{J}_{O1}^{l_1} & \boldsymbol{0}\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\\1 & 0\end{bmatrix}$$

For Link 2 haves

$$\dot{\boldsymbol{p}}_{l_2} = J_P^{l_2}\dot{\boldsymbol{q}} \quad \text{and} \quad \boldsymbol{\omega}_2 = J_O^{l_2}\dot{\boldsymbol{q}}$$

where

$$J_P^{l_2} = \begin{bmatrix} \boldsymbol{J}_{P1}^{l_2} & \boldsymbol{J}_{P2}^{l_2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{z}_0 \times (\boldsymbol{p}_{l_2} - \boldsymbol{p}_0) & \boldsymbol{z}_1 \times (\boldsymbol{p}_{l_2} - \boldsymbol{p}_1) \end{bmatrix}$$
$$J_O^{l_2} = \begin{bmatrix} \boldsymbol{J}_{O1}^{l_2} & \boldsymbol{J}_{O2}^{l_2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{z}_0 & \boldsymbol{z}_1 \end{bmatrix}$$

This is radius(es) from point of origin to point of study

$$J_P^{l_2} = \begin{bmatrix} \boldsymbol{J}_{P1}^{l_2} & \boldsymbol{J}_{P2}^{l_2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{z}_0 \times (\boldsymbol{p}_{l_2} - \boldsymbol{p}_0) & \boldsymbol{z}_1 \times (\boldsymbol{p}_{l_2} - \boldsymbol{p}_1) \end{bmatrix}$$

For Link 2 haves

$$\dot{\boldsymbol{p}}_{l_2} = J_P^{l_2}\dot{\boldsymbol{q}} \quad \text{and} \quad \boldsymbol{\omega}_2 = J_O^{l_2}\dot{\boldsymbol{q}}$$

where

$$J_P^{l_2} = \begin{bmatrix} \boldsymbol{J}_{P1}^{l_2} & \boldsymbol{J}_{P2}^{l_2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{z}_0 \times (\boldsymbol{p}_{l_2} - \boldsymbol{p}_0) & \boldsymbol{z}_1 \times (\boldsymbol{p}_{l_2} - \boldsymbol{p}_1) \end{bmatrix}$$
$$J_O^{l_2} = \begin{bmatrix} \boldsymbol{J}_{O1}^{l_2} & \boldsymbol{J}_{O2}^{l_2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{z}_0 & \boldsymbol{z}_1 \end{bmatrix}$$

This implies that

$$J_P^{l_2} = \left[\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} l_1c_1 + l_{c2}c_{12} \\ l_1s_1 + l_{c2}s_{12} \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \left(\begin{bmatrix} l_1c_1 + l_{c2}c_{12} \\ l_1s_1 + l_{c2}s_{12} \\ 0 \end{bmatrix} - \begin{bmatrix} l_1c_1 \\ l_1s_1 \\ 0 \end{bmatrix} \right) \right]$$
$$= \begin{bmatrix} -l_1s_1 - l_{c2}s_{12} & -l_{c2}s_{12} \\ l_1c_1 + l_{c2}c_{12} & l_{c2}c_{12} \\ 0 & 0 \end{bmatrix} \quad \text{and } J_O^{l_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

-

potential energy --> comes from position
kinetic energy --> comes from velocity

The potential energy should be expressed in an inertial frame e.g. the base frame, which does not accelerate. Then the potential energy can be computed as

$$E_{\text{pot}}(\boldsymbol{q}) = \sum_{i=1}^{n} E_{\text{pot},l_i}(\boldsymbol{q}) \qquad [\text{J}]$$

where E_{pot,l_i} is the potential energy for Link i [J].

-

The potential energy should be expressed in an inertial frame e.g. the base frame, which does not accelerate. Then the potential energy can be computed as

$$E_{\text{pot}}(\boldsymbol{q}) = \sum_{i=1}^{n} E_{\text{pot},l_i}(\boldsymbol{q}) \qquad \text{[J]}$$

where E_{pot,l_i} is the potential energy for Link i [J].

The total potential energy becomes

$$E_{\text{pot}}(\boldsymbol{q}) = -\sum_{i=1}^{n} m_{l_i} \boldsymbol{g}_0^T \boldsymbol{p}_{l_i}(\boldsymbol{q}) \qquad \text{[J]}$$

where m_{l_i} is the mass of Link i [kg], $\boldsymbol{g}_0$ is the gravitational acceleration in Base Frame [m/s^2] and $\boldsymbol{p}_{l_i}(\boldsymbol{q})$ is the position of the center of mass of Link i in Base Frame [m].

