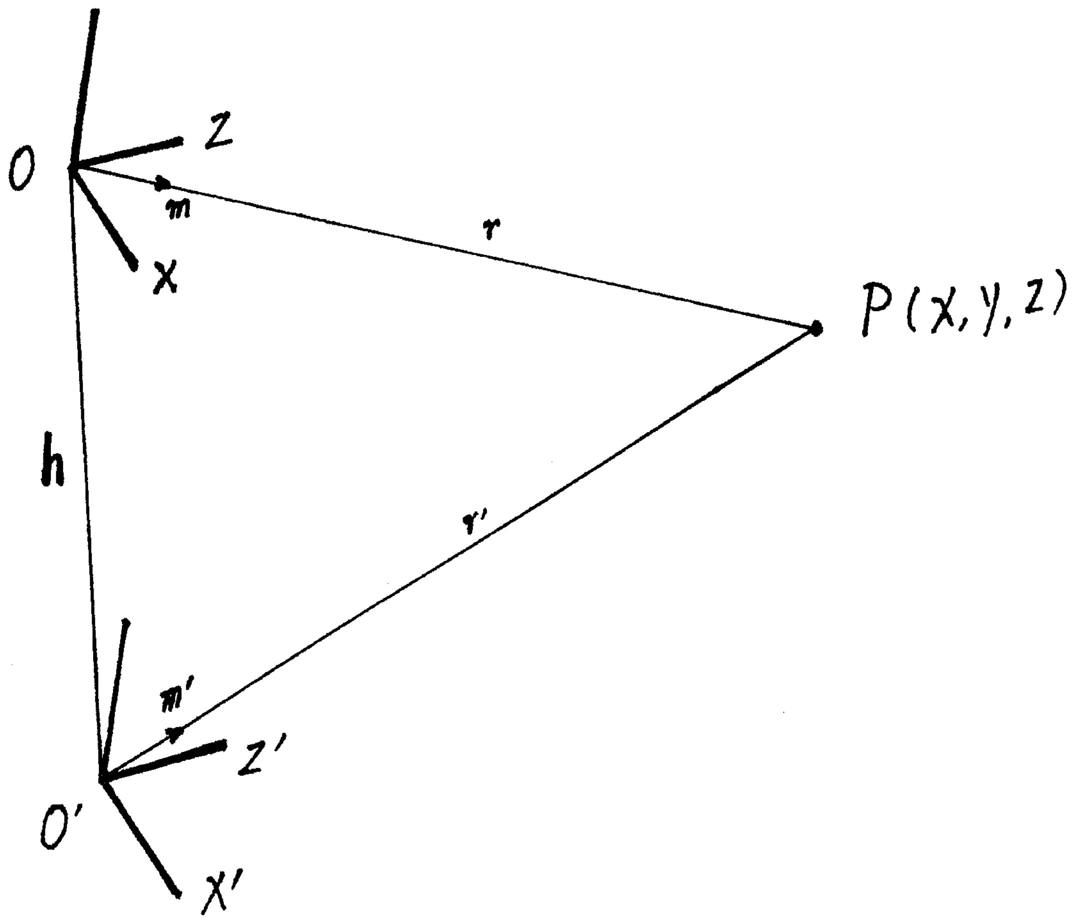
## 3.3 Motion Parallax

A translation of the camera causes an effective translation of the object relative to the camera, and the resulting image motion of points and lines reveal their 3-dimensional geometries. This fact is known as *motion parallax.*




The camera translates from $O$ to $O'$ with no rotation:

$$\vec{OO'} = \mathbf{h} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$$

A world point $P$ has been observed twice as $(x, y)$ and $(x', y')$. For convenience, we convert the image points into unit vectors:

$$\mathbf{m} = \frac{1}{\sqrt{x^2 + y^2 + f^2}} \begin{pmatrix} x \\ y \\ f \end{pmatrix} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

and

$$\mathbf{m}' = \frac{1}{\sqrt{x'^2 + y'^2 + f^2}} \begin{pmatrix} x' \\ y' \\ f \end{pmatrix} = \begin{pmatrix} m'_1 \\ m'_2 \\ m'_3 \end{pmatrix}.$$

Now, we can recover the distance $r$ or $r'$ (scalars) from the following vector relation:

$$\vec{OO'} + \vec{O'P} = \vec{OP}$$

or

$$\mathbf{h} + r'\mathbf{m}' = r\mathbf{m}$$

In effect, there are 3 equations here, being

$$\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} + r' \begin{pmatrix} m_1' \\ m_2' \\ m_3' \end{pmatrix} = r \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \qquad (1)$$

$r$ (or $r'$) can be solved by picking any 2 of the 3 eqations.

## (1) Parallel Stereo is a special case of motion parallax

The world point can be expressed as $(X, Y, Z) = (rm_1, rm_2, rm_3)$. We are looking for the depth

$$Z = rm_3$$

Let $\mathbf{h} = (h_1, 0, 0)$, because the camera translated only along the X axis. Rewrite eq. (1), we have,

$$\begin{cases} h_1 + r'm_1' &= rm_1 \\ r'm_2' &= rm_2 \\ r'm_3' &= rm_3 \end{cases} \qquad (2)$$

From the 3rd equation of eq. (2),

$$r' = r\,\frac{m_3}{m_3'}.$$

Substitute $r'$, into the first equation of eq. (2), we have,

$$\begin{aligned} h_1 &= r\, m_1 \;-\; r'\, m_1' \\ &= r\, m_1 - (r\, \frac{m_3}{m_3'})\, m_1' \\ &= r\, m_3 (\frac{m_1}{m_3} - \frac{m_1'}{m_3'}) \\ &= Z (\frac{m_1}{m_3} - \frac{m_1'}{m_3'}) \end{aligned}$$

or,

$$Z = \frac{h_1}{\frac{m_1}{m_3} - \frac{m_1'}{m_3'}}$$

From the perspective projection equation we know,

$$x_{left} = \frac{m_1}{m_3} f \qquad x_{right} = \frac{m_1'}{m_3'} f$$

Therefore,

$$Z = \frac{h_1 f}{x_{left} - x_{right}}$$

with $h_1$ being the effective baseline.

## (2) Motion parallax using 3-equations

We can compute the motion parallax by using all 3 equations. Let,

$$\mathbf{a} = \begin{pmatrix} a1 \\ a2 \\ a3 \end{pmatrix} = r\mathbf{m} - r'\mathbf{m}' - \mathbf{h}$$

In the presence of noise, $\mathbf{a}$ may not be a zero vector (error). So we proceed to look for the minimum value of $||\mathbf{a}||^2$ in an attempt to find the optimal solution for $r$ and $r'$. Define the residual $E$ as,

$$E = \mathbf{a}^T\mathbf{a} = (a1, a2, a3)\begin{pmatrix} a1 \\ a2 \\ a3 \end{pmatrix} \quad (= ||\mathbf{a}||^2)$$

Take the first derivative of $E$ with respect to $r$:

$$\begin{aligned} \frac{\partial E}{\partial r} &= \mathbf{a}^T \frac{\partial \mathbf{a}}{\partial r} + \frac{\partial(\mathbf{a}^T)}{\partial r}\mathbf{a} \\ &= \mathbf{a}^T\mathbf{m} + \mathbf{m}^T\mathbf{a} \\ &= \mathbf{a}^T\mathbf{m} + \mathbf{a}^T\mathbf{m} \\ &= 2\mathbf{a}^T\mathbf{m} \end{aligned}$$

Substitute $\mathbf{a} = r\mathbf{m} - r'\mathbf{m}' - \mathbf{h}$ into this equation, and set it to zero, we get,

$$r - r'(\mathbf{m}, \mathbf{m}') - (\mathbf{h}, \mathbf{m}) = 0$$

Repeat the above steps for $r'$, we get

$$r(\mathbf{m}, \mathbf{m}') - r' - (\mathbf{h}, \mathbf{m}') = 0$$

$r$ and $r'$ can be solved from the two equations as:

$$r = \frac{(\mathbf{h}, \mathbf{m}) - (\mathbf{m}, \mathbf{m}')(\mathbf{h}, \mathbf{m}')}{1 - (\mathbf{m}, \mathbf{m}')^2}$$

and

$$r' = \frac{(\mathbf{m}, \mathbf{m}')(\mathbf{h}, \mathbf{m}) - (\mathbf{h}, \mathbf{m}')}{1 - (\mathbf{m}, \mathbf{m}')^2}$$

## (3) Motion Parallax with Rotation

Full 3-dimensional montion can be described by translation and rotation. For example, when the camera is mounted on a vehicle or on a robot arm (manipulator), the motion is no longer a pure translation.

Suppose the camera undergoes a translation **h** followed by a rotation **R** (3x3 matrix), the following is true:

$$r'\mathbf{m}' = \mathbf{R}^{-1}(r\mathbf{m} - \mathbf{h})$$

or,

$$\mathbf{R}(r'\mathbf{m}') = r\mathbf{m} - \mathbf{h}$$
$$r'(\mathbf{R}\mathbf{m}') = r\mathbf{m} - \mathbf{h}$$

This is equivalent to substitute **m'** with **Rm'** in the motion paralax equation, hence

$$r = \frac{(\mathbf{h}, \mathbf{m}) - (\mathbf{m}, \mathbf{R}\mathbf{m}')(\mathbf{h}, \mathbf{R}\mathbf{m}')}{1 - (\mathbf{m}, \mathbf{R}\mathbf{m}')^2}$$

and

$$r' = \frac{(\mathbf{m}, \mathbf{R}\mathbf{m}')(\mathbf{h}, \mathbf{m}) - (\mathbf{h}, \mathbf{R}\mathbf{m}')}{1 - (\mathbf{m}, \mathbf{R}\mathbf{m}')^2}$$

