Sets, relations, and functions

Dave Dubin

February 20, 2017

Sets we've already encountered

- Propositional logic: A set of n proposition letters generates a set of 2^n states of affairs: ways the world might be.

Sets we've already encountered

- Propositional logic: A set of n proposition letters generates a set of 2^n states of affairs: ways the world might be.
- $\{p, q, r\}$ generates $\{pqr, pq\overline{r}, p\overline{q}r, p\overline{qr}, \overline{p}qr, \overline{p}q\overline{r}, \overline{pq}r, \overline{pqr}\}$

Sets we've already encountered

- Propositional logic: A set of n proposition letters generates a set of 2^n states of affairs: ways the world might be.
- $\{p, q, r\}$ generates $\{pqr, pq\overline{r}, p\overline{q}r, p\overline{qr}, \overline{p}qr, \overline{p}q\overline{r}, \overline{pq}r, \overline{pqr}\}$
- Grammar: let P be a set of proposition letters and let $p \in P$.

Sets we've already encountered

- Propositional logic: A set of n proposition letters generates a set of 2^n states of affairs: ways the world might be.
- $\{p, q, r\}$ generates $\{pqr, pq\overline{r}, p\overline{q}r, p\overline{qr}, \overline{p}qr, \overline{p}q\overline{r}, \overline{pq}r, \overline{pqr}\}$
- Grammar: let P be a set of proposition letters and let $p \in P$.
- Production set: $\varphi ::= p|\neg\varphi|(\varphi \wedge \varphi)|(\varphi \vee \varphi)|(\varphi \rightarrow \varphi)|(\varphi \leftrightarrow \varphi)$

Sets we've already encountered

- Propositional logic: A set of n proposition letters generates a set of 2^n states of affairs: ways the world might be.
- $\{p, q, r\}$ generates $\{pqr, pq\overline{r}, p\overline{q}r, p\overline{qr}, \overline{p}qr, \overline{p}q\overline{r}, \overline{pq}r, \overline{pqr}\}$
- Grammar: let P be a set of proposition letters and let $p \in P$.
- Production set: $\varphi ::= p|\neg\varphi|(\varphi \wedge \varphi)|(\varphi \vee \varphi)|(\varphi \rightarrow \varphi)|(\varphi \leftrightarrow \varphi)$
- Think $\{\varphi ::= p, \varphi ::= \neg\varphi, \varphi ::= (\varphi \wedge \varphi), \varphi ::= (\varphi \vee \varphi), \ldots\}$

Sets we've already encountered

- Propositional logic: A set of n proposition letters generates a set of 2^n states of affairs: ways the world might be.
- $\{p, q, r\}$ generates $\{pqr, pq\overline{r}, p\overline{q}r, p\overline{qr}, \overline{p}qr, \overline{p}q\overline{r}, \overline{pq}r, \overline{pqr}\}$
- Grammar: let P be a set of proposition letters and let $p \in P$.
- Production set: $\varphi ::= p|\neg\varphi|(\varphi \wedge \varphi)|(\varphi \vee \varphi)|(\varphi \rightarrow \varphi)|(\varphi \leftrightarrow \varphi)$
- Think $\{\varphi ::= p, \varphi ::= \neg\varphi, \varphi ::= (\varphi \wedge \varphi), \varphi ::= (\varphi \vee \varphi), \ldots\}$
- Logical consistency: A set of propositional logic statements is consistent if at least one state of affairs satisfies every statement in the set.

Sets we've already encountered

- Propositional logic: A set of n proposition letters generates a set of 2^n states of affairs: ways the world might be.
- $\{p, q, r\}$ generates $\{pqr, pq\overline{r}, p\overline{q}r, p\overline{qr}, \overline{p}qr, \overline{p}q\overline{r}, \overline{pq}r, \overline{pqr}\}$
- Grammar: let P be a set of proposition letters and let $p \in P$.
- Production set: $\varphi ::= p|\neg\varphi|(\varphi \wedge \varphi)|(\varphi \vee \varphi)|(\varphi \rightarrow \varphi)|(\varphi \leftrightarrow \varphi)$
- Think $\{\varphi ::= p, \varphi ::= \neg\varphi, \varphi ::= (\varphi \wedge \varphi), \varphi ::= (\varphi \vee \varphi), \ldots\}$
- Logical consistency: A set of propositional logic statements is consistent if at least one state of affairs satisfies every statement in the set.
- Inference: a conclusion is *valid* with respect to a set of premises if the conclusion is true in every sitation where the premises are true (van Benthem, et al, section 2-4).

From our syllabus

The course is thus simultaneously a foundations course and a survey course. There are several important cross-cutting themes:

- Data independence through abstraction.