-

The the considered robot manipulator's potential energy is

$$E_{\text{pot}}(\boldsymbol{q}) = -\sum_{i=1}^{2} m_{l_i} \boldsymbol{g}_0^T \boldsymbol{p}_{l_i}(\boldsymbol{q}) \qquad \text{[J]}$$

where

$$\boldsymbol{p}_{l_1} = \begin{bmatrix} l_{c1}c_1 \\ l_{c1}s_1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{p}_{l_2} = \begin{bmatrix} l_1 c_1 + l_{c2}c_{12} \\ l_1 s_1 + l_{c2}s_{12} \\ 0 \end{bmatrix}, \qquad \boldsymbol{g}_0 = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix}$$

- pl1 and pl2 are the two bodies
- g0 is the gravitational acceleration vector for x y and z

The the considered robot manipulator's potential energy is

$$E_{\text{pot}}(\boldsymbol{q}) = -\sum_{i=1}^{2} m_{l_i} \boldsymbol{g}_0^T \boldsymbol{p}_{l_i}(\boldsymbol{q}) \qquad \text{[J]}$$

where

$$\boldsymbol{p}_{l_1} = \begin{bmatrix} l_{c1}c_1 \\ l_{c1}s_1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{p}_{l_2} = \begin{bmatrix} l_1 c_1 + l_{c2}c_{12} \\ l_1 s_1 + l_{c2}s_{12} \\ 0 \end{bmatrix}, \qquad \boldsymbol{g}_0 = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix}$$

This gives

$$E_{\text{pot}}(\boldsymbol{q}) = m_{l_1} g l_{c1} s_1 + m_{l_2} g (l_1 s_1 + l_{c2} s_{12}) \qquad \text{[J]}$$

The kinetic energy should be computed in an inertial frame, e.g., Base Frame that does not accelerate; thus, the kinetic energy can be computed as

$$E_{\text{kin}}(\boldsymbol{q}) = \sum_{i=1}^{n} E_{\text{kin},l_i}(\boldsymbol{q}) \qquad \text{[J]}$$

where E_{kin,l_i} is the kinetic energy of Link i [J].

Robot with Two Joints
Kinetic Energy
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The kinetic energy should be computed in an inertial frame, e.g., Base Frame that does not accelerate; thus, the kinetic energy can be computed as

$$E_{\text{kin}}(\boldsymbol{q}) = \sum_{i=1}^{n} E_{\text{kin},l_i}(\boldsymbol{q}) \qquad \text{[J]}$$

where E_{kin,l_i} is the kinetic energy of Link i [J].

The kinetic energy can be expressed as the sum of translational and rotational kinetic energy

$$E_{\text{kin},l_i}(\boldsymbol{q}) = \frac{1}{2} m_{l_i} \dot{\boldsymbol{p}}_{l_i}^T \dot{\boldsymbol{p}}_{l_i} + \frac{1}{2} \boldsymbol{\omega}_i^T I_{l_i}(\boldsymbol{q}) \boldsymbol{\omega}_i$$

where both $\dot{\boldsymbol{p}}_i$, $\boldsymbol{\omega}_i$ and I_{l_i} are given in Base Frame.

Translational part

$$= \frac{1}{2} m_{l_i} \dot{\boldsymbol{p}}_{l_i}^T \dot{\boldsymbol{p}}_{l_i} -$$

Rotational part

$$+\frac{1}{2}\boldsymbol{\omega}_i^T I_{l_i}(\boldsymbol{q})\boldsymbol{\omega}_i$$

Constant in local coordinates but CAN have varying base coordinates

---

Robot with Two Joints

Kinetic Energy: Inertia Tensor

The inertia tensor I_{l_i} given in Base Frame can be computed by using an inertia tensor at the link's center of mass ($I_{l_i}^i$)

$$I_{l_i}(\boldsymbol{q}) = R_i^0(\boldsymbol{q}) I_{l_i}^i R_i^{0T}(\boldsymbol{q})$$

we rotate to fit base-frame axis by multiplying with base frame

---

Robot with Two Joints
Kinetic Energy: Inertia Tensor

The inertia tensor I_{l_i} given in Base Frame can be computed by using an inertia tensor at the link's center of mass ($I^i_{l_i}$)

$$I_{l_i}(\boldsymbol{q}) = R^0_i(\boldsymbol{q}) I^i_{l_i} R^{0T}_i(\boldsymbol{q})$$

