24 Spring Midterm Question

24 Spring Midterm Question 中英试题分离 分页留空版本
线性代数2023-2024学年春季学期期中考试

1. (共15分，每小题3分)选择题，只有一个选项是正确的.

(1) 假定

$$\alpha_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 7 \\ 3 \\ c \end{bmatrix}.$$

若$\alpha_1, \alpha_2, \alpha_3$线性相关，则$c$的取值为

$(A)5.$
$(B)6.$
$(C)7.$
$(D)8.$

(2) 设 A 为一个 $m \times n$ 实矩阵，b 为一个 m 维实列向量，以下说法一定是**正确**的是？

(A)若$A\mathbf{x} = \mathbf{b}$无解，则$A\mathbf{x} = \mathbf{0}$只有零解.
(B)若$A\mathbf{x} = \mathbf{0}$有无穷多解，则$A\mathbf{x} = \mathbf{b}$有无穷多解。
(C)若$m < n$,则$A\mathbf{x} = \mathbf{b}$和$A\mathbf{x} = \mathbf{0}$都有无穷多解。
(D)若A的秩为n,则$A\mathbf{x} = 0$只有零解.

(3) 如果以下线性方程组有两个自由变量

$$\begin{cases} x_1 + 2x_2 - 4x_3 + 3x_4 = 0, \\ x_1 + 3x_2 - 2x_3 - 2x_4 = 0, \\ x_1 + 5x_2 + (5-k)x_3 - 12x_4 = 0, \end{cases}$$

k的取值为

$(A)5.$
$(B)4.$
$(C)3.$
$(D)2.$

(4) 设 $u, v \in \mathbb{R}^3, \lambda \in \mathbb{R}$. 以下说法**错误**的是?

(A)如果u和v为满足$u^T v = 0$的非零向量，则u和v线性无关.
(B)如果$u+v$和$u-v$正交，则$||u|| = ||v||$.
$(C)u^T v = 0$当且仅当 $u = 0$ or $v = 0$.
$(D)\lambda v = 0$当且仅当 $v = 0$ or $\lambda = 0$.

(5) 设A和B都为n阶矩阵.以下说法**错误**的是?

(A)如果A, B为对称矩阵，则AB也为一个对称矩阵。
(B)如果A, B 为可逆矩阵，则 AB 也为一个可逆矩阵.
(C)如果A, B为置换矩阵，则AB也为一个置换矩阵.
(D)如果A, B为上三角矩阵，则AB也为上三角矩阵。

2. (20 points, 5 points each) 填空，共4题。

(1)$A=\begin{bmatrix}1 & 0 & 0\\ a & 1 & 0\\ b & 3 & 2\end{bmatrix}, a, b\in\mathbb{R}$, 则$A^{-1}=$______.

(2)设A为一个4×3的实矩阵，B为$\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ -1 & 0 & 3\end{bmatrix}$.

如果矩阵A的秩为 2,则AB的秩为______.

(3)设 $A=\begin{bmatrix}1 & -1 & 1\\ -1 & 1 & -1\\ 2 & -2 & 2\end{bmatrix}$, 则 $A^{2024}=$______.

(4)考虑以下线性方程组：

$$A\mathbf{x}=\mathbf{b}:\begin{cases} x &= 2\\ y &= 3\\ x+y &= 6\end{cases}$$

该线性方程组的最小二乘解为______.

3. (10points)设

$$A=\begin{bmatrix}1 & -2 & 3\\ 2 & -5 & 1\\ 1 & -4 & -7\end{bmatrix}.$$

求矩阵A的一个LU分解

4. 考虑以下 4× 5 矩阵 A 以及 4 维列向量 $\mathbf{b}$:

$$A = \begin{bmatrix} 0 & 2 & 4 & 1 & 6 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 4 & 10 & 1 & 2 \\ 0 & -1 & -5 & 1 & 7 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 10 \end{bmatrix}$$

(a)分别求矩阵 A 的四个基本子空间的一组基向量。

(b)求$Ax = b$的所有解.

5. (20 points)设 Let $A=\begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}$, T为按照以下方式定义的从$\mathbb{R}^{2\times 2}$到 $\mathbb{R}^{2\times 2}$线性变换：

$$T(X)=XA+AX, X\in\mathbb{R}^{2\times 2}.$$

其中$\mathbb{R}^{2\times 2}$表示所有2×2实矩阵构成的向量空间.

(a)求T在以下有序基

$$v_1=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, v_2=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, v_3=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, v_4=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

下的矩阵表示.