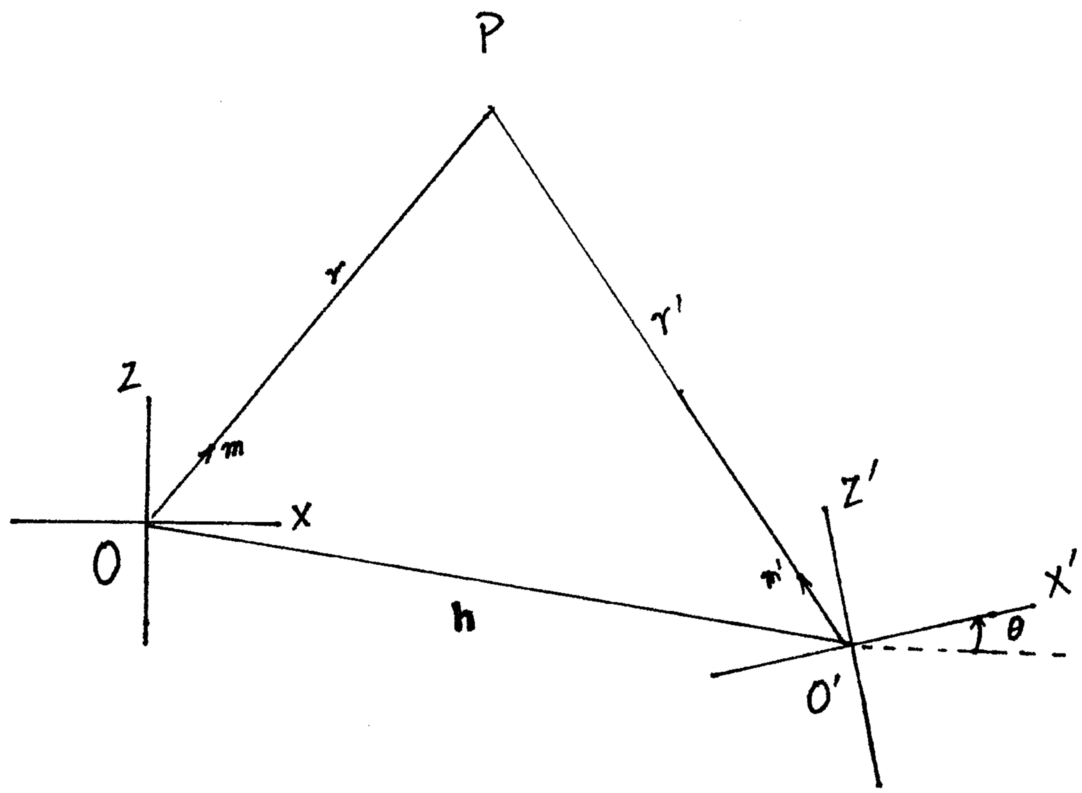



Figure 1: The camera undergoes translation $\mathbf{h}$ (with reference to the first frame) and rotation $\mathbf{R}$. The simplified case in this figure shows that the rotation is against the Y axis with angle $\theta$. The vector $\vec{O'P} = r'\mathbf{m}'$ is equally rotated $\mathbf{R}^{-1}$ with reference to the new frame.

# (3) More About 3D Rotation

$$\mathbf{R} = \mathbf{R_x} \cdot \mathbf{R_y} \cdot \mathbf{R_z} = \begin{pmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ r_7 & r_8 & r_9 \end{pmatrix}.$$

The matrix $\mathbf{R_x}$ describes a rotation about the $X$ axis by a certain angle $\alpha$ (*pan angle*). It holds that

$$\mathbf{R_x}(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\alpha) & sin(\alpha) \\ 0 & -sin(\alpha) & cos(\alpha) \end{pmatrix}.$$

The matrix $\mathbf{R_y}$ describes a rotation about the $Y$ axis by a certain angle $\beta$ (*tilt angle*). It holds

$$\mathbf{R_y}(\beta) = \begin{pmatrix} cos(\beta) & 0 & -sin(\beta) \\ 0 & 1 & 0 \\ sin(\beta) & 0 & cos(\beta) \end{pmatrix}.$$

The matrix $\mathbf{R_z}$ describes a rotation about the $Z$ axis by a certain angle $\gamma$ (*roll angle*). Here it holds that

$$\mathbf{R_z}(\gamma) = \begin{pmatrix} cos(\gamma) & sin(\gamma) & 0 \\ -sin(\gamma) & cos(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The rotation matrix is orthogonal,

$$\mathbf{R}^T\mathbf{R} = \mathbf{I}$$

or, the magnitude or each row or column is 1,

$$r_1^2 + r_2^2 + r_3^2 = 1$$

and

$\texttt{det}(\mathbf{R}) = 1$ (determinant);

$\mathbf{R}^{-1} = \mathbf{R}^T$ (the inverse rotation is the transpose);

$||\mathbf{Ra}|| = ||\mathbf{a}||$ (rotational transformation does not change the magnitude of a vector);

$(\mathbf{Ra}, \mathbf{Rb}) = (\mathbf{a}, \mathbf{b})$ (for arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$ the length and angle are preserved by a rotation).

# COMPUTATION OF 3-D ROTA/ TION

When an object is rotated in the 3-D space, the rotational motion can be computed via a *rotation matrix* $\mathbf{R}$:

$$\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

$\mathbf{R}$ is an *orthogonal* matrix, meaning:

$$\mathbf{R}^\top \mathbf{R} = \mathbf{I}$$

or, the magnitude of each row or column is 1,

$$r_{11}^2 + r_{12}^2 + r_{13}^2 = 1$$

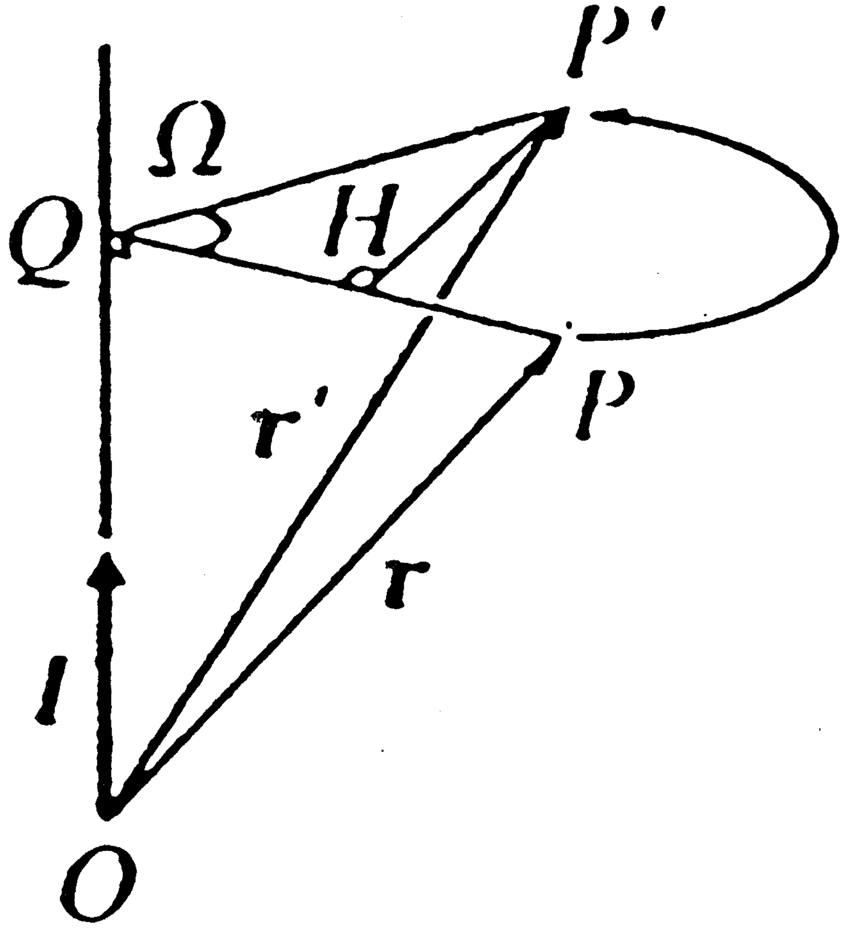
and

$$\begin{aligned} \det \mathbf{R} &= 1 \\ \mathbf{R}^{-1} &= \mathbf{R}^\top \\ ||\mathbf{Ra}|| &= ||\mathbf{a}|| \\ (\mathbf{Ra}, \mathbf{Rb}) &= (\mathbf{a}, \mathbf{b}) \quad \text{(inner product)} \end{aligned}$$

For arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$ the length and angle are preserved by a rotation.

(1) (Euler's theorem) Every rotation matrix represents a rotation around an axis by some angle.

## )MPUTATION OF 3-D ROTAT




## The axis and angle of a 3-D ı

**(2)** Let $\mathbf{l} = (l_1, l_2, l_3)^\top$ denote the unit vector of the rotation axis, $\Omega$ denote the angle, the rotation matrix $\mathbf{R}$ can be expressed as

$$\mathbf{R} = \begin{pmatrix} C + l_1^2 V & l_1 l_2 V - l_3 S & l_1 l_3 V + l_2 S \\ l_2 l_1 V + l_3 S & C + l_2^2 V & l_2 l_3 V - l_1 S \\ l_3 l_1 V - l_2 S & l_3 l_2 V + l_1 S & C + l_3^2 V \end{pmatrix} \tag{2}$$

where $C = \cos\Omega$, $S = \sin\Omega$, and $V = 1 - \cos\Omega$.

*Proof*. Point $P$ rotates by angle $\Omega$ around $\mathbf{l}$, Let $P'$ be the resulting new point. Let $Q$ be the orthogonal projection of point $P$ onto the axis. Then, $|OP| = |OP'|$ and $|QP| = |QP'|$. Let $H$ be the orthogonal projection of point $P'$ onto $QP$. If we put $\mathbf{r} = \vec{OP}$ and $\mathbf{r}' = \vec{OP'}$, we see that

$$\mathbf{r}' = \vec{OQ} + \vec{QH} + \vec{HP'} \tag{3}$$

Since $\vec{OQ}$ is the orthogonal projection of vector $\mathbf{r}$ onto the axis by $\mathbf{l}$, we have

$$\vec{OQ} = (\mathbf{r}, \mathbf{l})\mathbf{l}$$

Similarly, $\vec{QH}$ is the orthogonal projection of vector

$\vec{QP'}$ onto $\vec{QP}$. Noting that $|QP'| = |QP|$, we have

$$\begin{aligned}\vec{QH} &= \frac{\vec{QP}}{|QP|}|QP'|\cos\Omega = \vec{QP}\cos\Omega \\ &= (\vec{OP} - \vec{OQ})\cos\Omega \\ &= (\mathbf{r} - (\mathbf{r},\mathbf{l})\mathbf{l})\cos\Omega\end{aligned}$$

Vector $\vec{HP'}$ is orthogonal to both $\mathbf{l}$ and $\mathbf{r}$, and has length $|QP'|\sin\Omega$. Noting that

$$|QP'| = |QP| = ||\mathbf{l}\times\mathbf{r}||,$$

we have

$$\begin{aligned}\vec{HP'} &= \frac{\mathbf{l}\times\mathbf{r}}{||\mathbf{l}\times\mathbf{r}||}|QP'|\sin\Omega = \frac{|QP|}{||l\times r||}(l\times r)\sin\Omega \\ &= \mathbf{l}\times\mathbf{r}\sin\Omega\end{aligned}$$

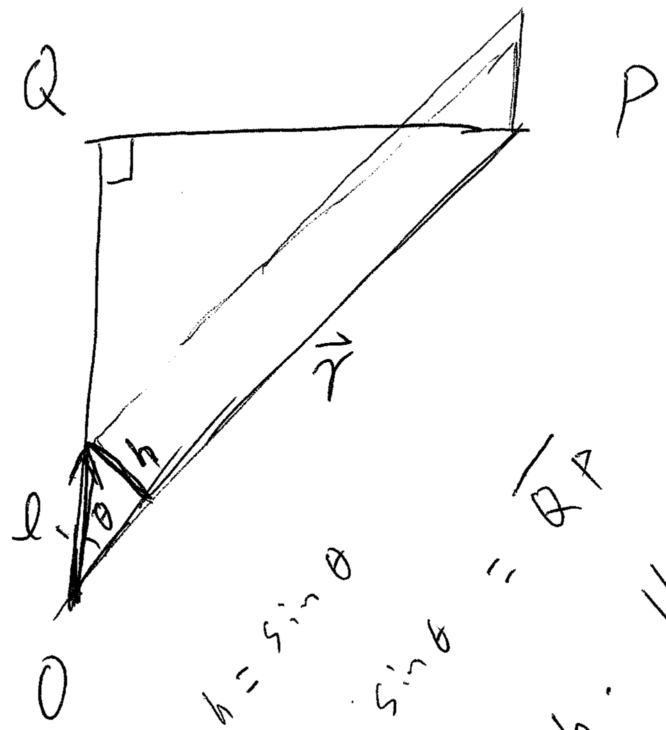
substituting $\vec{OQ}$, $\vec{QH}$ and $\vec{HP'}$ into equation (3), we have

$$\mathbf{r}' = \mathbf{r}\cos\Omega + \mathbf{l}\times\mathbf{r}\sin\Omega + (1-\cos\Omega)(\mathbf{l},\mathbf{r})\mathbf{l}$$

(*Rodrigues formula*). Rewriting this equation in matrix form as

$$\mathbf{r}' = \mathbf{R}\mathbf{r}$$

we obtain equation (2).




