Lecture 7: Generating Learning Algorithm

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October 21, 2025

1. Review of Last Week
   - Exponential Families
   - Generalized Linear Models

2. Generative Learning Algorithms
   - Discriminative vs. Generative Learning Algorithms
   - Gaussian Discriminant Analysis (GDA)
   - Linear Discriminant Analysis (LDA)
   - LDA and Logistic Regression
   - MLE for GDA
   - Naïve Bayes

Reading materials & Reference

Reading materials:

- Chapter 4, Stanford CS 229 Lecture Notes, `https://cs229.stanford.edu/notes2022fall/main_notes.pdf`
- Chapter 9, Probabilistic Machine Learning: an introduction.
- Learning From Data Lecture 4: Generative Learning Algorithms, `http://yangli-feasibility.com/`

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It is called the **link function**.

Exponential family — definition

A distribution belongs to the exponential family if it can be written in the form

$$p(y|\boldsymbol{\eta}) = \underbrace{h(y)}_{\geq 0} \exp\left[\boldsymbol{\eta}^\top \boldsymbol{\phi}(y) - A(\boldsymbol{\eta})\right] \tag{3}$$

[1] Assume that we are given independent samples from this distribution. We do know $\boldsymbol{\phi}(y)$ and $h(y)$ but not $\boldsymbol{\eta}$. In order to optimally estimate $\boldsymbol{\eta}$ given these samples, all we need is the empirical average of the $\boldsymbol{\phi}(y)$.

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$$\text{(we need)} \int p(y|\boldsymbol{\eta}) dy = \int h(y) \exp\left[\boldsymbol{\eta}^\top \boldsymbol{\phi}(y) - A(\boldsymbol{\eta})\right] dy = 1 \tag{4}$$

$$\implies \int h(y) \exp\left[\boldsymbol{\eta}^\top \boldsymbol{\phi}(y)\right] dy = \int h(y) \exp\left[A(\boldsymbol{\eta})\right] dy = \exp\left[A(\boldsymbol{\eta})\right] \tag{5}$$

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$$\boldsymbol{\eta} = \mathbf{g}\left(\boldsymbol{\mu} := \mathbb{E}\left[\boldsymbol{\phi}(y)\right]\right) \iff \boldsymbol{\mu} = \mathbf{g}^{-1}(\boldsymbol{\eta}) = \nabla A(\boldsymbol{\eta}) \tag{6}$$

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- The GLM framework extends these ideas to the general exponential family.

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The condition probability is thus modeled as:

$$p(y|\mathbf{x}; \mathbf{w}) = h(y_n) \exp\left(\eta\phi(y) - A(\eta)\right) \qquad \text{for} \quad \eta = g \circ f(\mathbf{x}^\top \mathbf{w}) \tag{7}$$

Negative log-likelihood estimation

Note that:

$$\mathcal{L}(\mathbf{w}) = -\frac{1}{N}\sum_{n=1}^{N} \ln p(y_n|\mathbf{x}_n^\top \mathbf{w}) = -\frac{1}{N}\sum_{n=1}^{N} (\ln(h(y_n)) + \eta_n \phi(y_n) - A(\eta_n)) \tag{8}$$

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In the case of Logistic Regression:

$$\nabla_{\mathbf{w}}\mathcal{L}(\mathbf{w}) = \frac{1}{N}\mathbf{X}^\top \left[\sigma(\mathbf{X}\mathbf{w}) - \mathbf{y}\right] \tag{12}$$

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This lecture:

- Discriminative vs. Generative learning algorithms
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Recap

Classification: we observe some data $\mathcal{S} = \{\mathbf{x}_n, y_n\}_{n=1}^{N} \in \mathcal{X} \times \underbrace{\mathcal{Y}}_{\text{Discrete set}}$

Goal: given a new observation $\mathbf{x}$, we want to predict its label y

How: relates input to a categorical variable

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In previous lectures, we assume that we know the joint distribution $p(\mathbf{x}, y)$.