(b)求一个矩阵B使得

$$T(B)=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.$$

(c)求一个矩阵C使得

$$T(C)=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}.$$

6. (5 points) 设A, B为满足$A^2 = A$和$B^2 = B$的n阶实矩阵.证明:
如果$(A+B)^2 = A+B$,则$AB = O$.其中O 表示n阶零矩阵。

7. (6 points)设A为 3$\times$2矩阵，B 为 2$\times$3矩阵，并且

$$AB=\begin{bmatrix} 8 & 0 & -4 \\ -\frac{3}{2} & 9 & -6 \\ -2 & 0 & 1 \end{bmatrix}.$$

(a)计算$(AB)^2$.

(b)求BA.

1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.
(共15分，每小题3分)选择题，只有一个选项是正确的.

$(1) Let$

$$\alpha_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 7 \\ 3 \\ c \end{bmatrix}.$$

If $\alpha_1, \alpha_2, \alpha_3$ are linearly dependent, then c equals
$(A)5.$
$(B)6.$
$(C)7.$
$(D)8.$

(2) let A be an $m \times n$ real matrix and b be an $m \times 1$ real column vector. Which of the following statements is correct?
(A) If $A\mathbf{x} = \mathbf{b}$ does not have any solution, then $A\mathbf{x} = \mathbf{0}$ has only the zero solution.
(B) If $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, then $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.
(C) If $m < n$, both $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$ have infinitely many solutions.
(D) If the rank of A is n, then $A\mathbf{x} = \mathbf{0}$ has only the zero solution.

(3) For which value of k does the system

$$\begin{cases} x_1 + 2x_2 - 4x_3 + 3x_4 = 0, \\ x_1 + 3x_2 - 2x_3 - 2x_4 = 0, \\ x_1 + 5x_2 + (5-k)x_3 - 12x_4 = 0, \end{cases}$$

have exactly two free variables?

$(A)5.$

$(B)4.$

$(C)3.$

$(D)2.$

(4) Let $u, v \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$. Which of the following statements is false?

(A) If u and v are nonzero vectors satisfying $u^T v = 0$, then u and v are linearly independent.

(B)If $u + v$ is orthogonal to $u - v$, then $||u|| = ||v||$.

$(C)u^T v = 0$ if and only if $u = 0$ or $v = 0$.

$(D)\lambda v = 0$ if and only if $v = 0$ or $\lambda = 0$.

(5) Let A and B be two $n \times n$ matrices. Which of the following assertions is **false**?

(A) If A, B are symmetric matrices, then AB is a symmetric matrix.

(B) If A, B are invertible matrices, then AB is an invertible matrix.

(C) If A, B are permutation matrices, then AB is a permutation matrix.

(D) If A, B are upper triangular matrices, then AB is an upper triangular matrix.

2. (20 points, 5 points each) Fill in the blanks.

(1)Let $A=\begin{bmatrix}1&0&0\\a&1&0\\b&3&2\end{bmatrix}, a,b\in\mathbb{R}$. Then $A^{-1}=$ ______.

(2) Let A be a 4×3 real matrix with rank 2 and $B=\begin{bmatrix}1&0&2\\0&2&0\\-1&0&3\end{bmatrix}$.

Then the rank AB is ______.

(3)Let $A=\begin{bmatrix}1&-1&1\\-1&1&-1\\2&-2&2\end{bmatrix}$. Then $A^{2024}=$ ______.

(4) Consider the system of linear equations:

$$Ax=b:\begin{cases}x&=&2\\y&=&3\\x+y&=&6\end{cases}$$

The least-squares solution for the system is ______.

3. (10points)Let

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 1 \\ 1 & -4 & -7 \end{bmatrix}.$$

Find an LU factorization of A.

4. (24 points) Consider the following 4×5 matrix A and 4-dimensional column vector b:

$$A = \begin{bmatrix} 0 & 2 & 4 & 1 & 6 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 4 & 10 & 1 & 2 \\ 0 & -1 & -5 & 1 & 7 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 10 \end{bmatrix}$$

(a) Find a basis for each of the four fundamental subspaces of A.

(b) Find the complete solution to $A\mathbf{x} = \mathbf{b}$.

5. (20 points) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ and T be the linear transformation from $R^{2\times 2}$ to $R^{2\times 2}$ defined by

$$T(X) = XA + AX,\ X \in \mathbb{R}^{2\times 2}.$$

Where $\mathbb{R}^{2\times 2}$ denotes the vector space consisting of all 2×2 real matrices.

