Problem Set 3. Econ 8108. Spring 2023

This problem set is due on Tuesday Apr 25, during class time. Please upload your electronic answer according to the instructions in Canvas posted by the TA.

Everyone should write their own individual answers (including your own version of the code used, with your own comments) but you are encouraged to work in groups.

1. Transition for Huggett's model: Debt, and Pareto Improvements

This question is going to guide you in generating a transition for the Huggett model, and is based in Aguiar-Amador-Arellano (2023) RPI paper. We are going to consider the model of Huggett analyzed in class with a borrowing limit that is equal to zero, $\phi = 0$. The provided code to this problem set describes how to solve the household problem for a given interest rate R and computes the stationary distribution. (This is slightly modified code from the one you used in Problem Set 2, with the same preference and output process parameters.)

(a) Compute $\overline{R}$:

$$\overline{R} = \frac{1}{\beta} \min_{s \in S} \frac{u'(y(s))}{\sum_{s' \in S} \pi(s'|s) u'(y(s'))}$$

(b) Plot the stationary distribution if the interest rate is $\overline{R}$. Does it make sense? Does this represent a stationary equilibrium? Explain.

We are going to assume that we start from the stationary equilibrium described in part (b). Next, we are going to introduce a government that issues debt and lump-sum transfers to households. The government budget constraint is

$$B_{t+1} - R_t B_t \geq T_t,$$

with $B_0 = 0$. The household budget constraint needs to be modified and becomes:

$$c + a' \leq R_t a + y(s) + T_t.$$

Starting from the equilibrium in part (b), we consider now a government policy that leads to a constant (higher) interest rate at all times, that is, $R_t = R' > \overline{R}$ for all t. We will assume that the government will set $T_t = 0$ for all t as well — which implies that some resources may be lost (e.g., thrown into the sea).

Let us first compute the new stationary equilibrium.

(c) Assume that the new interest rate is $R' = 1$, solve the household problem, and compute the associated stationary distribution for this new interest rate.

(d) Use the asset market clearing condition to compute the level of debt that the government must be issuing in this stationary equilibrium.

(e) Are there any resources lost in the new stationary equilibrium for $R' = 1$?

(f) How would your answer to part (e) change if $\overline{R} < R' < 1$?

Now, we are going to compute the transition from the stationary equilibrium in part (b) to the stationary equilibrium in part (d). Recall that along the transition we are assuming that the policy is such that $T_t = 0$ (there are no transfers); and $R_t = R' = 1$ for all t.

(g) Argue that the household problem *along the transition* is the same as in the new stationary equilibrium. As a result we can use the same policy functions we computed in part (c) to do the transition. (This means, we do not need to do the backward iteration step described in class)

(h) We however need to do the forward iteration. Write down the code for the following algorithm. Take a transition phase of T = 100 periods. Starting from the pdf at the initial stationary equilibrium (computed in part (b)), and given your answer to part (g), use the new policies (computed in part (c)) and solve for the evolution of the asset distribution for all $t \leq T$. You will need to store the 100 pdfs. *Hint: This is effectively the same as the algorithm used to compute the stationary distribution, but starting from the distribution in part (b), iterating only for 100 periods, and storing the iterations.*

(i) Using the stored pdfs for the transition, compute the aggregate savings of the households at every t in the transition phase. This must equal to the level of the government debt at every period, given asset market clearing; and should converge to your answer in part (d). Compute the aggregate endowment in the model, and plot the path of debt to output ratio, B_t/Y.

(j) Use the government budget constraint, and compute $B_{t+1} - R_t B_t$, which is the amount of resources the goverment must throw away every period given that $T_t = 0$. Do a plot of this time series as a fraction of Y.

(k) For the policy to be feasible it must be that the budget constraint of the government is satisfied, that is, $B_{t+1} - R_t B_t \geq 0$ (that is, no resources are added, but can be disposed of). Is this condition satisfied in your transition? Does the transition you have computed constitute an equilibrium?

(l) Compute the aggregate consumption of the household sector at every t in the transition and plot its time series. Is this equal to Y at every t? How is this related to part (k)?

(m) Assuming that your anwer to part (k) was affirmative, does the new allocation constitute a Pareto improvement over the stationary equilibrium in part (b)? Explain how this is possible given your answer in part (l).

(n) Using the computed value functions, confirm in your simulation that every household at the initial stationary distribution is weakly better off in the equilibrium with the new policy. Note that to answer this, you can use the value function computed in part (c). Why?

(o) Generalize now: Consider any Huggett model with no borrowing, that is $\phi = 0$. Suppose that in the stationary equilibrium, $R < 1$. Argue that there is always a feasible Pareto improvement where the government issues debt (and throws away resources) if the sequence of aggregate household savings when the interest rate is constant at $R_t = R' = 1$ is monotonically increasing starting from the stationary equilibrium distribution. That is, if $A_{t+1}(\{R_t = 1\}) \geq A_t(\{R_t = 1\})$ for all t.

