Propositional Logic

Dave Dubin

Fall Semester, 2018

Reasoning vs. representation

- In this class we'll give some attention to logical reasoning and inference.

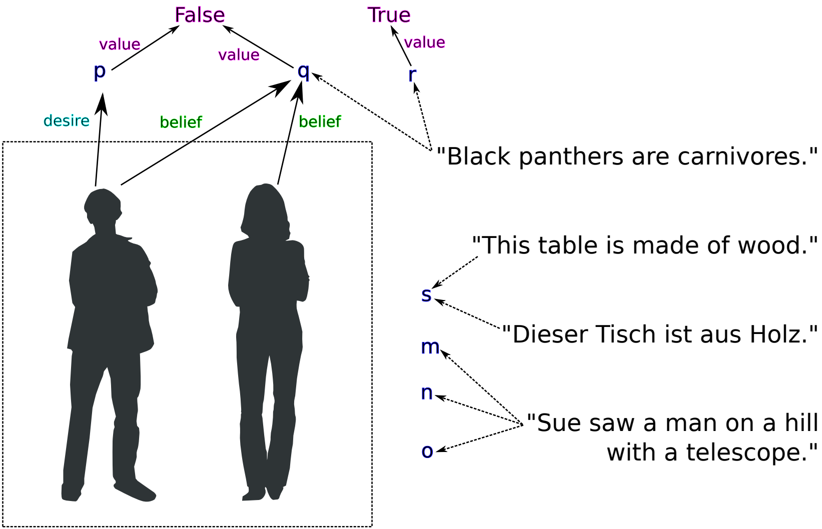
Reasoning vs. representation

- In this class we'll give some attention to logical reasoning and inference.
- The same rules that govern valid argumentation also enable information systems to answer interesting questions.

Reasoning vs. representation

- In this class we'll give some attention to logical reasoning and inference.
- The same rules that govern valid argumentation also enable information systems to answer interesting questions.
- Logic is also a means to describe a *domain* (some part of the world we're interested in).

Propositions, statements, and sentences



Propositions, statements, and sentences

- Propositions are the bearers of truth values: the kinds of things that can be true or false.

Propositions, statements, and sentences

- Propositions are the bearers of truth values: the kinds of things that can be true or false.
- States of affairs are the parts of reality responsible for making propositions true or false.

Propositional logic

- We represent propositions with lower case letters (typically p, q, and r).

Propositional logic

- We represent propositions with lower case letters (typically p, q, and r).
- A set of proposition letters generates a set of states of affairs: ways the world might be.

Propositional logic

- We represent propositions with lower case letters (typically p, q, and r).
- A set of proposition letters generates a set of states of affairs: ways the world might be.
- n proposition letters generate 2^n states of affairs.

Propositional logic

- We represent propositions with lower case letters (typically p, q, and r).
- A set of proposition letters generates a set of states of affairs: ways the world might be.
- n proposition letters generate 2^n states of affairs.
- $\{pqr, pq\overline{r}, p\overline{q}r, p\overline{qr}, \overline{p}qr, \overline{p}q\overline{r}, \overline{pq}r, \overline{pqr}\}$

Logical Operators

Table 1: 2.15 from van Benthem, et al.

Symbol	In natural language	Technical name
$\neg$	not	negation
$\wedge$	and	conjunction
$\vee$	or	disjunction
$\rightarrow$	if ... then	implication
$\leftrightarrow$	if and only if	equivalence

Logical Expressions

- A sentence constructed from proposition letters and operators is true or false in each state of affairs.

Logical Expressions

- A sentence constructed from proposition letters and operators is true or false in each state of affairs.
- Consider, for example: $((\neg p \vee q) \rightarrow r)$

Logical Expressions

- A sentence constructed from proposition letters and operators is true or false in each state of affairs.
- Consider, for example: $((\neg p \vee q) \rightarrow r)$
- The sentence is mapped to a truth value via the following tables

Semantics of the operators

φ	$\neg\varphi$
0	1
1	0

Table 3: 2.18 from van Benthem, et al.

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \to \psi$	$\varphi \leftrightarrow \psi$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

Exercise 2.6

- "I will only go to school if I get a cookie now."

