MA113 线性代数习题手册
Linear Algebra

期中考试

24春 23秋 23春

Mid-term Exams

24Spring 23Fall 23Spring

前言

妮可 MA113 线性代数 **24 Spring/23 Fall/23 Spring** 期中试题
扫描下方网盘二维码以获取完整资料（中英文试题，答案）：

目录（试题/答案）

24 Spring Midterm Question

24 Spring Midterm Question 中英试题分离 分页留空版本
线性代数2023-2024学年春季学期期中考试

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1. (共15分，每小题3分)选择题，只有一个选项是正确的.

(1) 假定

$$\alpha_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 7 \\ 3 \\ c \end{bmatrix}.$$

若$\alpha_1, \alpha_2, \alpha_3$线性相关，则$c$的取值为

(A)5.
(B)6.
(C)7.
(D)8.

(2) 设 A 为一个 $m \times n$ 实矩阵, b 为一个 m 维实列向量, 以下说法一定是 **正确** 的是？

(A)若$A\mathbf{x} = \mathbf{b}$无解，则$A\mathbf{x} = \mathbf{0}$只有零解.
(B)若$A\mathbf{x} = \mathbf{0}$有无穷多解，则$A\mathbf{x} = \mathbf{b}$有无穷多解。
(C)若$m < n$,则$A\mathbf{x} = \mathbf{b}$和$A\mathbf{x} = \mathbf{0}$都有无穷多解。
(D)若A的秩为n,则$A\mathbf{x} = 0$只有零解.

(3) 如果以下线性方程组有两个自由变量

$$\begin{cases} x_1 + 2x_2 - 4x_3 + 3x_4 = 0, \\ x_1 + 3x_2 - 2x_3 - 2x_4 = 0, \\ x_1 + 5x_2 + (5-k)x_3 - 12x_4 = 0, \end{cases}$$

k的取值为

(A)5.
(B)4.
(C)3.
(D)2.

(4) 设 $u, v \in \mathbb{R}^3, \lambda \in \mathbb{R}$. 以下说法**错误**的是?

(A)如果u和v为满足$u^T v = 0$的非零向量，则u和v线性无关.
(B)如果$u + v$和$u - v$正交，则$||u|| = ||v||$.
$(C)$$u^T v = 0$当且仅当 $u = 0$ or $v = 0$.
$(D)$$\lambda v = 0$当且仅当 $v = 0$ or $\lambda = 0$.

(5) 设A和B都为n阶矩阵.以下说法**错误**的是?

(A)如果A, B为对称矩阵，则AB也为一个对称矩阵。
(B)如果A, B 为可逆矩阵，则 AB 也为一个可逆矩阵.
(C)如果A, B为置换矩阵，则AB也为一个置换矩阵.
(D)如果A, B为上三角矩阵，则AB也为上三角矩阵。

2. (20 points, 5 points each) 填空，共4题。

(1)$A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 3 & 2 \end{bmatrix}, a, b \in \mathbb{R}$, 则 $A^{-1} = $ _____.

(2)设A为一个4×3的实矩阵，B为 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$.

如果矩阵A的秩为 2,则AB的秩为_____.

(3)设 $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 2 & -2 & 2 \end{bmatrix}$, 则 $A^{2024} = $ _____.

(4)考虑以下线性方程组：

$$A\mathbf{x} = \mathbf{b} : \begin{cases} x & = & 2 \\ y & = & 3 \\ x + y & = & 6 \end{cases}$$

该线性方程组的最小二乘解为_____.

3. (10points)设

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 1 \\ 1 & -4 & -7 \end{bmatrix}.$$

求矩阵A的一个LU分解

4. 考虑以下 4 × 5 矩阵 A 以及 4 维列向量 **b**:

$$A = \begin{bmatrix} 0 & 2 & 4 & 1 & 6 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 4 & 10 & 1 & 2 \\ 0 & -1 & -5 & 1 & 7 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 10 \end{bmatrix}$$

(a)分别求矩阵 A 的四个基本子空间的一组基向量。

(b)求 $Ax = b$ 的所有解.

5. (20 points)设 Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$，$T$为按照以下方式定义的从 $\mathbb{R}^{2 \times 2}$到 $\mathbb{R}^{2 \times 2}$线性变换：

$$T(X) = XA + AX, X \in \mathbb{R}^{2 \times 2}.$$

其中$\mathbb{R}^{2 \times 2}$表示所有2×2实矩阵构成的向量空间.

(a)求T在以下有序基

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

下的矩阵表示.

(b)求一个矩阵B使得

$$T(B) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(c)求一个矩阵C使得

$$T(C) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

6. (5 points) 设 A, B 为满足 $A^2 = A$ 和 $B^2 = B$ 的 n 阶实矩阵. 证明：如果 $(A+B)^2 = A+B$, 则 $AB = O$. 其中 O 表示 n 阶零矩阵。

7. (6 points)设A为 3×2矩阵，B 为 2×3矩阵，并且

$$AB = \begin{bmatrix} 8 & 0 & -4 \\ -\frac{3}{2} & 9 & -6 \\ -2 & 0 & 1 \end{bmatrix}.$$

(a)计算$(AB)^2$.
(b)求BA.

1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.

(共15分，每小题3分)选择题，只有一个选项是正确的.

$(1) Let$

$$\alpha_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 7 \\ 3 \\ c \end{bmatrix}.$$

If $\alpha_1, \alpha_2, \alpha_3$ are linearly dependent, then c equals

$(A) 5.$

$(B) 6.$

$(C) 7.$

$(D) 8.$

(2) let A be an $m \times n$ real matrix and b be an $m \times 1$ real column vector. Which of the following statements is correct?

