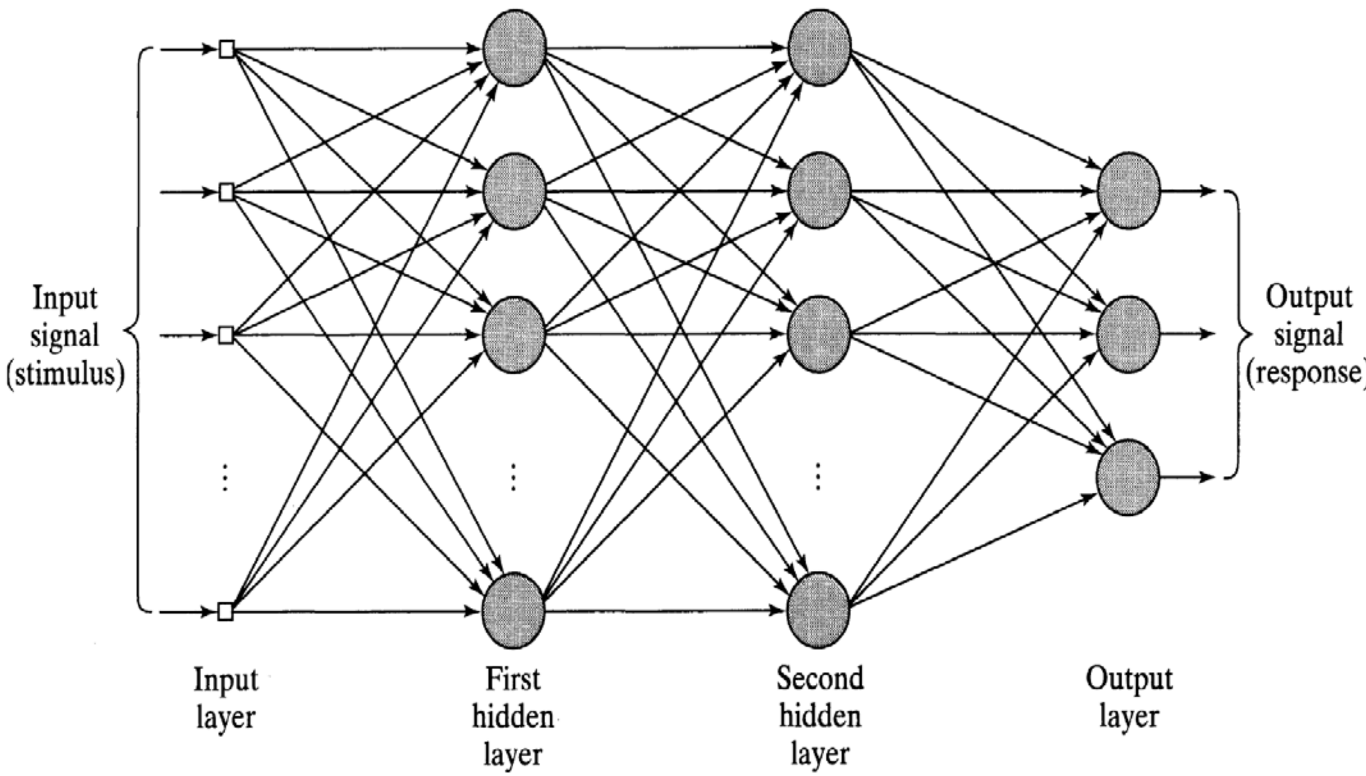
2. Feed-Forward Multi-layer Neural Networks

The architecture of a typical feed-forward multi-layer neural network is shown below:



Definitions

Input layer: the layer where input patterns are applied.

Output layer: the layer from which the output responses are obtained.

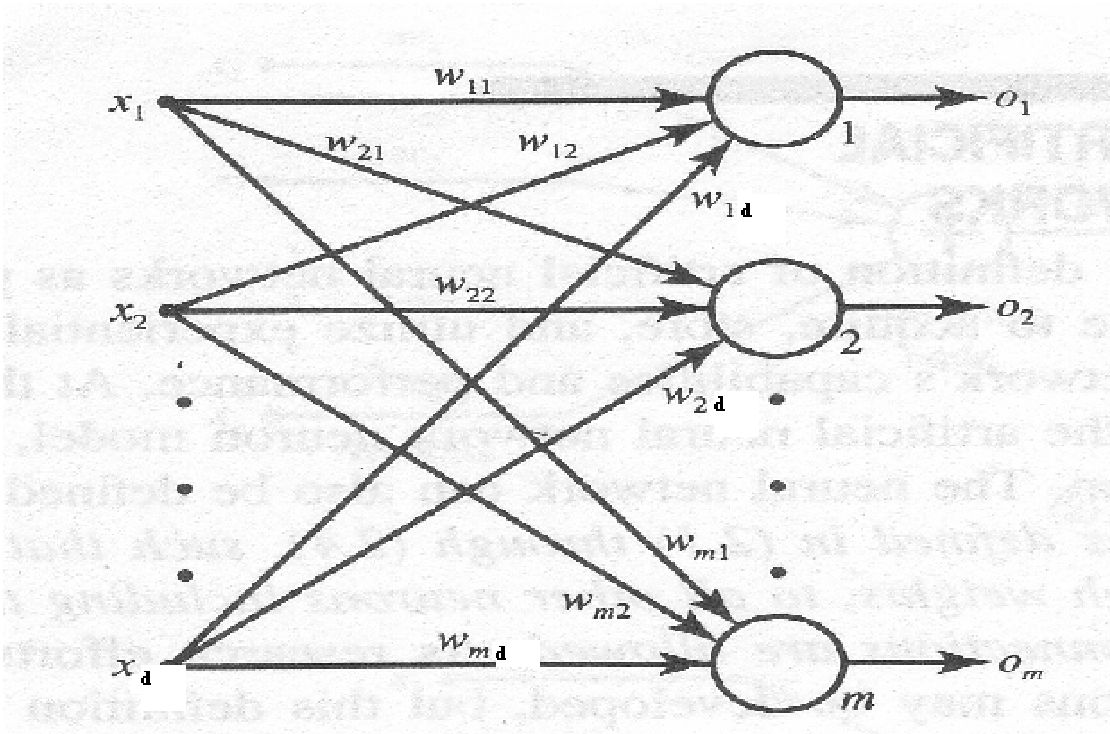
Hidden layer: the intermediate layers between the output and the input layers.

Neurons: represented by circles, neurons are the basic units that perform computations.

Connection and communication path: represented by arrows. Notice that in the forward neural networks, the communications are from one layer to the next and do not leapfrog layers.

Single-layer feed-forward network

The multi-layer neural networks consist of single-layer networks. The architecture of a single-layer feed-forward neural network is shown below:



The architecture of single layer feed-forward network

The network is with *d* inputs and *m* outputs. The input and output vectors are respectively:

$$\mathbf{x} = [x_1, x_2, \cdots, x_d]^T$$

$$\mathbf{o} = [o_1, o_2, \cdots, o_m]^T$$

where the input and output of the network are *d*-dimensional and *m*-dimensional vectors respectively. If we denote w_{ij} as the weight that connects output neuron *i* with input neuron *j*, the activation value for output neuron i can be written as:

$$v_i = \sum_{j=1}^{d} w_{ij} x_j$$

Note that the activation value is a scalar.

After receiving the activation, neurons will perform processing of the activation signal. The processing can be considered as a transform with strong non-linearity:

$$o_i = \varphi(v_i) = \varphi(\sum_{j=1}^{d} w_{ij} x_j)$$

For ease of representation, we define the weight vector:

$$\mathbf{w}_i = [w_{i1}, w_{i2}, \cdots, w_{id}]$$

Then the activation value to output layer neuron *i* is:

$$v_i = \mathbf{w}_i \mathbf{x}$$

The output of neuron i can be written as:

$$o_i = \varphi(v_i) = \varphi(\mathbf{w}_i \mathbf{x})$$

We next introduce a vector operator $\boldsymbol{\varphi}$ that maps the input pattern from input space to output space:

$$\mathbf{o} = \boldsymbol{\varphi}(\mathbf{Wx})$$

where weight matrix or connect matrix **W** is defined as:

$$\mathbf{W} = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1d} \\ w_{21} & w_{22} & \cdots & w_{2d} \\ \vdots & \vdots & \vdots & \vdots \\ w_{m1} & w_{m2} & \cdots & w_{md} \end{bmatrix}$$

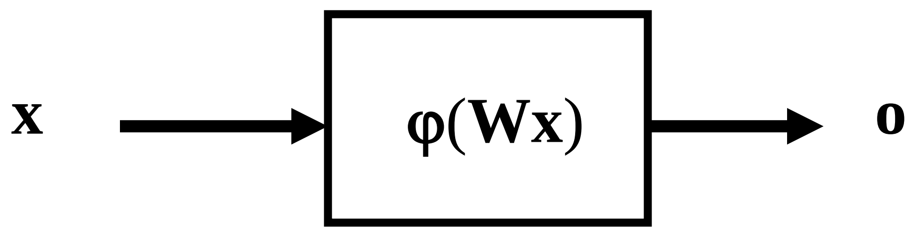
The vector operator $\boldsymbol{\varphi}$ is defined as:

$$\boldsymbol{\varphi}(\bullet) = [\varphi(v_1), \varphi(v_2), \cdots, \varphi(v_m)]^T$$

The mapping from the input to the output is instantaneous, with no time delay, so we have:

$$\mathbf{o} = \varphi(\mathbf{Wx})$$

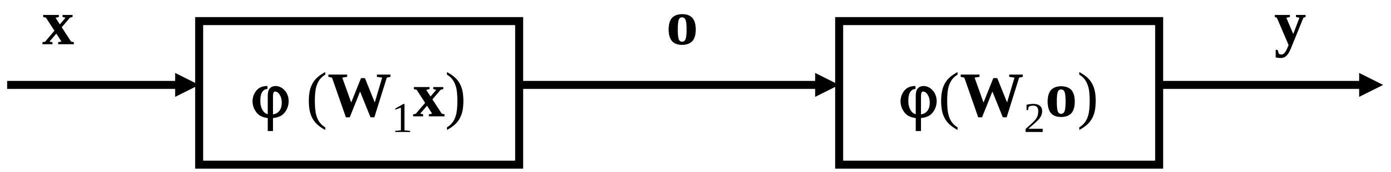
The block diagram of this realization is as follow:



The output vector "**o**" and input vector "**x**" are often called output pattern and input pattern respectively.

Multi-layer Feed-Forward Neural Network

A multi-layer feed-forward neural network consists of two or more connected single-layer feed-forward neural networks. The connection is in a cascade form, where the output of the preceding layer is used as the input of the following layer. The diagram of the cascade connection is shown below:

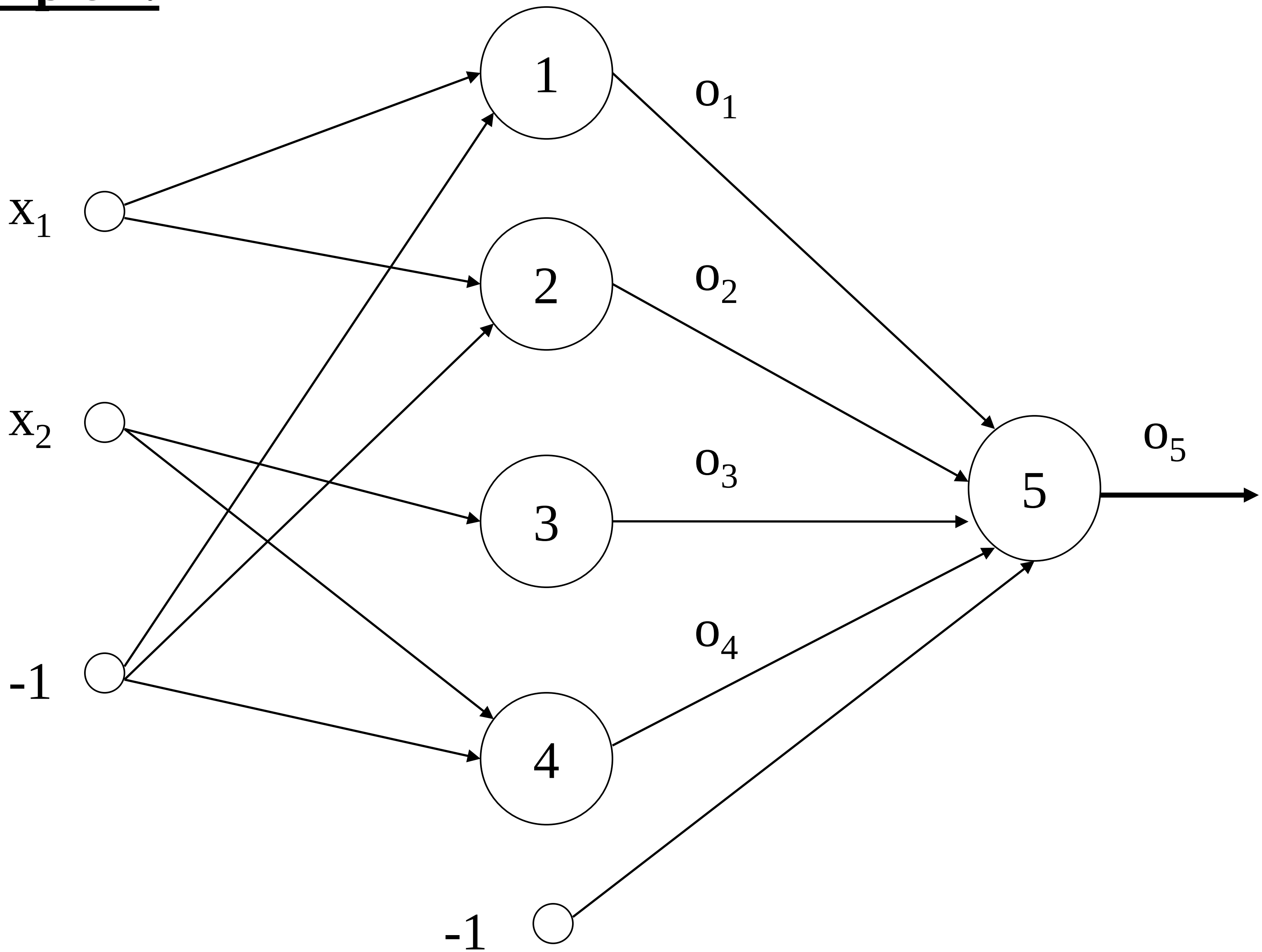


the first layer the second layer

Where $\mathbf{W}_1$ and $\mathbf{W}_2$ are the weight matrices of the first layer and the second layer respectively.