$h = \sin\theta$

$\|\vec{r}\| \cdot h = \|\vec{r}\| \cdot \sin\theta = \overline{QP}$

$\|l \times r\| = h \cdot \|\vec{r}\|$

$= \sin\theta \cdot \|\vec{r}\| = QP$

Conversely, giving the rotation matrix, we can compute its axis $\mathbf{l}$ and the rotation angle.

**(3)** The axis $\mathbf{l}$ and angle $\Omega$ $(0 \leq \Omega \leq \pi)$ of rotation $\mathbf{R} = (r_{ij})$ are given by

$$\Omega = \arccos \frac{trace\mathbf{R} - 1}{2}$$

$$\mathbf{l} = N[\begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix}]$$

*Proof*. Equation (2) implies the following relations:

$$trac\mathbf{R} = 1 + 2\cos\Omega$$

$$\frac{1}{2}(\mathbf{R} - \mathbf{R}^{\top}) = \begin{pmatrix} 0 & -l_3 \sin\Omega & l_2 \sin\Omega \\ -l_3 \sin\Omega & 0 & l_1 \sin\Omega \\ -l_2 \sin\Omega & l_1 \sin\Omega & 0 \end{pmatrix}$$

$$\begin{pmatrix} s & s & s \\ s & s & s \\ s & \frac{1}{2}(r_{32} - r_{23}) & s \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ & l & \end{pmatrix}$$

Consider a rotational motion $\mathbf{R}(\mathrm{t})$ around a fixed axis $\mathbf{l}$ with a constant angular velocity $\omega$. This rotational motion is specified by vector

$$\mathbf{w} = \omega \mathbf{l}$$

which is called the *rotation velocity*: the angular velocity is $\omega = ||\mathbf{w}||$, and the axis is $\mathbf{l} = N[\mathbf{w}]$.

$$R - R^T = \begin{pmatrix} \sim & \sim & \sim \\ \sim & \sim & \sim \\ \sim & r_{32} & \sim \end{pmatrix} - \begin{pmatrix} \sim & \sim & \sim \\ \sim & \sim & \sim \\ \sim & r_{23} & \sim \end{pmatrix}$$

$$= \begin{pmatrix} \sim & \sim & \sim \\ \sim & \sim & \sim \\ \sim & r_{32} - r_{23} & \sim \end{pmatrix}$$

$$= \begin{pmatrix} \sim & \sim & \sim \\ \sim & \sim & \sim \\ \sim & 2 \ell_1 \sin\Omega & \sim \end{pmatrix}$$

$$\ell_1 = (2 \sin\Omega)(r_{32} - r_{23})$$

$$\ell_2 = (2 \sin\Omega)(r_{13} - r_{31})$$

$$\ell_3 = (2 \sin\Omega)(r_{21} - r_{12})$$

# QUATERNION REPRESENTATION

We define a 4x1 vector $\mathbf{q} = (q_0,\ q_1,\ q_2,\ q_3)^T$ such that

$$||\mathbf{q}||^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$$

Rewrite (re-define) it as a pair

$$\mathbf{q} = (s, \mathbf{v})$$

and

$$s = q_0, \quad \mathbf{v} = (q_1, q_2, q_3)^T$$

A 3-D rotation can be represented as

$$s = \cos(\theta/2)$$

$$\mathbf{v} = \sin(\theta/2)\ \mathbf{u}$$

$\theta$ – angle rotated
$\mathbf{u}$ – rotation axis. (unit vector)

Note:
-$\mathbf{q}$ = $(-s, -\mathbf{v})$ represent the same rotation.

* The conjugate of a quaternion is

$$\bar{\mathbf{q}} = (s, -\mathbf{v})$$

$a \pm j\,b$

(similar to a complex number)

* The product of two quaternions **q, q'** is defined as

$$\mathbf{q}\mathbf{q}' = (ss' - (\mathbf{v},\ \mathbf{v}'),\ s\mathbf{v}' + s'\mathbf{v} + \mathbf{v} \times \mathbf{v}')$$

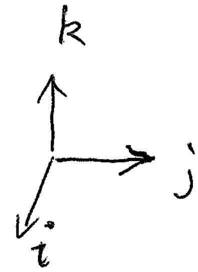
Given two rotations represented by **q1** and **q2**, the product of the two rotations ( apply rotation 2 first then rotation 1) corresponds to the products **q1q2** and -**q1q2**.

* Quaternion is not commutative:

$$\mathbf{q}_1\mathbf{q}_2 \neq \mathbf{q}_2\mathbf{q}_1$$

* An easy way to perform the quaternion product is by the algebraic expression (Hamilton)

$$\mathbf{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$




where

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$

$$\mathbf{jk} = -\mathbf{kj} = \mathbf{i},\ \ \mathbf{ki} = -\mathbf{ik} = \mathbf{j},\ \ \mathbf{ij} = -\mathbf{ji} = \mathbf{k}.$$

* Quaternion is associative:

$$(\mathbf{q}\mathbf{q}')\mathbf{q}'' = \mathbf{q}(\mathbf{q}'\mathbf{q}'')$$

$$(q_0 + q_1 i + q_2 j + q_3 k)(p_0 + p_1 i + p_2 j + p_3 k)$$

$$= \begin{pmatrix} q_0 p_0 \\ -q_1 p_1 \\ -q_2 p_2 \\ -q_3 p_3 \end{pmatrix} + (q_0 p_1 + \cdots \quad ) i + (\ldots) j + (\ldots) k$$

* The inverse of quaternion is

$$\mathbf{q}^{-1} = \bar{\mathbf{q}}/||\mathbf{q}||^2$$

* A rotation given by the orthogonal matrix $R$ can be expressed using the quaternion notation

$$(0, R\mathbf{m}) \;=\; \mathbf{q}(\mathbf{0}, \mathbf{m})\bar{\mathbf{q}}$$

or in short

$$R\mathbf{m} \;=\; \mathbf{q}\mathbf{m}\bar{\mathbf{q}}$$

The rotation matrix $R$ is given as

$$\begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_2q_1 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_3q_1 - q_0q_2) & 2(q_3q_2 + q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix}$$

**Problem**– Given two sets of unit vectors $\mathbf{m}_i$ and $\mathbf{m'}_i$, $i = 1,\ 2, \ldots,\ N$, compute a rotation $R$ such that

$$\mathbf{m_i} \;=\; \mathbf{Rm'_i}, \quad i = 1, \ldots,\ N.$$

This can be solved using least squares:

$$\sum_i^N W_i||\mathbf{m_i} - \mathbf{Rm'}||^2 \rightarrow min$$

for positive weights $W_i$.

A close form solution can be found in terms of maximising

$$trace(R^T K) \rightarrow max$$

where $K$ is the correlation matrix

$$K = \sum_{i}^{N} W_i \mathbf{m_i} \mathbf{m_i'}^{\mathbf{T}}$$

* Given corrrelation matrix $K$, define a four-dimensional symmetric matrix $\hat{K} =$

$$\left( \begin{array}{cccc} K11 + K22 + K33 & K32 - K23 & K13 - K31 & K21 - K12 \\ K32 - K23 & K11 - K22 - K33 & K12 + K21 & K31 + K13 \\ K13 - K31 & K12 + K21 & -K11 + K22 - K33 & K23 + K32 \\ K21 - K12 & K31 + K13 & K23 + K32 & -K11 - K22 + K33 \end{array} \right.$$

Let $\hat{\mathbf{q}}$ be the four-dimensional unit eigenvector of $\hat{K}$ for the largest eigenvalue. Then, $trace(R^T K)$ is maximised by the rotation $R$ represented by $\hat{\mathbf{q}}$.

The solution is unique if the largest eigen value of $\hat{K}$ is a simple root.

# Vanishing points & vanishing lines

1. Projections of parallel space lines meet at a common "vanishing point" on the image plane; or the *vanishing point* of a space line is the limit of the projection of a point that moves along the space line indefinitely in one direction.

2. A space line extending along unit vector $\mathbf{m}$ has, when projected a *vanishing* point of N-vector $\pm\mathbf{m}$.

3. A planar surface of unit surface normal $\mathbf{n}$ has, when projected, a vanishing line of N-vector $\pm\mathbf{n}$.

4. Projections of planar surfaces that are parallel in the scene define a common vanishing line.

(1) In summary, if a vanishing point is detected on the image plane, its N-vector indicates the 3-D orientation of the corresponding space line; (2) if a vanishing line is detected on the image plane, its N-vector indicates the surface normal to the corresponding planar surface. This 3-D interpretation of vanishing points and vanishing lines plays an essential role in 3-D scene analysis for machine vision.