(A) If $A\mathbf{x} = \mathbf{b}$ does not have any solution, then $A\mathbf{x} = \mathbf{0}$ has only the zero solution.

(B) If $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, then $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

(C) If $m < n$, both $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$ have infinitely many solutions.

(D) If the rank of A is n, then $A\mathbf{x} = \mathbf{0}$ has only the zero solution.

(3) For which value of k does the system

$$\begin{cases} x_1 + 2x_2 - 4x_3 + 3x_4 = 0, \\ x_1 + 3x_2 - 2x_3 - 2x_4 = 0, \\ x_1 + 5x_2 + (5 - k)x_3 - 12x_4 = 0, \end{cases}$$

have exactly two free variables?

$(A)5.$

$(B)4.$

$(C)3.$

$(D)2.$

(4) Let $u, v \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$. Which of the following statements is false?

(A) If u and v are nonzero vectors satisfying $u^T v = 0$, then u and v are linearly independent.

(B)If $u + v$ is orthogonal to $u - v$, then $||u|| = ||v||$.

$(C)u^T v = 0$ if and only if $u = 0$ or $v = 0$.

$(D)\lambda v = 0$ if and only if $v = 0$ or $\lambda = 0$.

(5) Let A and B be two $n \times n$ matrices. Which of the following assertions is **false**?

(A) If A, B are symmetric matrices, then AB is a symmetric matrix.

(B) If A, B are invertible matrices, then AB is an invertible matrix.

(C) If A, B are permutation matrices, then AB is a permutation matrix.

(D) If A, B are upper triangular matrices, then AB is an upper triangular matrix.

2. (20 points, 5 points each) Fill in the blanks.

(1) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 3 & 2 \end{bmatrix}, a, b \in \mathbb{R}$. Then $A^{-1} = $ _____.

(2) Let A be a 4×3 real matrix with rank 2 and $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$.

Then the rank AB is _____.

(3) Let $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 2 & -2 & 2 \end{bmatrix}$. Then $A^{2024} = $ _____.

(4) Consider the system of linear equations:

$$Ax = b: \begin{cases} x & = & 2 \\ y & = & 3 \\ x + y & = & 6 \end{cases}$$

The least-squares solution for the system is _____.

3. (10points)Let

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 1 \\ 1 & -4 & -7 \end{bmatrix}.$$

Find an LU factorization of A.

4. (24 points) Consider the following 4×5 matrix A and 4-dimensional column vector b:

$$A = \begin{bmatrix} 0 & 2 & 4 & 1 & 6 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 4 & 10 & 1 & 2 \\ 0 & -1 & -5 & 1 & 7 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 10 \end{bmatrix}$$

(a) Find a basis for each of the four fundamental subspaces of A.

(b) Find the complete solution to $A\mathbf{x} = \mathbf{b}$.

5. (20 points) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ and T be the linear transformation from $R^{2\times 2}$ to $R^{2\times 2}$ defined by

$$T(X) = XA + AX, \ X \in \mathbb{R}^{2\times 2}.$$

Where $\mathbb{R}^{2\times 2}$ denotes the vector space consisting of all 2×2 real matrices.

(a) Find the matrix representation of T with respect to the following ordered basis

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) Find a matrix B such that

$$T(B) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(c) Find a matrix C such that

$$T(C) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

6. (5 points) Let A, B be two $n \times n$ real matrices satisfying $A^2 = A$ and $B^2 = B$. Show that if $(A + B)^2 = A + B$, then $AB = O$. Where O denotes the $n \times n$ zero matrix.

7. (6 points) Let A be a 3×2 matrix, B be a 2×3 matrix such that

$$AB = \begin{bmatrix} 8 & 0 & -4 \\ -\frac{3}{2} & 9 & -6 \\ -2 & 0 & 1 \end{bmatrix}.$$

(a) Compute $(AB)^2$.

(b) Find BA.

24 Spring Midterm Answer

线性代数2023-2024学年春季学期期中考试

快速对答案（详解在之后）

一、D D C C A

二、

$$(1)A^{-1} = \begin{bmatrix} 1 & & \\ -a & 1 & \\ (3a-b)/2 & -3/2 & 1/2 \end{bmatrix} \qquad (2)2$$

$$(3)4^{2023}A = 4^{2023}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 2 & -2 & 2 \end{bmatrix} \qquad (4)\begin{bmatrix} \dfrac{7}{3} \\ \dfrac{10}{3} \end{bmatrix}$$

三、$A = LU = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ & -1 & -5 \\ & & 0 \end{bmatrix}$

四、(a)

$$(1)C(A) = span\left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 10 \\ -5 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 2 \\ 7 \end{bmatrix} \right\}$$

$$(2)\, C\left(A^T\right) = \mathrm{Span}\left[\begin{bmatrix} 0 \\ 2 \\ 4 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 10 \\ 1 \\ 2 \end{bmatrix}\right]$$

$$(3)\, X = K_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + K_4 \begin{bmatrix} 0 \\ -\frac{3}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \quad (k_1, k_2 \in \mathbb{R})$$

$$(4)\, y = k_0 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, k_0 \in R$$

(b)

$$x = x_p + x_n = \begin{bmatrix} 1 \\ -5 \\ 1 \\ 3 \\ 1 \end{bmatrix} + k_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + k_4 \begin{bmatrix} 0 \\ -3/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} \quad k_1, k_4 \in \mathbb{R}$$

