Propositional Logic

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- A set of proposition letters generates a set of states of affairs: ways the world might be.
- n proposition letters generate 2^n states of affairs.
- $\{pqr, pq\overline{r}, p\overline{q}r, p\overline{qr}, \overline{p}qr, \overline{p}q\overline{r}, \overline{pq}r, \overline{pqr}\}$

Logical Operators

Table 1: 2.15 from van Benthem, et al.

Symbol	In natural language	Technical name
$\neg$	not	negation
$\wedge$	and	conjunction
$\vee$	or	disjunction
$\rightarrow$	if ... then	implication
$\leftrightarrow$	if and only if	equivalence

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- The sentence is mapped to a truth value via the following tables

Semantics of the operators

ϕ	$\neg\phi$
0	1
1	0

Table 3: 2.18 from van Benthem, et al.

ϕ	ψ	$\phi \land \psi$	$\phi \lor \psi$	$\phi \to \psi$	$\phi \leftrightarrow \psi$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

Exercise 2.3

You are given the information that p-or-q and (not-p)-or-r. What can you conclude about q and r? What is the strongest valid conclusion you can draw? (A statement is stronger than another statement if it rules out more possibilities.)

Drawing truth tables for expressions

p	q	r	(($\sim$	p	$\vee$	q)	$\rightarrow$	r)
1	1	1	0	1	1	1	1	1
1	1	0	0	1	1	1	0	0
1	0	1	0	1	0	0	1	1
1	0	0	0	1	0	0	1	0
0	1	1	1	0	1	1	1	1
0	1	0	1	0	1	1	0	0
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0	1	1	1	0	1	1	1	1
0	1	0	1	0	1	1	0	0
0	0	1	1	0	1	0	1	1
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1	1	0		0	1	1	1		0	0	
1	0	1		0	1	0	0		1	1	
1	0	0		0	1	0	0		1	0	
0	1	1		1	0	1	1		1	1	
0	1	0		1	0	1	1		0	0	
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0	1	1		1	0	1	1		1	1	
0	1	0		1	0	1	1		0	0	
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Drawing truth tables for expressions

```
   (~p v q) -> r
------------------
      0    0      0
      0    0      1
      0    1      0
      0    1      1
      1    0      0
      1    0      1
      1    1      0
      1    1      1
```

Drawing truth tables for expressions

```
   (~p v q) -> r
-------------------
     10   0      0
     10   0      1
     10   1      0
     10   1      1
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Grammar of propositional logic

Let P be a set of proposition letters and let $p \in P$.
The following expression defines the recursive grammar for a logical expression ϕ in Backus–Naur Form:
$\phi ::= p|\neg\phi|(\phi \wedge \phi)|(\phi \vee \phi)|(\phi \rightarrow \phi)|(\phi \leftrightarrow \phi)$

Syntactically conforming expressions

Let $P = \{o, q, r, s\}$
Examples of grammatically conforming expressions include:

- r

Grammatically *incorrect* expressions would include:

How many correct expressions are consistent with the last one?

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- $\neg p \vee q \rightarrow r$

How many correct expressions are consistent with the last one?

Syntactic ambiguity

These conforming expressions are all consistent with $\neg p \lor q \to r$

- $((\neg p \lor q) \to r)$

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- $\neg((p \vee q) \rightarrow r)$
- $(\neg p \vee (q \rightarrow r))$

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- $(\neg p \vee (q \rightarrow r))$
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- For "$V(\phi) = 1$" we also write "$V \models \phi$" read as "V is a model of ϕ" or "V satisfies ϕ."
- If V doesn't satisfy ϕ we write "$V \not\models \phi$". In other words $V(\phi) = 0$.

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- A set of propositional logic statements is consistent if at least one state of affairs satisfies every statement in the set.
- A set of propositional logic statements is inconsistent if no state of affairs satisfies every statement in the set.

Inference and validity

- A conclusion is *valid* with respect to a set of premises if the conclusion is true in every sitation where the premises are true (van Benthem, et al, page 2-4).

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- A conclusion is *valid* with respect to a set of premises if the conclusion is true in every sitation where the premises are true (van Benthem, et al, page 2-4).
- One can validly infer a conclusion ϕ from a set of premises P if the negation of ϕ is inconsistent with the set of statements P.

Computation and expressive power

(From van Bentham, et al., chapter 2)

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1. Computing a truth value for a formula takes linear time.
2. Computing a truth table for validity takes exponential time.
3. The problem of testing for validity in propositional logic is decidable: there exists a mechanical method that computes the answer, at least in principle.