Dynamic Programming

Assignment 3: Nonlinear Asset Pricing

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In the long-run risk literature (e.g., Schorfheide et al ECMA 2018), the wealth-consumption ratio w obeys

$$\beta^\theta \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} \left(\frac{w(X_{t+1})}{w(X_t) - 1} \right)^\theta \right] = 1$$

where

- $(X_t)_{t \geqslant 0}$ is a Markov process state space X
- $(C_t)_{t \geqslant 0}$ is consumption
- $\theta := (1-\gamma)/(1-\psi)$
- γ measures risk-aversion and ψ measures EIS
- preference shocks are held constant (omitted)

Your aim is to solve this functional equation for w

Rearranging the previous expression gives

$$(w(X_t) - 1)^\theta = \beta^\theta \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} w(X_{t+1})^\theta \right]$$

Letting

$$G_{t+1} = \ln \frac{C_{t+1}}{C_t}$$

we can rewrite as

$$(w(X_t) - 1)^\theta = \beta^\theta \mathbb{E}_t \left[\exp\left((1-\gamma) G_{t+1}\right) w(X_{t+1})^\theta \right]$$

Let K be the linear operator defined by

$$(Kf)(x) = \mathbb{E}_x \, f(X_{t+1}) \, \exp\left((1-\gamma)G_{t+1}\right)$$

- $\mathbb{E}_x$ conditions on $X_t = x$

With this notation we can now rewrite

$$(w(X_t) - 1)^\theta = \beta^\theta \mathbb{E}_t \left[\exp\left((1-\gamma)G_{t+1}\right) w(X_{t+1})^\theta \right]$$

as

$$(w(x) - 1)^\theta = \beta^\theta (K w^\theta)(x)$$

Rearranging and using vector/function notation,

$$w = 1 + \beta (K w^\theta)^{1/\theta}$$

Your task is to solve

$$w = 1 + \beta(Kw^{\theta})^{1/\theta}$$

Equivalently, you need to find a fixed point of

$$Tw = 1 + \beta\left(Kw^{\theta}\right)^{1/\theta}$$

To fully specify this operator, we need to fully specify K

To do this we first need to define consumption growth G_{t+1}

Growth of consumption is given by

$$G_{t+1} = \mu_c + Z_t + \bar{\sigma} \exp(H_t) \varepsilon_{t+1}$$

where (Z_t) is a persistent component and

$$H_{t+1} = \rho_c H_t + \sigma_c \eta_{t+1}$$

Here $\{\eta_t, \varepsilon_t\}$ are IID and standard normal

The state process is

$$X_t = (H_t, Z_t)$$

We can now write K more explicitly as

$$(Kf)(h, z) = \mathbb{E}_{h,z}\, f(H_{t+1}, Z_{t+1}) \\ \exp\left((1-\gamma)\mu_c + Z_t + \bar\sigma \exp(H_t)\varepsilon_{t+1}\right)$$

Taking (Z_t) to be Q-Markov and discretizing (H_t) to be P-Markov, we can write K as

$$(Kf)(h, z) = \int \exp\left((1-\gamma)\mu_c + z + \bar\sigma \exp(h)\varepsilon\right) \nu(\mathrm{d}\varepsilon) \\ \sum_{h', z'} f(z', h')\, Q(z, z') P(h, h')$$

- ν is the standard normal distribution

The following result can be proved using a theorem in Ch. 7

Proposition If P, Q are irreducible and $r(K)^{1/\theta} < 1$, then

1. T has a unique fixed point w^* in $V := (0, \infty)^{\mathsf{X}}$
2. $Tw^k \to w^*$ as $k \to \infty$ for all $w \in V$

Using this definition of K complete the code in `wc_ratio.py` and solve for w^* using

1. successive approximiation and
2. Newton iteration

If your code is running correctly, it should produce