五、略，解析部分有方法

六、证明略

七、

$$(AB)^2 = \begin{bmatrix} 72 & 0 & -36 \\ -\frac{27}{2} & 81 & -54 \\ -18 & 0 & 9 \end{bmatrix}$$
$$BA = 9I$$

填空及大题详解

二、(1)

$$[A \quad I] = \begin{bmatrix} 1 & & & \vdots & 1 & & \\ a & 1 & & \vdots & & 1 & \\ b & 3 & 2 & \vdots & & & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & & & \vdots & 1 & & \\ 0 & 1 & & \vdots & -a & 1 & \\ 0 & 3 & 2 & \vdots & -b & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & & & \vdots & 1 & & \\ 0 & 1 & & \vdots & -a & 1 & \\ 0 & 0 & 2 & \vdots & 3a-b & -3 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & & & \vdots & 1 & & \\ & 1 & & \vdots & -a & 1 & \\ & & 1 & \vdots & (3a-b)/2 & -3/2 & 1/2 \end{bmatrix} = [I \quad A^{-1}]$$

(2)法1令

$$A_{4\times 3} = \begin{bmatrix} 1 & & \\ & & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & & \\ & & 1 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow R(AB) = 2$$

法2

$$R(A) = 2, R(B) = 3$$
$$R(AB) \geq R(A) + R(B) - n = 2 + 3 - 3 = 2$$
$$R(AB) \leq min\{R(A), R(B)\} = 2$$

$$\implies R(AB) = 2$$

(3)剥蒜（爆算）法 直接计算A^2, A^3的得出规律

(4)$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$

三、

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 1 \\ 4 & -4 & -7 \end{bmatrix}$$

$$E_{21}(-2)A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & -5 \\ 1 & -4 & -7 \end{bmatrix}$$

$$E_{31}(-1)E_{21}(-2)A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & -5 \\ 0 & -2 & -10 \end{bmatrix}$$

$$E_{32}(-2)E_{31}(-1)E_{31}(-2)A = \begin{bmatrix} 1 & -2 & -3 \\ 0 & -1 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A = E_{21}^{-1}(-2)E_{31}(-1)^{-1}E_{32}^{-1}(-2)\begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 0 \end{bmatrix} = LU$$

四、(a) (1)C(A),对A行变换

$$A \rightarrow \begin{bmatrix} 0 & \boxed{2} & 0 & 3 & 0 \\ 0 & 0 & \boxed{2} & -1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow C(A) = span \left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 10 \\ -5 \end{bmatrix}, \right.$$

(2)$C(A^T)$ 对A^T行变换

$$A^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 4 & -1 \\ 4 & 1 & 10 & -5 \\ 1 & 1 & 1 & 1 \\ 0 & 3 & 2 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & \cancel{3} & -\cancel{6} & \cancel{9} \\ 0 & 3 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \boxed{1} & 1 & 1 & 1 \\ & \boxed{1} & -2 & 3 \\ & & \boxed{10} & 10 \\ & & & 0 \\ & & & 0 \end{bmatrix} \Rightarrow C(A^T) = Span \left\{ \begin{bmatrix} 0 \\ 2 \\ 4 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 10 \\ 1 \\ 2 \end{bmatrix} \right\}$$

(3)$N(A)$

$$Ax = 0 \Rightarrow \begin{bmatrix} 0 & 2 & 0 & 3 & 0 \\ 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix} = 0 \Rightarrow \begin{cases} x_1, x_4 \in \mathbb{R} \\ x_2 = -\frac{3}{2}x_4 \\ x_3 = \frac{1}{2}x_4 \\ x_5 = 0 \end{cases}$$

$$\Rightarrow x = k_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ -3/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} (k_1, k_2 \in \mathbb{R})$$

$(4)N(A^T)$

$$A^Ty = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ & 1 & -2 & 3 \\ & & 10 & -10 \\ & & & 0 \\ & & & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} y_1 = -y_4 \\ y_2 = -y_4 \\ y_3 = y_4 \\ y_4 \in R \end{cases} \Rightarrow y = k_0 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, k_0 \in R$$

$(b)Ax = b$ 时 利用高斯消元化简增广矩阵(略)

令出一特解 $x_p = \begin{bmatrix} 1 & -5 & 1 & 3 & 1 \end{bmatrix}^T$

$$x = x_p + x_n = \begin{bmatrix} 1 \\ -5 \\ 1 \\ 3 \\ 1 \end{bmatrix} + k_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + k_4 \begin{bmatrix} 0 \\ -3/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} \quad k_1, k_4 \in \mathbb{R}$$

五、令 $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

则 $XA = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} a & a+2b \\ c & c+2a \end{bmatrix}$

$AX = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ 2c & 2d \end{bmatrix}$

$T(x) = XA + AX = \begin{bmatrix} 2a+c & a+3b+d \\ 3c & c+4d \end{bmatrix}$

$\qquad = (2a+c)V_1 + (a+3b+d)V_2 + (3c)V_3 + (c+4d)V_4$

$$T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = (V_1\ V_2\ V_3\ V_4) \begin{pmatrix} 2a+c \\ a+3b+d \\ 3c \\ c+4d \end{pmatrix}$$

六、 $(A+B)^2$

$= (A+B)(A+B) = A^2 + AB + BA + B^2 = A + B$

$\because A^2 = A, B^2 = B$

$\therefore AB + BA = 0 \quad \dots\dots\dots ①$

$B(AB + BA) = BAB + B^2A = (BA)B + B^2A = -AB^2 + B^2A$

又 $\because B^2 = B$

$\therefore -AB^2 + B^2A = -AB + BA = 0 \quad \dots\dots\dots ②$

联立①② $\begin{cases} AB + BA = 0 \\ -AB + BA = 0 \end{cases} \Rightarrow AB = 0,$ 得证

