Propositional Logic

Dave Dubin

October 10, 2016

Concepts from earlier in the semester

- Propositions are the bearers of truth values: the kinds of things that can be true or false.

Concepts from earlier in the semester

- Propositions are the bearers of truth values: the kinds of things that can be true or false.
- States of affairs are the parts of reality responsible for making propositions true or false.

Propositional logic

- We represent propositions with lower case letters (typically p, q, and r).

Propositional logic

- We represent propositions with lower case letters (typically p, q, and r).
- A set of proposition letters generates a set of states of affairs: ways the world might be.

Propositional logic

- We represent propositions with lower case letters (typically p, q, and r).
- A set of proposition letters generates a set of states of affairs: ways the world might be.
- n proposition letters generate 2^n states of affairs.

Propositional logic

- We represent propositions with lower case letters (typically p, q, and r).
- A set of proposition letters generates a set of states of affairs: ways the world might be.
- n proposition letters generate 2^n states of affairs.
- $\{pqr, pq\overline{r}, p\overline{q}r, p\overline{qr}, \overline{p}qr, \overline{p}q\overline{r}, \overline{pq}r, \overline{pqr}\}$

Logical Operators

Table 1: 2.15 from van Benthem, et al.

Symbol	In natural language	Technical name
$\neg$	not	negation
$\wedge$	and	conjunction
$\vee$	or	disjunction
$\rightarrow$	if ... then	implication
$\leftrightarrow$	if and only if	equivalence

Logical Expressions

- A sentence constructed from proposition letters and operators is true or false in each state of affairs.

Logical Expressions

- A sentence constructed from proposition letters and operators is true or false in each state of affairs.
- Consider, for example: $(\neg p \vee q) \rightarrow r$

Logical Expressions

- A sentence constructed from proposition letters and operators is true or false in each state of affairs.
- Consider, for example: $(\neg p \vee q) \rightarrow r$
- The sentence is mapped to a truth value via the following tables

Semantics of the operators

ϕ	$\neg\phi$
0	1
1	0

Table 3: 2.18 from van Benthem, et al.

ϕ	ψ	$\phi \wedge \psi$	$\phi \vee \psi$	$\phi \rightarrow \psi$	$\phi \leftrightarrow \psi$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

Exercise 2.3

You are given the information that p-or-q and (not-p)-or-r. What can you conclude about q and r? What is the strongest valid conclusion you can draw? (A statement is stronger than another statement if it rules out more possibilities.)

Drawing truth tables for expressions

p	q	r	(($\sim$	p	$\vee$	q)	$\rightarrow$	r)
1	1	1	0	1	1	1	1	1
1	1	0	0	1	1	1	0	0
1	0	1	0	1	0	0	1	1
1	0	0	0	1	0	0	1	0
0	1	1	1	0	1	1	1	1
0	1	0	1	0	1	1	0	0
0	0	1	1	0	1	0	1	1
0	0	0	1	0	1	0	0	0

Drawing truth tables for expressions

p	q	r	(($\sim$	p	$\vee$	q)	$\rightarrow$	r)
1	1	1	0	1	1	1	1	1
1	1	0	0	1	1	1	0	0
1	0	1	0	1	0	0	1	1
1	0	0	0	1	0	0	1	0
0	1	1	1	0	1	1	1	1
0	1	0	1	0	1	1	0	0
0	0	1	1	0	1	0	1	1
0	0	0	1	0	1	0	0	0

Drawing truth tables for expressions

p	q	r	(($\sim$	p	$\vee$	q)	$\rightarrow$	r)
1	1	1	0	1	1	1	1	1
1	1	0	0	1	1	1	0	0
1	0	1	0	1	0	0	1	1
1	0	0	0	1	0	0	1	0
0	1	1	1	0	1	1	1	1
0	1	0	1	0	1	1	0	0
0	0	1	1	0	1	0	1	1
0	0	0	1	0	1	0	0	0

Drawing truth tables for expressions

p	q	r	(($\sim$	p	$\vee$	q)	$\rightarrow$	r)
1	1	1	0	1	1	1	1	1
1	1	0	0	1	1	1	0	0
1	0	1	0	1	0	0	1	1
1	0	0	0	1	0	0	1	0
0	1	1	1	0	1	1	1	1
0	1	0	1	0	1	1	0	0
0	0	1	1	0	1	0	1	1
0	0	0	1	0	1	0	0	0

Drawing truth tables for expressions

```
   (~p v q) -> r
-------------------
     0    0      0
     0    0      1
     0    1      0
     0    1      1
     1    0      0
     1    0      1
     1    1      0
     1    1      1
```