(a) Find the matrix representation of T with respect to the following ordered basis

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

$(\mathbf{b})$ Find a matrix B such that

$$T(B) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(c) Find a matrix C such that

$$T\left(C\right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

6. (5 points) Let A, B be two $n \times n$ real matrices satisfying $A^2 = A$ and $B^2 = B$. Show that if $(A+B)^2 = A+B$, then $AB = O$. Where O denotes the $n \times n$ zero matrix.

7. (6 points) Let A be a 3×2 matrix, B be a 2×3 matrix such that

$$AB = \begin{bmatrix} 8 & 0 & -4 \\ -\frac{3}{2} & 9 & -6 \\ -2 & 0 & 1 \end{bmatrix}.$$

(a)Compute$(AB)^2$.

(b) Find BA.

24 Spring Midterm Answer

线性代数2023-2024学年春季学期期中考试

快速对答案（详解在之后）

一、D D C C A

二、

$$(1)A^{-1}=\begin{bmatrix}1 & & \\ -a & 1 & \\ (3a-b)/2 & -3/2 & 1/2\end{bmatrix}\qquad (2)2$$

$$(3)4^{2023}A=4^{2023}\begin{bmatrix}1 & -1 & 1\\ -1 & 1 & -1\\ 2 & -2 & 2\end{bmatrix}\qquad (4)\begin{bmatrix}\frac{7}{3}\\ \frac{10}{3}\end{bmatrix}$$

三、$A=LU=\begin{bmatrix}1 & & \\ 2 & 1 & \\ 1 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & -2 & 3\\ & -1 & -5\\ & & 0\end{bmatrix}$

四、(a)

$$(1)C(A)=span\left\{\begin{bmatrix}2\\1\\4\\-1\end{bmatrix},\begin{bmatrix}4\\1\\10\\-5\end{bmatrix},\begin{bmatrix}6\\3\\2\\7\end{bmatrix}\right\}$$

$$(2)\, C\left(A^T\right)=\operatorname{Span}\left[\begin{bmatrix}0\\2\\4\\1\\6\end{bmatrix},\begin{bmatrix}0\\1\\1\\1\\3\end{bmatrix},\begin{bmatrix}0\\4\\10\\1\\2\end{bmatrix}\right]$$

$$(3)\, X=K_1\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}+K_4\begin{bmatrix}0\\-\frac{3}{2}\\\frac{1}{2}\\1\\0\end{bmatrix}\left(k_1,k_2\in\mathbb{R}\right)$$

$$(4) y=k_0\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix},k_0\in R$$

(b)

$$x=x_p+x_n=\begin{bmatrix}1\\-5\\1\\3\\1\end{bmatrix}+k_1\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}+k_4\begin{bmatrix}0\\-3/2\\1/2\\1\\0\end{bmatrix}\quad k_1,k_4\in\mathbb{R}$$

五、略，解析部分有方法

六、证明略

七、

$$(AB)^2=\begin{bmatrix}72&0&-36\\-\frac{27}{2}&81&-54\\-18&0&9\end{bmatrix}$$
$$BA=9I$$

填空及大题详解

二、(1)

$$[A \quad I]=\begin{bmatrix}1 & & & \vdots & 1 & & \\ a & 1 & & \vdots & & 1 & \\ b & 3 & 2 & \vdots & & & 1\end{bmatrix}\to\begin{bmatrix}1 & & & \vdots & 1 & & \\ 0 & 1 & & \vdots & -a & 1 & \\ 0 & 3 & 2 & \vdots & -b & 0 & 1\end{bmatrix}$$

$$\to\begin{bmatrix}1 & & & \vdots & 1 & & \\ 0 & 1 & & \vdots & -a & 1 & \\ 0 & 0 & 2 & \vdots & 3a-b & -3 & 1\end{bmatrix}$$

$$\to\begin{bmatrix}1 & & & \vdots & 1 & & \\ & 1 & & \vdots & -a & 1 & \\ & & 1 & \vdots & (3a-b)/2 & -3/2 & 1/2\end{bmatrix}=[I \quad A^{-1}]$$

(2)法1令

$$A_{4\times 3}=\begin{bmatrix}1 & & \\ & & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}$$

$$AB=\begin{bmatrix}1 & & \\ & & 1 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots\end{bmatrix}\begin{bmatrix}1 & 0 & 2 \\ 0 & 2 & 0 \\ -1 & 0 & 3\end{bmatrix}=\begin{bmatrix}1 & 0 & 2 \\ -1 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}$$
$$\to R(AB)=2$$