Next, we present two examples to demonstrate the working principle of multi-layer feed-forward neural networks.

Example 1:



The connection weight vectors are:

$$\mathbf{w}_1 = \begin{bmatrix}1 & 0 & 1\end{bmatrix} \qquad \mathbf{w}_2 = \begin{bmatrix}-1 & 0 & -2\end{bmatrix}$$

$$\mathbf{w}_3 = \begin{bmatrix}0 & 1 & 0\end{bmatrix} \qquad \mathbf{w}_4 = \begin{bmatrix}0 & -1 & -3\end{bmatrix}$$

$$\mathbf{w}_5 = \begin{bmatrix}1 & 1 & 1 & 1 & 3.5\end{bmatrix}$$

The activation function is a bi-polar binary function as follow:

$$\varphi(\mathrm{v}) = \begin{cases} 1 & \text{if } \mathrm{v} \geq 0 \\ -1 & \text{if } \mathrm{v} < 0 \end{cases}$$

Based on the connections and the weights, we have the following relationships:

$$v_1 = x_1 - 1$$
$$v_2 = -x_1 + 2$$
$$v_3 = x_2$$
$$v_4 = -x_2 + 3$$

Then:

$$o_1 = \varphi(x_1 - 1)$$
$$o_2 = \varphi(-x_1 + 2)$$
$$o_3 = \varphi(x_2)$$
$$o_4 = \varphi(-x_2 + 3)$$

The output vector of the hidden layer is then used as the input of the output layer. The output of the overall network is as follow:

$$o_5 = \varphi(o_1 + o_2 + o_3 + o_4 - 3.5)$$

The value of o_5 is 1 if and only if

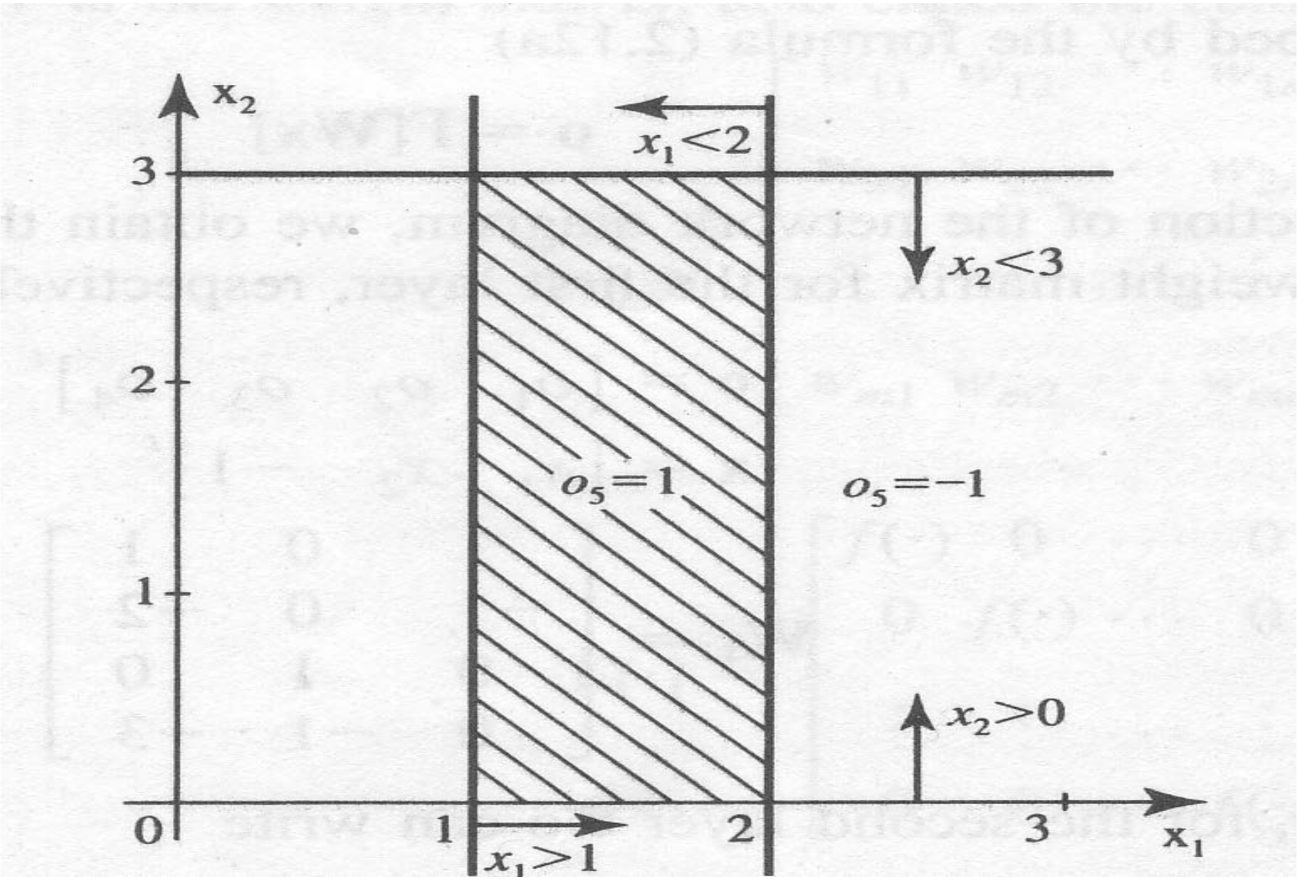
$$o_1 + o_2 + o_3 + o_4 \geq 3.5$$

This actually requires that:

$$o_1 = o_2 = o_3 = o_4 = 1$$

which is equivalent to:

$$x_1 \geq 1$$
$$x_1 \leq 2$$
$$x_2 \geq 0$$
$$x_2 \leq 3$$



If the activation function is a bi-polar sigmoid function, the activation values for the first layer neurons are the same, but the outputs are different:

$$o_1 = \frac{2}{1+\exp[(1-x_1)\lambda]} - 1$$

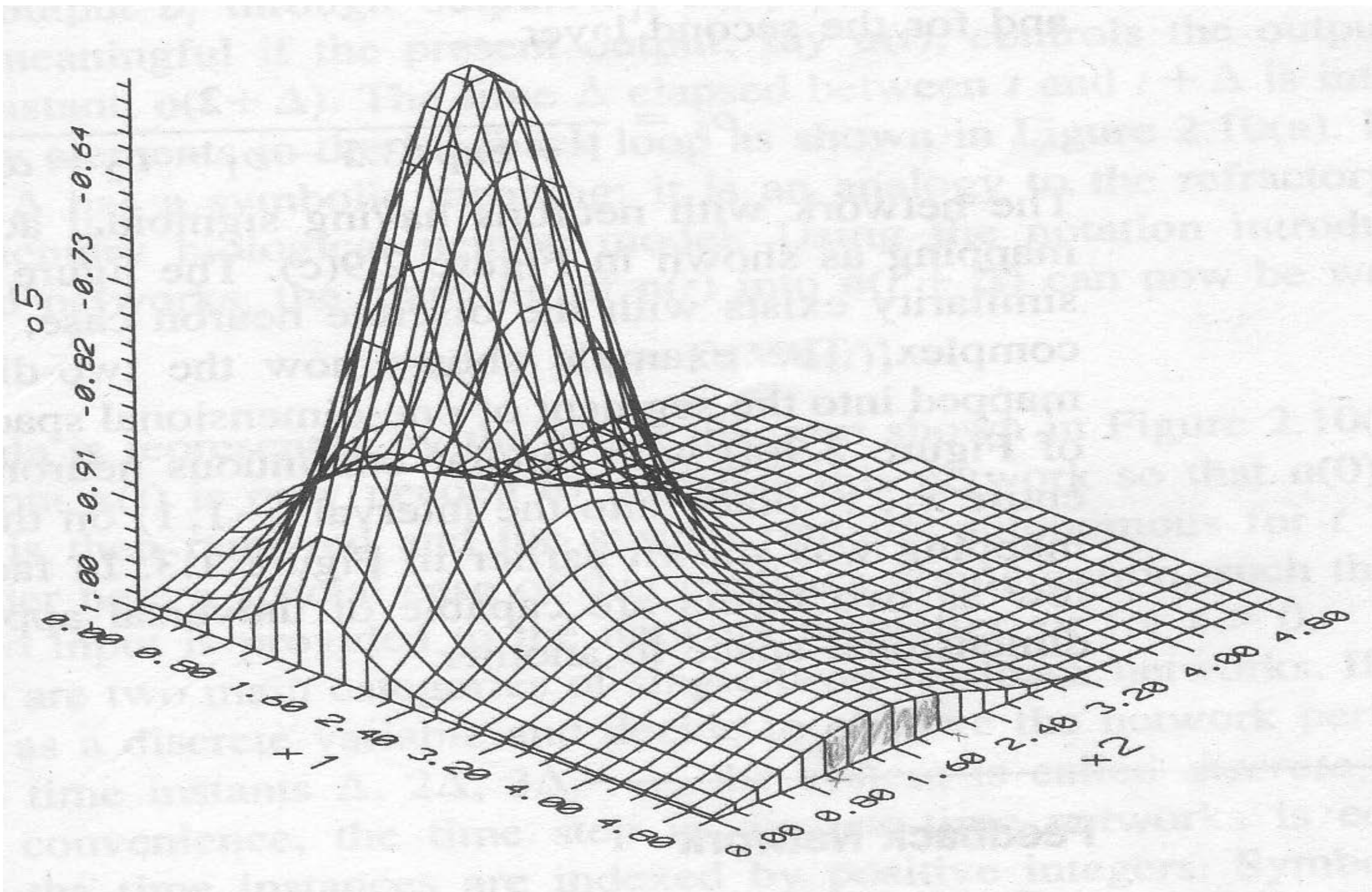
$$o_2 = \frac{2}{1+\exp[(x_1-2)\lambda]} - 1$$

$$o_3 = \frac{2}{1+\exp[(-x_2)\lambda]} - 1$$

$$o_4 = \frac{2}{1+\exp[(x_2-3)\lambda]} - 1$$

And the output of the network is as follow:

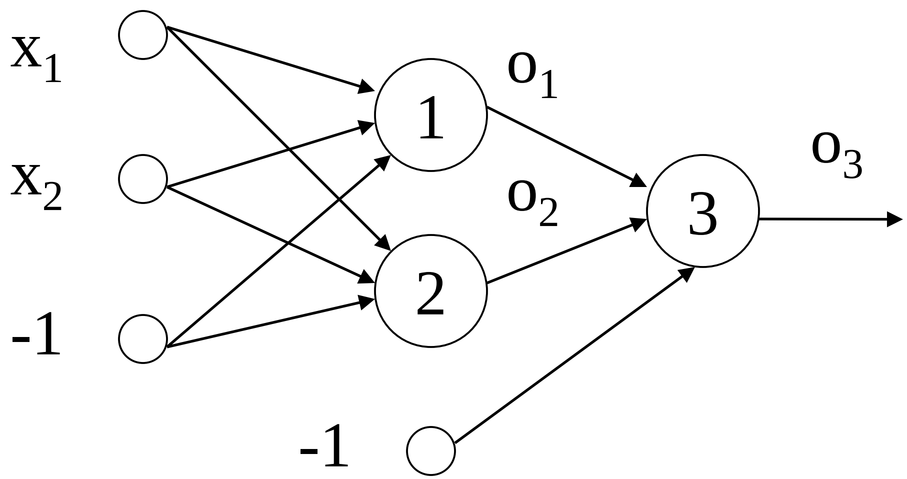
$$o_5 = \frac{2}{1 + \exp[(3.5 - o_1 - o_2 - o_3 - o_4)\lambda]} - 1$$



The mapping of the overall network

Example 2:

Consider the following 2-layer neural network:



The weight vectors are:

$$\mathbf{w}_1 = \begin{bmatrix} 1 & 1 & 1.5 \end{bmatrix} \qquad \mathbf{w}_2 = \begin{bmatrix} 1 & 1 & 0.5 \end{bmatrix} \qquad \mathbf{w}_3 = \begin{bmatrix} -2 & 1 & 0.5 \end{bmatrix}$$