This gives the following expression for the kinetic energy

$$E_{\mathsf{kin},l_i} = \frac{1}{2} m_{l_i} \dot{\boldsymbol{p}}^T_{l_i} \dot{\boldsymbol{p}}_{l_i} + \frac{1}{2} \boldsymbol{\omega}^T_i R^0_i I^i_{l_i} R^{0T}_i \boldsymbol{\omega}_i$$

where both $\dot{\boldsymbol{p}}_i$ and $\boldsymbol{\omega}_i$ are given in Base Frame.

pi and wi are given in terms of base frame

Robot with Two Joints
Kinetic Energy: Inertia Tensor

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The inertia tensor I_{l_i} given in Base Frame can be computed by using an inertia tensor at the link's center of mass ($I^i_{l_i}$)

$$I_{l_i}(\boldsymbol{q}) = R^0_i(\boldsymbol{q}) I^i_{l_i} R^{0T}_i(\boldsymbol{q})$$

This gives the following expression for the kinetic energy

$$E_{\mathsf{kin},l_i} = \frac{1}{2} m_{l_i} \dot{\boldsymbol{p}}^T_{l_i} \dot{\boldsymbol{p}}_{l_i} + \frac{1}{2} \boldsymbol{\omega}^T_i R^0_i I^i_{l_i} R^{0T}_i \boldsymbol{\omega}_i$$

where both $\dot{\boldsymbol{p}}_i$ and $\boldsymbol{\omega}_i$ are given in Base Frame.

We intend to express $\dot{\boldsymbol{p}}_i$ and $\boldsymbol{\omega}_i$ by the use of generalized coordinates $\boldsymbol{q}$.

Robot with Two Joints

Kinetic Energy: Jacobian

By using the Jacobian, the kinetic energy is expressed as

$$E_{\mathsf{kin}}(\boldsymbol{q}) = \sum_{i=1}^{n} E_{\mathsf{kin},l_i} \qquad \text{[J]}$$

where

$$E_{\mathsf{kin},l_i} = \frac{1}{2} m_{l_i} \dot{\boldsymbol{q}}^T J_P^{l_i T} J_P^{l_i} \dot{\boldsymbol{q}} + \frac{1}{2} \dot{\boldsymbol{q}}^T J_O^{l_i T} R_i^0 I_{l_i}^i R_i^{0T} J_O^{l_i} \dot{\boldsymbol{q}}$$

This ensures we can compute Ekin in terms of kinematics directly

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Robot with Two Joints

Example: Kinetic Energy (I)

For Link 1 the kinetic energy is

$$\begin{aligned}
E_{\mathsf{kin},l_1} &= \frac{1}{2} m_{l_1} \dot{\boldsymbol{q}}^T J_P^{l_1 T} J_P^{l_1} \dot{\boldsymbol{q}} + \frac{1}{2} \dot{\boldsymbol{q}}^T J_O^{l_1 T} R_1^0 I_{l_1}^1 R_1^{0T} J_O^{l_1} \dot{\boldsymbol{q}} \\
&= \frac{1}{2} m_{l_1} \dot{\boldsymbol{q}}^T \begin{bmatrix} -l_{c1} s_1 & l_{c1} c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -l_{c1} s_1 & 0 \\ l_{c1} c_1 & 0 \\ 0 & 0 \end{bmatrix} \dot{\boldsymbol{q}} + \frac{1}{2} \begin{bmatrix} 0 & 0 & \dot{q}_1 \end{bmatrix} I_{l_1}^1 \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix} \\
&= \frac{1}{2} m_{l_1} l_{c1}^2 \dot{q}_1^2 + \frac{1}{2} \begin{bmatrix} 0 & 0 & \dot{q}_1 \end{bmatrix} I_{l_1}^1 \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix} \\
&= \frac{1}{2} \dot{\boldsymbol{q}}^T \begin{bmatrix} m_{l_1} l_{c1}^2 + I_{l_1,zz}^1 & 0 \\ 0 & 0 \end{bmatrix} \dot{\boldsymbol{q}}
\end{aligned}$$