(p) Now suppose we would like to introduce back T_t into the environment and have the *government budget constraint holding with equality* at all times. It is no longer the case that we can skip the backward iteration step. Explain (but do not write the code for this) how you will go about solving for the transition in this case.

(q) What will be wrong if we were to use the value function of part (c) to evaluate welfare in part (o) for this case?

2. One sided lack of commitment - $\beta R < 1$

This problem walks through the money lender model with lack of commitment problem but relaxing the assumption that $\beta R = 1$.

Suppose that income is $i.i.d$ and have only two possible states: 1 and $\theta > 1$. The probability of each state is 1/2. The consumer has a intra-period utility function given by

$$u(c) = \log c$$

The money lender is risk neutral and discounts the future at rate R. The consumer's discount factor is β.

The consumer can at any point in time deviate from the contract and remain in autarky forever. Let $P(v)$ be the value to the money lender in an optimal contract that promises the consumer a lifetime utility equal to v. Let us assume that $P(v)$ is strictly concave, and differentiable function.

(a) Show that autarky is given by:

$$v_{aut} = \frac{\log(\theta)}{2(1-\beta)}$$

The value function P be the solution to the following Bellman equation:

$$P(v) = \max_{c_1, v_1, c_2, v_2} \frac{1}{2}\Big\{(1 - c_1) + \frac{1}{R}P(v_1) + (\theta - c_2) + \frac{1}{R}P(v_2)\Big\}$$

subject to

$$\frac{1}{2}\Big(u(c_1) + \beta v_1 + u(c_2) + \beta v_2\Big) \geq v \tag{1}$$

$$u(c_1) + \beta v_1 \geq u(1) + \beta v_{aut} \tag{2}$$

$$u(c_2) + \beta v_2 \geq u(\theta) + \beta v_{aut} \tag{3}$$

$$v_1, v_2 \geq v_{aut} \tag{4}$$

where (1) is the promise-keeping constraint and (2) and (3) are participation constraints.

Let us denote by $c_1(v), c_2(v), v_1(v), v_2(v)$ the optimal policies that solves the Bellman problem. Assume that $P(v)$ is strictly decreasing, strictly concave and differentiable for $v \geq v_{aut}$. You can also assume that the policies are continuous.

(b) Compute the first order conditions and state the envelope.

(c) Show that in the optimal solution to the lack of commitment problem, if the participation constraint at state 1 does not bind, then $v_1(v) < v$.

(d) Argue that both c_1, c_2, v_1 and v_2 are weakly increasing in v.

(e) Argue that if no participation constraint binds, then $c_1(v) = c_2(v)$ and $v_1(v) = v_2(v) < v$.

(f) Show that if the participation constraint for state θ binds, this implies $v_2(v) = \tilde{v}_2$ (independent of v) where $\tilde{v}_2$ is given by the solution to

$$-\log(\beta R) + \log P'(\tilde{v}_2) + \beta \tilde{v}_2 = \log(\theta) + \beta v_{aut}$$

Show that this solution is unique.

(g) Similarly to part (f), show that if the participation constraint at state 1 binds, then the policy function is $v_1(v) = \tilde{v}_1$ where

$$-\log(\beta R) + \log P'(\tilde{v}_1) + \beta \tilde{v}_1 = \beta v_{aut}$$

Show that $\tilde{v}_1 < \tilde{v}_2$.

(h) Show that there exists a $v^\star$ such for all $v > v^\star$, both participation constraints do not bind. So it follows that for all $v > v^\star$, $v_1(v) = v_2(v) < v$.

(i) Show that for $v < v^\star$ the participation constraint for θ always binds.

(j) Show that if both participation constraints bind, then it has to be the case that $v = v_{aut}$. Argue that this implies then that for $v = v_{aut}$ both participation constraints bind; for $v \in (v_{aut}, v^\star)$, only the participation constraint for θ binds; and for $v \geq v^\star$ no participation constraint binds.

(k) Show that $\tilde{v}_1 = v_{aut}$. To do this use continuity of the policy function $v_1(v)$ to argue that if $\tilde{v}_1 > v_{aut}$ then this means that there exists a value $v_o > v_{aut}$ such that $v_1(v_o) > v_o$, which is a contradiction given that the participation constraint is not binding at $v_o > v_{aut}$ by part (j). Together with $\tilde{v}_1 \geq v_{aut}$ (why is this?) the result follows.

(l) Plot the policy graphs $v_1(v)$ and $v_2(v)$ and the 45% line. What happens to consumption in the long run? Does it converge to a number?