Assume x_1 and x_2 are in the following range:

$$0 \le x_1 \le 1$$

$$0 \le x_2 \le 1$$

Based on the weight vectors given, we have:

$$v_1 = x_1 + x_2 - 1.5$$

$$v_2 = x_1 + x_2 - 0.5$$

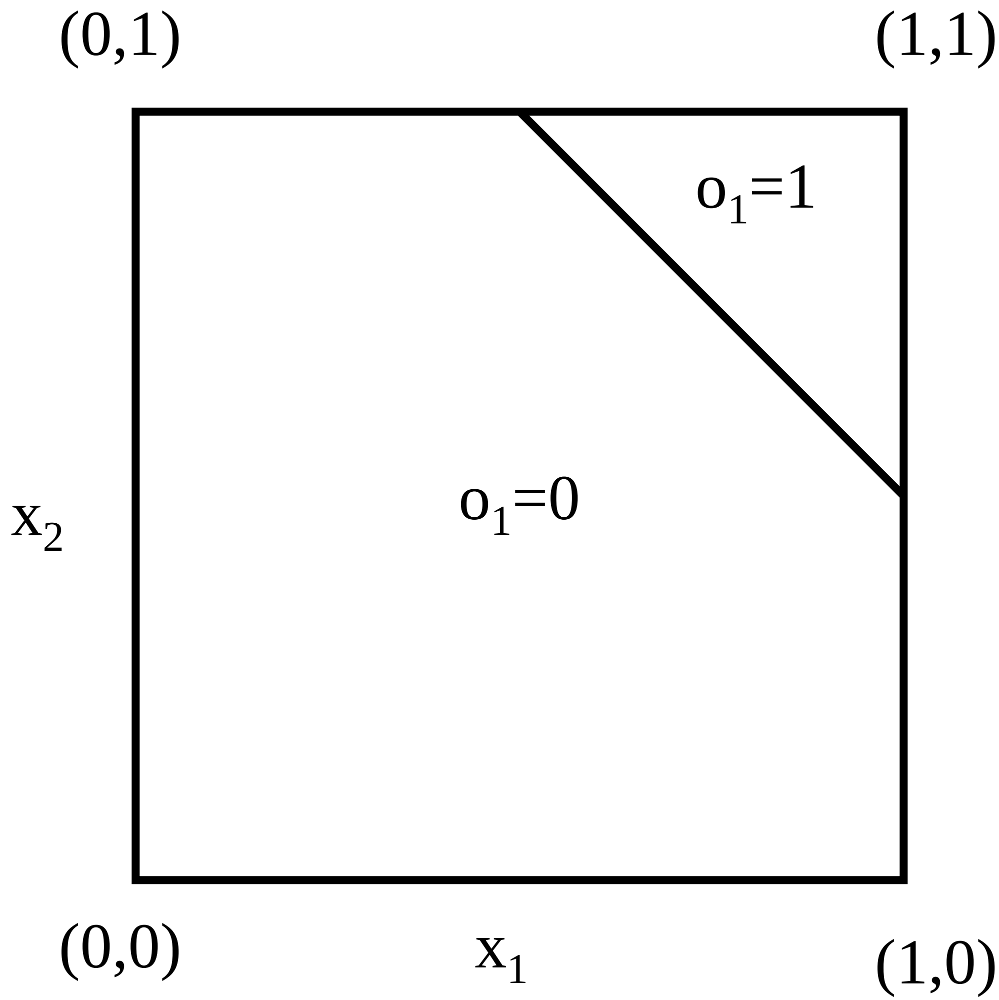
Assume the activation function is an uni-polar binary function:

$$\varphi(v) = \begin{cases} 1 & if\ v \geq 0 \\ 0 & if\ v < 0 \end{cases}$$

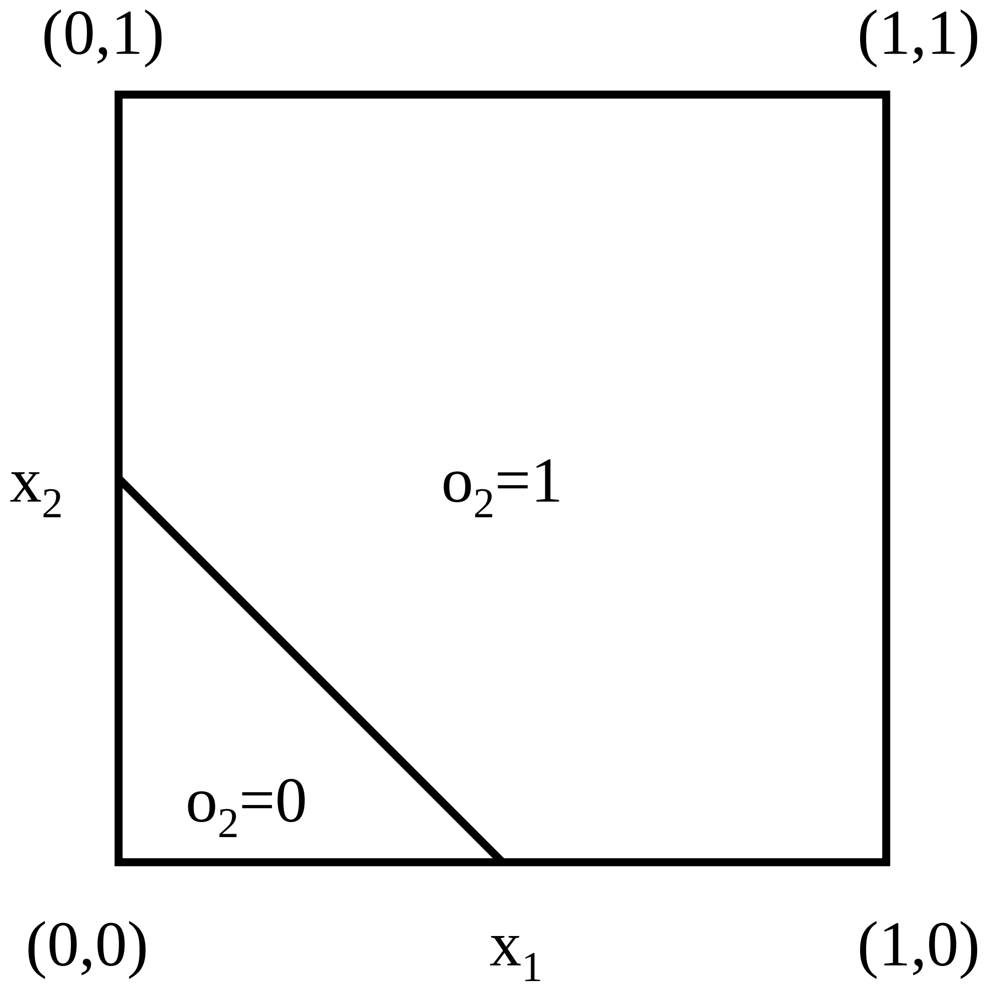
Then we have:

$$o_1 = \begin{cases} 1 & if\ x_1 + x_2 - 1.5 \geq 0 \\ 0 & if\ x_1 + x_2 - 1.5 < 0 \end{cases}$$

$$o_2 = \begin{cases} 1 & if\ x_1 + x_2 - 0.5 \geq 0 \\ 0 & if\ x_1 + x_2 - 0.5 < 0 \end{cases}$$



Output of neuron 1



Output of neuron 2

The activation of neuron 3 is:

$$v_3 = -2o_1 + o_2 - 0.5$$

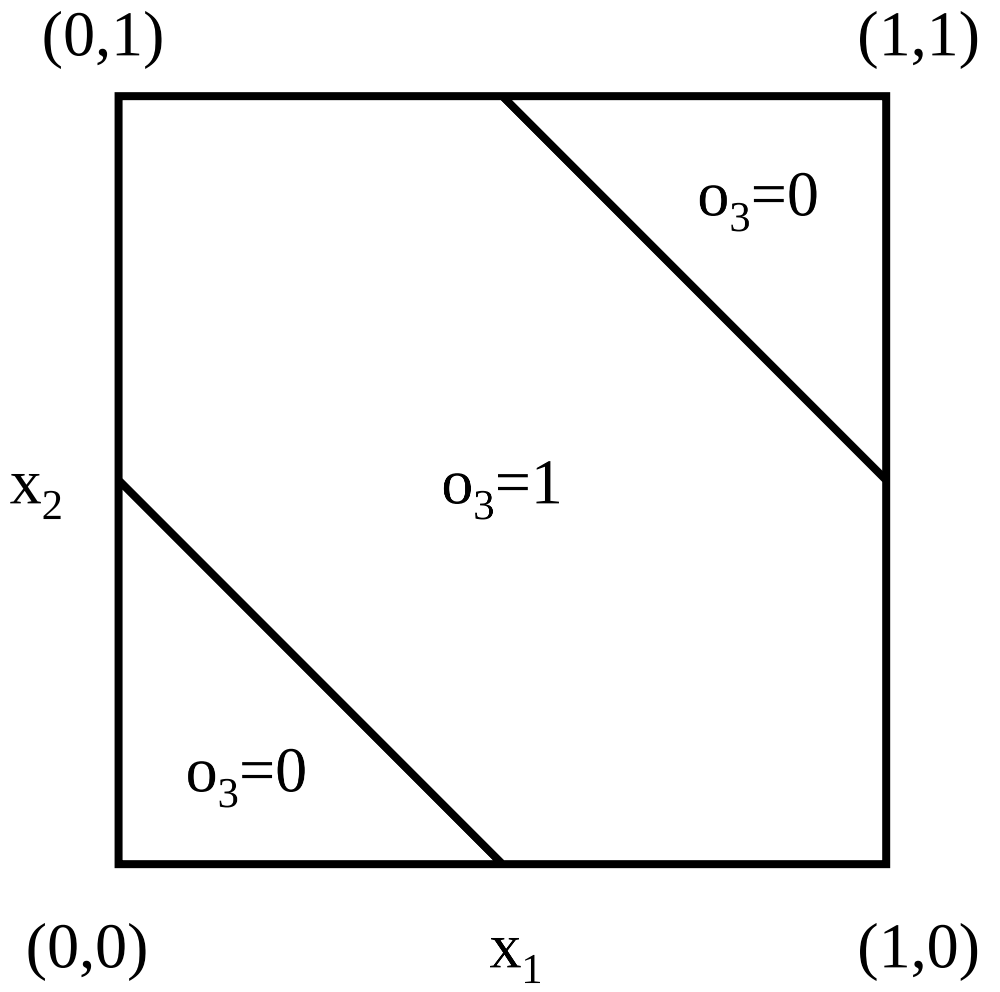
The output of neuron 3 is:

$$o_3 = \begin{cases} 1 & if\ o_2 - 2o_1 - 0.5 \geq 0 \\ 0 & if\ o_2 - 2o_1 - 0.5 < 0 \end{cases}$$

Since the values of o_1 and o_2 are either 1 or 0, thus only when o_1 and o_2 satisfy the following conditions, the value of output o_3 could be 1:

$$\begin{cases} o_1 = 0 \\ o_2 = 1 \end{cases}$$

The region in which o_3=1 is the intersection of the regions where o_1=0 and o_2=1.



Output of the network

If the values of input x_1 and x_2 are either 1 or 0, we can see from the above diagram that:

$$f(1,1) = 0$$
$$f(0,0) = 0$$
$$f(0,1) = 1$$
$$f(1,0) = 1$$

Where *f* is the nonlinear mapping realized by the above neural network. Actually, f realizes the XOR logic operation:

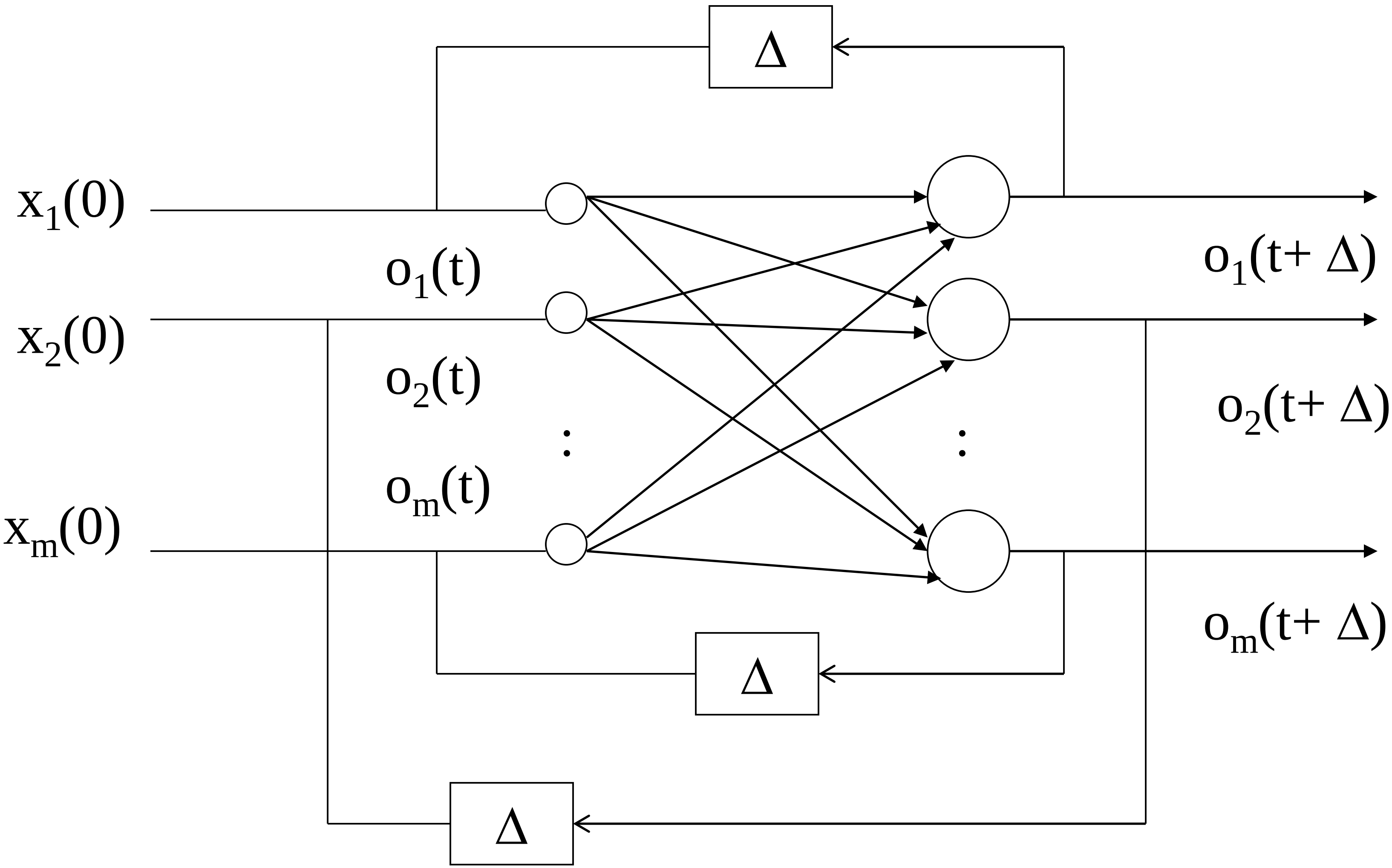
$$1 \oplus 1 = 0$$
$$0 \oplus 0 = 0$$
$$0 \oplus 1 = 1$$
$$1 \oplus 0 = 1$$

Where $\oplus$ denotes the exclusive OR (XOR) operator.