-

For Link 2 the kinetic energy is

$$
\begin{aligned}
E_{\text{kin},l_2} &= \frac{1}{2} m_{l_2} \dot{\boldsymbol{q}}^T J_P^{l_2 T} J_P^{l_2} \dot{\boldsymbol{q}} + \frac{1}{2} \dot{\boldsymbol{q}}^T J_O^{l_2 T} R_2^0 I_{l_2}^2 R_2^{0T} J_O^{l_2} \dot{\boldsymbol{q}} \\
&= \frac{1}{2} m_{l_2} \dot{\boldsymbol{q}}^T \begin{bmatrix} -l_1 s_1 - l_{c2} s_{12} & l_1 c_1 + l_{c2} c_{12} & 0 \\ -l_{c2} s_{12} & l_{c2} c_{12} & 0 \end{bmatrix} \begin{bmatrix} -l_1 s_1 - l_{c2} s_{12} & -l_{c2} s_{12} \\ l_1 c_1 + l_{c2} c_{12} & l_{c2} c_{12} \\ 0 & 0 \end{bmatrix} \dot{\boldsymbol{q}} \\
&+ \frac{1}{2} \begin{bmatrix} 0 & 0 & \dot{q}_1 + \dot{q}_2 \end{bmatrix} I_{l_2}^2 \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 + \dot{q}_2 \end{bmatrix} \\
&= \frac{1}{2} m_{l_2} \dot{\boldsymbol{q}}^T \begin{bmatrix} l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} c_2 & l_{c2}^2 + l_1 l_{c2} c_2 \\ l_{c2}^2 + l_1 l_{c2} c_2 & l_{c2}^2 \end{bmatrix} \dot{\boldsymbol{q}} + \frac{1}{2} \begin{bmatrix} 0 & 0 & \dot{q}_1 + \dot{q}_2 \end{bmatrix} I_{l_2}^2 \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 + \dot{q}_2 \end{bmatrix} \\
&= \frac{1}{2} \dot{\boldsymbol{q}}^T \begin{bmatrix} m_{l_2}(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} c_2) + I_{l_2,zz}^2 & m_{l_2}(l_{c2}^2 + l_1 l_{c2} c_2) + I_{l_2,zz}^2 \\ m_{l_2}(l_{c2}^2 + l_1 l_{c2} c_2) + I_{l_2,zz}^2 & m_{l_2} l_{c2}^2 + I_{l_2,zz}^2 \end{bmatrix} \dot{\boldsymbol{q}}
\end{aligned}
$$

-

The potential and kinetic energy are

$$
\begin{aligned}
E_{\text{pot}} &= m_{l_1} g l_{c1} s_1 + m_{l_2} g (l_1 s_1 + l_{c2} s_{12}) \\
E_{\text{kin}} &= \frac{1}{2} \dot{\boldsymbol{q}}^T \underbrace{\begin{bmatrix} m_{l_1} l_{c1}^2 + I_{l_1,zz}^1 + m_{l_2}(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} c_2) + I_{l_2,zz}^2 & m_{l_2}(l_{c2}^2 + l_1 l_{c2} c_2) + I_{l_2,zz}^2 \\ m_{l_2}(l_{c2}^2 + l_1 l_{c2} c_2) + I_{l_2,zz}^2 & m_{l_2} l_{c2}^2 + I_{l_2,zz}^2 \end{bmatrix}}_{=B(\boldsymbol{q})} \dot{\boldsymbol{q}}
\end{aligned}
$$

where $B(\boldsymbol{q})$ is the inertia tensor expressed in Base Frame.

Lagrange–D'Alembert's Principle can be used for modelling the system, where $\boldsymbol{q}$ is a vector of the two joint angles, and τ_i is the torque applied at Joint i

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} - \frac{\partial \mathcal{L}}{\partial q_1} = \tau_1$$
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2} - \frac{\partial \mathcal{L}}{\partial q_2} = \tau_2$$

where

$$\mathcal{L} = E_{\text{kin}} - E_{\text{pot}}$$

Procedure for Modelling
Overview

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The following procedure can be used for setting up a dynamical model of a robot with n degrees of freedom