七、

$(a)(AB)^2 = (AB)(AB) = \begin{bmatrix} 8 & 0 & -4 \\ -\frac{3}{2} & 9 & -6 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 & 0 & -4 \\ -\frac{3}{2} & 9 & -6 \\ -2 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 72 & 0 & -36 \\ -\frac{27}{2} & 81 & -54 \\ -18 & 0 & 9 \end{bmatrix}$

(b)

$(AB)^2 = 9(AB)$

$R(A) \geq R(AB) = 2 \Rightarrow R(A) = 2,$ 同理 $R(B) = 2$

A 行满秩, 令 $XA = I_2, X$ 为 A 左逆

B 列满秩, 令 $BY = I_2, Y$ 为 B 右逆

$\therefore BA = (XA)(BA)(BY) = X(AB)^2Y = 9XABY = 9I$

23 Fall Midterm Question

线代23秋期中试题 发布版 中英分离

1.(共 15 分，每小题 3 分) 选择题，只有一个选项是正确的.

(1)设$\alpha_1, \alpha_2, \alpha_3$为矩阵$A$的零空间$N(A)$的一组基. 下列哪一组向量也是矩阵A的零空间的一组基？

(A)$\alpha_1 + \alpha_2 - \alpha_3, \alpha_1 + \alpha_2 + 5\alpha_3, 4\alpha_1 + \alpha_2 - 2\alpha_3$.

(B)$\alpha_1 + 2\alpha_2 + \alpha_3, 2\alpha_1 + \alpha_2 + 2\alpha_3, \alpha_3 + \alpha_1 + \alpha_2$,

(C)$\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3$.

(D)$\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \alpha_3 - \alpha_1$.

(2)以下说法一定是正确的是？

(A)如果矩阵A的列向量线性无关，那么对任意的$\mathbf{b}, A\mathbf{x} = \mathbf{b}$有唯一的解.

(B)任意5×7矩阵的列向量一定是线性相关的.

(C)如果矩阵A 的列向量线性相关，该矩阵的行向量也线性相关.

(D)一个10×12矩阵的行空间和列空间可能具有不同的维数

(3)设 $\alpha_1 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 6 \\ 2 \\ -16 \end{bmatrix}, \beta = \begin{bmatrix} 2 \\ t \\ 3 \end{bmatrix}$
当$t = ()$时，β可用$\alpha_1, \alpha_2, \alpha_3$线性表示

(A)1.

(B)3.

(C)6.

(D)9.

(4) 以下说法一定是正确的事?

(A) 设E为一个可逆矩阵.如果A, B矩阵满足$EA = B$,则A和B的列空间相同

(B) 设A为秩为1的n阶的方阵,则$A^n = cA$,其中 n 为 正 整 数 ,c 为 实 数 .

(C) 如果A, B 为对称矩阵, 则AB为对称矩阵. 如果矩阵 A 为一个行满秩矩阵, 那么 $Ax = 0$ 只 有 零 解 .

(D) 如果矩阵A为一个列满秩矩阵,那么$Ax = 0$只有零解。

设A与B都为n阶矩阵,A为非零矩阵,且AB=0,则

(1)$BA = 0$

(2)$B = 0$

(3)$(A + B)(A - B) = A^2 - B^2$

(4)rank $B < n$.

2.(共 25 分,每小题 5 分)填空题.

(1)记所有7×7实矩阵构成的向量空间为$M_{7 \times 7}(\mathbb{R}), W$为$M_{7 \times 7}(\mathbb{R})$中所有斜对称矩阵构成的子空间, 则 $\dim W =$ _____.
如果$A^T = -A, A$就称之为斜对称的。

(2)设A, B为两个 可逆矩阵, 假设的逆矩阵为, 期中O为的零矩阵, 则$D =$ _____.

(3)设 $A = \begin{bmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{bmatrix}$ 且 $rank(A) < 4$, 则 $a =$ _____.

(4)考虑一下线性方程组:

$$A\mathbf{x} = \mathbf{b} : \begin{cases} x & + & 2y & = & 1 \\ x & - & y & = & 2 \\ & & y & = & -1. \end{cases}$$

该线性方程组的最小二乘解为＿＿＿＿＿．

(5)设H为如下定义的一个R^3中的子空间

$$H = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Bigg| x_1 + 2x_2 + x_3 = 0 \right\}.$$

一个和子空间H正交的**单位向量**为＿＿＿＿＿．

3.(24 points)考虑以下这个4×5矩阵A以及他的简化阶梯形矩阵R:

$$A = \begin{bmatrix} 1 & 2 & * & 1 & * \\ 0 & 1 & * & 1 & * \\ -1 & 1 & * & 3 & * \\ 2 & 0 & * & 1 & * \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a)分别求矩阵 A 的四个基本子空间的一组基向量.

(b)求出矩阵 A 的第三个列向量.

4.(15 points) 设

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & a & 1 \\ -1 & 1 & a \end{bmatrix}, B = \begin{bmatrix} 2 & 2 \\ 1 & a \\ -a-1 & -2 \end{bmatrix}.$$

当 a 为何值时，矩阵方程 $AX = B$ 无解、有唯一解、有无穷多解？在有解时，求解此方程，这里的 X 为一个 3×2 矩阵

5.（15 points）设$M_{2\times2}(\mathbb{R})$为所有2×2实矩阵构成的向量空间，设

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

考虑以下映射

$$T : M_{2\times2}(\mathbb{R}) \to \mathbb{R}^3, \ T(X) = \begin{bmatrix} tr(A^TX) \\ tr(B^TX) \\ tr(C^TX) \end{bmatrix},$$

对任意的 2×2实矩阵, 其中 $tr(D)$ 表示n阶矩阵D的迹.