**Example**: Show that if a planar surface in the scene with unit normal $\mathbf{n} = (n_1, n_2, n_3)^\top$ is not parallel to the image plane, its vanishing line is

$$n_1 x + n_2 y + n_3 f = 0.$$

The equation of the planar surface passing through $(X_0, Y_0, Z_0)$ and having unit surface normal $\mathbf{n}$ is

$$n_1(X - X_0) + n_2(Y - Y_0) + n_3(Z - Z_0) = 0.$$

The scene coordinates $(X, Y, Z)$ and image coordinates $(x, y)$ are related by the projection equations, that is

$$X = xZ/f, \qquad Y = yZ/f.$$

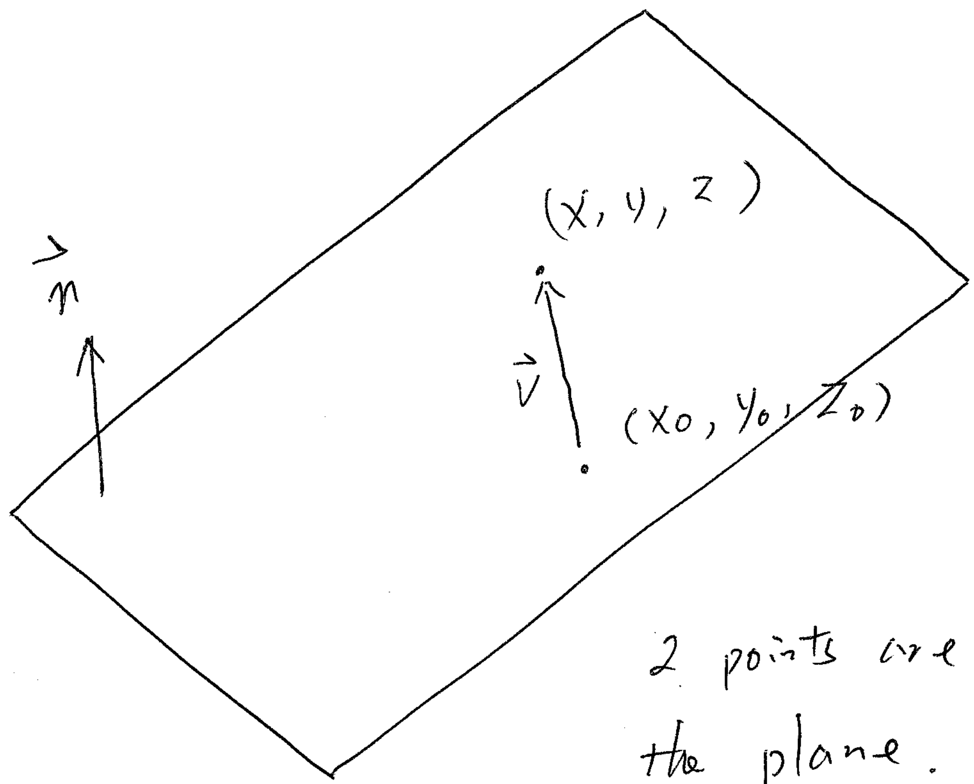
Substituting these into the surface equation, we obtain

$$n_1 x + n_2 y + n_3 f = f \frac{n_1 X_0 + n_2 Y_0 + n_3 Z_0}{Z}.$$

The image coordinates $(x, y)$ of all the image points for which the Z-coordinate of the corresponding space points is infinity ($Z \rightarrow \pm\infty$) satisfy, irrespective of $(X_0, Y_0, Z_0)$,

$$n_1 x + n_2 y + n_3 f = 0,$$

which defines the vanishing line on the image plane. Since the plane is not parallel to the image plane, $n_1$ and $n_2$ are not both zero.




2 points are on the plane.

$$V = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

$$V \perp \vec{n}$$

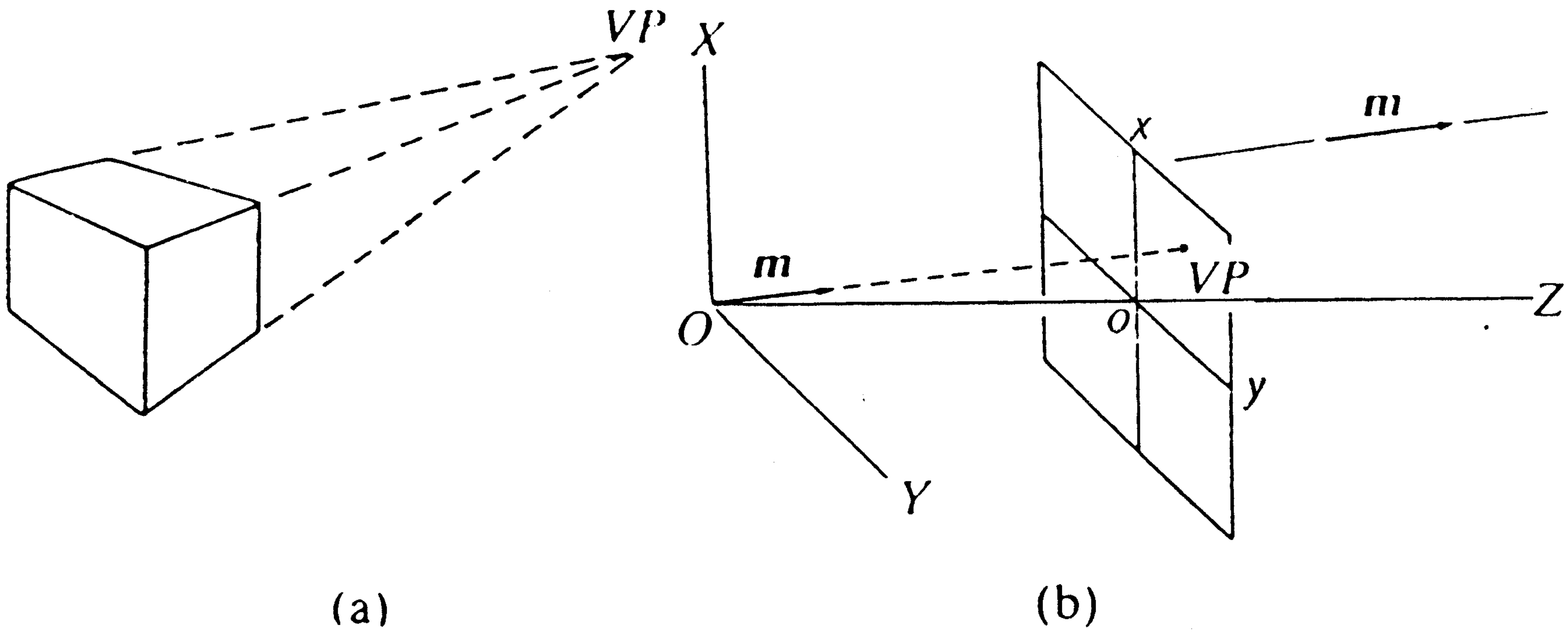
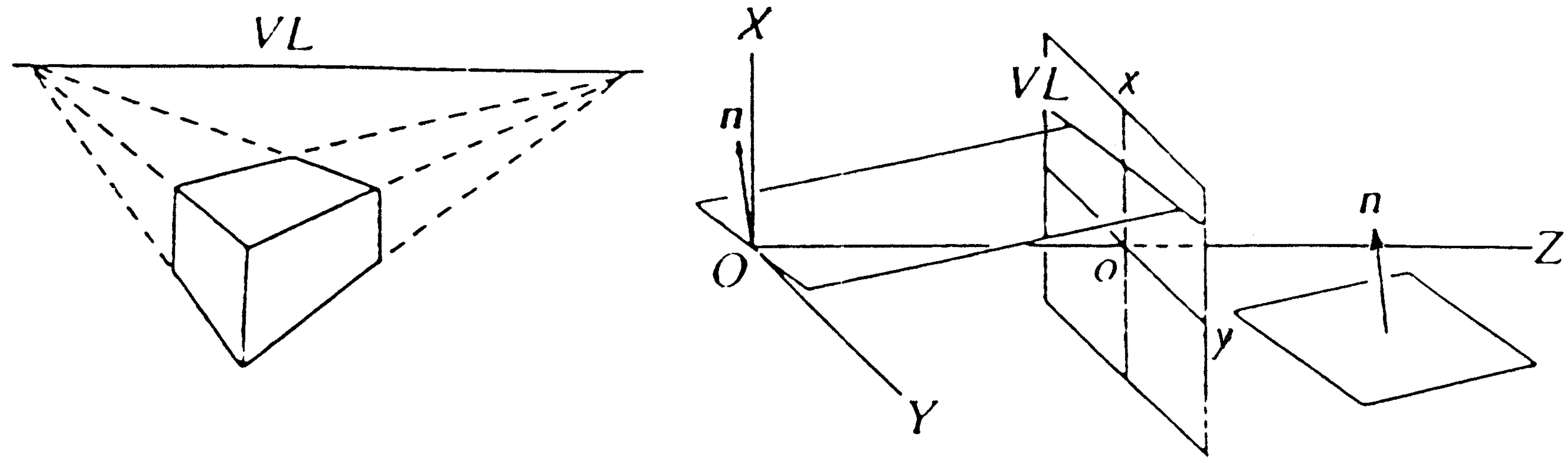



FIG. 2.3. (a) Vanishing point. (b) The N-vector $\boldsymbol{m}$ of the vanishing point indicates the 3-D orientation of the line.




(a) (b)

FIG. 2.4. (a) Vanishing line. (b) The N-vector $\boldsymbol{n}$ of the vanishing line indicates the unit normal to the surface.

## Cross Ratio

Let $A, B, C$ and $D$ be distinct points on line $\ell$. Their *cross ratio* $[ABCD]$ is defined by

$$[ABCD] = \frac{AC}{BC} / \frac{AD}{BD}$$

where $AC$, $BC$... are signed distances with respect to an arbitrary fixed orientation of the line $\ell$, hence

$$AC = -CA, \ \ldots$$

The following relations are obvious

$$[ABCD] = [BADC] = [CDAB] = [DCBA]$$

$$[ABDC] = 1/[ABCD]$$

$$[ACBD] = 1 - [ABCD]$$

$$[ACDB] = \frac{1}{1 - [ABCD]}$$

$$[ADBC] = \frac{[ABCD] - 1}{[ABCD]}$$

The cross ratio of four collinear space points is equal to the cross ratio of their projections on the image plane. (perspective invariance of cross ratio)