法2

$$R(A)=2, R(B)=3$$
$$R(AB) \geq R(A)+R(B)-n=2+3-3=2$$
$$R(AB) \leq min\{R(A), R(B)\}=2$$

$$\Longrightarrow R(AB)=2$$

(3)剥蒜（爆算）法 直接计算A^2, A^3的得出规律

(4)$A=\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, b=\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$

三、

$$A=\begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 1 \\ 4 & -4 & -7 \end{bmatrix}$$
$$E_{21}(-2)A=\begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & -5 \\ 1 & -4 & -7 \end{bmatrix}$$
$$E_{31}(-1)E_{21}(-2)A=\begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & -5 \\ 0 & -2 & -10 \end{bmatrix}$$

$$E_{32}(-2)E_{31}(-1)E_{31}(-2)A=\begin{bmatrix} 1 & -2 & -3 \\ 0 & -1 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$
$$A=E_{21}^{-1}(-2)E_{31}(-1)^{-1}E_{32}^{-1}(-2)\begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$
$$=\begin{bmatrix} 1 & & \\ 2 & 1 & \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 0 \end{bmatrix}=LU$$

四、(a) (1)C(A),对A行变换

$$A \to \begin{bmatrix} 0 & \boxed{2} & 0 & 3 & 0 \\ 0 & 0 & \boxed{2} & -1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow C(A) = span\left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 10 \\ -5 \end{bmatrix}, \right.$$

(2)$C(A^T)$　对A^T行变换

$$A^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 4 & -1 \\ 4 & 1 & 10 & -5 \\ 1 & 1 & 1 & 1 \\ 0 & 3 & 2 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & \cancel{3} & -\cancel{6} & \cancel{9} \\ 0 & 3 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} \boxed{1} & 1 & 1 & 1 \\ & \boxed{1} & -2 & 3 \\ & & \boxed{10} & 10 \\ & & & 0 \\ & & & 0 \end{bmatrix} \Rightarrow C(A^T) = Span\left\{ \begin{bmatrix} 0 \\ 2 \\ 4 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 10 \\ 1 \\ 2 \end{bmatrix} \right\}$$

(3)$N(A)$

$$Ax = 0 \Rightarrow \begin{bmatrix} 0 & 2 & 0 & 3 & 0 \\ 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix} = 0 \Rightarrow \begin{cases} x_1, x_4 \in \mathbb{R} \\ x_2 = -\frac{3}{2}x_4 \\ x_3 = \frac{1}{2}x_4 \\ x_5 = 0 \end{cases}$$

$$\Rightarrow x = k_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ -3/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} (k_1, k_2 \in \mathbb{R})$$

$(4) N(A^T)$

$$A^T y = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ & 1 & -2 & 3 \\ & & 10 & -10 \\ & & & 0 \\ & & & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} y_1 = -y_4 \\ y_2 = -y_4 \\ y_3 = y_4 \\ y_4 \in R \end{cases} \Rightarrow y = k_0 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, k_0 \in R$$

$(b) Ax = b$ 时 利用高斯消元化简增广矩阵(略)

令出一特解 $x_p = [1 \ -5 \ 1 \ 3 \ 1]^T$

$$x = x_p + x_n = \begin{bmatrix} 1 \\ -5 \\ 1 \\ 3 \\ 1 \end{bmatrix} + k_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + k_4 \begin{bmatrix} 0 \\ -3/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} \quad k_1, k_4 \in \mathbb{R}$$

五、令 $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\text{则 XA} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} a & a+2b \\ c & c+2a \end{bmatrix}$$

$$AX = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ 2c & 2d \end{bmatrix}$$

$$T(x) = XA + AX = \begin{bmatrix} 2a+c & a+3b+d \\ 3c & c+4d \end{bmatrix}$$

$$= (2a+c)V_1 + (a+3b+d)V_2 + (3c)V_3 + (c+4d)V_4$$

$$T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = (V_1\ V_2\ V_3\ V_4)\begin{pmatrix} 2a+c \\ a+3b+d \\ 3c \\ c+4d \end{pmatrix}$$