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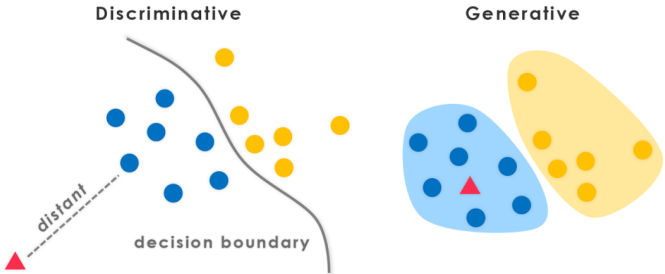
Discriminative v/s Generative Learning Algorithms



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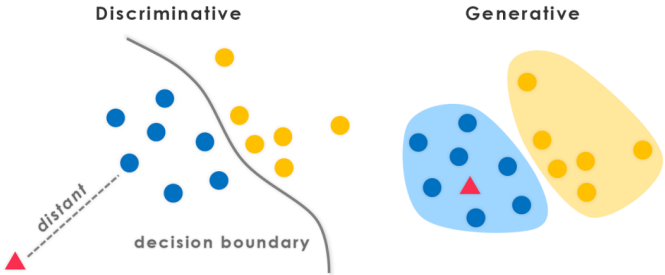

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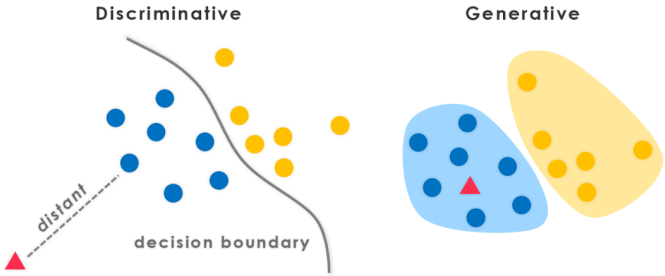

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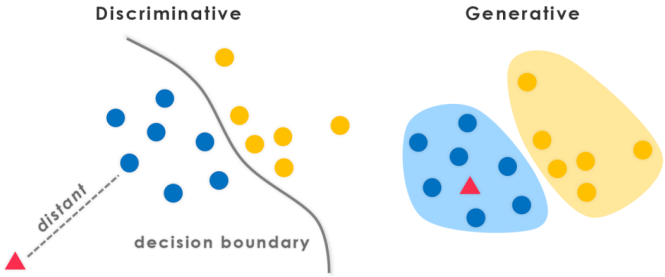

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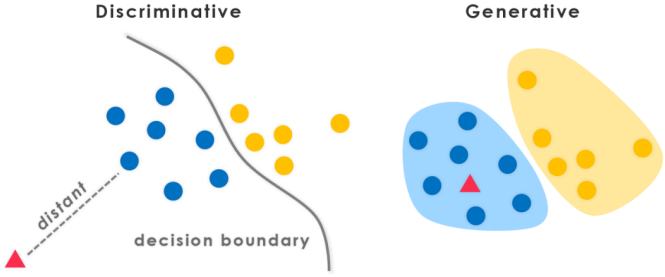

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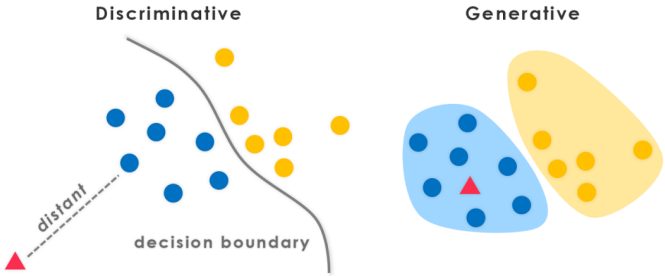

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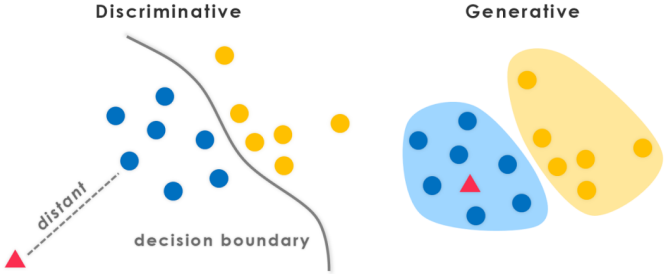

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- learned models are transformed to $p(y|\mathbf{x}, \mathbf{w})$ later to classify data using Bayes' rule:
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The posterior distribution on y given $\mathbf{x}$:

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Make predictions in a generative model (exercise):

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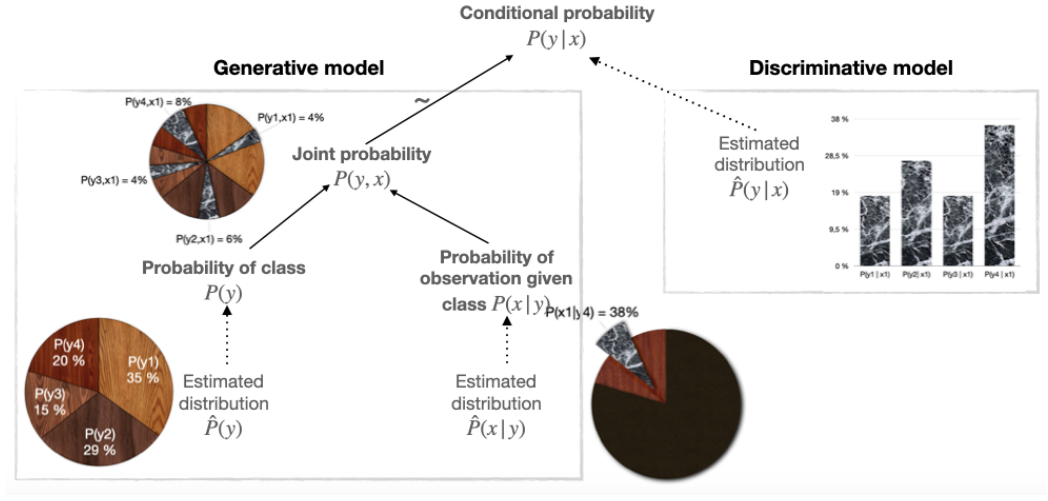
Discriminative v/s Generative Learning Algorithms (an alternative view)



From https://towardsdatascience.com/introduction-to-generative-and-discriminative-models-9c9ef152b9af/

Discriminative Models v.s. Generative Models (an alternative view)

Discriminative classification algorithms

- focus on the decision boundary
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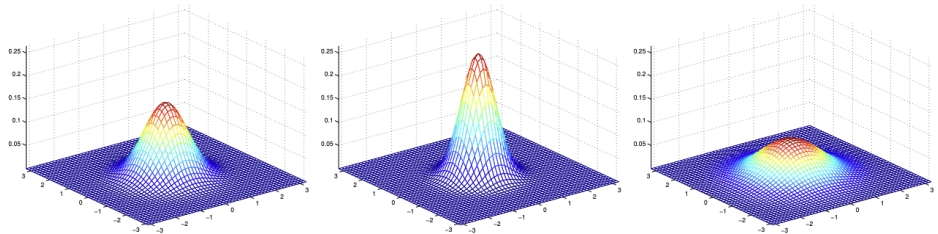

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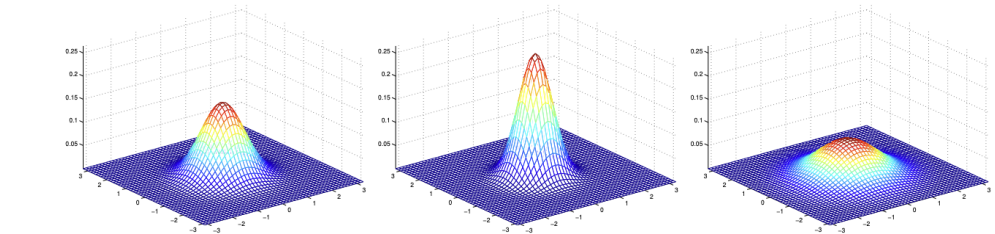

(Left): $\Sigma = \mathbf{I}$; (Middle): $\Sigma = 0.6\mathbf{I}$; (Right): $\Sigma = 2\mathbf{I}$. We can see that as

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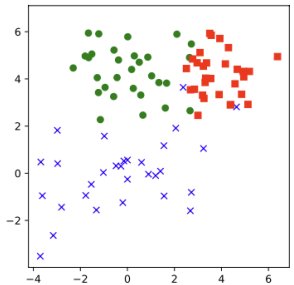

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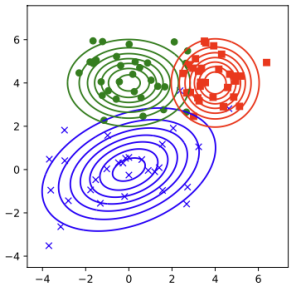

(b)

2d data from 3 different classes in (a), and we fit the full covariance Gaussian class-conditionals in (b).