方阵D的迹是指D的对角元之和，也即

$$tr(D) = d_{11} + d_{22} + \cdots + d_{nn}$$

(a)证明T是一个线性变换

(b)求T在$M_{2\times2}(R)$的一组基

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

以及R^3的标准基

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

下的矩阵表示.

(c)是否可以找到一个矩阵 X 使得$T(X) = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$?如果可以,请求出

一个符合要求的矩阵X. 如果不存在,请说明理由.

6. (6 points) 设 A 为 $m \times n$ 矩阵，B 为 $m \times p$ 矩阵, C 为 $q \times p$ 矩阵. 证明:

$$\operatorname{rank} \begin{bmatrix} A & B \\ O & C \end{bmatrix} \geq \operatorname{rank} A + \operatorname{rank} C,$$

其中 O 为 $q \times n$ 的零矩阵

1.(15 points, 3 points each) Multiple Choice. Only one choice is correct.

(1)Suppose that $\alpha_1, \alpha_2, \alpha_3$ are a basis for nullspace of a matrix $A, N(A)$. Which of the following lists of vectors is also a basis for $N(A)$?
(A)$\alpha_1 + \alpha_2 - \alpha_3, \alpha_1 + \alpha_2 + 5\alpha_3, 4\alpha_1 + \alpha_2 - 2\alpha_3$.
(B)$\alpha_1 + 2\alpha_2 + \alpha_3, 2\alpha_1 + \alpha_2 + 2\alpha_3, \alpha_3 + \alpha_1 + \alpha_2$,
(C)$\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3$.
(D)$\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \alpha_3 - \alpha_1$.

(2)Which of the following statements is correct?
(A) If the columns of A are linearly independent, then $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every b.
(B) Any 5×7 matrix has linearly dependent columns.
(C) If the columns of a matrix A are linearly dependent, so are the rows.
(D) The column space and row space of a 10×12 matrix may have different dimensions.

(3)Let

$$\alpha_1 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 6 \\ 2 \\ -16 \end{bmatrix}, \beta = \begin{bmatrix} 2 \\ t \\ 3 \end{bmatrix}.$$

β can be written as a linear combination of $\alpha_1, \alpha_2, \alpha_3$ if $t = (\ )$
(A)1.
(B)3.

(C)6.

(D)9.

(4) Which of the following statements is correct?

(A) Suppose that $EA = B$ and E is an invertible matrix, then the column space of A and the column space of B are the same.

(B) Let A be a $n \times n$ matrix with rank 1, then $A^n = cA$, where n is a positive integer and c is a real number.

(C) Let A, B be symmetric matrices, then AB is symmetric.

(D) If A is of full row rank, then $Ax = 0$ has only the zero solution.

(5)Let A and B be two $n \times n$ matrices. If A is a non-zero matrix and AB = 0, then

(1)$BA = 0$

(2)$B = 0$

(3)$(A + B)(A - B) = A^2 - B^2$

(4)rank $B < n$.

2.(25 points, 5 points each) Fill in the blanks.

(1)Denote the vector space of 7×7 real matrices by $M_{7\times 7}(\mathbb{R})$, and let W be the subspace of $M_{7\times 7}(\mathbb{R})$ consisting of skew-symmetric real matrices, then dim $W = $ _____.

A matrix A is called skew symmetric if $A^T = -A$.

(2)Let A, B be two $n \times n$ invertible matrices. Suppose the inverse of $\begin{bmatrix} A & C \\ O & B \end{bmatrix}$ is $\begin{bmatrix} A^{-1} & D \\ O & B^{-1} \end{bmatrix}$, where O is the $n \times n$ zero matrix. Then $D = $ _____.

(3)Let $A = \begin{bmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{bmatrix}$ with $rank(A) < 4$. Then $a = $ _____ .

(4)Consider the system of linear equations

$$Ax = b : \begin{cases} x & + & 2y & = & 1 \\ x & - & y & = & 2 \\ & & y & = & -1. \end{cases}$$

The least-squares solution for the system is_____ .

(5)

Let H be the subspace of R^3 be defined as follows:

$$H = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_1 + 2x_2 + x_3 = 0 \right\}.$$

A **unit** vector orthogonal to H is _____ .

3.(24 points) Consider the following 4×5 matrix A with its reduced row echelon form R:

$$A = \begin{bmatrix} 1 & 2 & * & 1 & * \\ 0 & 1 & * & 1 & * \\ -1 & 1 & * & 3 & * \\ 2 & 0 & * & 1 & * \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for each of the four fundamental subspaces of A.

(b) Find the third column of matrix A.

4.(15points)Let

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & a & 1 \\ -1 & 1 & a \end{bmatrix}, \ B = \begin{bmatrix} 2 & 2 \\ 1 & a \\ -a-1 & -2 \end{bmatrix}.$$

For which value(s) of a, the matrix equation $AX = B$ has no solution, a unique solution, or infinitely many solutions? Where X is a 3×2 matrix. Solve $AX = B$ if it has at least one solution.