From our syllabus

The course is thus simultaneously a foundations course and a survey course. There are several important cross-cutting themes:

- Data independence through abstraction.
- The interdefinability of fundamental modeling constructs.

From our syllabus

The course is thus simultaneously a foundations course and a survey course. There are several important cross-cutting themes:

- Data independence through abstraction.
- The interdefinability of fundamental modeling constructs.
- Deep vs. superficial differences in modeling languages.

From our syllabus

The course is thus simultaneously a foundations course and a survey course. There are several important cross-cutting themes:

- Data independence through abstraction.
- The interdefinability of fundamental modeling constructs.
- Deep vs. superficial differences in modeling languages.
- The expressiveness vs. tractability tradeoff.

From our syllabus

The course is thus simultaneously a foundations course and a survey course. There are several important cross-cutting themes:

- Data independence through abstraction.
- The interdefinability of fundamental modeling constructs.
- Deep vs. superficial differences in modeling languages.
- The expressiveness vs. tractability tradeoff.
- The fundamental role of a very small set of inter-related concepts.

Formal set-theoretic accounts

- It's important to our authors that we can reduce complex concepts to simpler ones.

Formal set-theoretic accounts

- It's important to our authors that we can reduce complex concepts to simpler ones.
- They're willing to sacrifice conceptual richness and subtlety for precision and rigor.

Formal set-theoretic accounts

- It's important to our authors that we can reduce complex concepts to simpler ones.
- They're willing to sacrifice conceptual richness and subtlety for precision and rigor.
- They aim to capture common sense understanding and domain knowledge without appeal to either of those.

Formal set-theoretic accounts

- It's important to our authors that we can reduce complex concepts to simpler ones.
- They're willing to sacrifice conceptual richness and subtlety for precision and rigor.
- They aim to capture common sense understanding and domain knowledge without appeal to either of those.
- Sometimes a demonstration that we *can* account for something in terms of simple sets is seen as valuable, even if it's not convenient or practical to ever use the reduction again.

From the Partee reading

The notion of set is taken as "undefined", "primitive", or "basic", so we don't try to define what a set is, but we can give an informal description, describe important properties of sets, and give examples. All other notions of mathematics can be built up based on the notion of set.

Description: a set is a collection of objects which are called the members or elements of that set. If we have a set we say that some objects belong (or do not belong) to this set, are (or are not) in the set. We say also that sets consist of their elements.

Implications

- Sets are abstract objects, even if their members are concrete or social objects.

Implications

- Sets are abstract objects, even if their members are concrete or social objects.
- No two distinct sets can have the same members: same members implies same set.

Implications

- Sets are abstract objects, even if their members are concrete or social objects.
- No two distinct sets can have the same members: same members implies same set.
- No set can change with respect to its members: different members implies different sets.

Implications

- Sets are abstract objects, even if their members are concrete or social objects.
- No two distinct sets can have the same members: same members implies same set.
- No set can change with respect to its members: different members implies different sets.
- Be careful of set descriptions that pick out different things at different times.

Implications

- Sets are abstract objects, even if their members are concrete or social objects.
- No two distinct sets can have the same members: same members implies same set.
- No set can change with respect to its members: different members implies different sets.
- Be careful of set descriptions that pick out different things at different times.
- When specifying a set in predicate notation, different descriptions may pick out exactly the same set.

Implications

- Sets are abstract objects, even if their members are concrete or social objects.
- No two distinct sets can have the same members: same members implies same set.
- No set can change with respect to its members: different members implies different sets.
- Be careful of set descriptions that pick out different things at different times.
- When specifying a set in predicate notation, different descriptions may pick out exactly the same set.
- But properties are not defined by their instances. Therefore, properties and sets are different.

Elements vs. subsets

- We use the $\in$ operator to relate an element to the set it belongs to.

Elements vs. subsets

- We use the $\in$ operator to relate an element to the set it belongs to.
- For example, $b \in \{a, b, c\}$

Elements vs. subsets

- We use the $\in$ operator to relate an element to the set it belongs to.
- For example, $b \in \{a, b, c\}$
- $\{a, b\} \in \{\{a, b\}, \{b, c, d\}, \{a, f, g\}\}$

Elements vs. subsets

- We use the $\in$ operator to relate an element to the set it belongs to.
- For example, $b \in \{a, b, c\}$
- $\{a, b\} \in \{\{a, b\}, \{b, c, d\}, \{a, f, g\}\}$
- But $\{a, b\} \subseteq \{a, b, c\}$