show : $[ACBD] = 1 - [ABCD]$

① $[ACBD] = \dfrac{AB}{CB} \Big/ \dfrac{AD}{CD}$

$$= \frac{AB \cdot CD}{CB \cdot AD}$$

② $1 - [ABCD] = 1 - \dfrac{AC}{BC} \Big/ \dfrac{AD}{BD}$

$$= 1 - \frac{AC \cdot BD}{BC \cdot AD}$$

$$= 1 + \frac{AC \cdot BD}{CB \cdot AD}$$

$$= \frac{CB \cdot AD + AC \cdot BD}{CB \cdot AD}$$

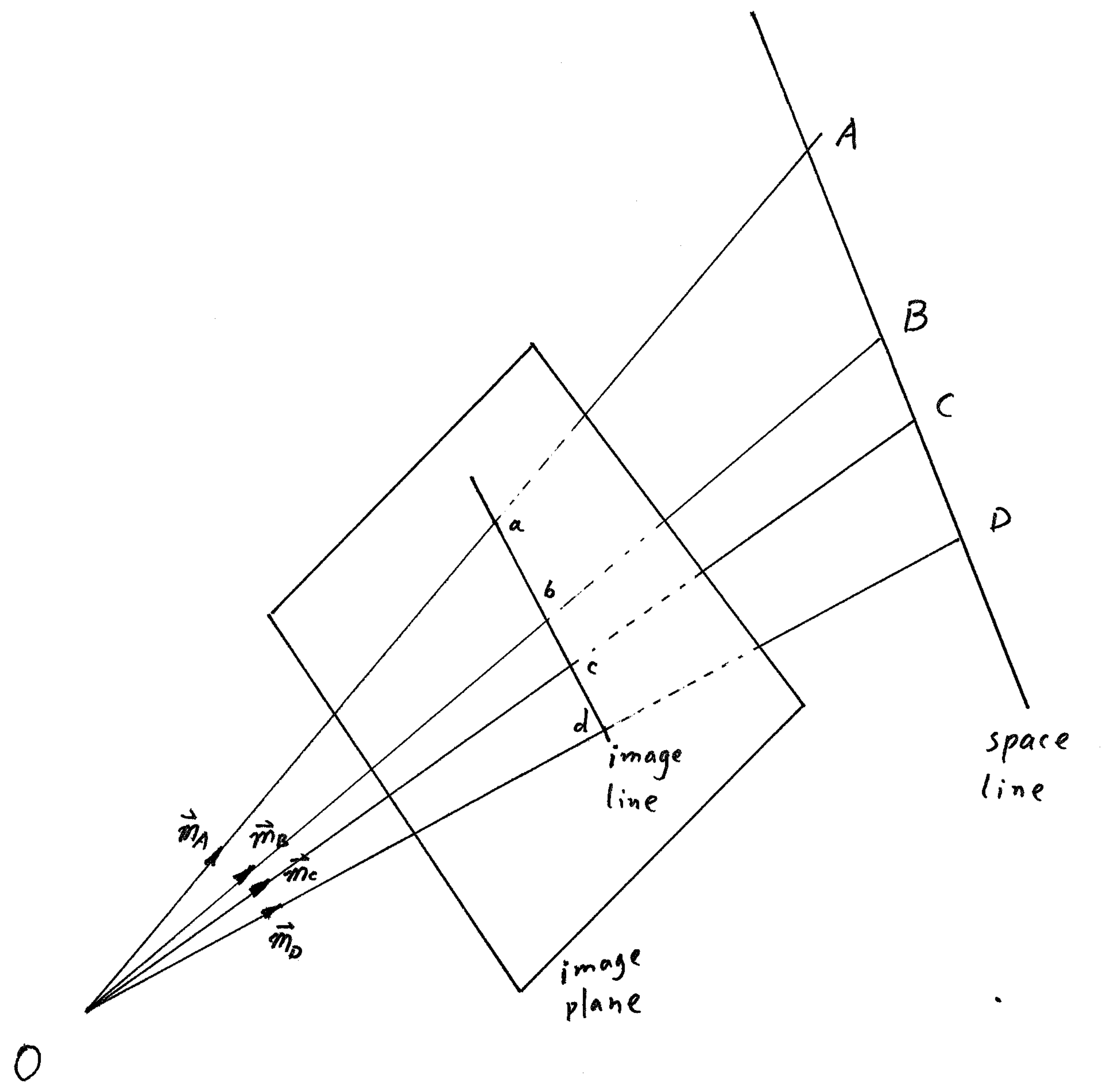
$$= [CB \cdot (AB + BD) + (AB + BC) \cdot BD] \,/\, CB \cdot AD$$

$$= [AB \cdot (CB + BD) + BD \cdot (CB + BC)] \,/\, CB \cdot AD$$

$$= [AB \cdot (BD - BC)] \,/\, CB \cdot AD$$

$$= AB \cdot CD \,/\, CB \cdot AD$$

## Computation of Cross Ratio :




* A, B, C & D space points

  a, b, c & d image points

  $\vec{m}_A$, $\vec{m}_B$, $\vec{m}_C$, & $\vec{m}_D$ N-vectors

- Given 4 collinear points on the image
- The cross ratio can be found as

$$[ABCD] = \frac{\|\vec{m}_A \times \vec{m}_C\|}{\|\vec{m}_B \times \vec{m}_C\|} \Big/ \frac{\|\vec{m}_A \times \vec{m}_D\|}{\|\vec{m}_B \times \vec{m}_D\|}$$

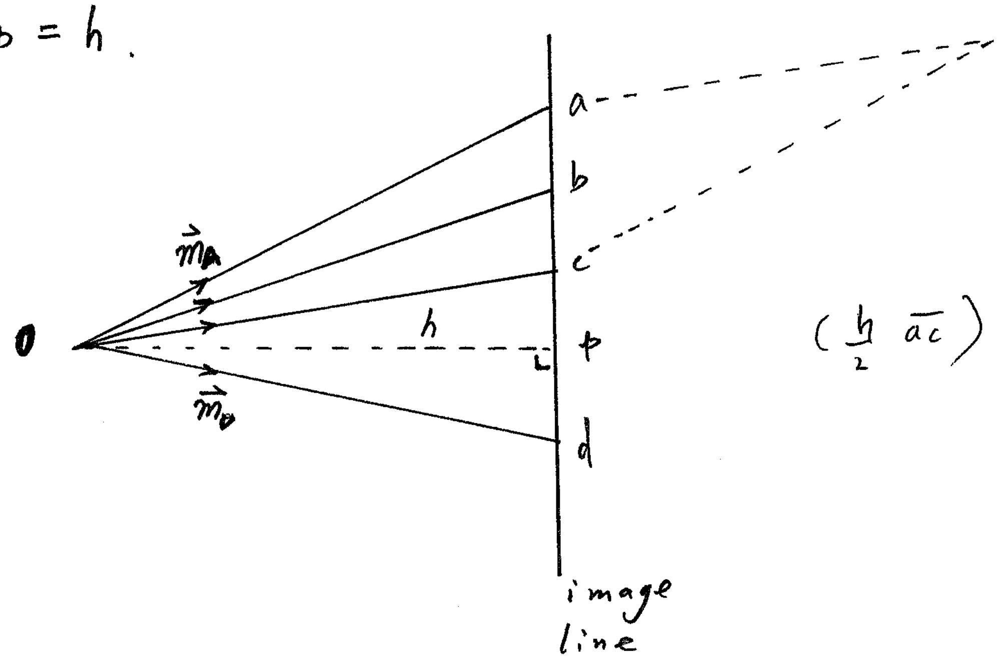
(1) Knowing $a, b, c$ & $d$ is equivalent to knowing

$\vec{m}_A$ , $\vec{m}_B$ , $\vec{m}_C$ & $\vec{m}_D$

Using the normalisation operator $N[\cdot]$

(2) Draw a line $Op$ ; make sure $p$ is on the image line and $Op \perp ad$.

let $Op = h$.




$(\frac{h}{2}\ \overline{ac})$

(3) Consider the area of the triangle Oac

$$\frac{1}{2} h \cdot |ac| = \frac{1}{2} \| \vec{Oa} \times \vec{Oc} \|$$

where $\|\cdot\|$ represents the magnitude of the vector.

(4)

$$|ac| = \frac{1}{h} \| \vec{oa} \times \vec{oc} \|$$

similarly

$$|bc| = \frac{1}{h} \| \vec{ob} \times oc \|$$

$$|ad| = \cdots$$

$$|bd| = \cdots$$

(5) $\vec{Oa} = |Oa| \vec{m}_A$ , $\vec{Ob} = |O_b| \vec{m}_B$

$\vec{Oc} = |Oc| \vec{m}_c$ , $\vec{Od} = |Od| \vec{m}_D$

(6) $[ABCD] = [abcd] = \dfrac{|ac|}{|bc|} \Big/ \dfrac{|ad|}{|bd|}$

$$= \frac{\frac{|Oa| \cdot |Oc|}{h} \| \vec{m}_A \times \vec{m}_c \|}{\frac{|Ob| \cdot |Oc|}{h} \| \vec{m}_B \times \vec{m}_c \|} \Bigg/ \frac{\frac{|Oa| \cdot |Od|}{h} \| \vec{m}_A \times \vec{m}_d \|}{\frac{|Ob| \cdot |Od|}{h} \| \vec{m}_B \times \vec{m}_d \|}$$

$\square$

# FOCUS OF EXPANSION (FOE)

Projections of translating space points seem to be moving on the image plane away from (or toward) a fixed point, this is known as the *focus of expansion.*

Since the focus of expansion is simply the "vanishing point" of the trajectories in the scene. Thus,

**(1)** A space point translating in the direction of unit vector $\mathbf{u}$ has, when projected onto the image plane, a focus of expansion whose N-vector is $\pm\mathbf{u}$.

**(2)** Projections of rigidly translating space points have a common focus of expansion.

**(3)** If two image points of N-vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ in the first frame move to image points of N-vectors $\mathbf{m}'_1$ and $\mathbf{m}'_2$ in the second frame, respectively, the N-vector of the focus of expansion is given by

$$\mathbf{u} = \pm N[N[\mathbf{m}_1 \times \mathbf{m}'_1] \times N[\mathbf{m}_2 \times \mathbf{m}'_2]],$$

provided that the four image points are all distinct.

*Proof.* Let $P_1, P_2, P'_1$ and $P'_2$ be the image points of N-vectors $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}'_1$ and $\mathbf{m}'_2$, respectively. The N-vector of the trajectory defined by $P_1$ and $P'_1$ is

$\pm N[\mathbf{m}_1 \times \mathbf{m}'_1]$, and the N-vector of the trajectory defined by $P_2$ and $P'_2$ is $\pm N[\mathbf{m}_2 \times \mathbf{m}'_2]$. The N-vector of their intersection is given by the cross product of these two lines.

**(4)** If two image points of N-vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ are moving on the image plane with N-velocities $\dot{\mathbf{m}}_1$ and $\dot{\mathbf{m}}_2$, respectively, the N-vector of the focus of expansion is given by

$$\mathbf{u} = \pm N[N[\mathbf{m}_1 \times \dot{\mathbf{m}}_1] \times N[\mathbf{m}_2 \times \dot{\mathbf{m}}_2]],$$

provided that the trajectories of the two image points are distinct.

*Proof.* The focus of expansion is the intersection of the tracjectories on the image plane. The N-vectors of the trjactories of the two image points are $\pm N[\mathbf{m}_1 \times \dot{\mathbf{m}}_1]$ and $\pm N[\mathbf{m}_2 \times \dot{\mathbf{m}}_2]]$. Hence the N-vector of their intersection is give by the above equation of the cross product of the two vectors.

In practice, noise are present and the computation of FOE is not accurate. This can be improved by additonal observations of more than two points.