Exercise 2.6

- "I will only go to school if I get a cookie now."
- Compare with "I will go to school if I get a cookie"

Exercise 2.6

- "I will only go to school if I get a cookie now."
- Compare with "I will go to school if I get a cookie"
- The latter is cookie implies school, but the former is school implies cookie.

Exercise 2.6

- "I will only go to school if I get a cookie now."
- Compare with "I will go to school if I get a cookie"
- The latter is cookie implies school, but the former is school implies cookie.
- You want $(s \rightarrow c)$

Exercise 2.6

- "I will only go to school if I get a cookie now."
- Compare with "I will go to school if I get a cookie"
- The latter is cookie implies school, but the former is school implies cookie.
- You want $(s \to c)$
- Gloss: s = "I will go to school" and c = "I get a cookie."

Exercise 2.6

- "I will only go to school if I get a cookie now."
- Compare with "I will go to school if I get a cookie"
- The latter is cookie implies school, but the former is school implies cookie.
- You want $(s \rightarrow c)$
- Gloss: s = "I will go to school" and c = "I get a cookie."
- John and Mary are running

Exercise 2.6

- "I will only go to school if I get a cookie now."
- Compare with "I will go to school if I get a cookie"
- The latter is cookie implies school, but the former is school implies cookie.
- You want $(s \rightarrow c)$
- Gloss: s = "I will go to school" and c = "I get a cookie."
- John and Mary are running
- $(j \wedge m)$

Exercise 2.6

- "I will only go to school if I get a cookie now."
- Compare with "I will go to school if I get a cookie"
- The latter is cookie implies school, but the former is school implies cookie.
- You want $(s \rightarrow c)$
- Gloss: s = "I will go to school" and c = "I get a cookie."
- John and Mary are running
- $(j \wedge m)$
- A foreign national is entitled to social security if he has legal employment or if he has had such less than three years ago, unless he is currently also employed abroad.

Exercise 2.6

- "I will only go to school if I get a cookie now."
- Compare with "I will go to school if I get a cookie"
- The latter is cookie implies school, but the former is school implies cookie.
- You want $(s \rightarrow c)$
- Gloss: s = "I will go to school" and c = "I get a cookie."
- John and Mary are running
- $(j \wedge m)$
- A foreign national is entitled to social security if he has legal employment or if he has had such less than three years ago, unless he is currently also employed abroad.
- $(((e \vee l) \wedge \neg a) \rightarrow s)$

Exercise 2.6

- "I will only go to school if I get a cookie now."
- Compare with "I will go to school if I get a cookie"
- The latter is cookie implies school, but the former is school implies cookie.
- You want $(s \rightarrow c)$
- Gloss: s = "I will go to school" and c = "I get a cookie."
- John and Mary are running
- $(j \wedge m)$
- A foreign national is entitled to social security if he has legal employment or if he has had such less than three years ago, unless he is currently also employed abroad.
- $(((e \vee l) \wedge \neg a) \rightarrow s)$
- $e =$ "A foreign national has legal employment."

Exercise 2.6

- "I will only go to school if I get a cookie now."
- Compare with "I will go to school if I get a cookie"
- The latter is cookie implies school, but the former is school implies cookie.
- You want $(s \to c)$
- Gloss: s = "I will go to school" and c = "I get a cookie."
- John and Mary are running
- $(j \wedge m)$
- A foreign national is entitled to social security if he has legal employment or if he has had such less than three years ago, unless he is currently also employed abroad.
- $(((e \vee l) \wedge \neg a) \to s)$
- e = "A foreign national has legal employment."
- l = "He has had legal employment less than three years ago."

Exercise 2.6

- "I will only go to school if I get a cookie now."
- Compare with "I will go to school if I get a cookie"
- The latter is cookie implies school, but the former is school implies cookie.
- You want $(s \to c)$
- Gloss: s = "I will go to school" and c = "I get a cookie."
- John and Mary are running
- $(j \wedge m)$
- A foreign national is entitled to social security if he has legal employment or if he has had such less than three years ago, unless he is currently also employed abroad.
- $(((e \vee l) \wedge \neg a) \to s)$
- $e =$ "A foreign national has legal employment."
- $l =$ "He has had legal employment less than three years ago."
- $a =$ "He is currently employed abroad."