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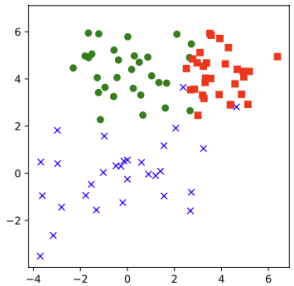

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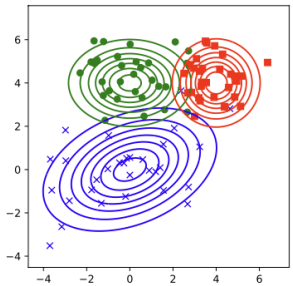

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$\Longrightarrow$ in which models $p(\mathbf{x}|y, \mathbf{w})$ uses a multi-variate normal distribution:

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The corresponding class posterior ((Gaussian Discriminant Analysis: GDA)) therefore has the form

$$p(y = c|\mathbf{x}, \mathbf{w}) \propto \pi_c \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_c, \boldsymbol{\Sigma}_c)\,, \tag{20}$$

where $\pi_c = p(y = c|\mathbf{w})$.

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Write out the Probability Density Functions (PDFs):

$$p(y) = \phi^y (1-\phi)^{1-y} \tag{21}$$

$$p(\mathbf{x}|y=0) = \frac{1}{\sqrt{(2\pi)^d \, det(\mathbf{\Sigma}_0)}} \exp\left\{ -\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu}_0)^\top \mathbf{\Sigma}_0^{-1} (\mathbf{x} - \boldsymbol{\mu}_0) \right\} \tag{22}$$

$$p(\mathbf{x}|y=1) = \frac{1}{\sqrt{(2\pi)^d \, det(\mathbf{\Sigma}_1)}} \exp\left\{ -\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu}_1)^\top \mathbf{\Sigma}_1^{-1} (\mathbf{x} - \boldsymbol{\mu}_1) \right\} \tag{23}$$

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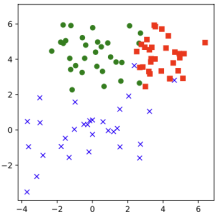
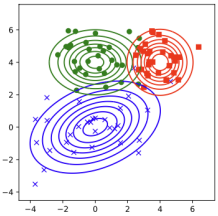
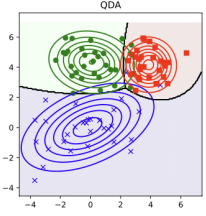


(a)



(b)



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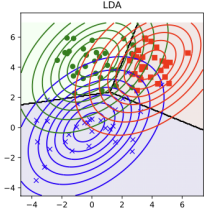

(b)

(a): GDA fits to data, where unconstrained covariances induce quadratic decision boundaries;
(b): LDA fits to data, where tied covariances induce linear decision boundaries.

What is LDA?

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Linear Discriminant Analysis (LDA)

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The connection between LDA and logistic regression

Recall that in LDA, we have

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To gain further insights, let's consider the binary case.

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This has the same form as binary logistic regression, and the MAP decision rule is

$$\hat{y}(\mathbf{x}) = 1 \text{ iff } \mathbf{w}^\top \mathbf{x} > c\,, \text{ where } c = \mathbf{w}^\top \mathbf{x}_0$$

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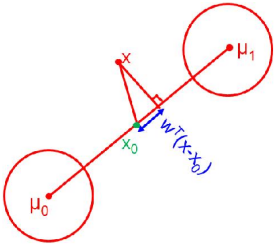

Geometry of LDA in the 2 class case where $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2 = \mathbf{I}$.

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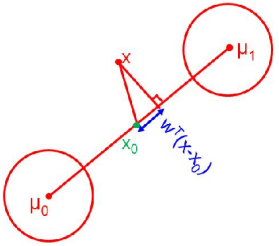
$\Longrightarrow$ we can classify a point by projecting it onto this line, and check its distance to $\boldsymbol{\mu}_0$ and $\boldsymbol{\mu}_1$.