Drawing truth tables for expressions

```
   (~p v q) -> r
-------------------
    10   0      0
    10   0      1
    10   1      0
    10   1      1
    01   0      0
    01   0      1
    01   1      0
    01   1      1
```

Drawing truth tables for expressions

```
   (~p v q) -> r
-------------------
    10 1 0     0
    10 1 0     1
    10 1 1     0
    10 1 1     1
    01 0 0     0
    01 0 0     1
    01 1 1     0
    01 1 1     1
```

Drawing truth tables for expressions

```
   (~p v q) -> r
-------------------
    10 1 0  0  0
    10 1 0  1  1
    10 1 1  0  0
    10 1 1  1  1
    01 0 0  1  0
    01 0 0  1  1
    01 1 1  0  0
    01 1 1  1  1
```

Grammar of propositional logic

Let P be a set of proposition letters and let $p \in P$.
The following expression defines the recursive grammar for a logical expression ϕ in Backus–Naur Form:
$\phi ::= p|\neg\phi|(\phi \wedge \phi)|(\phi \vee \phi)|(\phi \rightarrow \phi)|(\phi \leftrightarrow \phi)$

Syntactically conforming expressions

Let $P = \{o, q, r, s\}$
Examples of grammatically conforming expressions include:

- r

Grammatically *incorrect* expressions would include:

How many correct expressions are consistent with the last one?

Syntactically conforming expressions

Let $P = \{o, q, r, s\}$
Examples of grammatically conforming expressions include:

- r
- $\neg q$

Grammatically *incorrect* expressions would include:

How many correct expressions are consistent with the last one?

Syntactically conforming expressions

Let $P = \{o, q, r, s\}$
Examples of grammatically conforming expressions include:

- r
- $\neg q$
- $(s \leftrightarrow o)$

Grammatically *incorrect* expressions would include:

How many correct expressions are consistent with the last one?

Syntactically conforming expressions

Let $P = \{o, q, r, s\}$
Examples of grammatically conforming expressions include:

- r
- $\neg q$
- $(s \leftrightarrow o)$
- $(\neg(s \leftrightarrow \neg\neg\neg o) \rightarrow (q \wedge q))$

Grammatically *incorrect* expressions would include:

How many correct expressions are consistent with the last one?

Syntactically conforming expressions

Let $P = \{o, q, r, s\}$
Examples of grammatically conforming expressions include:

- r
- $\neg q$
- $(s \leftrightarrow o)$
- $(\neg(s \leftrightarrow \neg\neg\neg o) \rightarrow (q \wedge q))$

Grammatically *incorrect* expressions would include:

- $\neg \vee p$

How many correct expressions are consistent with the last one?

Syntactically conforming expressions

Let $P = \{o, q, r, s\}$
Examples of grammatically conforming expressions include:

- r
- $\neg q$
- $(s \leftrightarrow o)$
- $(\neg(s \leftrightarrow \neg\neg\neg o) \rightarrow (q \wedge q))$

Grammatically *incorrect* expressions would include:

- $\neg \vee p$
- $\vee) p \neg$

How many correct expressions are consistent with the last one?

Syntactically conforming expressions

Let $P = \{o, q, r, s\}$
Examples of grammatically conforming expressions include:

- r
- $\neg q$
- $(s \leftrightarrow o)$
- $(\neg(s \leftrightarrow \neg\neg\neg o) \rightarrow (q \wedge q))$

Grammatically *incorrect* expressions would include:

- $\neg \vee p$
- $\vee)p\neg$
- $\neg p \vee q \rightarrow r$

How many correct expressions are consistent with the last one?

Syntactic ambiguity

These conforming expressions are all consistent with $\neg p \vee q \rightarrow r$

- $((\neg p \vee q) \rightarrow r)$

Syntactic ambiguity

These conforming expressions are all consistent with $\neg p \vee q \rightarrow r$

- $((\neg p \vee q) \rightarrow r)$
- $(\neg(p \vee q) \rightarrow r)$

Syntactic ambiguity

These conforming expressions are all consistent with $\neg p \vee q \rightarrow r$

- $((\neg p \vee q) \rightarrow r)$
- $(\neg(p \vee q) \rightarrow r)$
- $\neg((p \vee q) \rightarrow r)$

Syntactic ambiguity

These conforming expressions are all consistent with $\neg p \lor q \rightarrow r$

- $((\neg p \lor q) \rightarrow r)$
- $(\neg(p \lor q) \rightarrow r)$
- $\neg((p \lor q) \rightarrow r)$
- $(\neg p \lor (q \rightarrow r))$

Syntactic ambiguity

These conforming expressions are all consistent with $\neg p \vee q \rightarrow r$

- $((\neg p \vee q) \rightarrow r)$
- $(\neg(p \vee q) \rightarrow r)$
- $\neg((p \vee q) \rightarrow r)$
- $(\neg p \vee (q \rightarrow r))$
- $\neg(p \vee (q \rightarrow r))$

More on semantics

- Semantics is the relationship of a language to the part of the world that we're modeling.