Elements vs. subsets

- We use the $\in$ operator to relate an element to the set it belongs to.
- For example, $b \in \{a, b, c\}$
- $\{a, b\} \in \{\{a, b\}, \{b, c, d\}, \{a, f, g\}\}$
- But $\{a, b\} \subseteq \{a, b, c\}$
- $\{b\} \subseteq \{a, b, c\}$

Elements vs. subsets

- We use the $\in$ operator to relate an element to the set it belongs to.
- For example, $b \in \{a, b, c\}$
- $\{a, b\} \in \{\{a, b\}, \{b, c, d\}, \{a, f, g\}\}$
- But $\{a, b\} \subseteq \{a, b, c\}$
- $\{b\} \subseteq \{a, b, c\}$
- $\{a, c\} \subseteq \{a, b, c\}$

Elements vs. subsets

- We use the $\in$ operator to relate an element to the set it belongs to.
- For example, $b \in \{a, b, c\}$
- $\{a, b\} \in \{\{a, b\}, \{b, c, d\}, \{a, f, g\}\}$
- But $\{a, b\} \subseteq \{a, b, c\}$
- $\{b\} \subseteq \{a, b, c\}$
- $\{a, c\} \subseteq \{a, b, c\}$
- $\{\{a, b\}, \{a, f, g\}\} \subseteq \{\{a, b\}, \{b, c, d\}, \{a, f, g\}\}$

Elements vs. subsets

- We use the $\in$ operator to relate an element to the set it belongs to.
- For example, $b \in \{a, b, c\}$
- $\{a, b\} \in \{\{a, b\}, \{b, c, d\}, \{a, f, g\}\}$
- But $\{a, b\} \subseteq \{a, b, c\}$
- $\{b\} \subseteq \{a, b, c\}$
- $\{a, c\} \subseteq \{a, b, c\}$
- $\{\{a, b\}, \{a, f, g\}\} \subseteq \{\{a, b\}, \{b, c, d\}, \{a, f, g\}\}$
- $\emptyset \subseteq \{a, b, c\}$ but $\emptyset \notin \{a, b, c\}$

Sets and properties are different things

- The property of being an odd integer less than 8 and greater than 2 is different from the property of being a prime integer less than 8 and greater than 2, because "being prime" and "being odd" are different properties.

Sets and properties are different things

- The property of being an odd integer less than 8 and greater than 2 is different from the property of being a prime integer less than 8 and greater than 2, because “being prime” and “being odd” are different properties.
- But the set of odd integers between 2 and 8 is the same as the set of primes between 2 and 8. This is not just any equivalence relationship: we’re talking about only one set.

Sets and properties are different things

- The property of being an odd integer less than 8 and greater than 2 is different from the property of being a prime integer less than 8 and greater than 2, because "being prime" and "being odd" are different properties.
- But the set of odd integers between 2 and 8 is the same as the set of primes between 2 and 8. This is not just any equivalence relationship: we're talking about only one set.
- You might specify a category "students enrolled at the iSchool," but its *extension* would be different sets at different times.

Set operations

- Union: $A \cup B =_{def} \{x | x \in A \text{ or } x \in B\}$

Set operations

- Union: $A \cup B =_{def} \{x | x \in A \text{ or } x \in B\}$
- Intersection: $A \cap B =_{def} \{x | x \in A \text{ and } x \in B\}$

Set operations

- Union: $A \cup B =_{def} \{x | x \in A \text{ or } x \in B\}$
- Intersection: $A \cap B =_{def} \{x | x \in A \text{ and } x \in B\}$
- Complement: $A' =_{def} \{x | x \notin A\}$

Set operations

- Union: $A \cup B =_{def} \{x | x \in A \text{ or } x \in B\}$
- Intersection: $A \cap B =_{def} \{x | x \in A \text{ and } x \in B\}$
- Complement: $A' =_{def} \{x | x \notin A\}$
- Relative Complement: $A - B =_{def} \{x | x \in A \text{ and } x \notin B\}$

Tuples are ordered sets

- $\langle a, b \rangle =_{def} \{\{a\}, \{a, b\}\}$

Tuples are ordered sets

- $\langle a, b \rangle =_{def} \{\{a\}, \{a, b\}\}$
- $\langle a, b, c \rangle =_{def} \langle \langle a, b \rangle, c \rangle$

Tuples are ordered sets

- $\langle a, b \rangle =_{def} \{\{a\}, \{a, b\}\}$
- $\langle a, b, c \rangle =_{def} \langle \langle a, b \rangle, c \rangle$
- Cartesian products are sets of tuples.