Exercise 2.6

- "I will only go to school if I get a cookie now."
- Compare with "I will go to school if I get a cookie"
- The latter is cookie implies school, but the former is school implies cookie.
- You want $(s \to c)$
- Gloss: s = "I will go to school" and c = "I get a cookie."
- John and Mary are running
- $(j \land m)$
- A foreign national is entitled to social security if he has legal employment or if he has had such less than three years ago, unless he is currently also employed abroad.
- $(((e \lor l) \land \neg a) \to s)$
- e = "A foreign national has legal employment."
- l = "He has had legal employment less than three years ago."
- a = "He is currently employed abroad."
- s = "He is entitled to social security."

Drawing truth tables for expressions

p	q	r	((	¬	p	∨	q	)	→	r	)
1	1	1		0	1	1	1		1	1	
1	1	0		0	1	1	1		0	0	
1	0	1		0	1	0	0		1	1	
1	0	0		0	1	0	0		1	0	
0	1	1		1	0	1	1		1	1	
0	1	0		1	0	1	1		0	0	
0	0	1		1	0	1	0		1	1	
0	0	0		1	0	1	0		0	0	

Drawing truth tables for expressions

p	q	r	((	¬	p	∨	q	)	→	r	)
1	1	1		0	1	1	1		1	1	
1	1	0		0	1	1	1		0	0	
1	0	1		0	1	0	0		1	1	
1	0	0		0	1	0	0		1	0	
0	1	1		1	0	1	1		1	1	
0	1	0		1	0	1	1		0	0	
0	0	1		1	0	1	0		1	1	
0	0	0		1	0	1	0		0	0	

Drawing truth tables for expressions

p	q	r	((	¬	p	∨	q	)	→	r	)
1	1	1		0	1	1	1		1	1	
1	1	0		0	1	1	1		0	0	
1	0	1		0	1	0	0		1	1	
1	0	0		0	1	0	0		1	0	
0	1	1		1	0	1	1		1	1	
0	1	0		1	0	1	1		0	0	
0	0	1		1	0	1	0		1	1	
0	0	0		1	0	1	0		0	0	

Drawing truth tables for expressions

p	q	r	((	$\neg$	p	$\vee$	q	)	$\rightarrow$	r	)
1	1	1		0	1	1	1		1	1	
1	1	0		0	1	1	1		0	0	
1	0	1		0	1	0	0		1	1	
1	0	0		0	1	0	0		1	0	
0	1	1		1	0	1	1		1	1	
0	1	0		1	0	1	1		0	0	
0	0	1		1	0	1	0		1	1	
0	0	0		1	0	1	0		0	0	

Exercise 2.11

p	q	((p	$\to$	q)	$\vee$	(q	$\to$	p))
1	1	1	1	1	1	1	1	1
1	0	1	0	0	1	0	1	1
0	1	0	1	1	1	1	0	0
0	0	0	1	0	1	0	1	0

Exercise 2.11

p	q	r	$(((p$	$\vee$	$\neg$	$q)$	$\wedge$	$r)$	$\leftrightarrow$	$(\neg$	$(p$	$\wedge$	$r)$	$\vee$	$q))$
1	1	1	1	1	0	1	1	1	1	0	1	1	1	1	1
1	1	0	1	1	0	1	0	0	0	1	1	0	0	1	1
1	0	1	1	1	1	0	1	1	0	0	1	1	1	0	0
1	0	0	1	1	1	0	0	0	0	1	1	0	0	1	0
0	1	1	0	0	0	1	0	1	0	1	0	0	1	1	1
0	1	0	0	0	0	1	0	0	0	1	0	0	0	1	1
0	0	1	0	1	1	0	1	1	1	1	0	0	1	1	0
0	0	0	0	1	1	0	0	0	0	1	0	0	0	1	0

Grammar of propositional logic

Let P be a set of proposition letters and let $p \in P$.

The following expression defines the recursive grammar for a logical expression φ in Backus–Naur Form:

$$\varphi \boldsymbol{p} | \neg\varphi | (\varphi \wedge \varphi) | (\varphi \vee \varphi) | (\varphi \rightarrow \varphi) | (\varphi \leftrightarrow \varphi)$$

Syntactically conforming expressions

Let $P = \{o, q, r, s\}$

Examples of grammatically conforming expressions include:

- r

Grammatically *incorrect* expressions would include:

How many correct expressions are consistent with the last one?

