Conservative System: total mechanical energy is constant --> no energy dissipation
This is "ideal" - a system with no friction, the basis of modelling

Non-conservative systems models friction using Euler-Lagrange modelling

Hamiltons Principle

Euler-Lagrange modelling
Hamilton's Principle

The motion of a mechanical system from time a to b is such that the integral

$$I(t, q, \dot{q}) = \int_a^b \mathcal{L}(t, q, \dot{q})dt,$$

where $\mathcal{L} = E_{kin} - E_{pot}$ has a stationary value. The function $\mathcal{L}$ is called the **Lagrangian**.

Euler-Lagrange modelling can be used for finding the equations of motion of e.g. mechanical systems using the system's potential energy E_{pot} and kinetic energy E_{kin}.

instead of free-body diagrams we use Ekin and Epot as base for our model

Consider a mechanical system with n degrees of freedom. The system is modelled with n **generalized coordinates** $q_1, \ldots, q_n$.

Should be:
minimal
independant
complete

Eksempel:

The rotational mass-spring system has dynamics given by

$$I_1\ddot{\theta}_1 = -K_1\theta_1 - K_2(\theta_1 - \theta_2) \qquad [\text{Nm}]$$
$$I_2\ddot{\theta}_2 = -K_2(\theta_2 - \theta_1) \qquad [\text{Nm}]$$

where I_1, I_2 are moments of inertia [kgm^2] and K_1, K_2 are stiffnesses [N/rad].

From Euler-Lagrange Equation with generalized coordinates $q = (q_1, q_2) = (\theta_1, \theta_2)$ we obtain

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}} - \frac{\partial\mathcal{L}}{\partial q} = 0$$

where

$$\mathcal{L} = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 - \left(\frac{1}{2}K_1\theta_1^2 + \frac{1}{2}K_2(\theta_1 - \theta_2)^2\right)$$

This can be written as

$$\begin{bmatrix} I_1\ddot{\theta}_1 + K_1\theta_1 + K_2(\theta_1 - \theta_2) \\ I_2\ddot{\theta}_2 - K_2(\theta_1 - \theta_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

SO the Euler-Lagrange Equation is used in place of Newtons second law of motion (F = am)

If q is a trajectory of a conservative mechanical system then

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}} - \frac{\partial\mathcal{L}}{\partial q} = 0$$

where q is an n-dimensional vector of generalized coordinates and $\mathcal{L}$ is the *Lagrangian* given by

$$\mathcal{L} = E_{\text{kin}} - E_{\text{pot}} \qquad [\text{J}]$$

where E_{pot} is the system's potential energy and E_{kin} is the system's kinetic energy.

D'Alamberts Principle

If q is a trajectory of a mechanical system that is affected by a generalized force Q then

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = Q$$

where Q is an n-dimensional vector of generalized forces. **Lagrange–D'Alembert's Principle** can be written as (for $q = (q_1, q_2, \ldots, q_n)$)
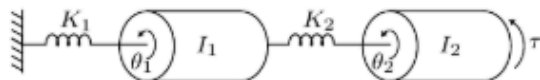
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} - \frac{\partial \mathcal{L}}{\partial q_1} = Q_1$$
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2} - \frac{\partial \mathcal{L}}{\partial q_2} = Q_2$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_n} - \frac{\partial \mathcal{L}}{\partial q_n} = Q_n$$

Only dissipitative forces go into capital Q
otherwise they are modelled twice

External force tau

The above rotational mass-spring system has dynamics

$$I_1\ddot{\theta}_1 = -K_1\theta_1 - K_2(\theta_1 - \theta_2) \qquad \text{[Nm]}$$
$$I_2\ddot{\theta}_2 = -K_2(\theta_2 - \theta_1) + \tau \qquad \text{[Nm]}$$

where I_1, I_2 are moments of inertia [kgm^2] and K_1, K_2 are stiffnesses [N/rad].

From Euler-Lagrange Equation with generalized coordinates $q = (q_1, q_2) = (\theta_1, \theta_2)$ and generalized force $Q = (0, \tau)$ we obtain

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = Q$$

where

$$\mathcal{L} = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 - \left(\frac{1}{2}K_1\theta_1^2 + \frac{1}{2}K_2(\theta_1 - \theta_2)^2\right)$$

This can be written as

$$\begin{bmatrix} I_1\ddot{\theta}_1 + K_1\theta_1 + K_2(\theta_1 - \theta_2) \\ I_2\ddot{\theta}_2 - K_2(\theta_1 - \theta_2) \end{bmatrix} = \begin{bmatrix} 0 \\ \tau \end{bmatrix}$$

Lagrange D'Alamberts principle

If q is a trajectory of a mechanical system that is affected by a generalized force Q then

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = Q$$

where Q is an n-dimensional vector of generalized forces and $q = (q_1, q_2, \ldots, q_n)$ is the generalized coordinate and the Lagrangian is given by

$$\mathcal{L}(q, \dot{q}) = E_{kin}(q, \dot{q}) - E_{pot}(q).$$

Kinetic and Potential Energy calculation

Properties of Dynamical Robot Models
Kinetic and Potential Energies

The gravitational potential energy is given by

$$E_{pot}(q) = -\sum_{i=1}^{n} m_{l_i} g_0^T p_{l_i}(q) \qquad [J]$$

where m_{l_i} is the mass of Link i [kg], g_0 is the gravitational acceleration in Base Frame [m/s²] and $p_{l_i}(q)$ is the position of the center of mass of Link i in Base Frame [m]; and

$$E_{kin}(q, \dot{q}) = \frac{1}{2}\dot{q}^T B(q)\dot{q} \qquad [J]$$

where $B(q)$ is the inertia tensor in Base Frame.