(5) If image points of N-vectors $\mathbf{m}_i$ move to image points of N-vectors $\mathbf{m}'_i$, $i = 1, ..., N$, and if $\mathbf{u}$ is the N-vector of the focus of expansion, then

$$|\mathbf{u},\ \mathbf{m}_i,\ \mathbf{m}'_i| = 0, \quad i = 1, ..., N.$$

This is called the *epipolar equation.* From this we can robustly compute the N-vector $\mathbf{u}$ of the focus of expansion by

$$\sum_{i=1}^{N} W_i |\mathbf{u},\ \mathbf{m}_i,\ \mathbf{m}'_i|^2 \rightarrow min,$$

where $W_i$ are positive weights.

We can re-arrange the above equation,

$$\begin{aligned}\sum_{i=1}^{N} W_i|\mathbf{u},\ \mathbf{m_i},\ \mathbf{m_i'}|^2 &= \sum_{i=1}^{N} W_i(\mathbf{u},\ \mathbf{m_i} \times \mathbf{m_i'})^2 \\ &= \sum_{i=1}^{N} W_i \mathbf{u}^\top (\mathbf{m_i} \times \mathbf{m_i'})(\mathbf{m_i} \times \mathbf{m_i'})^\top \mathbf{u} \\ &= (\mathbf{u}, \underbrace{\left(\sum_{i=1}^{N} W_i(\ \mathbf{m_i} \times \mathbf{m_i'})(\ \mathbf{m_i} \times \mathbf{m_i'})\top\right)}_{A} \mathbf{u}). \\ &= (\mathbf{u}, A\mathbf{u}).\end{aligned}$$

where $A$ is a $3 \times 3$ matrix. The problem is now reduced to

$$(\mathbf{u}, A\mathbf{u}) \ \rightarrow \ min.$$

The solution of $\mathbf{u}$ is given by the unit eigenvector of $A$ for the smallest eigenvalue.

# REPRESENTATION OF SPACE LINE

Define $H$ the centre of a space line $\ell$ as the point closest to the viewpoint O on $\ell$, $\mathbf{u}$ (unit vector) as its orientation, and $\mathbf{n}$ as the N-vector of $\ell$.

The sign of $\mathbf{u}$ is chosen so that the three vectors $\{\mathbf{n},\ \vec{OH},\ \mathbf{u}\}$ form a right-handed system; or a space line is oriented so that it "positively circulates" around its N-vector.

Lines passing through the viewpoint O is invisible and therefore are not considered.

$$\mathbf{p} = \frac{\mathbf{u}}{|OH|}.$$

is called the *P-vector*. The space line $\ell$ is completely defined by $\{\mathbf{n},\ \mathbf{p}\}$

$$\mathbf{u} = N[\mathbf{p}], \quad ||\mathbf{p}|| = \frac{1}{|OH|}$$

and $H$ is found in the direction of $\mathbf{p} \times \mathbf{n}$. The 3-D position of the centre $H$ is given by

$$\vec{OH} = \frac{N[\mathbf{p} \times \mathbf{n}]}{||\mathbf{p}||} = \frac{\mathbf{p} \times \mathbf{n}}{||\mathbf{p}||^2}$$

Proof :

① By definition

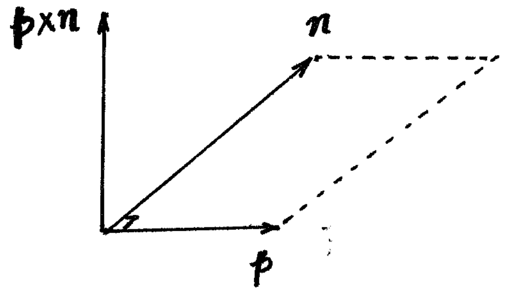
$$N[\mathbf{p} \times \mathbf{n}] = \frac{\mathbf{p} \times \mathbf{n}}{\|\mathbf{p} \times \mathbf{n}\|}$$

② $\mathbf{m} = N[\mathbf{p} \times \mathbf{n}]$

$$\vec{OH} = \frac{1}{\|\mathbf{p}\|}\,\mathbf{m} = \frac{N[\mathbf{p} \times \mathbf{n}]}{\|\mathbf{p}\|}$$

$$= \frac{\mathbf{p} \times \mathbf{n}}{\|\mathbf{p}\| \cdot \|\mathbf{p} \times \mathbf{n}\|}$$

③ $\|\mathbf{p} \times \mathbf{n}\| = \|\mathbf{p}\|$, because $\|\mathbf{n}\| = 1$ and $\mathbf{p} \perp \mathbf{n}$.
(fundamental identity)



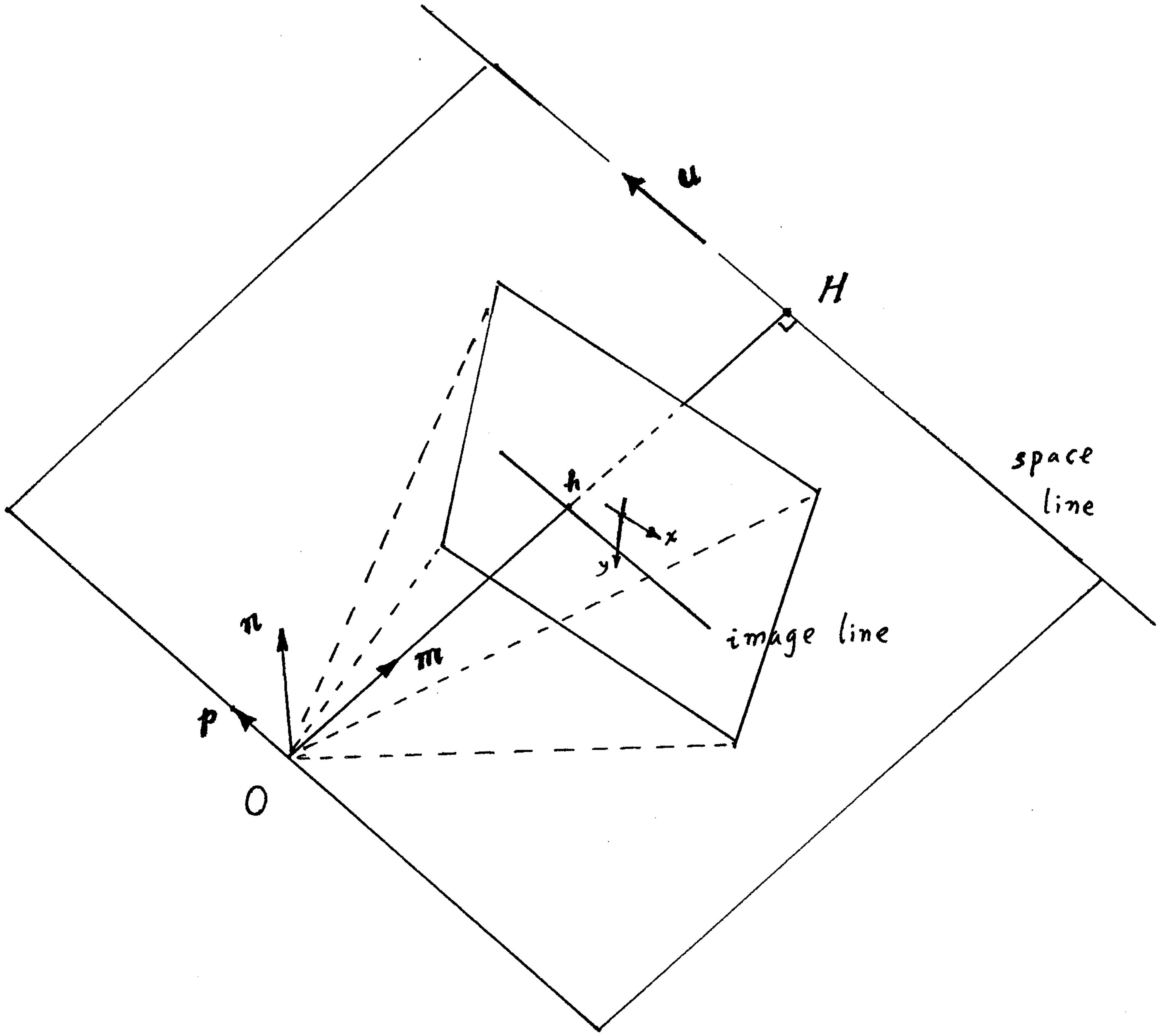

$\square$

$$\vec{OH} = |OH| \cdot \overbrace{N[p \times n]}^{\text{unit vec.}}$$

$$= \frac{N[p \times n]}{\|p\|} = \frac{1}{\|p\|} \cdot \underbrace{\left[\frac{p}{\|p\|} \times n\right]}_{= N[p \times n]}$$

$$= \left[\frac{1}{\|p\|^2}\right] \cdot p \times n$$

$$\|p\| = \frac{1}{d}$$




Definition of space line and P-vector

**(1)** For any space line $\{\mathbf{n}, \mathbf{p}\}$, its N-vector $\mathbf{n}$ and P-vector $\mathbf{p}$ are mutually orthogonal.

$$(\mathbf{n}, \mathbf{p}) = 0.$$

This is known as the *fundamental identity* of a space line in algebraic geometry.

**(2)** The equation of space line $\{\mathbf{n},\ \mathbf{p}\}$ is

$$\mathbf{r} \times \mathbf{p} \ = \ \mathbf{n}.$$

*Proof.* Let $H$ be the centre of space line $\{\mathbf{n},\ \mathbf{p}\}$. Let $d = |OH|$, $\mathbf{m} = \vec{OH}/d$, and $\mathbf{u} = d\mathbf{p}$. A point is on this line iff vector $\mathbf{r} = \vec{OP}$ satisfies

$$\mathbf{r} = d\mathbf{m} + t\mathbf{u}$$

for some number $t$, or

$$\mathbf{r} \times \mathbf{p} = (d\mathbf{m} + t\mathbf{u}) \times \mathbf{p}$$

$$\mathbf{r} \times \mathbf{p} = d\mathbf{m} \times \mathbf{p} + 0$$

$$\mathbf{r} \times \mathbf{p} = \mathbf{m} \times d\mathbf{p}$$

$$\mathbf{r} \times \mathbf{p} = \mathbf{m} \times \mathbf{u}$$

since $\mathbf{m}$ is orthogonal to $\mathbf{u}$ and $\mathbf{m},\ \mathbf{u}$ are all unit vectors,

$$\mathbf{m} \times \mathbf{u} = \mathbf{n}$$

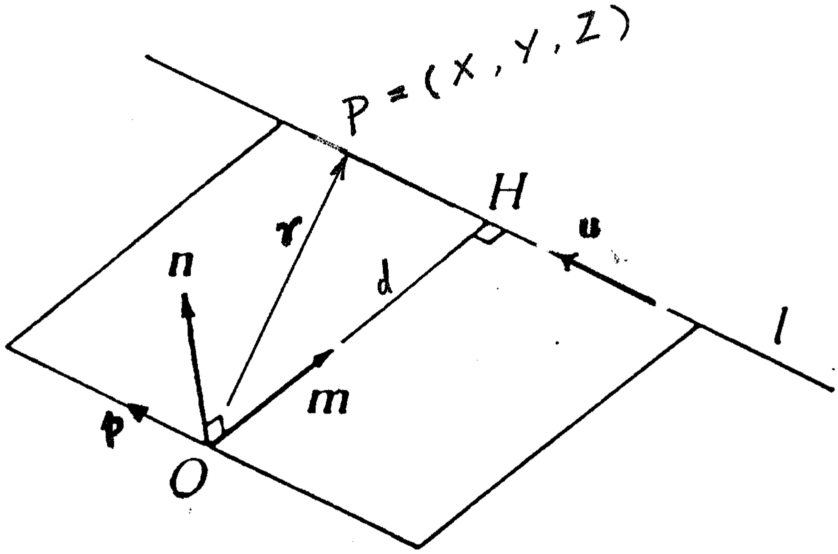
hence

$$\mathbf{r} \times \mathbf{p} \ = \ \mathbf{n}.$$

a. P is a space point on the space line $\ell$.

