Labs
Machine Learning Course
Fall 2025

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https://github.com/LINs-lab/course_machine_learning

Problem Set Lab 01 (graded), Sept. 04, 2025 (Mathematical Foundation of Machine Learning)

Goals. The goals of this lab are to:

- Familiarize yourself with the mathematical foundations of the machine learning course.

Submission instructions:

- Please submit a PDF file to canvas.
- Deadline: 23.59 on Sept. 10, 2025

Review of Linear Algebra

Problem 1 (Idempotent Matrices and Rank Inequality):

Given A and B are idempotent $n \times n$ matrices (i.e., $A^2 = A$ and $B^2 = B$), and they commute with each other ($AB = BA$):

1. Prove that $A + B - AB$ is also an idempotent matrix.
2. Further prove that

$$\operatorname{rank}(A + B - AB) \leq \operatorname{rank}(A) + \operatorname{rank}(B)\,. \tag{1}$$

Problem 2 (Diagonalizability and Eigenvalues of a Linear Transformation):

Let V be a four-dimensional vector space, and let $T : V \to V$ be a linear transformation satisfying $T^3 - 2T^2 + T - 2I = 0$, where I is the identity transformation on V. Prove that T is diagonalizable and determine all its eigenvalues.

Problem 3 (Existence of Real Matrix Roots for Positive Eigenvalue Matrices):

If a real matrix A has all eigenvalues as positive real numbers, then for any positive integer m, there exists a real matrix B such that $B^m = A$.

Problem 4 (Eigenvalue Equivalence under Commutator-like Condition):

Let A and B be $n \times n$ square matrices satisfying

$$AB - BA = A - B\,. \tag{2}$$

Then, A and B have the same eigenvalues.

Problem 5 (Matrix Determinant and Commutator):

Let A and B be two $n \times n$ matrices satisfying the equation

$$AB - BA = A\,. \tag{3}$$

Prove that $\det(A) = 0$.

Review of Probability Theory

Problem 6 (Moment Bound for a Standard Normal Random Variable):

For a standard normal random variable X, there exists a constant C such that for all $p > 1$,

$$\left(\mathbb{E}\left[|X|^p\right]\right)^{1/p} \leq C\sqrt{p}\,. \tag{4}$$

Problem 7 (Bounded Random Variable and Exponential Expectation):

Let X be a bounded random variable with $\mathbb{E}[X] = 0$ and $|X|_\infty \leq a$ for some $a > 0$. Prove that

$$\mathbb{E}\left[e^X\right] \leq \cosh(a)\,. \tag{5}$$

Problem 8 (Almost Sure Convergence of Scaled Random Walks):

Let X be a random variable with distribution $P(X = 1) = P(X = -1) = \frac{1}{2}$. Define the partial sum $S_n = X_1 + X_2 + \cdots + X_n$, where $X_1, X_2, \ldots, X_n$ are independent and identically distributed (i.i.d.) copies of X. For any $\alpha > \frac{1}{2}$, prove that

$$P\left(\lim_{n\to\infty} \frac{S_n}{n^\alpha} = 0\right) = 1\,. \tag{6}$$

Problem 9 (Probability Bound for the Standardized Sum of Uniform Random Variables):

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (i.i.d.) random variables uniformly distributed on the interval $(-1, 1)$. For any $r > 0$, prove that

$$P\left(\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}} < r\right) > 1 - \frac{1}{3r^2}\,. \tag{7}$$

Problem 10 (Central Limit Theorem and Standardized Sum Convergence):

Let $\{X_1, X_2, \ldots, X_n\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with finite mean $\mu = \mathbb{E}[X_i]$ and finite variance $\sigma^2 = \mathrm{Var}(X_i) > 0$. Define the standardized sum:

$$Z_n = \frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}\,. \tag{8}$$

Then, as $n \to \infty$, the distribution of Z_n converges to the standard normal distribution $\mathcal{N}(0, 1)$. Formally, for all real numbers z:

$$\lim_{n\to\infty} P(Z_n \leq z) = \Phi(z)\,, \tag{9}$$

where $\Phi(z)$ is the cumulative distribution function (CDF) of the standard normal distribution.