3. Feedback/Recurrent Neural Networks

In the multi-layer feed-forward neural network, the layers are connected in the form of cascade, where the outputs of preceding layer neurons are fed to the following layers. The direction of information communication is single and forward: from the preceding layer to the following layer. A feedback neural network is a feed-forward neural network plus a feedback loop which feeds the outputs of output layer neurons back to the input neurons.

The architecture of a typical feedback neural network is shown below:



The architecture of a single-layer discrete-time feedback neural network

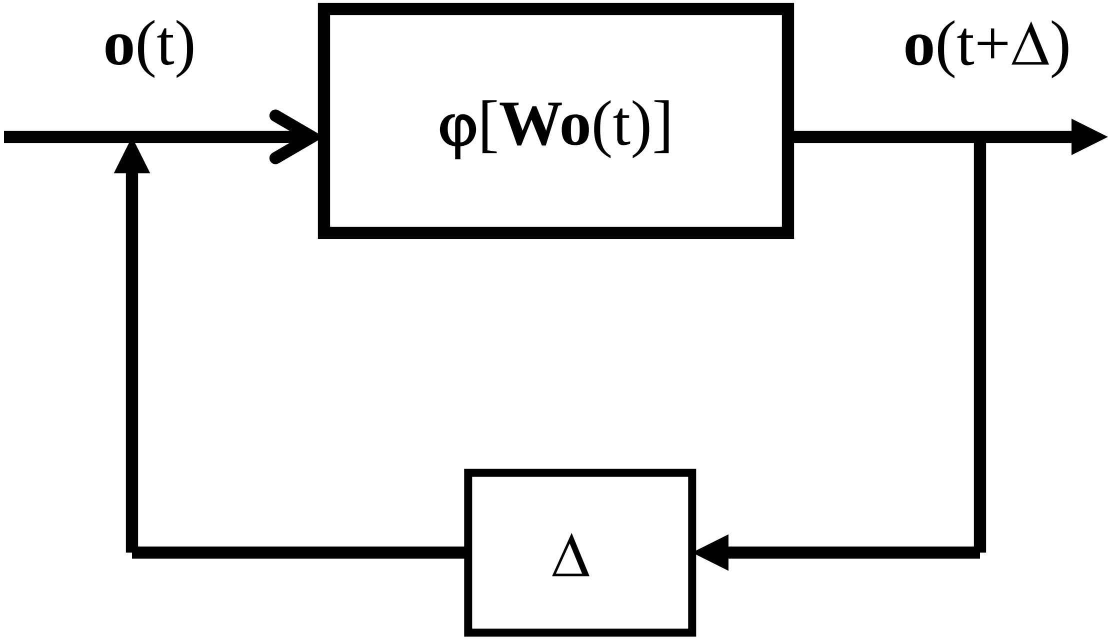
From the architecture of the feedback network, we can see that the network outputs at the current time instant are influenced by the outputs at previous time. This property of the feedback neural network makes it suitable for dynamic system description.

A dynamic system can be described by the following discrete-time state space equation:

$$\mathbf{z}(n) = \boldsymbol{\varphi}[\mathbf{z}(n-1)] + \mathbf{g}[\mathbf{u}(n)]$$

where **z**(n) is the state variable vector, **u**(n) is an external input vector to the system. **z**(n), the current time state of the variable depends on both the state at previous time n-1 and the input variable at the current time n, and the relationship is determined by functions **φ** and **g**.

The feedback neural network can be considered as a dynamic system, whose block diagram is shown bellow:



The block diagram of the feedback network

The formula of the diagram can be written as:

$$\mathbf{o}(t+\Delta) = \boldsymbol{\varphi}[\mathbf{W}\mathbf{o}(t)]$$

Note that the input is only needed to initialize this network so that $\mathbf{o}(0)=\mathbf{x}(0)$. The input is then removed and the system becomes autonomous. If the time is considered as discrete, and the time delay is a unity delay, the above formula can be rewritten as:

$$\mathbf{o}(\mathrm{n}+1) = \boldsymbol{\varphi}[\mathbf{W}\mathbf{o}(\mathrm{n})]$$

where n is an integer and

$$\mathbf{t} = \mathbf{n}\Delta$$
$$\Delta = 1$$

The above discrete-time model of the single layer feedback network can be expanded to a series of nested solutions:

$$\mathbf{o}(1) = \varphi[\mathbf{W}\mathbf{x}(0)]$$

$$\mathbf{o}(2) = \varphi[\mathbf{W}\mathbf{o}(1)] = \varphi[\mathbf{W}[\varphi[\mathbf{W}\mathbf{x}(0)]]]$$

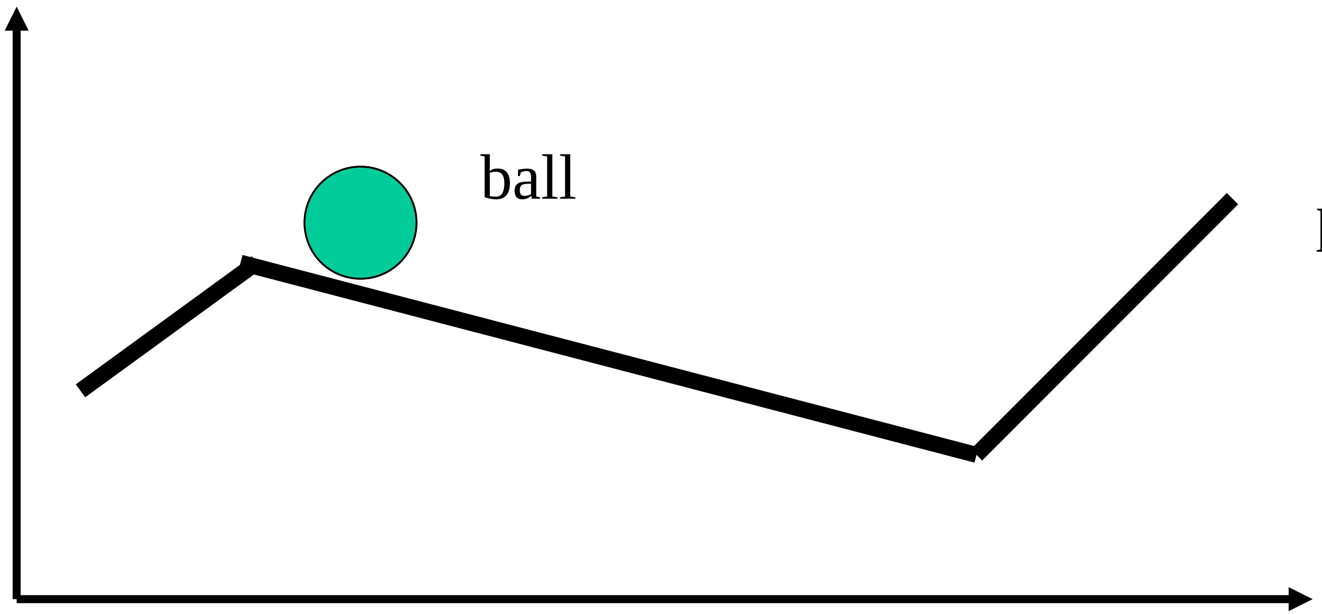
$$\vdots$$

$$\mathbf{o}(\mathrm{n}) = \varphi[\mathbf{W}\varphi[\cdots\varphi[\mathbf{W}\mathbf{x}(0)]\cdots]]$$

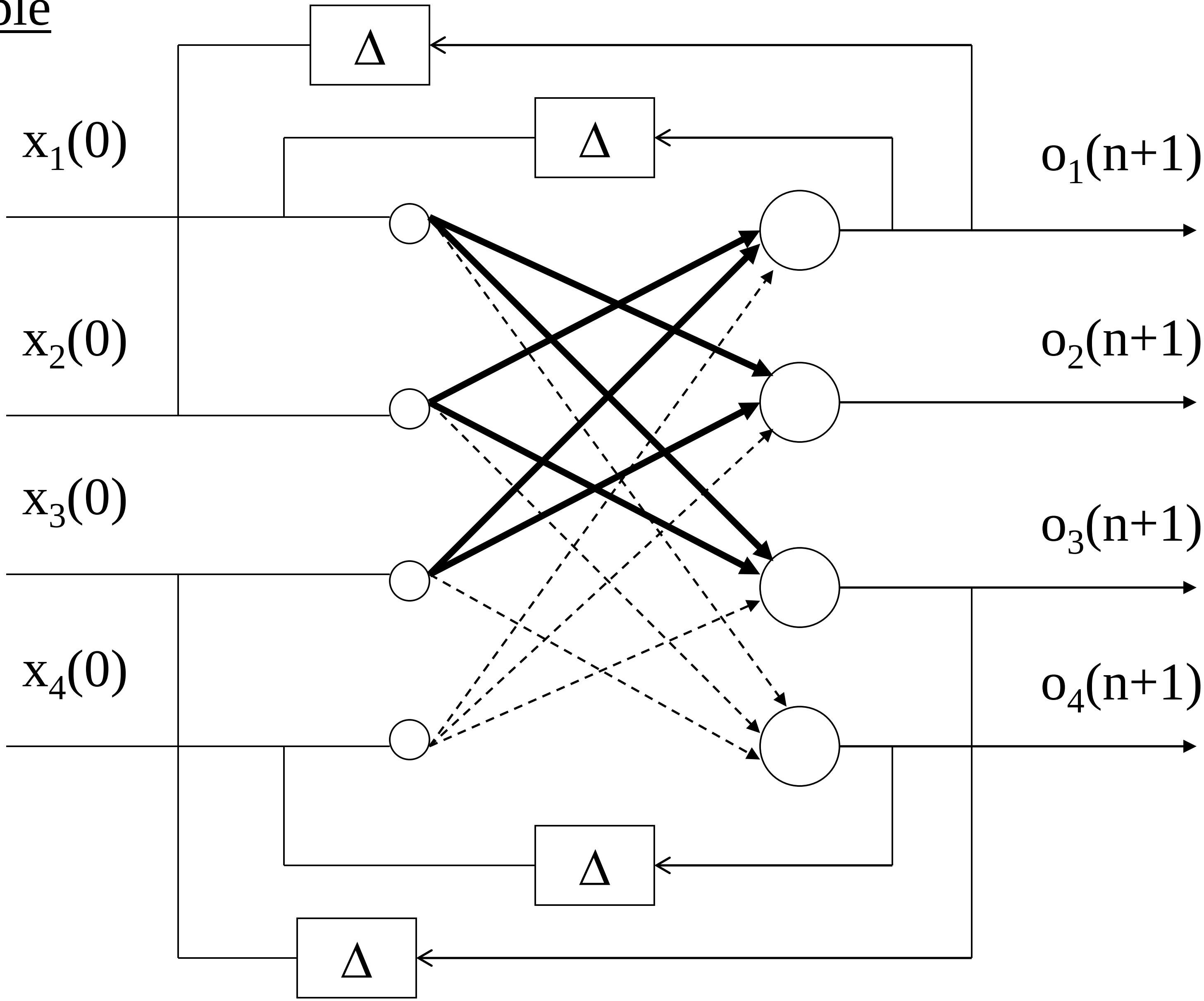
Obviously, the output at time instant n depends on the entire history of the network starting from n=0. The above network is therefore called recurrent neural network. Typically, the recurrent neural network operates with a discrete representation of data and employs neurons with a bi-polar binary activation function.

The above equations describe the outputs of the network at time instants n=1,2,3,…N. The output vector can be considered as the state variables of the network, these equations therefore describe state transitions.

The network begins state transition once the initial state is given. The transition goes on until it reaches to a stable state, called the equilibrium state, as illustrated below:



Example



Where the solid lines denote weight to be 1, the dashed lines to be -1. By inspecting the network, we obtain the following matrix of weight:

$$\mathbf{W} = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$$

If we begin with the following initial setting:

$$\mathbf{x}(0) = \begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix}^T$$

The output vector at time instant n=1 is:

$$\mathbf{o}(1) = \varphi[\mathbf{Wx}(0)] = \varphi(\begin{bmatrix} 3 & 3 & 3 & -3 \end{bmatrix}^T) = \begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix}^T$$

Which is equal to the initial state:

$$\mathbf{o}(1) = \mathbf{x}(0)$$

To compute the state at time instant n=2, we can take the output at time instant n=1 as the initial state setting. Again, we have:

$$\mathbf{o}(2) = \varphi[\mathbf{Wo}(1)] = \begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix}^T = \mathbf{o}(1)$$

This means that no state transition occurs, and $[1\ 1\ 1\ -1]^T$ is an equilibrium state. If we start with another initial state:

$$\mathbf{x}(0) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$$

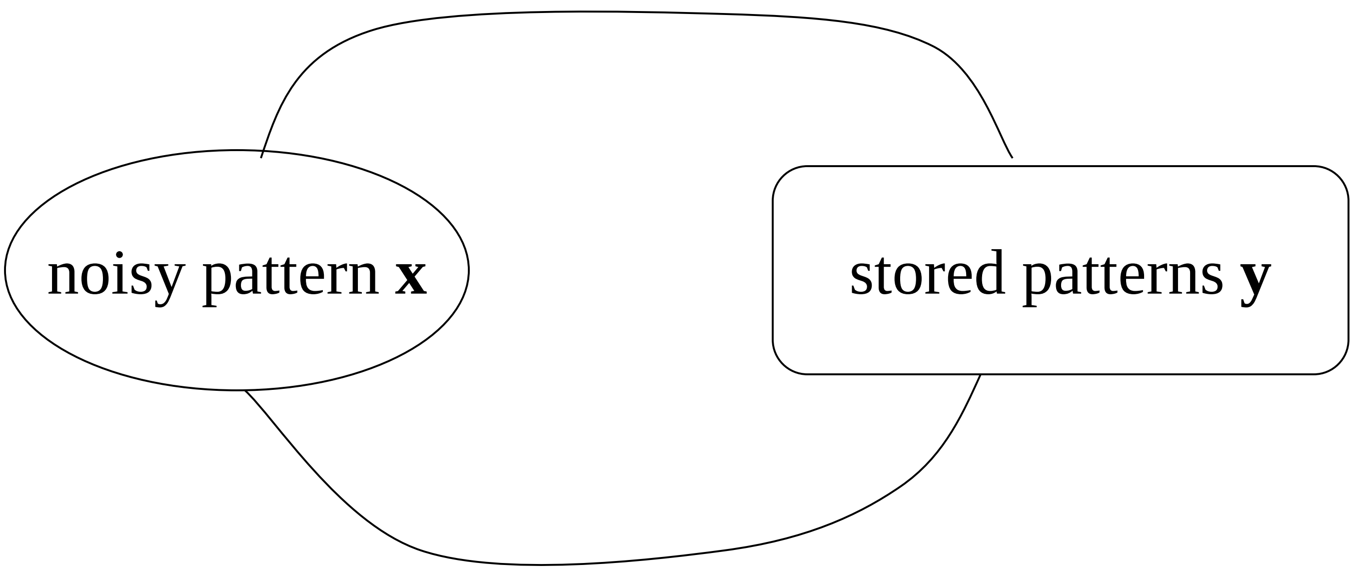
The output at time instant n=1 is:

$$\mathbf{o}(1) = \varphi(\begin{bmatrix} 1 & 1 & 1 & -3 \end{bmatrix}^T) = \begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix}^T$$

The output transits from a non-equilibrium to an equilibrium state. Once the state reaches an equilibrium state, no transitions would occur unless external force is applied to the system.

