# Dynamic Programming

## Assignment 1: Working with Julia

Thomas J. Sargent and John Stachurski

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# Julia Exercise

Consider the problem

$$\max \sum_{t \geqslant 0} \beta^t u(C_t)$$

subject to $W_{t+1} = R(W_t - C_t)$ and $0 \leqslant C_t \leqslant W_t$

Let's consider how to solve this problem in Julia

We will use two methods:

1. Value function iteration (VFI)
2. Optimistic policy iteration (OPI)

VFI is standard:

1. Set up the Bellman operator

$$(Tv)(w) = \max_{0 \leqslant c \leqslant w} \{u(c) + \beta v(R(w-c))\}$$

2. Approximate $v^*$ via $v^* = \lim_{k\to\infty} T^k v$

3. Choose optimal consumption at wealth $w$ via

$$c^* \in \operatorname*{argmax}_{0 \leqslant c \leqslant w} \{u(c) + \beta v^*(R(w-c))\}$$

Alternatively, we can use **optimistic policy iteration** (OPI)

To introduce it we define a **feasible policy** to be a map $\sigma \colon \mathbb{R}_+ \to \mathbb{R}_+$ with

$$0 \leqslant \sigma(w) \leqslant w \quad \text{for all} \quad w \in \mathbb{R}_+$$

- given current wealth $w$, choose consumption $c = \sigma(w)$
- $\Sigma :=$ all feasible policies

A feasible policy $\sigma$ is called **$v$-greedy** if

$$\sigma(w) \in \operatorname*{argmax}_{0 \leqslant c \leqslant w} \{u(c) + \beta v(R(w-c))\}$$

The **lifetime value** $v_\sigma$ of policy $\sigma$ is the unique $v$ that solves

$$v(w) = u(\sigma(w)) + \beta v(R(w - \sigma(w)))$$

To compute it we introduce the **policy operator**

$$(T_\sigma \, v)(w) = u(\sigma(w)) + \beta v(R(w - \sigma(w)))$$

Facts:

1. $v_\sigma$ is the unique fixed point of $T_\sigma$
2. $T_\sigma^k v \to v_\sigma$ as $k \to \infty$ for all reasonable $v$

The OPI algorithm is stated on the next slide

For this model, OPI converges to an optimal policy

The intuition and proof of convergence will be discussed later in the lectures

**Algorithm 1:** OPI

input $v_0$, an initial guess of $v^*$

input $\tau$, a tolerance level for error

input $m \in \mathbb{N}$, a step size

$k \leftarrow 0$

$\varepsilon \leftarrow +\infty$

**while** $\varepsilon > \tau$ **do**

  $\sigma_k \leftarrow$ a $v_k$-greedy policy

  $v_{k+1} \leftarrow T_{\sigma_k}^m v_k$

  $\varepsilon \leftarrow \|v_k - v_{k+1}\|_\infty$

  $k \leftarrow k + 1$

**end**

**return** $\sigma_k$, $v_k$

**Ex.** Start with the file `opt_savings.jl` in this directory

Implement optimistic policy iteration

Visually compare the run time to convergence for

1. VFI
2. OPI for a range of $m$ values

Use the same initial condition and error tolerance for VFI and OPI