Syntactically conforming expressions

Let $P = \{o, q, r, s\}$

Examples of grammatically conforming expressions include:

- r
- $\neg q$

Grammatically *incorrect* expressions would include:

How many correct expressions are consistent with the last one?

Syntactically conforming expressions

Let $P = \{o, q, r, s\}$

Examples of grammatically conforming expressions include:

- r
- $\neg q$
- $(s \leftrightarrow o)$

Grammatically *incorrect* expressions would include:

How many correct expressions are consistent with the last one?

Syntactically conforming expressions

Let $P = \{o, q, r, s\}$

Examples of grammatically conforming expressions include:

- r
- $\neg q$
- $(s \leftrightarrow o)$
- $(\neg(s \leftrightarrow \neg\neg\neg o) \rightarrow (q \wedge q))$

Grammatically *incorrect* expressions would include:

How many correct expressions are consistent with the last one?

Syntactically conforming expressions

Let $P = \{o, q, r, s\}$

Examples of grammatically conforming expressions include:

- r
- $\neg q$
- $(s \leftrightarrow o)$
- $(\neg(s \leftrightarrow \neg\neg\neg o) \rightarrow (q \wedge q))$

Grammatically *incorrect* expressions would include:

- $\neg \vee p$

How many correct expressions are consistent with the last one?

Syntactically conforming expressions

Let $P = \{o, q, r, s\}$

Examples of grammatically conforming expressions include:

- r
- $\neg q$
- $(s \leftrightarrow o)$
- $(\neg(s \leftrightarrow \neg\neg\neg o) \rightarrow (q \wedge q))$

Grammatically *incorrect* expressions would include:

- $\neg \vee p$
- $\vee) p \neg$

How many correct expressions are consistent with the last one?

Syntactically conforming expressions

Let $P = \{o, q, r, s\}$

Examples of grammatically conforming expressions include:

- r
- $\neg q$
- $(s \leftrightarrow o)$
- $(\neg(s \leftrightarrow \neg\neg\neg o) \rightarrow (q \wedge q))$

Grammatically *incorrect* expressions would include:

- $\neg \vee p$
- $\vee)p\neg$
- $\neg p \vee q \rightarrow r$

How many correct expressions are consistent with the last one?

Syntactic ambiguity

These conforming expressions are all consistent with $\neg p \vee q \rightarrow r$

- $((\neg p \vee q) \rightarrow r)$

Syntactic ambiguity

These conforming expressions are all consistent with $\neg p \vee q \rightarrow r$

- $((\neg p \vee q) \rightarrow r)$
- $(\neg(p \vee q) \rightarrow r)$

Syntactic ambiguity

These conforming expressions are all consistent with $\neg p \lor q \rightarrow r$

- $((\neg p \lor q) \rightarrow r)$
- $(\neg(p \lor q) \rightarrow r)$
- $\neg((p \lor q) \rightarrow r)$

Syntactic ambiguity

These conforming expressions are all consistent with $\neg p \vee q \rightarrow r$

- $((\neg p \vee q) \rightarrow r)$
- $(\neg(p \vee q) \rightarrow r)$
- $\neg((p \vee q) \rightarrow r)$
- $(\neg p \vee (q \rightarrow r))$

Syntactic ambiguity

These conforming expressions are all consistent with $\neg p \vee q \rightarrow r$

- $((\neg p \vee q) \rightarrow r)$
- $(\neg(p \vee q) \rightarrow r)$
- $\neg((p \vee q) \rightarrow r)$
- $(\neg p \vee (q \rightarrow r))$
- $\neg(p \vee (q \rightarrow r))$

Exercise 2.7

Which of the following are formulas in propositional logic?

- $p \rightarrow \neg q$

Exercise 2.7

Which of the following are formulas in propositional logic?

- $p \rightarrow \neg q$
- $\neg\neg \wedge q \vee p$

Exercise 2.7

Which of the following are formulas in propositional logic?

- $p \rightarrow \neg q$
- $\neg\neg \wedge q \vee p$
- $p\neg q$

More on semantics

- Semantics is the relationship of a language to the part of the world that we're modeling.