Hopfield Network

Hopfield network is a typical example of the recurrent network that embodies a profound physical principle of storing information in a dynamically stable configuration. The Hopfield network can also be considered as a nonlinear associative memory, the function of which is to retrieve a pattern stored in memory in response to the presentation of an incomplete or noisy version of that pattern.



Characteristics of the Hopfield Network

(1) Single layer;

(2) The output of each neuron is fed back to the input of other neurons except itself;

(3) The network uses bi-polar binary function as its activation function.

The Hopfield network uses bi-polar binary activation function, thus each such neuron has two states described by the level of activation. The "on" state of a neuron is denoted by the output +1, while the "off" state of a neuron is represented by –1. For a network made up of 4 neurons, a typical example of the state of the network is:

$$\begin{bmatrix} 1 & 1 & -1 & 1 \end{bmatrix}^T$$

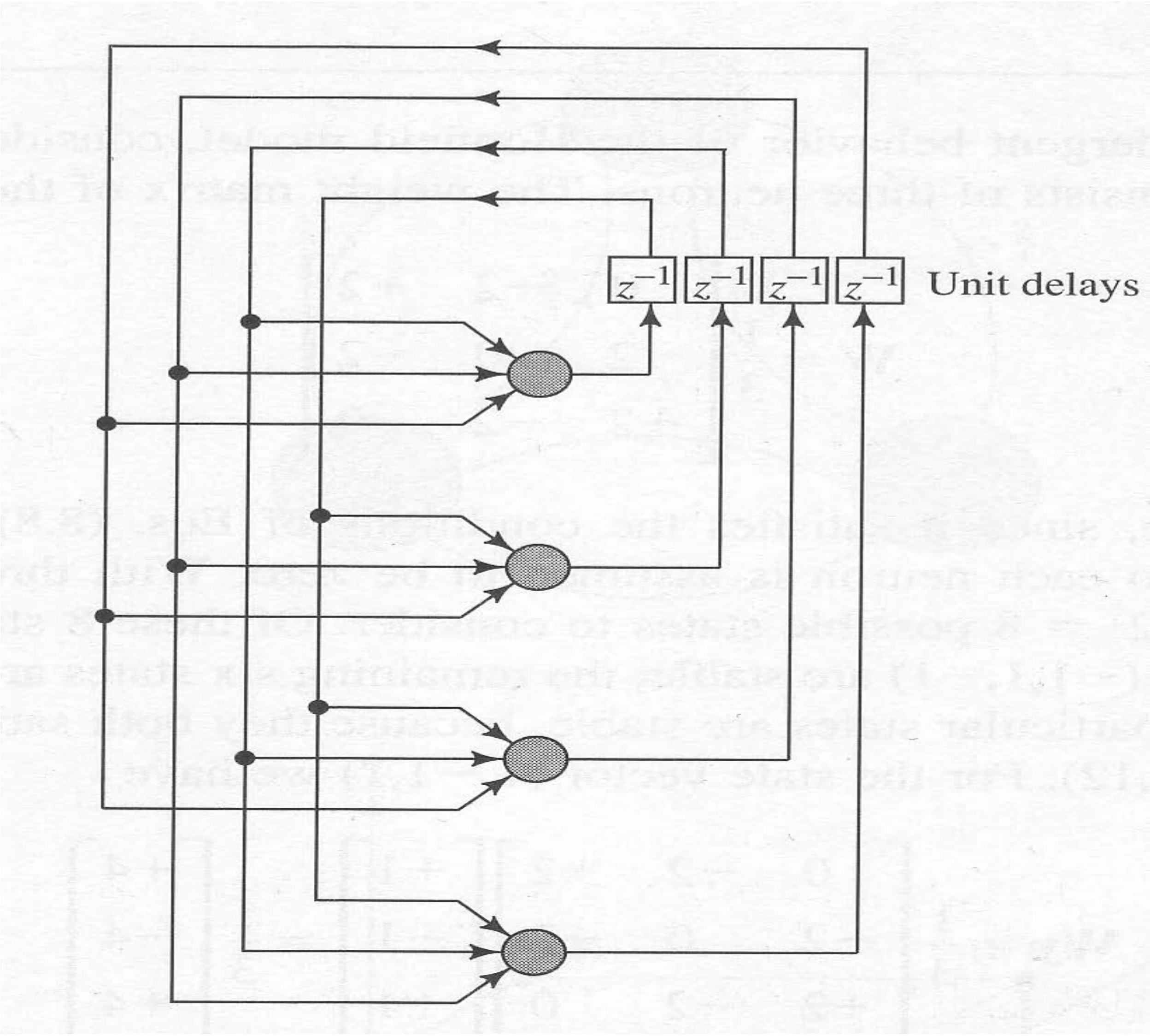


Diagram of the Hopfield network (4 neurons)

Example:

Assume the weight matrix is as follow:

$$\mathbf{W} = \frac{1}{3}\begin{bmatrix} 0 & -2 & 2 \\ -2 & 0 & -2 \\ +2 & -2 & 0 \end{bmatrix}$$

It is assumed that the threshold of the activation function is zero, and the activation function is:

$$\varphi(v) = \begin{cases} +1 & v \geq 0 \\ -1 & v < 0 \end{cases}$$

Assume the initial state is:

$$\mathbf{x}(0) = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T$$

The output at time instant n=1 is then given by:

$$\begin{aligned}
\mathbf{o}(1) &= \varphi[\mathbf{W}\mathbf{x}(0)] \\
&= \varphi(\frac{1}{3}\begin{bmatrix} 4 & -4 & 4 \end{bmatrix}^T) \\
&= \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T \\
&= \mathbf{x}(0)
\end{aligned}$$

This means that the state [1,-1,1] is a stable state.

Let's consider another initial state:

$$\mathbf{x}(0) = \begin{bmatrix} -1 & 1 & -1 \end{bmatrix}^T$$

The output at time instant n=1:

$$\mathbf{o}(1) = \varphi[\mathbf{W}\mathbf{x}(0)] = \varphi(\frac{1}{3}\begin{bmatrix} -4 & 4 & -4 \end{bmatrix}^T) = \begin{bmatrix} -1 & 1 & -1 \end{bmatrix}^T$$

The output at time instant n=2:

$$\mathbf{o}(2) = \varphi[\mathbf{W}\mathbf{o}(1)] = \begin{bmatrix} -1 & 1 & -1 \end{bmatrix}^T = \mathbf{o}(1)$$

Obviously, state transition would no loner occur once the state of the network reaches $[-1\ 1\ -1]^T$. Hence $[-1\ 1\ -1]^T$ is a stable state.

The network has 3 outputs, and each output has 2 states, the total number of states of the network is 8.

(+1 -1 -1), (+1 -1 1), (+1 +1 -1), (+1 +1 +1),

(-1 -1 -1), (-1 -1 +1), (-1 +1 -1), (-1 +1 +1)

Of the 8 possible state, only the two states (1 –1 1) and (-1 1 –1) are stable, the remaining 6 states are all unstable.

For a general case, the condition of stability of a state is:

$$\mathbf{o}(n+1) = \phi[\mathbf{W}\mathbf{o}(n)] = \mathbf{o}(n)$$

For a stable network, the state will eventually move to the stable state.

Energy function

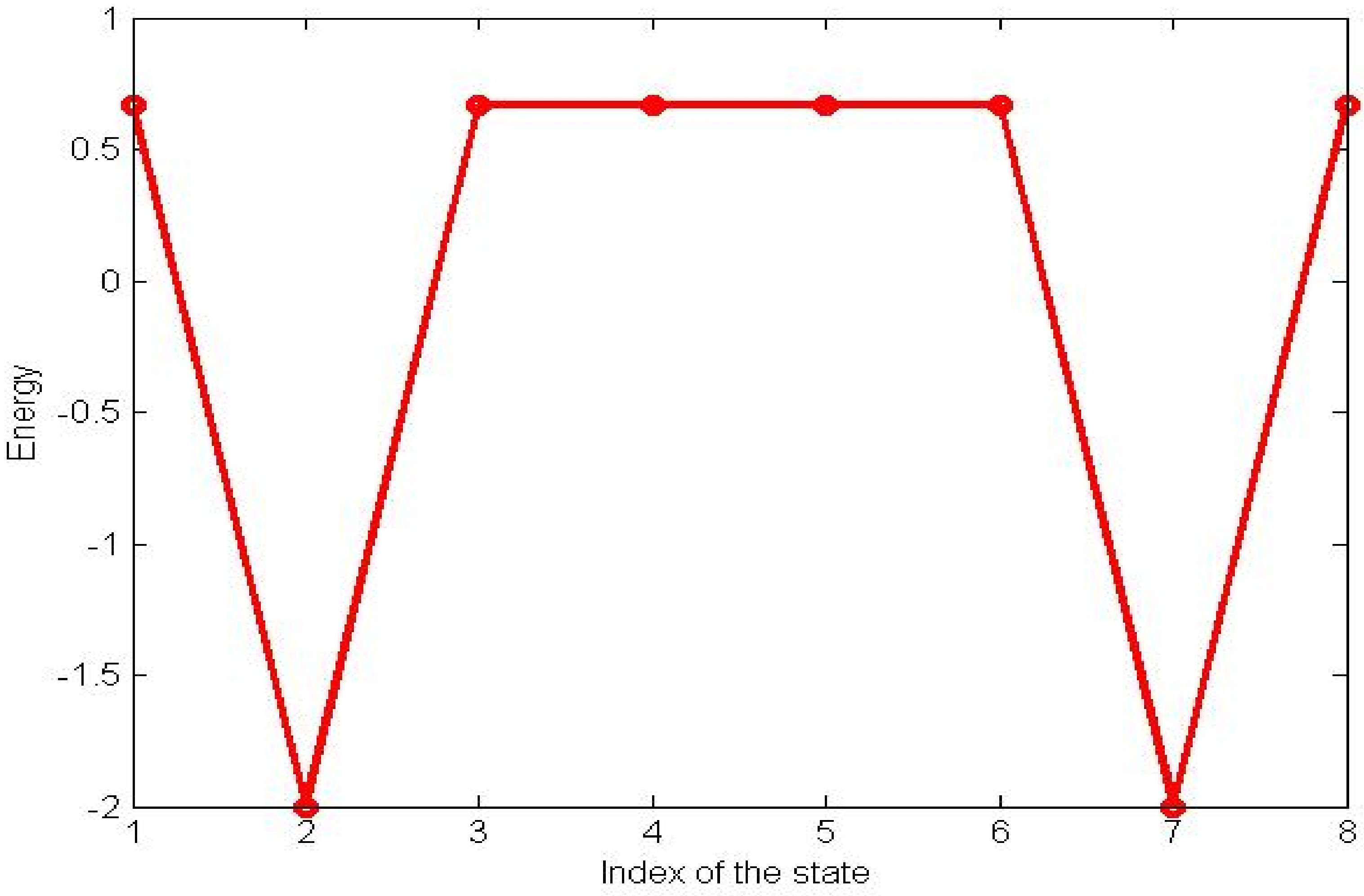
Considerable insight into the Hopfield neural network can be gained by evaluating its respective energy function, which is defined as:

$$E(\mathbf{o}) = -\frac{1}{2}\mathbf{o}^T\mathbf{W}\mathbf{o}$$

Based on the above energy definition, we evaluate the energy of the 8 states. For example, for state [1 -1 -1], the corresponding energy is:

$$E = -\frac{1}{2} \times \begin{bmatrix} 1 & -1 & -1 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} 0 & -2 & 2 \\ -2 & 0 & -2 \\ +2 & -2 & 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = 2/3$$

Similarly, we can evaluate the energy of the remaining 7 states. The energies of the eight states are shown below



Weight determination—Training nonlinear mapping using linear solution technique

The presence of a nonlinear activation function does not necessarily require a nonlinear solution formation. In Hebbian learning, the correlation between the pre-synaptic and post-synaptic signals have been used for weight determination. Next, we will develop a procedure to form W based on the above idea.