5.(15 points) Let $M_2 \times 2(\mathbb{R})$ be the vector space of 2×2 real matrices. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Consider the map

$$T : M_{2\times2}(\mathbb{R}) \to \mathbb{R}^3, \ T(X) = \begin{bmatrix} tr(A^T X) \\ tr(B^T X) \\ tr(C^T X) \end{bmatrix},$$

for any 2×2 real matrix X, where $tr(D)$ denotes the trace of a matrix D.

The trace of an $n \times n$ matrix D is defined to be the sum of all the diagonal entries of D, in other words,

$$tr(D) = d_{11} + d_{22} + \cdots + d_{nn}.$$

(a) Show that T is a linear transformation.

(b) Find the matrix representation of T with respect to the ordered basis

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ v_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \ v_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ v_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

for $M_{2\times2}(\mathbb{R})$ and the standard basis

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

for $\mathbb{R}^3$.

(c) Can we find a matrix X such that $T(X) = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$? If yes, please find one such matrix. Otherwise, give an explanation.

6.(6 points) Let A be an $m \times n$ matrix, B be an $m \times p$ matrix, and C be an $q \times p$ matrix. Show that

$$\text{rank} \begin{bmatrix} A & B \\ O & C \end{bmatrix} \geq \text{rank } A + \text{rank } C,$$

where O is the $q \times n$ zero matrix.

23 Fall Midterm Answer

线代23秋期中试题答案 发布版

Q1 (1)A (2)B（存疑, 一说D） (3)D (4)B (5)D

Q2(1)21

$(2) - A^{-1}CB^{-1}$

(3)1or-3 （存疑，一说 1 or -2)

$(4)\begin{bmatrix} -\frac{19}{11} \\ -\frac{5}{11} \end{bmatrix}$

$(5)\dfrac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ or } -\dfrac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Q3
A basis for $C(A)$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

A basis for $C(A^T)$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ or } \left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\}$$

A basis for $N(A)$

$$\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

A basis for $N(A^T)$

$$\left\{ \begin{bmatrix} -5 \\ 13 \\ -3 \\ 1 \end{bmatrix} \right\}.$$

$$(b) \begin{bmatrix} 5 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

Q4

Gaussian Eliminations give:

$$\begin{bmatrix} 1 & -1 & -1 & \vdots & 2 & 2 \\ 2 & a & 1 & \vdots & 1 & a \\ -1 & 1 & a & \vdots & -a-1 & -2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & -1 & \vdots & 2 & 2 \\ 0 & a+2 & 3 & \vdots & -3 & a-4 \\ 0 & 0 & a-1 & \vdots & 0 & 0 \end{bmatrix}$$

If $a = -2$, then $rank A = 2 \neq 3 = rank(A \vdots B)$, $AX = B$ has no solution.

If $a \neq 1$ and $a \neq -1$, $AX = B$ has a unique solution.

$$\begin{bmatrix} 1 & -1 & -1 & 1 & 2 \\ 0 & a+2 & 3 & 1 & -3 \\ 0 & 0 & a-1 & 1 & 1-a \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & -1 & -1 & \vdots & 2 \\ 0 & a+2 & 3 & \vdots & -3 \\ 0 & 0 & a-1 & \vdots & 1-a \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & -1 & -1 & \vdots & z \\ 0 & a+2 & 3 & \vdots & a-4 \\ 0 & 0 & a-1 & \vdots & 0 \end{bmatrix} \Rightarrow X = \begin{bmatrix} \frac{3a}{a+2} \\ \frac{a-4}{a+2} \\ 0 \end{bmatrix}.$$

$$X = \begin{bmatrix} 1 & \frac{3a}{a+2} \\ 0 & \frac{a-4}{a+2} \\ -1 & 0 \end{bmatrix}$$

If $a = 1$, $Ax = B$ has infinitely many solutions

$$\begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 3 & 3 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix} 1 & 1 \\ -k_1 - 1 & -k_2 - 1 \\ k_1 & k_2 \end{bmatrix}, \quad k_1, k_2 \quad \text{anbitrary constants.}$$

Q5

(a)Let $X, Y \in M_{2\times2}(R)$ and $C \in R$, than we have

$$T(CX + Y) = \begin{bmatrix} tr & A^T(CX+Y) \\ tr & B^T(CX+Y) \\ tr & C^T(CX+Y) \end{bmatrix}$$

$$= c \begin{bmatrix} tr(A^TX) \\ tr(B^TX) \\ tr(C^TX) \end{bmatrix} + \begin{bmatrix} tr(A^TX) \\ tr(B^TY) \\ tr(C^TY) \end{bmatrix}$$

$$= cT(X) + T(Y)$$

(b)

$$T(V_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{=w_1} + 0 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{=w_2} + 0 \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{=w_3}$$

$$T(V_2) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T(V_3) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T(V_4) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + -1 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore,the matix representation of T with respect to $V_1, V_2, V_3, V_4, V_4,$ and W_1, V_2, W_3 , is:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

(c)Since $T(A) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, T(B) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, T(C) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$,We can take X to be

$$\frac{1}{2}A - 2B + C$$
$$= \begin{bmatrix} y_2 & 0 \\ 0 & -y_2 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ -0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} y_2 & -2 \\ 1 & -y_2 \end{bmatrix}.$$

Q6 Apply Elementary Row and Column Operations to A and C to obtain $D_1 = \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix}$ for A and $D_2 = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$ for C.

Where $r = rankA, s = rankC$.

Let $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$. Then M can be converted to $M_1 = \begin{bmatrix} D_1 & C_1 \\ 0 & D_2 \end{bmatrix}$ via elementary row and column operations.