More on semantics

- Semantics is the relationship of a language to the part of the world that we're modeling.
- Valuations are functions from expressions to truth values.

More on semantics

- Semantics is the relationship of a language to the part of the world that we're modeling.
- Valuations are functions from expressions to truth values.
- "$V(\varphi) = 1$" means the formula (or sentence) φ is true in the state of affairs represented by the function V. "$V(\varphi) = 0$" means that φ is false in the state of affairs represented by the function V.

More on semantics

- Semantics is the relationship of a language to the part of the world that we're modeling.
- Valuations are functions from expressions to truth values.
- "$V(\varphi) = 1$" means the formula (or sentence) φ is true in the state of affairs represented by the function V. "$V(\varphi) = 0$" means that φ is false in the state of affairs represented by the function V.
- For "$V(\varphi) = 1$" we also write "$V \models \varphi$" read as "V is a model of φ" or "V satisfies φ."

More on semantics

- Semantics is the relationship of a language to the part of the world that we're modeling.
- Valuations are functions from expressions to truth values.
- "$V(\varphi) = 1$" means the formula (or sentence) φ is true in the state of affairs represented by the function V. "$V(\varphi) = 0$" means that φ is false in the state of affairs represented by the function V.
- For "$V(\varphi) = 1$" we also write "$V \models \varphi$" read as "V is a model of φ" or "V satisfies φ."
- If V doesn't satisfy φ we write "$V \not\models \varphi$". In other words $V(\varphi) = 0$.

Logical truth and logical falsity.

- A statement φ is logically true if it is true in every state of affairs generated by its propositional variables.

Logical truth and logical falsity.

- A statement φ is logically true if it is true in every state of affairs generated by its propositional variables.
- A statement φ is logically false if it is false in every state of affairs generated by its propositional variables.

Logical truth and logical falsity.

- A statement φ is logically true if it is true in every state of affairs generated by its propositional variables.
- A statement φ is logically false if it is false in every state of affairs generated by its propositional variables.
- If a statement φ is neither logically true or logically false then it is contingent.

Logical truth and logical falsity.

- A statement φ is logically true if it is true in every state of affairs generated by its propositional variables.
- A statement φ is logically false if it is false in every state of affairs generated by its propositional variables.
- If a statement φ is neither logically true or logically false then it is contingent.
- Examples:

Logical truth and logical falsity.

- A statement φ is logically true if it is true in every state of affairs generated by its propositional variables.
- A statement φ is logically false if it is false in every state of affairs generated by its propositional variables.
- If a statement φ is neither logically true or logically false then it is contingent.
- Examples:
  1. $(q \vee \neg q)$ is logically true.

Logical truth and logical falsity.

- A statement φ is logically true if it is true in every state of affairs generated by its propositional variables.
- A statement φ is logically false if it is false in every state of affairs generated by its propositional variables.
- If a statement φ is neither logically true or logically false then it is contingent.
- Examples:
  1. $(q \vee \neg q)$ is logically true.
  2. $(q \wedge \neg q)$ is logically false.

Consistency

- A set of propositional logic statements is consistent if at least one state of affairs satisfies every statement in the set.

Consistency

- A set of propositional logic statements is consistent if at least one state of affairs satisfies every statement in the set.
- A set of propositional logic statements is inconsistent if no state of affairs satisfies every statement in the set.

Inference and validity

- A conclusion is *valid* with respect to a set of premises if the conclusion is true in every sitation where the premises are true (van Benthem, et al, page 2-4).

Inference and validity

- A conclusion is *valid* with respect to a set of premises if the conclusion is true in every sitation where the premises are true (van Benthem, et al, page 2-4).
- One can validly infer a conclusion φ from a set of premises P if the negation of φ is inconsistent with the set of statements P.

Computation and expressive power

(From van Benthem, et al., chapter 2)

1. Computing a truth value for a formula takes linear time.

Computation and expressive power

(From van Benthem, et al., chapter 2)

1. Computing a truth value for a formula takes linear time.
2. Computing a truth table for validity takes exponential time.

Computation and expressive power

(From van Benthem, et al., chapter 2)

1. Computing a truth value for a formula takes linear time.
2. Computing a truth table for validity takes exponential time.
3. The problem of testing for validity in propositional logic is decidable: there exists a mechanical method that computes the answer, at least in principle.