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Some remarks

GDA

- maximizes the **joint likelihood** $\prod_{i=1}^{n} p(\mathbf{x}^{(i)}, y^{(i)})$
- modeling assumptions:
  $\mathbf{x}|y = b \sim \mathcal{N}(\boldsymbol{\mu}_b, \boldsymbol{\Sigma}), y \sim \text{Bernoulli}(\phi)$
- when modeling assumptions are correct, GDA is asymptotically efficient[a] and data efficient.

[a] in the limit of very large training sets (large n), there is no algorithm that is strictly better than GDA.

Logistic Regression

- maximizes the **conditional likelihood** $\prod_{i=1}^{n} p(y^{(i)}|\mathbf{x}^{(i)})$
- modeling assumptions:
  $p(y|\mathbf{x})$ is a logistic function; no restriction on $p(\mathbf{x})$
- more robust and less sensitive to incorrect modeling assumptions.

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Maximimum Likelihood Estimation for GDA

The likelihood function of a GDA model is as follows:

$$p(\mathcal{D}|\mathbf{w}) = \prod_{i=1}^{N} \text{Cat}(y_i|\boldsymbol{\pi}) \prod_{c=1}^{C} \mathcal{N}(\mathbf{x}_i|\boldsymbol{\mu}_c, \boldsymbol{\Sigma}_c)^{1\{y_i=c\}}$$

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MLE estimation:

- $\hat{\pi}_c = \frac{N_c}{N}$
- $\hat{\boldsymbol{\mu}}_c = \frac{1}{N_c} \sum_{i:y_i=c} \mathbf{x}_n$
- $\hat{\boldsymbol{\Sigma}}_c = \sum_{i:y_i=c} (\mathbf{x}_i - \hat{\boldsymbol{\mu}}_c)(\mathbf{x}_i - \hat{\boldsymbol{\mu}}_c)^\top$

A different version of derivatives for MLE of LDA

Log-likelihood of the data:

$\ell(\phi, \boldsymbol{\mu}_0, \boldsymbol{\mu}_1, \boldsymbol{\Sigma})$

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A different version of derivatives for MLE of LDA

Log-likelihood of the data:

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&= \sum_{i=1}^{n} \log \frac{1}{\sqrt{(2\pi)^d \, det(\boldsymbol{\Sigma})}} \exp\left\{ -\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu}_{y^i})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}_{y^{(i)}}) \right\} + \sum_{i=1}^{n} y^{(i)} \log \phi + \sum_{i=1}^{n} (1 - y^{(i)}) \log(1 - \phi)
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By maximizing ℓ w.r.t. the parameters, the maximumn likelihood estimate of the parameters:

$$
\begin{aligned}
\phi &= \frac{1}{n} \sum_{i=1}^{n} 1\{y^{(i)} = 1\} \\
\boldsymbol{\mu}_b &= \frac{\sum_{i=1}^{n} 1\{y^{(i)} = b\} \mathbf{x}^{(i)}}{\sum_{i=1}^{n} 1\{y^{(i)} = b\}} \text{ for } b = 0, 1 \\
\boldsymbol{\Sigma} &= \frac{1}{n} \sum_{i=1}^{n} (\mathbf{x}^{(i)} - \boldsymbol{\mu}_{y^{(i)}})(\mathbf{x}^{(i)} - \boldsymbol{\mu}_{y^{(i)}})^\top
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Table of Contents

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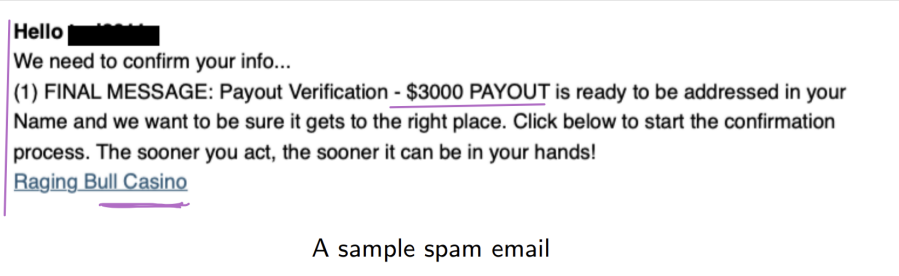
$\Longrightarrow$ **Naïve Bayes**

Naïve Bayes: Motivating Example

Example 2 (Spam filter (document classification))

Classify email message $\mathbf{x}$ to spam ($y = 1$) and non-spam ($y = 0$) classes.