Consider a single stimulus-response pair (**s**, **r**), with the d-dimensional stimulus vector **s** normalized to unity length:

$$\|\mathbf{s}\| = \sqrt{\mathbf{s}^T \mathbf{s}} = 1$$

The formation of a weight matrix using the following form is proposed:

$$\mathbf{W} = \mathbf{r}\mathbf{s}^T$$

Next, we examine the storage property of such a weight formation. For an arbitrary stimulus vector **s**, the activation signals of the neurons are given by:

$$\mathbf{W}\mathbf{s} = \mathbf{r}\mathbf{s}^T\mathbf{s} = \mathbf{r}$$

Thus, the output of the Hopfield neural network is obtained by applying the bi-polar binary function on the activation signals:

$$\mathbf{o} = \varphi(\mathbf{W}\mathbf{s}) = \varphi(\mathbf{r}) = \mathbf{r}$$

Next, we examine the storage property under a distorted stimulus. Assuming the distorted version of the stimulus is denoted by **s’,** the activation signals of the neurons are given by:

$$\mathbf{W}\mathbf{s}' = \mathbf{r}\mathbf{s}^T\mathbf{s}' = (\mathbf{s}^T\mathbf{s}')\mathbf{r}$$

For normalized **s** and **s’**, we can verify that:

$$\mathbf{s}^T\mathbf{s}' < \mathbf{s}^T\mathbf{s} = 1$$

(i) Case 1, if

$$0 \le \mathbf{s}^T\mathbf{s}' < 1$$

Then the output is:

$$\mathbf{o} = \varphi(\mathbf{W}\mathbf{s}) = \varphi[(\mathbf{s}^T\mathbf{s}')\mathbf{r}] = \mathbf{r}$$

(ii) Case 2, if

$$\mathbf{s}^T \mathbf{s}' < 0$$

Then the output is:

$$\mathbf{o} = \varphi(\mathbf{W}\mathbf{s}) = \varphi[(\mathbf{s}^T \mathbf{s}')\mathbf{r}] \neq \mathbf{r}$$

This indicates the generalization capability of Hopfield neural network under this learning algorithm. If the stimulus is seriously distorted, the Hopfield neural network may not restore it.

Suppose N d-dimensional vectors (fundamental memories) are to be stored. The N fundamental memories are denoted by:

$$\{\mathbf{s}_{\mathrm{i}} \mid \mathrm{i} = 1, 2, \cdots, \mathrm{N}\}$$

In the normal operation of the Hopfield network, the stimulus and the response are the same, and the neurons have no self-feedback. We set:

$$\mathbf{r}_i = \mathbf{s}_i$$

$$w_{ii} = 0$$

The matrix equation is given by:

$$\mathbf{W} = \frac{1}{d}\sum_{i=1}^{N}\mathbf{s}_i\mathbf{s}_i^T - \frac{N}{d}\mathbf{I}$$

where I denotes the identity matrix with proper dimensionality.

Operational Features of the Hopfield Network

There are two phases to the operation of the Hopfield network, namely, the storage phase and the retrieval phase.

Storage Phase (Design Phase)

(1) The number of neurons is equal to the dimension of vectors to store.
(2) The output of each neuron is fed back to all other neurons.
(3) There is no self-feedback in the network.
(4) The weight matrix of the network is symmetric, and is given by:

$$\mathbf{W} = \frac{1}{d}\sum_{i=1}^{N}\mathbf{s}_i\mathbf{s}_i^T - \frac{N}{d}\mathbf{I}$$

2. Retrieval Phase

During the retrieval phase, an d-dimensional vector **x**, called probe, is imposed on the Hopfield network as its input. The vector has elements +1 or –1. Often the probe is an incomplete or noisy version of a fundamental memory. Information retrieval then proceeds in accordance with dynamic rule, in which the state of the network is updated at each iteration. The updating of the state is continued until no further changes occur:

$$o_j(n+1) = \varphi[\sum_{i=1}^{n} w_{ji} o_i(n)]$$

If we write the above equation in matrix form, we have:

$$\mathbf{o}(n+1) = \varphi[\mathbf{W}\mathbf{o}(n)]$$

<u>3. Summary of the Hopfield network</u>

The operation procedure for the Hopfield network may now be summarized as follows:

Step 1: Construct a Hopfield structure that has *d* neurons, where *d* is the number of the dimension of pattern representation.

Step 2: Learning the weights. Assume the *N* memories are:

$$\{\mathbf{s}_i \mid i = 1, 2, \cdots, N\}$$

$$\mathbf{s}_i = [s_{1i}, s_{2i}, \cdots, s_{di}]^T$$

Then the weight matrix is given by:

$$\mathbf{W} = \frac{1}{d}\sum_{i=1}^{N} \mathbf{s}_i \mathbf{s}_i^T - \frac{N}{d}\mathbf{I}$$

Step 3: Initialization. The initial state of the Hopfield network can be set to the input pattern:

$$\mathbf{o}(0) = \mathbf{x}$$

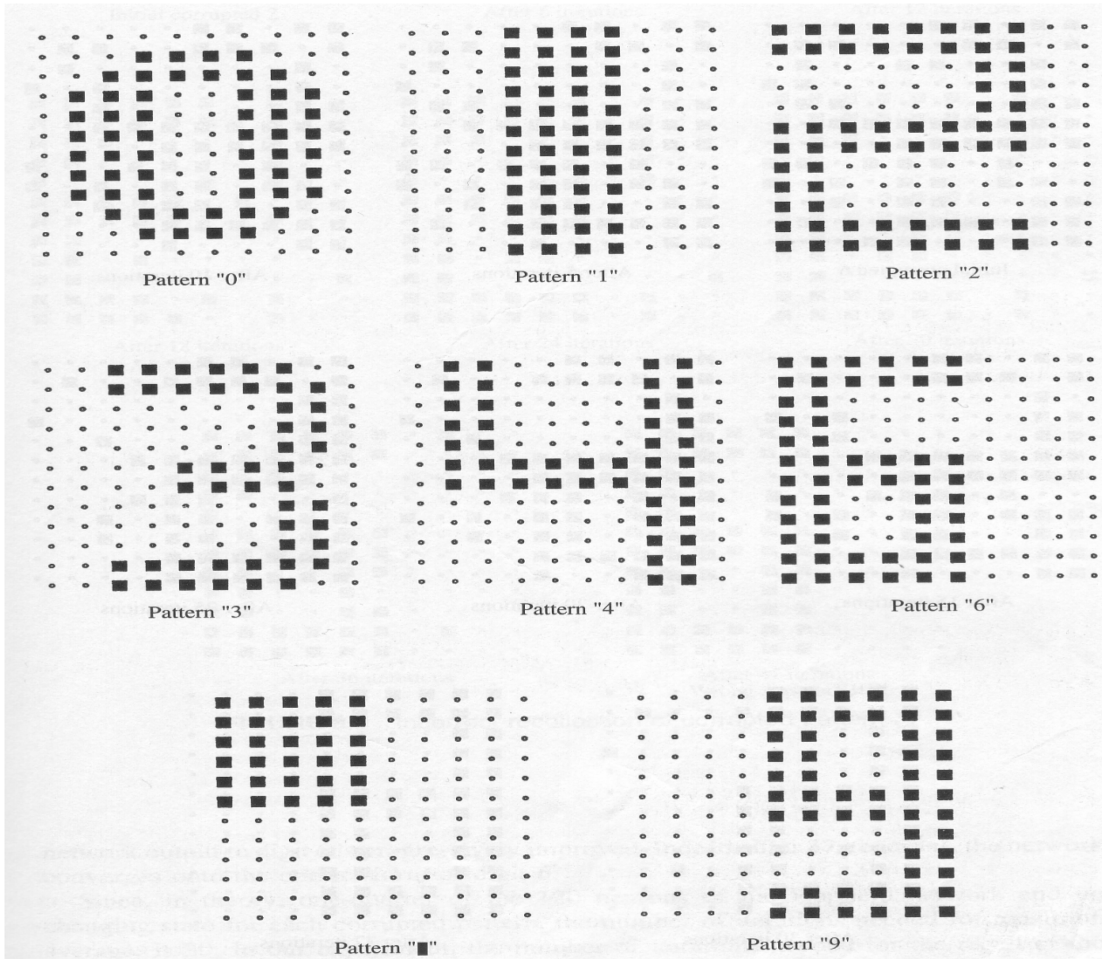
Step 4: Iteration. Update the output of the network according to the following rule:

$$\mathbf{o}(n+1) = \varphi[\mathbf{W}\mathbf{o}(n)]$$

Step 5: Output. The iteration at Step 4 is continued until the state of the network is unchanged. Assume totally L iterations are repeated, the resulting output of the network is given by:

$$\mathbf{y} = \mathbf{o}(\mathrm{L})$$

Example



Totally 8 patterns 0,1,2,3,4,6,.,9

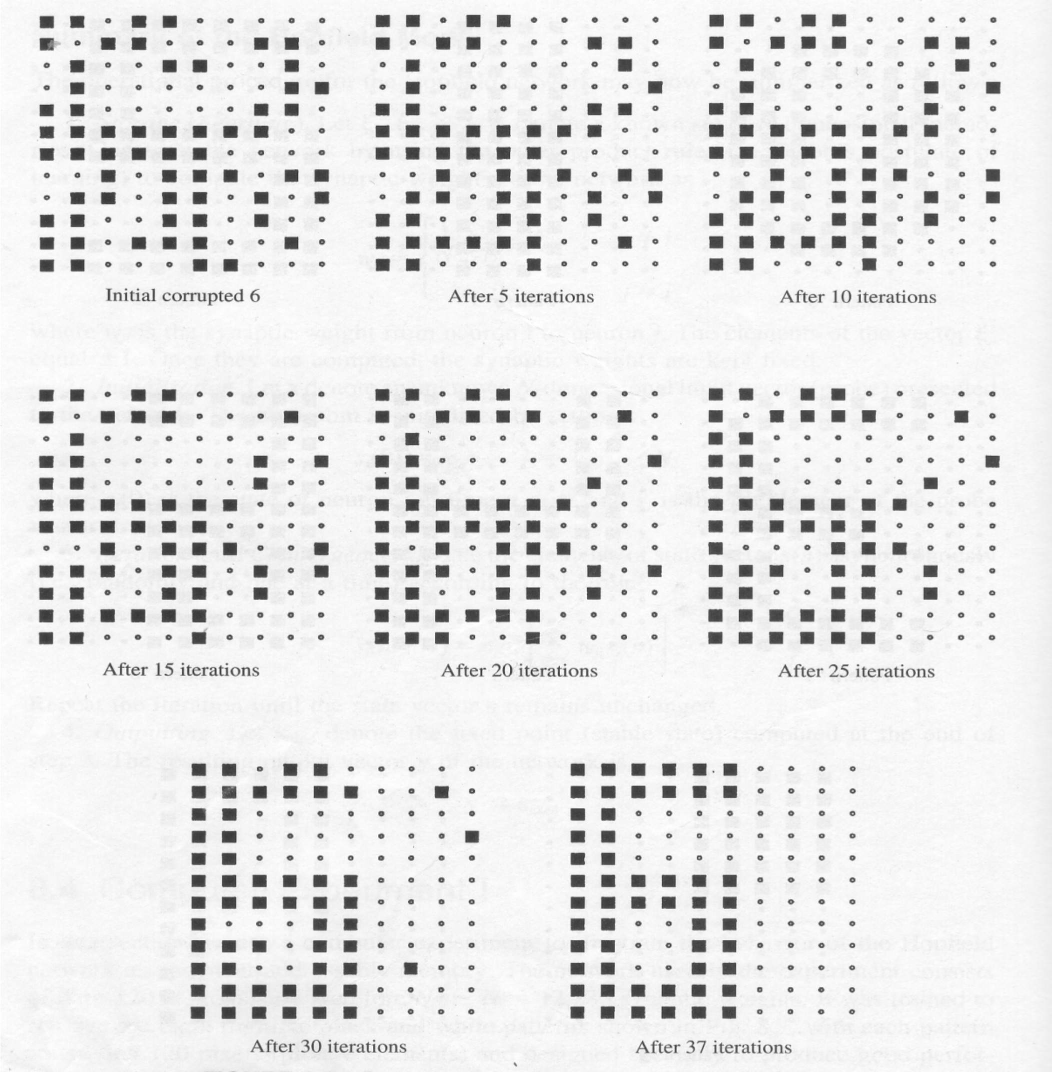
Each number is represented by 120 points, therefore the pattern vector is 120-dimensional, and network should have 120 neurons. The total number of weights is 120*(120-1)=12280.

Procedure to design and retrieve:

Step 1: Define a Hopfield network with 120 neurons.

Step 2: Compute the 12280 weights of the network using the 8 patterns (fundamental memories) provided;

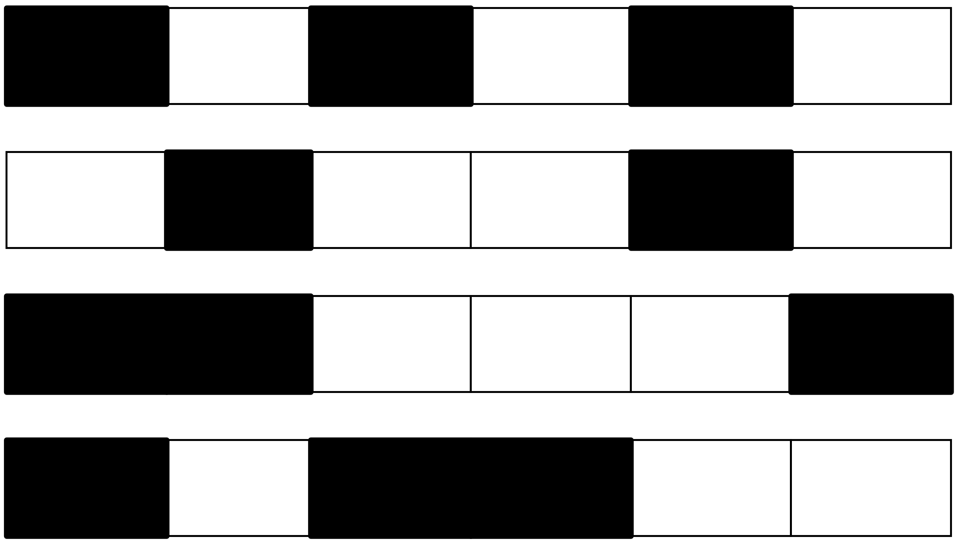
Step 3: Retrieve. Let the initial state be the pattern to be retrieved. Repeatedly update the state of the network until the state is stabilized.