0. Find the DH-parameters of the robot $(a_i, d_i, \alpha_i, \theta_i)$ for $i = 1, 2, \ldots, n$.
1. Set up a kinematic model $T_n^0(\boldsymbol{q})$ of the robot.
2. Compute the coordinates $\boldsymbol{p}_{ci}^0(q)$ for center of mass for each link (given in Base frame).
3. Compute the angular velocities $\boldsymbol{\omega}_i^0(\boldsymbol{q}, \dot{\boldsymbol{q}})$ for each link (given in Base frame).
4. Compute velocities $\boldsymbol{v}_{ci}^0(\boldsymbol{q}, \dot{\boldsymbol{q}})$ for center of mass of each link (given in Base frame).
5. Compute the inertia-tensor $I_{l_i}^0(\boldsymbol{q})$ for each link (given in Base frame).
6. Compute the potential energy of the system $E_{\text{pot}}(\boldsymbol{q})$.
7. Compute the kinetic energy of the system $E_{\text{kin}}(\boldsymbol{q}, \dot{\boldsymbol{q}})$.
8. Set up the equations of motion for the system using Lagrange D'Alembert's principle.

- 0 is using dh-parameters as that is standard in robotics

Summary

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-

If q is a trajectory of a conservative mechanical system then

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} = 0$$

where q is an n-dimensional vector of generalized coordinates and $\mathcal{L}$ is the *Lagrangian* given by

$$\mathcal{L} = E_{\text{kin}} - E_{\text{pot}} \qquad [\text{J}]$$

where E_{pot} is the system's potential energy and E_{kin} is the system's kinetic energy.

If q is a trajectory of a mechanical system that is affected by a generalized force Q then

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} = \boldsymbol{Q}$$

where Q is an n-dimensional vector of generalized forces. This is called ***Lagrange–D'Alembert's Principle***.

Summary
Procedure for Deriving Robot Manipulator Dynamics

The following procedure can be used for setting up a dynamical model of a serial robot manipulator with n degrees of freedom

0. Find the DH-parameters of the robot $(a_i, d_i, \alpha_i, \theta_i)$ for $i = 1, 2, \ldots, n$.
1. Set up a kinematic model $T_n^0(\boldsymbol{q})$ of the robot.
2. Compute the coordinates $\boldsymbol{p}_{ci}^0(q)$ for center of mass for each link (given in Base frame).
3. Compute the angular velocities $\boldsymbol{\omega}_i^0(\boldsymbol{q}, \dot{\boldsymbol{q}})$ for each link (given in Base frame).
4. Compute velocities $\boldsymbol{v}_{ci}^0(\boldsymbol{q}, \dot{\boldsymbol{q}})$ for center of mass of each link (given in Base frame).
5. Compute the inertia-tensor $I_{l_i}^0(\boldsymbol{q})$ for each link (given in Base frame).
6. Compute the potential energy of the system $E_{\text{pot}}(\boldsymbol{q})$.
7. Compute the kinetic energy of the system $E_{\text{kin}}(\boldsymbol{q}, \dot{\boldsymbol{q}})$.
8. Set up the equations of motion for the system using Lagrange D'Alembert's principle.

Model of (Underactuated) System

If tau is not completely "filled" it is underactuated --> so m < n (number of spaces in tau)

Model of robots with serial kinematics can be formulated as

$$B(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau$$

where $B(q)$ is the mass matrix, $C(q, \dot{q})$ is a matrix containing Coriolis and centrifugal terms, $g(q)$ is the gravity vector, and τ is the actuator torque. Here

$$E_{\text{kin}} = \frac{1}{2}\dot{\boldsymbol{q}}^T B(\boldsymbol{q})\dot{\boldsymbol{q}}$$

$$E_{\text{pot}} = -\sum_{i=1}^{n} m_{l_i} \boldsymbol{g}_0^T \boldsymbol{p}_{l_i}(\boldsymbol{q})$$

$$\boldsymbol{g}(\boldsymbol{q}) = \begin{bmatrix} \frac{\partial E_{\text{pot}}}{\partial q_1} & \frac{\partial E_{\text{pot}}}{\partial q_2} & \cdots & \frac{\partial E_{\text{pot}}}{\partial q_n} \end{bmatrix}^T$$

$$c_{ij} = \sum_{k=1}^{n} \frac{1}{2}\left(\frac{\partial b_{ij}}{\partial q_k} + \frac{\partial b_{ik}}{\partial q_j} - \frac{\partial b_{jk}}{\partial q_i}\right)\dot{q}_k$$