More on semantics

- Semantics is the relationship of a language to the part of the world that we're modeling.
- Valuations are functions from expressions to truth values.

More on semantics

- Semantics is the relationship of a language to the part of the world that we're modeling.
- Valuations are functions from expressions to truth values.
- "$V(\phi) = 1$" means the formula (or sentence) ϕ is true in the state of affairs represented by the function V. "$V(\phi) = 0$" means that ϕ is false in the state of affairs represented by the function V.

More on semantics

- Semantics is the relationship of a language to the part of the world that we're modeling.
- Valuations are functions from expressions to truth values.
- "$V(\phi) = 1$" means the formula (or sentence) ϕ is true in the state of affairs represented by the function V. "$V(\phi) = 0$" means that ϕ is false in the state of affairs represented by the function V.
- For "$V(\phi) = 1$" we also write "$V \models \phi$" read as "V is a model of ϕ" or "V satisfies ϕ."

More on semantics

- Semantics is the relationship of a language to the part of the world that we're modeling.
- Valuations are functions from expressions to truth values.
- "$V(\phi) = 1$" means the formula (or sentence) ϕ is true in the state of affairs represented by the function V. "$V(\phi) = 0$" means that ϕ is false in the state of affairs represented by the function V.
- For "$V(\phi) = 1$" we also write "$V \models \phi$" read as "V is a model of ϕ" or "V satisfies ϕ."
- If V doesn't satisfy ϕ we write "$V \not\models \phi$". In other words $V(\phi) = 0$.

- A statement ϕ is logically true if it is true in every state of affairs generated by its propositional variables.

Logical truth and logical falsity.

- A statement ϕ is logically true if it is true in every state of affairs generated by its propositional variables.
- A statement ϕ is logically false if it is false in every state of affairs generated by its propositional variables.

Logical truth and logical falsity.

- A statement ϕ is logically true if it is true in every state of affairs generated by its propositional variables.
- A statement ϕ is logically false if it is false in every state of affairs generated by its propositional variables.
- If a statement ϕ is neither logically true or logically false then it is contingent.

Logical truth and logical falsity.

- A statement ϕ is logically true if it is true in every state of affairs generated by its propositional variables.
- A statement ϕ is logically false if it is false in every state of affairs generated by its propositional variables.
- If a statement ϕ is neither logically true or logically false then it is contingent.
- Examples:

Logical truth and logical falsity.

- A statement ϕ is logically true if it is true in every state of affairs generated by its propositional variables.
- A statement ϕ is logically false if it is false in every state of affairs generated by its propositional variables.
- If a statement ϕ is neither logically true or logically false then it is contingent.
- Examples:
  1. $(q \vee \neg q)$ is logically true.

Logical truth and logical falsity.

- A statement ϕ is logically true if it is true in every state of affairs generated by its propositional variables.
- A statement ϕ is logically false if it is false in every state of affairs generated by its propositional variables.
- If a statement ϕ is neither logically true or logically false then it is contingent.
- Examples:
  1. $(q \vee \neg q)$ is logically true.
  2. $(q \wedge \neg q)$ is logically false.

Consistency

- A set of propositional logic statements is consistent if at least one state of affairs satisfies every statement in the set.

Consistency

- A set of propositional logic statements is consistent if at least one state of affairs satisfies every statement in the set.
- A set of propositional logic statements is inconsistent if no state of affairs satisfies every statement in the set.

Inference and validity

- A conclusion is *valid* with respect to a set of premises if the conclusion is true in every sitation where the premises are true (van Benthem, et al, page 2-4).

- A conclusion is *valid* with respect to a set of premises if the conclusion is true in every sitation where the premises are true (van Benthem, et al, page 2-4).
- One can validly infer a conclusion ϕ from a set of premises P if the negation of ϕ is inconsistent with the set of statements P.

(From van Bentham, et al., chapter 2)

1. Computing a truth value for a formula takes linear time.

Computation and expressive power

(From van Bentham, et al., chapter 2)

1. Computing a truth value for a formula takes linear time.
2. Computing a truth table for validity takes exponential time.

(From van Bentham, et al., chapter 2)

1. Computing a truth value for a formula takes linear time.
2. Computing a truth table for validity takes exponential time.
3. The problem of testing for validity in propositional logic is decidable: there exists a mechanical method that computes the answer, at least in principle.