Iteration process

Storage Capacity of the Hopfield Network

Let's first investigate the following example. Assume we have four 6-pixel binary images as shown below:



(a)

(b)

(c)

(d)

Assume the black pixel and the white pixel are represented by +1 and -1 respectively. Design a Hopfield network to store the 4 images. Assume the threshold is zero, the activation function is as follow:

$$\varphi(v) = \begin{cases} +1 & v \geq 0 \\ -1 & v < 0 \end{cases}$$

The four patterns can be represented by the following 4 vectors:

$$\mathbf{s}_1 = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}^T$$

$$\mathbf{s}_2 = \begin{bmatrix} -1 & 1 & -1 & -1 & 1 & -1 \end{bmatrix}^T$$

$$\mathbf{s}_3 = \begin{bmatrix} 1 & 1 & -1 & -1 & -1 & 1 \end{bmatrix}^T$$

$$\mathbf{s}_4 = \begin{bmatrix} 1 & -1 & 1 & 1 & -1 & -1 \end{bmatrix}^T$$

Weight matrix is:

$$\mathbf{W} = \frac{1}{3}\begin{bmatrix} 0 & -1 & 1 & 0 & -1 & 0 \\ -1 & 0 & -2 & -1 & 0 & 1 \\ 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 & -1 & 0 \end{bmatrix}$$

Consider an input pattern $\mathbf{x}=\mathbf{s}_1$, we have:

$$\mathbf{o}(1) = \varphi(\mathbf{Wx}) = \begin{bmatrix}1 & -1 & 1 & 1 & 1 & -1\end{bmatrix}^T$$

$$\mathbf{o}(2) = \varphi[\mathbf{Wo}(1)] = \begin{bmatrix}1 & -1 & 1 & 1 & -1 & -1\end{bmatrix}^T = \mathbf{s}_4$$

$$\mathbf{o}(3) = \varphi[\mathbf{Wo}(2)] = \begin{bmatrix}1 & -1 & 1 & 1 & -1 & -1\end{bmatrix}^T = \mathbf{s}_4$$

Obviously, $\mathbf{x}$ converges to $\mathbf{s}_4$ rather than $\mathbf{s}_1$.

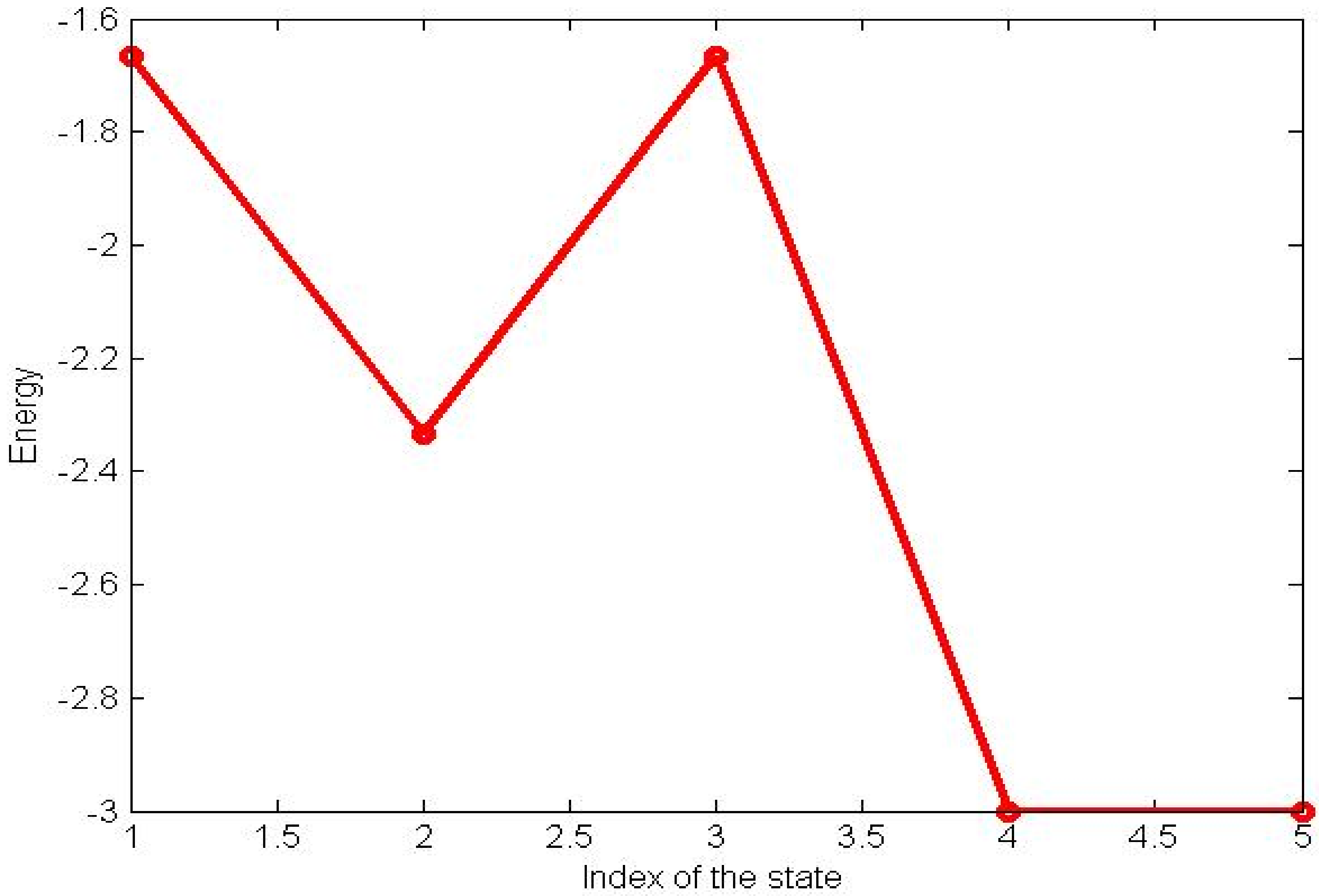
Consider another input pattern $\mathbf{x}=\mathbf{s}_2$, we have:

$$\mathbf{o}(1) = \varphi(\mathbf{Wx}) = \begin{bmatrix}-1 & 1 & -1 & -1 & 1 & 1\end{bmatrix}^T$$

$$\mathbf{o}(2) = \phi[\mathbf{Wo}(1)] = \begin{bmatrix}-1 & 1 & -1 & -1 & 1 & 1\end{bmatrix}^T$$

Obviously, $\mathbf{x}$ will converge to a state that is not a fundamental memory of the Hopfield network built.

The energies corresponding to the 4 states and the spurious state are shown below:



The above example reveals that the storage capacity problem of Hopfield neural networks. Two points related to this issue are:

(1) The fundamental memories of a Hopfield network are not always stable;
(2) Spurious state representing other stable states that are different from the fundamental memories can arise.

These two phenomena tend to decrease the efficiency of Hopfield neural network as a contend-addressable memory. Studies show that the quality of memory retrieval of the Hopfield network deteriorates with the increasing load parameter, which is defined as follow:

$$\alpha = N / d$$

Where N is the number of fundamental memories and d is the number of neurons of the Hopfield network

The storage capacity with error on retrieval is defined as:

$$N_c = 0.14d$$

The storage capacity almost without error on retrieval is defined as:

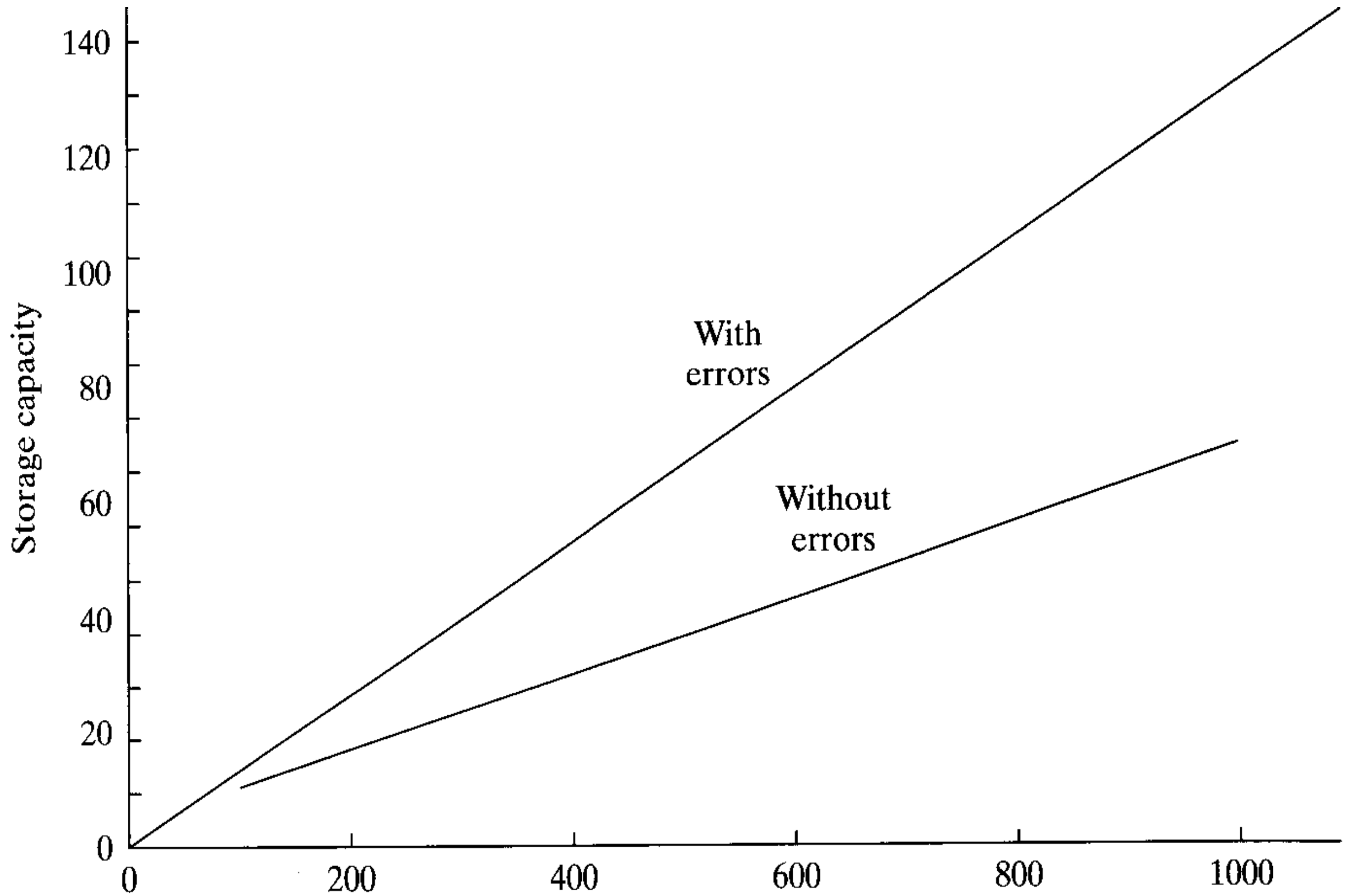
$$N_{\max} = \frac{d}{2\ln(d)}$$

$N_{\max}$ is the largest number of fundamental memories that can be Stored in the network and most of them will be retrieved correctly.

Two conclusions:

(1) Storage capacity of the Hopfield network scales linearly with the size of network;

(2) A major limitation of the Hopfield network is that its storage capacity must be maintained small for the fundamental memories to be recoverable.



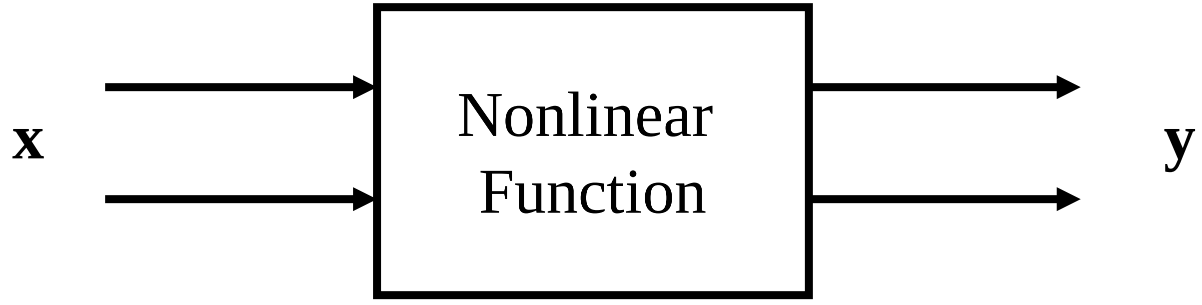
Number of neurons

4. Bi-directional Associative Memories

Recurrent neural network can be used to store information. The is because the dynamic behavior of recurrent neural network exhibits stable state. The iteration procedure actually moves the state from the initial point to the fundamental memory. The fundamental memory is also called prototype or stored memory.

Neural networks of this class are called associative memories. The recurrent neural network is an example of associative memories. An efficient associative memory can store a large set of patterns as memories. The set of patterns to be stored need to be acquired *a priori*.

Essentially, the associative memory is a kind of mapping:



which maps the input pattern from a *d*-dimensional space to a m-dimensional space:

$$\mathbf{x} \in R^d \rightarrow \mathbf{y} \in R^m$$

The associative memories Hopfield neural network just has one layer, and the information transmission is in a single direction. Next we introduce bi-directional associative memories (BAM), which consists of two layers and transmits information in both forward and backward directions.

Bi-directional Associative Memory (BAM)

Bi-directional associative memory consists of two layers. It uses the forward and backward information to produce an associative search for stored association.