Tuples are ordered sets

- $\langle a, b \rangle =_{def} \{\{a\}, \{a, b\}\}$
- $\langle a, b, c \rangle =_{def} \langle \langle a, b \rangle, c \rangle$
- Cartesian products are sets of tuples.
- $A \times B =_{def} \{\langle x, y \rangle \mid x \in A \ and \ y \in B\}$

Tuples are ordered sets

- $\langle a, b \rangle =_{def} \{\{a\}, \{a, b\}\}$
- $\langle a, b, c \rangle =_{def} \langle \langle a, b \rangle, c \rangle$
- Cartesian products are sets of tuples.
- $A \times B =_{def} \{\langle x, y \rangle \mid x \in A \text{ and } y \in B\}$
- Relations are subsets of cartesian products.

Tuples are ordered sets

- $\langle a, b \rangle =_{def} \{\{a\}, \{a, b\}\}$
- $\langle a, b, c \rangle =_{def} \langle \langle a, b \rangle, c \rangle$
- Cartesian products are sets of tuples.
- $A \times B =_{def} \{\langle x, y \rangle \mid x \in A \textit{ and } y \in B\}$
- Relations are subsets of cartesian products.
- If A and B are sets and $R \subseteq A \times B$ we say that R is a binary relation from A to B.

Tuples are ordered sets

- $\langle a, b \rangle =_{def} \{\{a\}, \{a, b\}\}$
- $\langle a, b, c \rangle =_{def} \langle \langle a, b \rangle, c \rangle$
- Cartesian products are sets of tuples.
- $A \times B =_{def} \{\langle x, y \rangle \mid x \in A \; and \; y \in B\}$
- Relations are subsets of cartesian products.
- If A and B are sets and $R \subseteq A \times B$ we say that R is a binary relation from A to B.
- $A \times B \times C =_{def} ((A \times B) \times C)$

Domain, range, functions, and composition.

- The set **dom** $R = \{a \mid \langle a, b\rangle \in R \text{ for some } b\}$ is called the *domain* of the relation R and ...

Domain, range, functions, and composition.

- The set **dom** $R = \{a \mid \langle a, b\rangle \in R$ for some $b\}$ is called the *domain* of the relation R and ...
- ... the set **range** $R = \{b \mid \langle a, b\rangle \in R$ for some $a\}$ is called the *range* of relation R.

Domain, range, functions, and composition.

- The set **dom** $R = \{a \mid \langle a, b \rangle \in R \text{ for some } b\}$ is called the *domain* of the relation R and ...
- ... the set **range** $R = \{b \mid \langle a, b \rangle \in R \text{ for some } a\}$ is called the *range* of relation R.
- If A and B are sets and $R \subseteq A \times B$ we say that $R^{-1} \subseteq B \times A$ is the *inverse* of R, where $R^{-1} =_{def} \{\langle b, a \rangle \mid \langle a, b \rangle \in R\}$.

Domain, range, functions, and composition.

- The set **dom** $R = \{a \mid \langle a, b\rangle \in R \text{ for some } b\}$ is called the *domain* of the relation R and ...
- ... the set **range** $R = \{b \mid \langle a, b\rangle \in R \text{ for some } a\}$ is called the *range* of relation R.
- If A and B are sets and $R \subseteq A \times B$ we say that $R^{-1} \subseteq B \times A$ is the *inverse* of R, where $R^{-1} =_{def} \{\langle b, a\rangle \mid \langle a, b\rangle \in R\}$.
- Given relations $R \subseteq A \times B$ and $S \subseteq B \times C$ the composite of R and S, written $S \circ R =_{def} \{\langle x, z\rangle | \text{ for some } y, \langle x, y\rangle \in R \text{ and } \langle y, z\rangle \in S\}$

Domain, range, functions, and composition.

- The set **dom** $R = \{a \mid \langle a, b\rangle \in R$ for some $b\}$ is called the *domain* of the relation R and ...
- ... the set **range** $R = \{b \mid \langle a, b\rangle \in R$ for some $a\}$ is called the *range* of relation R.
- If A and B are sets and $R \subseteq A \times B$ we say that $R^{-1} \subseteq B \times A$ is the *inverse* of R, where $R^{-1} =_{def} \{\langle b, a\rangle \mid \langle a, b\rangle \in R\}$.
- Given relations $R \subseteq A \times B$ and $S \subseteq B \times C$ the composite of R and S, written $S \circ R =_{def} \{\langle x, z\rangle \mid$ for some $y, \langle x, y\rangle \in R$ and $\langle y, z\rangle \in S\}$
- A relation $F \subseteq A \times B$ is a *function* (or mapping) $F : A \to B$ if and only if the domain of F is A and F pairs every element in that domain with exactly one element in the range, i.e. $\langle a, b\rangle \in F$ and $\langle a, c\rangle \in F$ implies $b = c$.