$$\vec{OP} = \boldsymbol{r}$$

b. $\vec{OP} = \vec{OH} + \vec{HP}$




c. $\vec{OH} = d\,\boldsymbol{m}$

d. $\vec{HP} = t\,\boldsymbol{u}$

Note : $\boldsymbol{r} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$

$\ell_1$ $\ell_2$

**(3)** Space lines $\{\mathbf{n_1}, \mathbf{p_1}\}$ and $\{\mathbf{n_2}, \mathbf{p_2}\}$ intersect iff

$$(\mathbf{n_1}, \mathbf{p_2}) + (\mathbf{n_2}, \mathbf{p_1}) = 0$$

*Proof.* The projections of these space lines intersect at an image point of N-vector $\mathbf{m} = \pm N[\mathbf{n_1} \times \mathbf{n_2}]$. The two space lines intersect iff there exist a real number $s$ such that the end point of vector $\mathbf{r} = \mathbf{n_1} \times \mathbf{n_2}/s$ is on both lines; if $s = 0$, the intersection is interpreted to be at infinity. From **(2)**, this condition is written as

$$\frac{\mathbf{n_1} \times \mathbf{n_2}}{s} \times \mathbf{p_1} = \mathbf{n_1},$$

$$\frac{\mathbf{n_1} \times \mathbf{n_2}}{s} \times \mathbf{p_2} = \mathbf{n_2}.$$

Note: $(a \times b) \times c = (a, c)b - (b, c)a$

Since

$$(\mathbf{n_1}, \mathbf{p_1}) = 0, \quad (\mathbf{n_2}, \mathbf{p_2}) = 0,$$

(the fundamental identities), these are equivalently written as

$$-(\mathbf{n_2}, \mathbf{p_1})\mathbf{n_1} = s\mathbf{n_1},$$

$$(\mathbf{n_1}, \mathbf{p_2})\mathbf{n_2} = s\mathbf{n_2},$$

or

$$s = -(\mathbf{n}_2, \mathbf{p}_1),$$
$$s = (\mathbf{n}_1, \mathbf{p}_2),$$

Such an $s$ exists iff $(\mathbf{n}_1, \mathbf{p}_2) \ + \ (\mathbf{n}_2, \mathbf{p}_1) \ = \ 0.$

$$\frac{1}{s}(\overset{a}{n_1} \times \overset{b}{n_2}) \times \overset{c}{p_1}$$

$$= \frac{1}{s}\left[\underline{(n_1, p_1)\, n_2} - (n_2, p_1)\, n_1\right]$$

$$= \left[-\frac{1}{s}(n_2, p_1)\right] n_1 = n_1$$

$$\therefore \text{(1)} \quad s = -(n_2, p_1)$$

$$\text{(2)} \quad s = (n_1, p_2)$$

**(4)** The space line $\{\mathbf{n},\ \mathbf{p}\}$ that passes through two space points at $\mathbf{r}_1$ and $\mathbf{r}_2$ ($\mathbf{r}_1 \times \mathbf{r}_2 \neq \mathbf{0}$) is given by

$$\mathbf{n} = \pm N[\mathbf{r}_1 \times \mathbf{r}_2]$$

$$\mathbf{p} = \mp \frac{\mathbf{r}_1 - \mathbf{r}_2}{||\mathbf{r}_1 \times \mathbf{r}_2||}$$

*Proof.* Since the N-vector should be orthogonal to both $\mathbf{r}_1$ and $\mathbf{r}_2$, we obtain the first equation

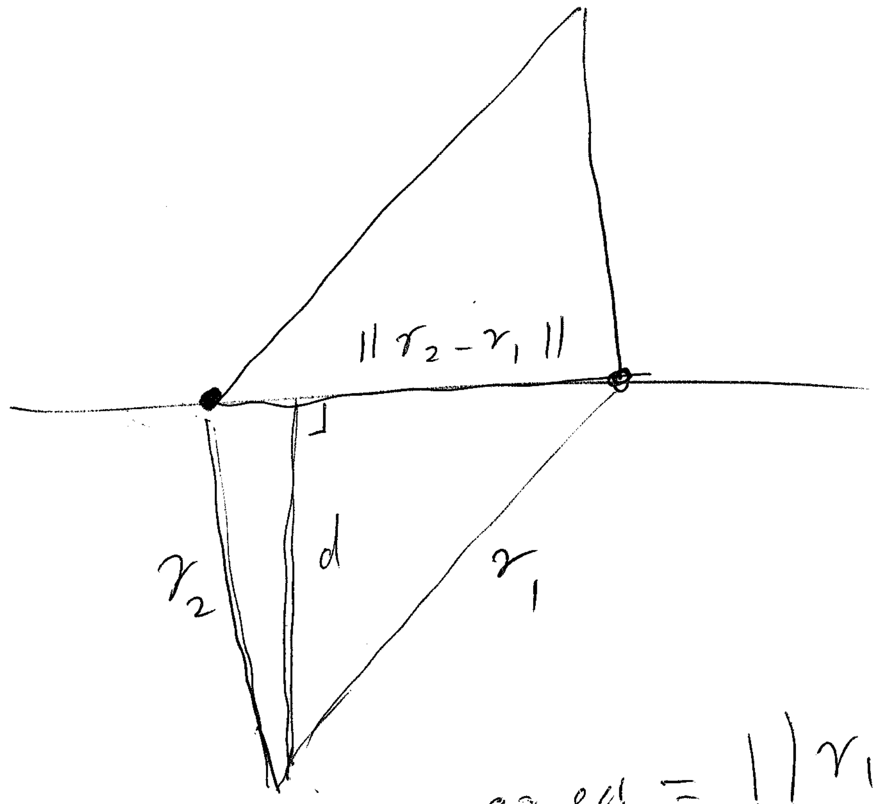
$$\mathbf{n} = \pm N[\mathbf{r}_1 \times \mathbf{r}_2]$$

If $d$ is the distance of the space line from the viewpoint $O$, its P-vector is

$$\mathbf{p} = \mp \frac{\overbrace{N[\mathbf{r}_1 - \mathbf{r}_2]}^{u}}{d} = \frac{\mathbf{r}_1 - \mathbf{r}_2}{||\mathbf{r}_1 - \mathbf{r}_2||d}$$

according to our sign convention. Consider the triangle defined by the two points at $\mathbf{r}_1$ and $\mathbf{r}_2$ and the viewpoint $O$. Its area is $||\mathbf{r}_1 - \mathbf{r}_2||d/2$. It is also equal to $||\mathbf{r}_1 \times \mathbf{r}_2||/2$. Hence,

$$d = \frac{||\mathbf{r}_1 \times \mathbf{r}_2||}{||\mathbf{r}_1 - \mathbf{r}_2||}.$$




$$\text{area} = \|r_1 \times r_2\|$$

$$= 2\left[\frac{1}{2}\|r_1 - r_2\| d\right]$$

$$= d \cdot \|r_1 - r_2\|$$

$$d = \frac{\|r_1 \times r_2\|}{\|r_1 - r_2\|}$$

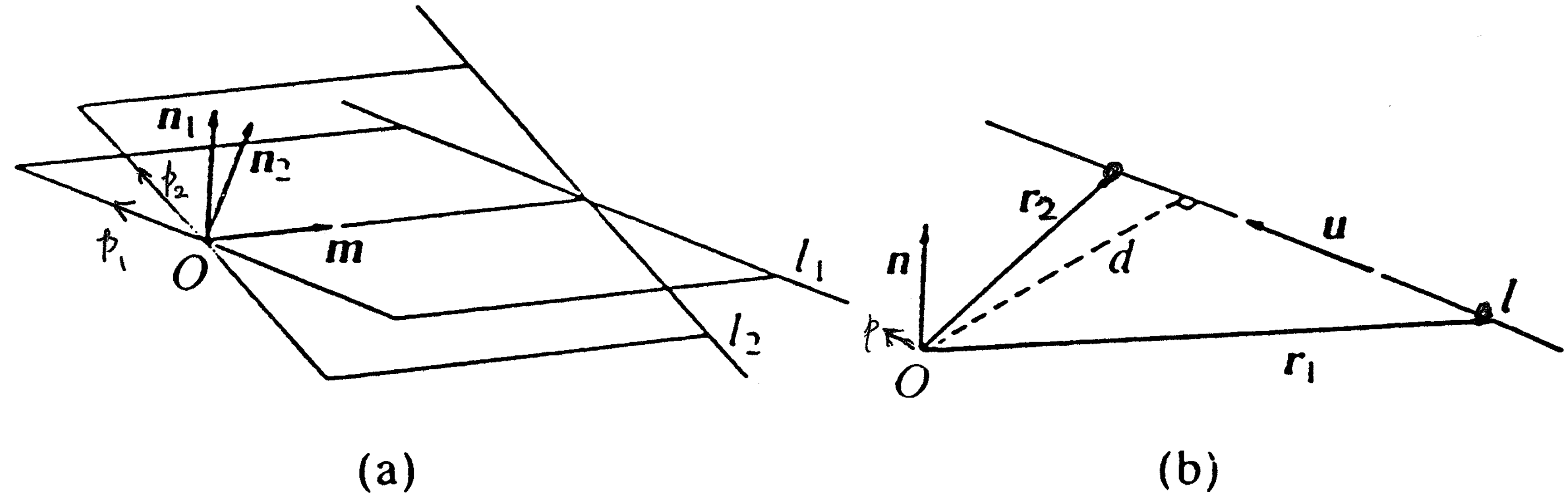



FIG. 4.4. (a) Intersection of space lines. (b) A space line passing through two space points.