Assume N vector association pairs are stored in the memory:

$$(\mathbf{x}_1, \mathbf{y}_1) \quad (\mathbf{x}_2, \mathbf{y}_2) \quad \ldots \quad (\mathbf{x}_\mathrm{d}, \mathbf{y}_\mathrm{d})$$

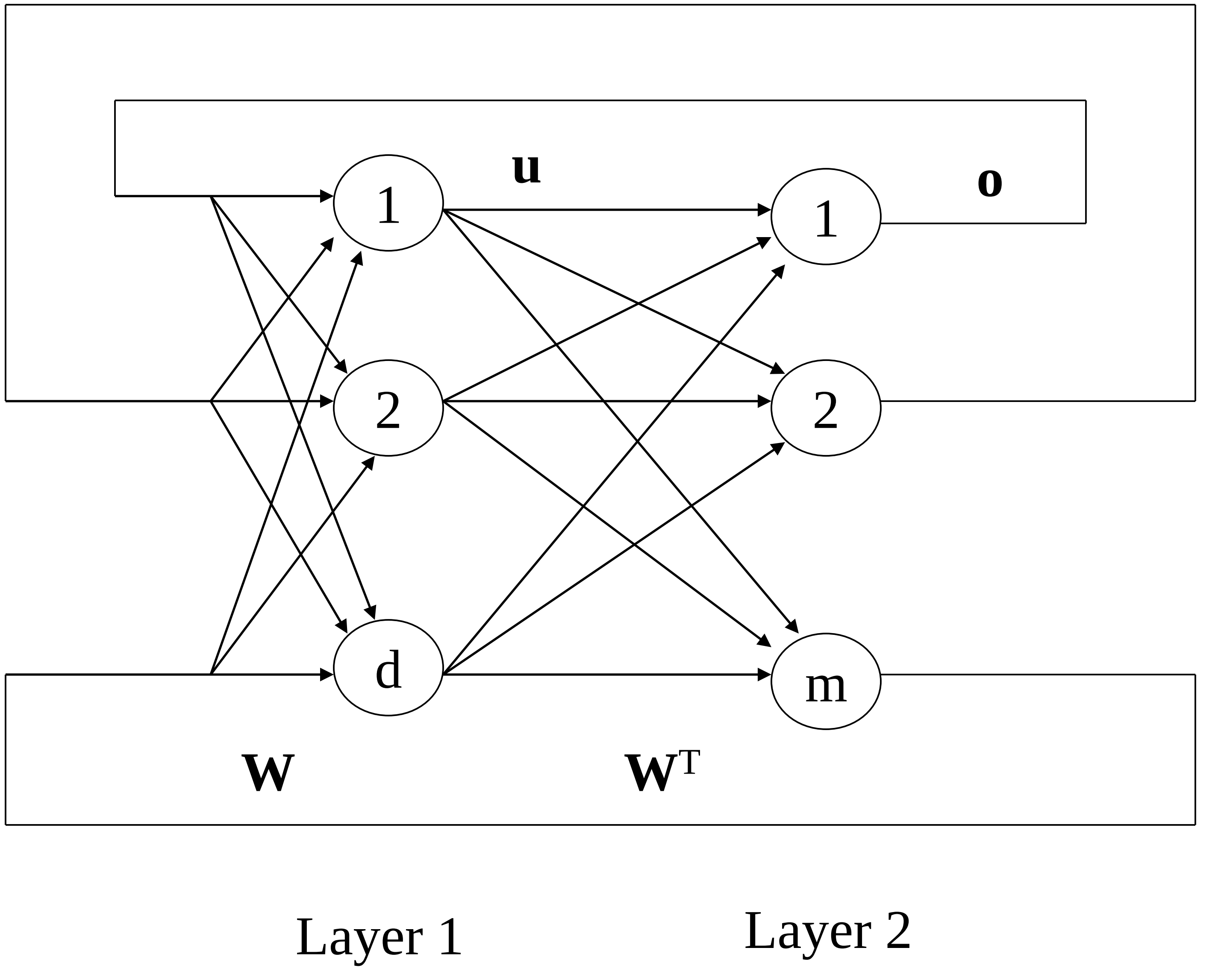
Where **x** and **y** are d-dimensional and m dimensional vectors:

$$\mathbf{x}_k = [x_{k1}, x_{k2}, \cdots, x_{kd}]^T$$

$$\mathbf{y}_k = [y_{k1}, y_{k2}, \cdots, y_{km}]^T$$

When the memory neurons are activated, the network evolves to a stable state of two-pattern reverberation, each pattern at the output of one layer.

The architecture of a BAM



Assume the activation functions of the two layer neurons are bipolar binary. The output of neuron i of the first layer is:

$$u_i = \varphi(\sum_{j=1}^{m} w_{ij} o_j)$$

The output of neuron j of the second layer is:

$$o_j = \varphi(\sum_{i=1}^{n} w_{ij} u_i)$$

In vector form:

$$\mathbf{u} = \varphi(\mathbf{W}\mathbf{o})$$

$$\mathbf{o} = \varphi(\mathbf{W}^T \mathbf{u})$$

Where:

$$\mathbf{u} = [u_1, u_2, \cdots, u_n]^T$$

$$\mathbf{o} = [o_1, o_2, \cdots, o_m]^T$$

Given an input pattern **x**, the iteration starts from the second layer:

$$o_j(1) = \varphi(\sum_{i=1}^{d} w_{ij} x_i)$$

In the second iteration, the first layer gives:

$$u_j(2) = \varphi[\sum_{i=1}^{m} w_{ji} o_i(1)]$$

And the second layer gives:

$$o_j(2) = \varphi[\sum_{i=1}^{d} w_{ij} u_i(2)]$$

The iteration is continued until no updating occurs to all the neurons in the first and the second layers.

The capacity of the bi-directional associative memory depends of the weights including those of the first and the second layers. The weights can be computed using the following formula:

$$w_{ij} = \sum_{k=1}^{N} x_{ki} y_{kj}$$

In matrix form:

$$\mathbf{W} = \sum_{i=1}^{N} \mathbf{x}_i \mathbf{y}_i^T$$

<u>Example 1:</u>

Consider two pairs of patterns

$$\mathbf{x}_1 = [+1,+1,-1]^T \qquad \mathbf{y}_1 = [-1,+1,-1,+1]^T$$

$$\mathbf{x}_2 = [+1,-1,+1]^T \qquad \mathbf{y}_2 = [+1,-1,+1,-1]^T$$

Then the matrix **W** is computed as follow:

$$\mathbf{W} = \sum_{i=1}^{2} \mathbf{x}_i \mathbf{y}_i^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -2 & +2 & -2 & +2 \\ +2 & -2 & +2 & -2 \end{bmatrix}$$

If we have an input pattern **x**:

$$\mathbf{x} = [+1, +1, -1]^T$$

In the first iteration, the output of feedback layer is:

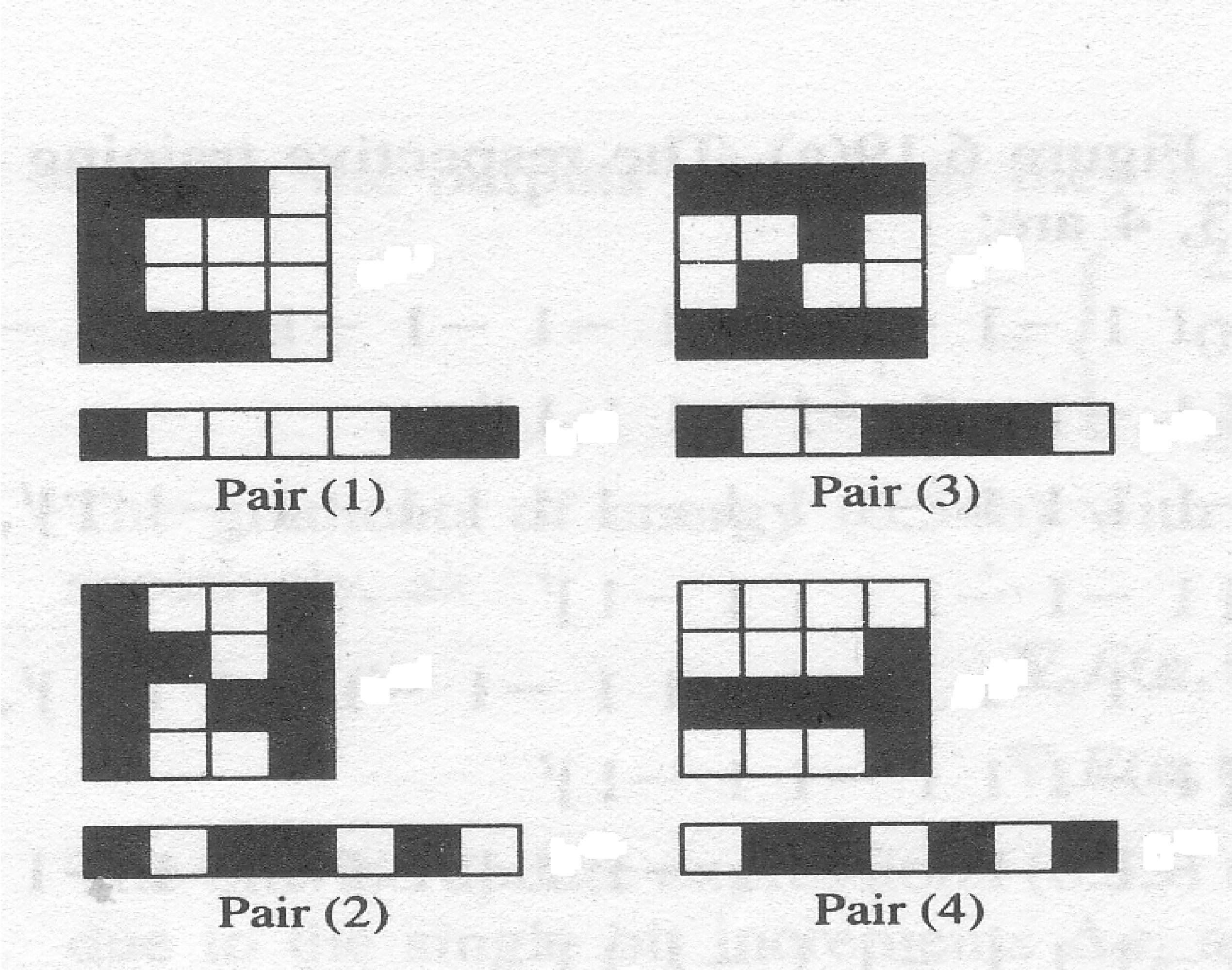
$$\mathbf{o}(1) = \varphi(\mathbf{W}^T \mathbf{x}) = \varphi([-4, +4, -4, +4]^T) = [-1, +1, -1, +1]^T$$

In the second iteration:

$$\mathbf{u}(2) = \varphi[\mathbf{W}\mathbf{o}(1)] = \varphi([0, 8, -8]^T) = [1, 1, -1]^T$$

$$\mathbf{o}(2) = \varphi[\mathbf{W}^T \mathbf{u}(2)] = \varphi([-4,4,-4,4]^T) = [-1,1,-1,1]^T$$

Example 2



The objective is to design a BAM to store the associations.

The association is from a 16-pixel map of letter characters to a 7-bit binary vector.

$$\mathbf{x}_1 = [1,1,1,-1,1,-1,-1,-1,1,-1,-1,-1,1,1,1,-1]^T$$

$$\mathbf{y}_1 = [1,-1,-1,-1,-1,1,1]^T$$

$$\mathbf{x}_2 = [1,-1,-1,1,1,1,-1,1,1,-1,1,1,1,-1,-1,1]^T$$

$$\mathbf{y}_2 = [1,-1,1,1,-1,1,-1]^T$$

$$\mathbf{x}_3 = [1,1,1,1,-1,-1,1,-1,-1,1,-1,-1,1,1,1,1]^T$$

$$\mathbf{y}_3 = [1,-1,-1,1,1,1,-1]^T$$

$$\mathbf{x}_4 = [-1,-1,-1,-1,-1,-1,-1,1,1,1,1,1,-1,-1,-1,1]^T$$

$$\mathbf{y}_4 = [-1,1,1,-1,1,-1,1]^T$$

Because the input and output are vectors of dimensional 16 and 7 respectively, the structure of the BAM is 16×7. The weight matrix is obtained as:

$$\mathbf{W}=\begin{bmatrix} 4 & -4 & -2 & 2 & -2 & 4 & -2 \\ 2 & -2 & -4 & 0 & 0 & 2 & 0 \\ 2 & -2 & -4 & 0 & 0 & 2 & 0 \\ 2 & -2 & 0 & 4 & 0 & 2 & -4 \\ 2 & -2 & 0 & 0 & -4 & 2 & 0 \\ 0 & 0 & 2 & 2 & -2 & 0 & -2 \\ 0 & 0 & -2 & 2 & 2 & 0 & -2 \\ -2 & 2 & 4 & 0 & 0 & -2 & 0 \\ 0 & 0 & 2 & -2 & -2 & 0 & 2 \\ -2 & 2 & 0 & 0 & 4 & -2 & 0 \\ -2 & 2 & 4 & 0 & 0 & -2 & 0 \\ -2 & 2 & 4 & 0 & 0 & -2 & 0 \\ 4 & -4 & -2 & 2 & -2 & 4 & -2 \\ 2 & -2 & -4 & 0 & 0 & 2 & 0 \\ 2 & -2 & -4 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 2 & 2 & 0 & -2 \end{bmatrix}$$

Consider a distorted $\mathbf{x}_1$:

$$\mathbf{x} = [1, 1, 1, 1, 1, -1, -1, -1, 1, -1, -1, -1, 1, 1, 1, -1]^T$$

Then:

$$\mathbf{o}(1) = \varphi(\mathbf{W}^T \mathbf{x}) = [1, -1, -1, -1, -1, 1, 1]^T$$

However, the error tolerating ability of the network is not unlimited.

Consider another distorted $\mathbf{x}_1$:

$$\mathbf{x} = [-1, -1, -1, 1, -1, 1, -1, 1, 1, -1, 1, 1, -1, -1, 1, 1]^T$$

$$\mathbf{o}(1) = \phi(\mathbf{W}^T \mathbf{x}) = [-1, 1, 1, 0, 0, -1, 1]^T$$

$$\mathbf{u}(2) = \phi[\mathbf{W}\mathbf{o}(1)] = [-1, -1, -1, -1, -1, 1, -1, 1, 1, 1, 1, 1, -1, -1, -1, 1]^T$$

$$\mathbf{o}(2) = \phi[\mathbf{W}^T \mathbf{u}(2)] = [-1, 1, 1, -1, 1, -1, 1]^T$$

$$\mathbf{u}(3) = \varphi[\mathbf{Wo}(2)] = [-1,-1,-1,-1,-1,1,-1,1,1,1,1,1,-1,-1,-1,1]^T$$

$$\mathbf{o}(3) = \varphi[\mathbf{W}^T\mathbf{u}(3)] = [-1,1,1,-1,1,-1,1]^T$$

The distorted $\mathbf{x}_1$ converges to $\mathbf{y}_4$.