A sample spam email

Example: Spam Filter

We would like to represent an email via a feature vector.

Binary text features: Given a dictionary of size n, represent a message composed of dictionary words as $\mathbf{x} \in \{0,1\}^n$:

$$x_i = \begin{cases} 1 & i\text{-th dictionary word is in message} \\ 0 & \text{otherwise} \end{cases}$$

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \text{a} \\ \text{aardvark} \\ \text{aardwolf} \\ \vdots \\ \text{buy} \\ \vdots \\ \text{zygmurgy} \end{matrix}$$

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Naïve Bayes assumption and Naïve Bayes classifier

Probability of observing email $x_1, \ldots, x_n$ given spam class y:

$$p(x_1, \ldots, x_n|y) = p(x_1|y)p(x_2|x_1, y), \ldots, p(x_n|x_1, \ldots x_{n-1}, y)$$

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To model $p(\mathbf{x}|y)$, we will therefore make a very strong assumption.

Assumption 1 (Naïve Bayes assumption)

x_i's are conditionally independent given y:

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As a result,

$$p(x_1, \ldots, x_n|y) = p(x_1|y)p(x_2|y) \ldots p(x_n|y) = \prod_{i=1}^{n} p(x_i|y)$$

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Model parameters:

- $\phi_y \in \mathbb{R}$
- $\phi_{i|y=1}, \phi_{i|y=0}$ for $i = 1, \ldots, n$

Naïve Bayes Parameter Learning

Likelihood of i.i.d. training data $(\mathbf{x}^{(1)}, y^{(1)}), \ldots, (\mathbf{x}^{(n)}, y^{(n)})$:

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Maximum likelihood estimation of parameters:

$$\phi_y = \frac{1}{n} \sum_{i=1}^{n} 1\{y^{(i)} = 1\} \qquad \text{(\% of spam emails)}$$

$$\phi_{i|y=b} = \frac{\sum_{i=1}^{n} 1\{x_j^{(i)} = 1, y^{(i)} = b\}}{\sum_{i=1}^{n} 1\{y^{(i)} = b\}} \text{ for } b = 1, 0$$

(\% of spam (non-spam) emails containing j-th dictionary word)

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Choose label $y=1$ (spam) if $p(y=1|x) > T$, where $T \in [0,1]$ is a threshold, e.g., $T=0.5$.

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In case of taking the x_j value in $\{1,\ldots,k_j\}$,
we can model $p(x_j|y)$ as multi-nomial distribution, rather than as Bernoulli.

Laplace smoothing

Issue with Naïve Bayes prediction:

- Suppose word x_j hasn't been seen in the training data,

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$$p(y = 1|\mathbf{x}) = \frac{\prod_{j=1}^{d} p(x_j|y = 1)p(y = 1)}{\left(\prod_{j=1}^{d} p(x_j|y = 1)\right) p(y = 1) + \left(\prod_{j=1}^{d} p(x_j|y = 0)\right) p(y = 0)} = \frac{0}{0}$$

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  with the risk that some of the ϕ_j's might end up as zero.
- Laplace smoothing can be used to eliminate this issue:

$$\phi_j = \frac{1 + \sum_{i=1}^{n} 1\{z^{(i)} = j\}}{k + n} \qquad \text{where}$$

  - $\phi_j \neq 0$ for all j
  - $\sum_{j=1}^{k} \phi_j = 1$

Naïve Bayes with Laplace smoothing

Apply Laplace smoothing to $\phi_{j|y=b}$ for $b \in \{0, 1\}$

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- In practice we don't apply Laplace smoothing to $\phi_y = p(y = 1)$, which is greater than 0.

Last lecture:

- Exponential Families
- Generalized Linear Models

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This lecture:

- Discriminative vs. Generative learning algorithms
- Gaussian Discriminant Analysis (GDA)
- Linear Discriminant Analysis (LDA)
- LDA and Logistic regression
- Naïve Bayes