Makes it possible to use kinematics --> this is a vector [x, y, z]
, g_0 is the gravitational acceleration in Base Frame
systematic approach
first write down kinematics
then we have center of mass
compute potential and kinetic energies

$B(q)$

this is invertable (?) so the kinetic energy is positive(?)

Coriolis and Centrigual terms

The final two terms of

$$B(q)\ddot{q} + \dot{B}(q)\dot{q} - \frac{\partial E_{kin}}{\partial q} + g(q) = Q$$

can be written as (the chain rule has been applied)

$$(\dot{B}(q)\dot{q})_i = \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial b_{ij}}{\partial q_k} \dot{q}_k \dot{q}_j$$

and

$$\frac{\partial E_{kin}}{\partial q_i} = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial b_{jk}}{\partial q_i} \dot{q}_k \dot{q}_j$$

Euler-Lagrange Model

The robot model given by the Euler-Lagrange equation can be formulated as

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

where $B(q)$ is the inertia tensor, $C(q,\dot{q})$ is a matrix containing Coriolis and centrifugal terms, $g(q)$ is the gravity vector, and τ is the actuator torque.

We need to isolate acceleration

- $\ddot{q} = B^{-1}(q)(\tau - C(q,\dot{q})\dot{q} - g(q))$

Kinematik og DH parametre

Potential Energy

The potential energy should be expressed in an inertial frame e.g. the base frame, which does not accelerate. Then the potential energy can be computed as

$$E_{pot}(q) = \sum_{i=1}^{n} E_{pot,l_i}(q) \qquad [J]$$

where E_{pot,l_i} is the potential energy for Link i [J].

The total potential energy becomes

$$E_{pot}(q) = -\sum_{i=1}^{n} m_{l_i} g_0^T p_{l_i}(q) \qquad [J]$$

where m_{l_i} is the mass of Link i [kg], g_0 is the gravitational acceleration in Base Frame [m/s^2] and $p_{l_i}(q)$ is the position of the center of mass of Link i in Base Frame [m].

If q is a trajectory of a conservative mechanical system then

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0$$

where q is an n-dimensional vector of generalized coordinates and $\mathcal{L}$ is the *Lagrangian* given by

$$\mathcal{L} = E_{\text{kin}} - E_{\text{pot}} \quad [\text{J}]$$

where E_{pot} is the system's potential energy and E_{kin} is the system's kinetic energy.

If q is a trajectory of a mechanical system that is affected by a generalized force Q then

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = Q$$

where Q is an n-dimensional vector of generalized forces. This is called **Lagrange–D'Alembert's Principle**.

Model of (Underactuated) System

If tau is not completely "filled" it is underactuated --> so m < n (number of spaces in tau)

Model of robots with serial kinematics can be formulated as

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

where $B(q)$ is the mass matrix, $C(q,\dot{q})$ is a matrix containing Coriolis and centrifugal terms, $g(q)$ is the gravity vector, and τ is the actuator torque. Here

$$E_{\text{kin}} = \frac{1}{2}\dot{q}^T B(q)\dot{q}$$

$$E_{\text{pot}} = -\sum_{i=1}^{n} m_{l_i} g_0^T p_{l_i}(q)$$

$$g(q) = \left[\frac{\partial E_{\text{pot}}}{\partial q_1} \quad \frac{\partial E_{\text{pot}}}{\partial q_2} \quad \cdots \quad \frac{\partial E_{\text{pot}}}{\partial q_n}\right]^T$$

$$c_{ij} = \sum_{k=1}^{n} \frac{1}{2}\left(\frac{\partial b_{ij}}{\partial q_k} + \frac{\partial b_{ik}}{\partial q_j} - \frac{\partial b_{jk}}{\partial q_i}\right)\dot{q}_k$$

The following procedure can be used for setting up a dynamical model of a serial robot manipulator with n degrees of freedom

0. Find the DH-parameters of the robot $(a_i, d_i, \alpha_i, \theta_i)$ for $i = 1, 2, \ldots, n$.
1. Set up a kinematic model $T_n^0(\boldsymbol{q})$ of the robot.
2. Compute the coordinates $\boldsymbol{p}_{ci}^0(q)$ for center of mass for each link (given in Base frame).
3. Compute the angular velocities $\boldsymbol{\omega}_i^0(\boldsymbol{q}, \dot{\boldsymbol{q}})$ for each link (given in Base frame).
4. Compute velocities $\boldsymbol{v}_{ci}^0(\boldsymbol{q}, \dot{\boldsymbol{q}})$ for center of mass of each link (given in Base frame).
5. Compute the inertia-tensor $I_{l_i}^0(\boldsymbol{q})$ for each link (given in Base frame).
6. Compute the potential energy of the system $E_{\text{pot}}(\boldsymbol{q})$.
7. Compute the kinetic energy of the system $E_{\text{kin}}(\boldsymbol{q}, \dot{\boldsymbol{q}})$.
8. Set up the equations of motion for the system using Lagrange D'Alembert's principle.