*Proof.* Since the N-vector $\boldsymbol{n}$ should be orthogonal to both $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ (Fig. 4.4(b)), we obtain the first of eqs. (4.42). If $d$ is the distance of the space line from the viewpoint $O$, its P-vector is

$$\boldsymbol{p} = \mp \frac{N[\boldsymbol{r}_1 - \boldsymbol{r}_2]}{d} = \frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{\|\boldsymbol{r}_1 - \boldsymbol{r}_2\| d}, \tag{4.43}$$

according to our sign convention. Consider the triangle defined by the two

# MOTION PARALLAX OF A LINE

**(1)** If the camera is translated by $h$, the representation $\{\mathbf{n},\mathbf{p}\}$ of a space line changes into $\{\mathbf{n}', \mathbf{p}'\}$ in the form

$$\mathbf{n}' = N[\mathbf{n} - \mathbf{h} \times \mathbf{p}],$$
$$\mathbf{p}' = \frac{\mathbf{p}}{||\mathbf{n} - \mathbf{h} \times \mathbf{p}||}$$

*Proof*. Let $\ell$ be the space line in question. Its orientation $\mathbf{u} = \mathbf{p}/||\mathbf{p}||$ is the same for both frames. Let $O$ and $O'$ be the origins of the first and second frames, respectively. Let $H$ be the centre of $\ell$. The N-vector $\mathbf{n}'$ of $\ell$ for the second frame is orthogonal to both $\mathbf{u}$ and $\overrightarrow{O'H}$. Noting that $\{\mathbf{n}', \overrightarrow{O'H}, \mathbf{u}\}$ is a right-handed system, we obtain

$$\mathbf{n}' = N[\overrightarrow{O'H} \times \mathbf{u}].$$

It is easy to see that the distance of $\ell$ from $O'$ is given by $||\overrightarrow{O'H} \times \mathbf{u}||$. (let $\theta$ represent the angle between $OP$ and line $\ell$, $d = |OP| \sin\theta$; hence $||\overrightarrow{OP} \times \mathbf{u}|| = |OP| \sin\theta = d$).

$u = \frac{p}{||p||}$

$$\mathbf{p}' = \frac{\mathbf{u}}{||\overrightarrow{O'H} \times \mathbf{u}||} = \frac{\mathbf{p}}{||\mathbf{p}|| \cdot ||\overrightarrow{O'H} \times \mathbf{u}||}. \qquad (1)$$

Further,

$$
\begin{aligned}
\vec{O'H} \times \mathbf{u} &= (\vec{OH} - \mathbf{h}) \times \mathbf{u} \\
&= \left(\frac{\mathbf{p} \times \mathbf{n}}{||\mathbf{p}||^2} - \mathbf{h}\right) \times \frac{\mathbf{p}}{||\mathbf{p}||} \\
&= \frac{\mathbf{n} - \mathbf{h} \times \mathbf{p}}{||\mathbf{p}||}
\end{aligned}
$$

replace it into equation (1), the result is obtained.

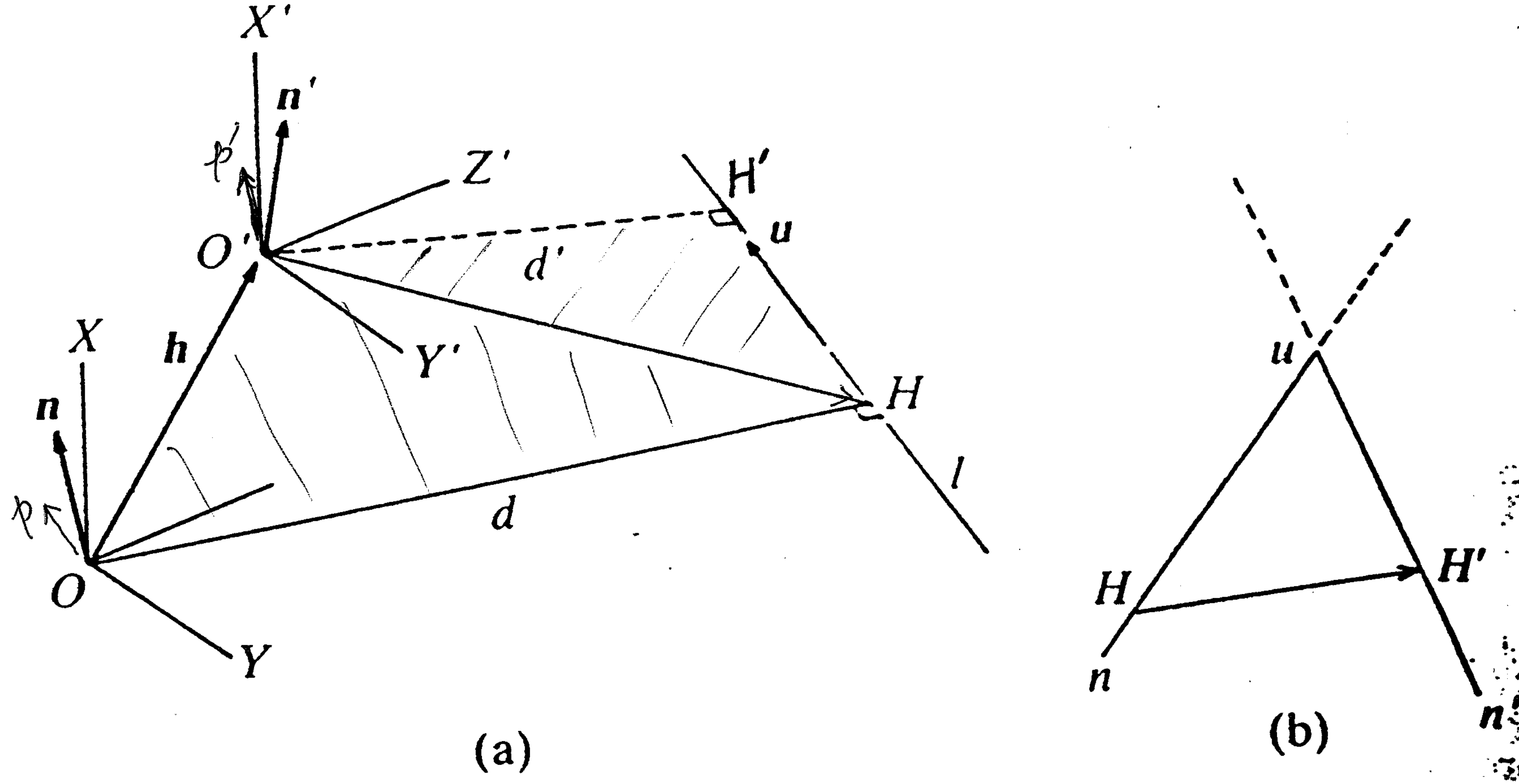
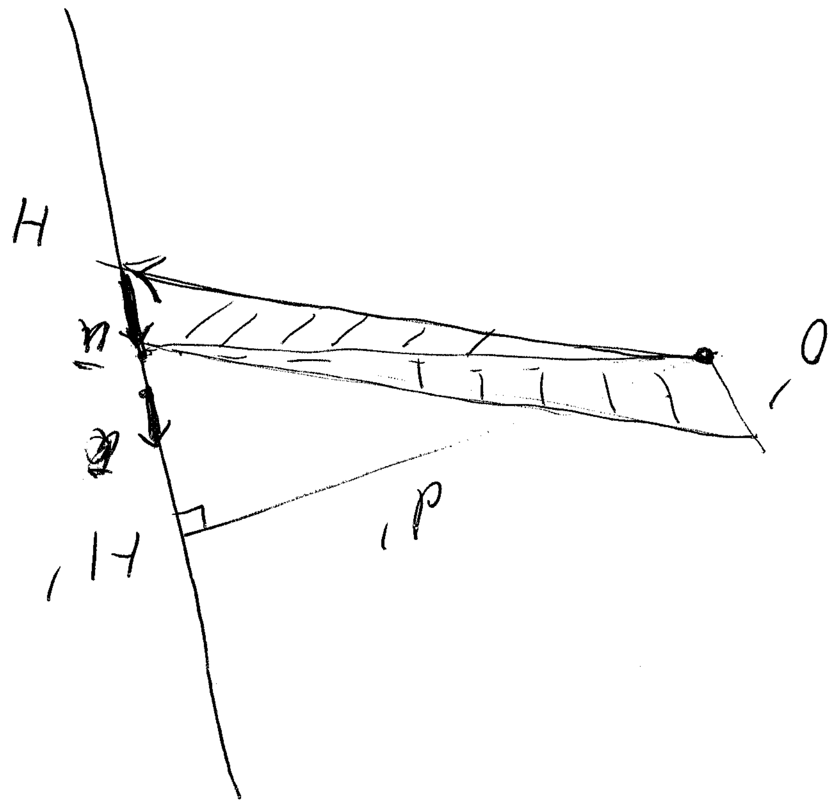



FIG. 4.6. (a) A space line and camera translation. (b) Motion parallax of a line.

$\boldsymbol{n}'$ of $l$ for the second frame is orthogonal to both $\boldsymbol{u}$ and $\vec{O'H}$. Noting that $\{\boldsymbol{n}', \vec{O'H}, \boldsymbol{u}\}$ is a right-handed system, we obtain

$$\left(u \times \frac{\|H\|}{d}\right)\frac{\|H\|}{1} = HO$$

$$\overline{,p = \left[ 1 \cdot ,p \frac{2}{1} \right] \cdot \tau} = \| n \times H,O \|$$




$1 = \| n \|$

**(2)** If a space line of N-vector $\mathbf{n}$ moves to a space line of N-vector $\mathbf{n}'$ by a camera translation $\mathbf{h}$ and if $(\mathbf{h}, \mathbf{n}) \neq 0$, and $(\mathbf{h}, \mathbf{n}') \neq 0$, the P-vectors $\mathbf{p}$ and $\mathbf{p}'$ for the first and second are respectively given by

$$\mathbf{p} = \frac{\mathbf{n} \times \mathbf{n}'}{(\mathbf{h}, \mathbf{n}')}, \quad \mathbf{p}' = \frac{\mathbf{n} \times \mathbf{n}'}{(\mathbf{h}, \mathbf{n})}.$$

(*motion parallax equations of a space line.*)

(3) with rotation

$$p = \frac{n \times R n'}{(h, R n')},$$

$$p' = \frac{R^T (n \times R n')}{(h, n)}$$

(4) Research

Find $\{R, h\}$ from space line correspondence.

# Summary

1. Camera params : intrinsics $f$, image centre
   extrinsics $R, T$

2. N-vectors
   ① image point
   ② image lines $N\left[\begin{pmatrix} A \\ B \\ C/f \end{pmatrix}\right]$

3. p-vectors
   $\{n, p\}$ — space lines

4. 2D → 3D
   - motion parallax
   - points
   - lines

5. 3D → 3D
   ego-motion, SVD

6. Epipolar geom.
   - E-matrix, epipolar lines
   - F-matrix, pixel-pixel