Propositional Logic

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- $\{pqr, pq\overline{r}, p\overline{q}r, p\overline{qr}, \overline{p}qr, \overline{p}q\overline{r}, \overline{pq}r, \overline{pqr}\}$

Logical Operators

Table 1: 2.15 from van Benthem, et al.

Symbol	In natural language	Technical name
$\neg$	not	negation
$\wedge$	and	conjunction
$\vee$	or	disjunction
$\rightarrow$	if ... then	implication
$\leftrightarrow$	if and only if	equivalence

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- The sentence is mapped to a truth value via the following tables

Semantics of the operators

φ	$\neg\varphi$
0	1
1	0

Table 3: 2.18 from van Benthem, et al.

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \rightarrow \psi$	$\varphi \leftrightarrow \psi$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

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- a = "He is currently employed abroad."
- s = "He is entitled to social security."

Drawing truth tables for expressions

p	q	r	(($\sim$	p	$\vee$	q)	$\rightarrow$	r)
1	1	1	0	1	1	1	1	1
1	1	0	0	1	1	1	0	0
1	0	1	0	1	0	0	1	1
1	0	0	0	1	0	0	1	0
0	1	1	1	0	1	1	1	1
0	1	0	1	0	1	1	0	0
0	0	1	1	0	1	0	1	1
0	0	0	1	0	1	0	0	0

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p	q	r	((	~	p	∨	q	)	→	r	)
1	1	1		0	1	1	1		1	1	
1	1	0		0	1	1	1		0	0	
1	0	1		0	1	0	0		1	1	
1	0	0		0	1	0	0		1	0	
0	1	1		1	0	1	1		1	1	
0	1	0		1	0	1	1		0	0	
0	0	1		1	0	1	0		1	1	
0	0	0		1	0	1	0		0	0	

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1	0	0	0	1	0	0	1	0
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1	0	1		0	1	0	0		1	1	
1	0	0		0	1	0	0		1	0	
0	1	1		1	0	1	1		1	1	
0	1	0		1	0	1	1		0	0	
0	0	1		1	0	1	0		1	1	
0	0	0		1	0	1	0		0	0	

Exercise 2.11

p	q	((p	$\to$	q)	$\vee$	(q	$\to$	p))
1	1	1	1	1	1	1	1	1
1	0	1	0	0	1	0	1	1
0	1	0	1	1	1	1	0	0
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p	q	r	(((p	$\vee$	$\sim$	q)	&	r)	$\leftrightarrow$	($\sim$	(p	&	r)	$\vee$	q))
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1	1	0	1	1	0	1	0	0	0	1	1	0	0	1	1
1	0	1	1	1	1	0	1	1	0	0	1	1	1	0	0
1	0	0	1	1	1	0	0	0	0	1	1	0	0	1	0
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0	0	1	0	1	1	0	1	1	1	1	0	0	1	1	0
0	0	0	0	1	1	0	0	0	0	1	0	0	0	1	0

Grammar of propositional logic

Let P be a set of proposition letters and let $p \in P$.
The following expression defines the recursive grammar for a logical expression φ in Backus–Naur Form:
$\varphi ::= p|\neg\varphi|(\varphi \wedge \varphi)|(\varphi \vee \varphi)|(\varphi \rightarrow \varphi)|(\varphi \leftrightarrow \varphi)$

Syntactically conforming expressions

Let $P = \{o, q, r, s\}$
Examples of grammatically conforming expressions include:

- r

Grammatically *incorrect* expressions would include:

How many correct expressions are consistent with the last one?

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Syntactic ambiguity

These conforming expressions are all consistent with $\neg p \vee q \to r$

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- $\neg((p \vee q) \rightarrow r)$
- $(\neg p \vee (q \rightarrow r))$

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These conforming expressions are all consistent with $\neg p \lor q \rightarrow r$

- $((\neg p \lor q) \rightarrow r)$
- $(\neg(p \lor q) \rightarrow r)$
- $\neg((p \lor q) \rightarrow r)$
- $(\neg p \lor (q \rightarrow r))$
- $\neg(p \lor (q \rightarrow r))$

Exercise 2.7

Which of the following are formulas in propositional logic?

- $p \rightarrow \neg q$

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- $\neg\neg \wedge q \vee p$

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- If V doesn't satisfy φ we write "$V \not\models \varphi$". In other words $V(\varphi) = 0$.

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- A set of propositional logic statements is consistent if at least one state of affairs satisfies every statement in the set.
- A set of propositional logic statements is inconsistent if no state of affairs satisfies every statement in the set.

Inference and validity

- A conclusion is *valid* with respect to a set of premises if the conclusion is true in every sitation where the premises are true (van Benthem, et al, page 2-4).

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- A conclusion is *valid* with respect to a set of premises if the conclusion is true in every sitation where the premises are true (van Benthem, et al, page 2-4).
- One can validly infer a conclusion φ from a set of premises P if the negation of φ is inconsistent with the set of statements P.

Computation and expressive power

(From van Bentham, et al., chapter 2)

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1. Computing a truth value for a formula takes linear time.
2. Computing a truth table for validity takes exponential time.
3. The problem of testing for validity in propositional logic is decidable: there exists a mechanical method that computes the answer, at least in principle.