Furthermore, the pivots in D_1 and D_2 can be used to eliminate the nonzero entries in C_1, to obtain

$$M_2 = \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & C_2 \\ 0 & 0 & I_s & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In conclusion,
$$rankM = rankM_1 = rankM_2 = r + s + rankC_2$$
$$\geq r + s = rankA + rankC$$

23 Spring Midterm Question

线性代数 23春季 期中试题 发布版

Q1.(20 points, 4 points each)
暂无选择题。

Q2.(25 points, 5 points each) Fill in the blanks

(1)Let $u, v \in \mathbb{R}^n$ with $\|u\| = 2$, $\|v\| = 4$ and $u^T v = 6$. Then
$\|3u - v\| = $ _____.

(2) Let A be an $n \times n$ matrix with $A^2 = -A$ and let I be the $n \times n$ identity matrix. Then $(A - I)^{-1} = $ _____.

(3)Let $A = \begin{bmatrix} 1 & a & a & a \\ a & 1 & a & a \\ a & a & 1 & a \\ a & a & a & 1 \end{bmatrix}$ with $rank\,(A) = 1$. Then $a = $ _____.

(4)Let α be a nonzero 3-dimensional real column vector in $\mathbb{R}^3$ with $\alpha^T \alpha \neq 1$, and I_3 be the 3×3 identity matrix. Then rank $\left(I_3 - \alpha\alpha^T\right) = $ _____.

(5) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.
Then the least squares solution to $Ax = b$ is $\widehat{x} = $ _____.

Q3 (15 points) Let $\alpha \in R$, and

$$A_\alpha = \begin{bmatrix} 1 & -\alpha & 1+\alpha \\ \alpha & \alpha^2 & \alpha \\ -\alpha & 1 & -2 \end{bmatrix}.$$

(a) By applying row operations, determine for which values of α is the matrix A_α invertible?

(b) Find the values of α such that the nullspace of A_α, $N(A_\alpha)$, has dimension 1?

(c) Let $\alpha = 2$. Write down the matrix inverse of A_α.

Q4.(10points)

Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 9 & -3 & 1 \\ -1 & 2 & 2 \end{bmatrix}.$$

Find an LU factorization of A.

Q5.(10 points)

Consider the following system ofline are quations:

$$(I) : \begin{cases} x_1 + x_2 = 0, \\ x_2 - x_4 = 0. \end{cases}$$

Note that the above system (I) has four variables x_1, x_2, x_3, x_4.
Suppose another homogeneous
system of linear equations (II) has special solutions

$$u = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, v = \begin{bmatrix} -1 \\ 2 \\ 2 \\ 1 \end{bmatrix}.$$

Find the common nonzero solutions of systems (I) and (II).

Q6.(8 points)

Let $R^{2\times 2}$ be the vector space consisting of all 2×2 real matrices.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$E = \left\{ E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

(a) Show that E is a basis for $\mathbb{R}^{2\times 2}$.

(b) Show that $T:\mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}, X \mapsto XA$ is a linear transformation.

(c)Find the matrix representation of T with respect to the ordered basis $E_{11}, E_{12}, E_{21}, E_{22}$.

Q7.(6 points) Let A,B be two $m \times n$ matrices. Prove

(a)$rank\,(A + B) \leq rank A + rank B$

(b)$rank\,(A + B) \geq rank A - rank B$

Q8.(6 points)

Let A be an $m \times n$ matrix with rank r. Show that there exist an $m \times r$ matrix B and an $r \times n$ matrix C such that $A = BC$ and both B and C have rank r.

(共25分，每小题5分)填空题。

(1)设 $u, v \in \mathbb{R}^n$ 且 $\|u\| = 2, \|v\| = 4$ 以及 $u^T v = 6$. 则
$\|3u - v\| = $ _____.

(2)设 A 为一个 n 阶矩阵，且 $A^2 = -A, I$ 表示 n 阶单位矩阵。则
$(A - I)^{-1} = $ _____.

(3)设 $A = \begin{bmatrix} 1 & a & a & a \\ a & 1 & a & a \\ a & a & 1 & a \\ a & a & a & 1 \end{bmatrix}$ 且 $rank(A) = 1$. 则 $a = $ _____.

(4)设 $\alpha \in \mathbb{R}^3$ 为一个非零列向量且 $\alpha^T \alpha \neq 1$, I_3 为 3×3 单位矩阵. 则
$\mathrm{rank}\left(I_3 - \alpha \alpha^T\right) = $ _____

(5)
令 $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.
则 $Ax = b$ 的最小二乘解 $\widehat{x} = $ _____.

Q3 (15 points)设α为实数，A_α为

$$A_\alpha = \begin{bmatrix} 1 & -\alpha & 1+\alpha \\ \alpha & \alpha^2 & \alpha \\ -\alpha & 1 & -2 \end{bmatrix}.$$

(a)对矩阵A_α做初等行变换，α为何时时，A_α为可逆矩阵？

(b)α取何值时，矩阵A_α的零空间的维数等于1？

(c)设 $\alpha = 2$ ，求矩阵A_α的逆矩阵.

Q4.(10 points)设

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 9 & -3 & 1 \\ -1 & 2 & 2 \end{bmatrix}.$$

求A的一个LU分解

Q5.(10 points) 考虑以下线性方程组：

$$(I) : \begin{cases} x_1 + x_2 = 0, \\ x_2 - x_4 = 0. \end{cases}$$

注意上述方程组(I)有四个变量x_1, x_2, x_3, x_4。假设另一个齐次线性方程组(II)有特殊解

$$u = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, v = \begin{bmatrix} -1 \\ 2 \\ 2 \\ 1 \end{bmatrix}.$$

找出方程组(I)和(II)的共同非零解。

Q6.(8 points)

设$R^{2\times 2}$为所有2×2实矩阵构成的向量空间. 设$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, 且

$$E = \left\{ E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

(a)证明：E为$R^{2\times 2}$的一组基。

(b)证明：$T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}, X \mapsto XA$为线性变换

(c)求 T 在有序基$E_{11}, E_{12}, E_{21}, E_{22}$下的矩阵表示

Q7.(6 points)设A, B都为$m \times n$矩阵，证明：

(a)$rank\,(A + B) \leq rankA + rankB$

(b)$rank\,(A + B) \geq rankA - rankB$

Q8.(6 points)

设 A 为一个秩为r的$m \times n$矩阵. 证明：存在一个$m \times r$矩阵B和一个$r \times n$矩阵C,使得 $A = BC$,其中B, C的秩都为r.

23 Spring Midterm Answer

线性代数 23春季 期中试题答案 发布版

Q1 (1)A (2)D (3)C (4)B (5)B

Q2

(1)4

(2)$-\dfrac{1}{2}A - I$

(3)1

(4)3

(5)$\begin{bmatrix} \dfrac{5}{3} \\ -\dfrac{1}{3} \end{bmatrix}$

Q3

(a) $\alpha \neq 0, 1, -3$

(b) $\alpha = 0, 1, -3$

(c) $\alpha = 2$

$$A_\alpha^{-1} = \begin{bmatrix} -\dfrac{1}{2} & -\dfrac{1}{20} & -\dfrac{4}{5} \\ 0 & \dfrac{1}{5} & \dfrac{1}{5} \\ -\dfrac{1}{2} & \dfrac{3}{20} & \dfrac{2}{5} \end{bmatrix}$$

Q4

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 9 & 1 & 0 \\ -1 & -\frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -12 & -8 \\ 0 & 0 & 1 \end{bmatrix}$$

Q5

$$k \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, k \neq 0.$$

Q6 (a)

- linear independent
- E spans $\mathbf{R}^{2 \times 2}$.

(b)

$$\cdot \, T(X + Y) = T(X) + T(Y),$$
$$\cdot \, T(\lambda X) = \lambda T(X).$$

(c)

$$M = \begin{bmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & 0 & d \end{bmatrix}.$$

Q7(a)

Pivot columns of A: $a_1, a_2, \cdots, a_r$;

Pivot columns of $B : b_1, b_2, \cdots, b_s$;

$rankA = r, rankB = s.$

$$V = span\,(a_1, \cdots, a_s, b_1, \cdots, b_s).\ dimV \leq r + s$$
$$= span\,(a_1, \cdots, a_s, b_1, \cdots, b_s) \supseteq C\,(A + B)$$
$$\implies dimC(A + B) \leq dimV$$
$$\implies rank(A + B) \leq rank(A) + rank(B)$$

(b)

$$A + B - B = A$$

$$rank(A + B - B) \leq rank(A + B) + rank(-B) \ldots \text{by (a)}$$

$$rank(A + B) + rank(-B) = rank(B)$$

$$\implies rankA - rankB \leq rank(A + B)$$

Q8 P_1, Q_1 invertible.

$$A = P_1 \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q_1$$

$$= P_1 \underbrace{\begin{bmatrix} I_r \\ 0 \end{bmatrix}}_{B} \underbrace{[I_r \quad 0]Q_1}_{C}$$

后记

感谢您使用这本小册子。您可通过github issue进行反馈，您的意见对我们改进服务和拓展其他科目非常重要。

- Github Repo: Open_Notes_SUSTech: 南方科技大学一位23级本科生的学习笔记，论文和项目
  https://github.com/LIUBINfighter/Open_Notes_SUSTech
- 电子书：MA113 线性代数 学习指南
  https://liubinfighter.github.io/Jay_Survival_Manual/#/doc/ma113/

目前的工程文件以及草稿不定期上传到仓库线性代数栏目。你也可以下载往期结项的文件了解我的工作方式，欢迎来戳。

同时，向各位同学和高年级助教征求之后表格中留空的材料，包括照片，扫描件，手写件，markdown或TeX，演示文稿等文件，二版时会将您加入贡献者栏并赠与样书，如果你是愿意帮助的热心人，助教或互助课堂的主讲人，能够予以手稿提供，OCR，排版，校对，答案审核一类的协助就更好了。

项目进度

线性代数期中考试试题

	原卷	答案	完整度	发布时间
20Fall	可用	可用	☑	与印刷版同时
21Spring	可用	可用	☑	与印刷版同时

	原卷	答案	完整度	发布时间
21Fall	可用	无	❌	搁置
22Spring	无	无	❌	搁置
22Fall	相片质量	无	❌	搁置
23Spring	无选择题	手写答案	❌	与印刷版同时
23Fall	Released	Released	✅	24.9.14
24Spring	Released	Released	✅	24.9.10

致谢

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感谢Obsidian，几乎完美潇洒的笔记管理软件和写作工作流。

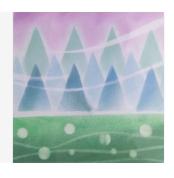感谢SimpleTex提供准确高速LaTeX/OCR服务. 希望**SimpleTex**能被更多人认识和使用，发展越来越好！

感谢Pandoc提供便捷快速的markdown转pdf服务. 这在优化排版上起了非同小可的帮助。

感谢我的三位舍友，宽容我在寝室里每天睡过头和半夜敲键盘的行为，并和我在闲暇之余一起玩游戏解压。

以及我自己，做了一点微小的工作。

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