六、$(A+B)^2$

$= (A+B)(A+B) = A^2 + AB + BA + B^2 = A + B$

$\because A^2 = A, B^2 = B$

$\therefore AB + BA = 0 \quad \ldots\ldots\ldots ①$

$B(AB+BA) = BAB + B^2A = (BA)B + B^2A = -AB^2 + B^2A$

又 $\because B^2 = B$

$\therefore -AB^2 + B^2A = -AB + BA = 0 \quad \ldots\ldots\ldots ②$

联立①② $\begin{cases} AB+BA=0 \\ -AB+BA=0 \end{cases} \Rightarrow AB = 0$, 得证

七、

$$(a)(AB)^2 = (AB)(AB) = \begin{bmatrix} 8 & 0 & -4 \\ -\frac{3}{2} & 9 & -6 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 8 & 0 & -4 \\ -\frac{3}{2} & 9 & -6 \\ -2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 72 & 0 & -36 \\ -\frac{27}{2} & 81 & -54 \\ -18 & 0 & 9 \end{bmatrix}$$

(b)

$(AB)^2 = 9(AB)$

$R(A) \geq R(AB) = 2 \Rightarrow R(A) = 2$, 同理 $R(B) = 2$

A行满秩, 令 $XA = I_2$, X为A左逆

B列满秩, 令 $BY = I_2$, Y为B右逆

$\therefore BA = (XA)(BA)(BY) = X(AB)^2Y = 9XABY = 9I$

23 Fall Midterm Question

线代23秋期中试题 发布版 中英分离

1.(共 15 分，每小题 3 分) 选择题，只有一个选项是正确的.

(1)设$\alpha_1, \alpha_2, \alpha_3$为矩阵$A$的零空间$N(A)$的一组基. 下列哪一组向量也是矩阵A的零空间的一组基?

(A)$\alpha_1 + \alpha_2 - \alpha_3, \alpha_1 + \alpha_2 + 5\alpha_3, 4\alpha_1 + \alpha_2 - 2\alpha_3.$

(B)$\alpha_1 + 2\alpha_2 + \alpha_3, 2\alpha_1 + \alpha_2 + 2\alpha_3, \alpha_3 + \alpha_1 + \alpha_2,$

(C)$\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3.$

(D)$\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \alpha_3 - \alpha_1.$

(2)以下说法一定是正确的是?

(A)如果矩阵A的列向量线性无关，那么对任意的$\mathbf{b}$, $A\mathbf{x} = \mathbf{b}$有唯一的解.

(B)任意5×7矩阵的列向量一定是线性相关的.

(C)如果矩阵A 的列向量线性相关，该矩阵的行向量也线性相关.

(D)一个10×12矩阵的行空间和列空间可能具有不同的维数

$$(3)\text{设}\,\alpha_1 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 6 \\ 2 \\ -16 \end{bmatrix}, \beta = \begin{bmatrix} 2 \\ t \\ 3 \end{bmatrix}$$

当$t = ()$时, β可用$\alpha_1, \alpha_2, \alpha_3$线性表示

(A)1.

(B)3.

(C)6.

(D)9.

(4) 以下说法一定是正确的事?

(A) 设E为一个可逆矩阵.如果A, B矩阵满足$EA = B$,则A和B的列空间相同

(B) 设A为秩为1的n阶的方阵, 则$A^n = cA$, 其 中 n 为 正 整 数 , c 为 实 数 .

(C) 如果A, B 为对称矩阵, 则AB为对称矩阵. 如果矩阵 A 为一个行满秩矩阵, 那么 $Ax = 0$ 只 有 零 解 .

(D) 如果矩阵A为一个列满秩矩阵，那么$Ax = 0$只有零解。

设A与B都为n阶矩阵，A为非零矩阵，且AB=0，则

(1)$BA = 0$

(2)$B = 0$

(3)$(A+B)(A-B) = A^2 - B^2$

(4)$\mathrm{rank}\ B < n$.

2.(共 25 分，每小题 5 分)填空题.

(1)记所有7×7实矩阵构成的向量空间为$M_{7\times 7}(\mathbb{R})$, W为$M_{7\times 7}(\mathbb{R})$中所有斜对称矩阵构成的子空间，则 $\dim W =$ ______.

如果$A^T = -A$, A就称之为斜对称的。

(2)设A, B为两个 可逆矩阵，假设的逆矩阵为，期中O为的零矩阵，则$D =$ ______.

(3)设 $A = \begin{bmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{bmatrix}$且 $rank(A) < 4$，则 $a =$ ______.

(4)考虑一下线性方程组:

$$A\mathbf{x}=\mathbf{b}:\begin{cases} x & + & 2y & = & 1 \\ x & - & y & = & 2 \\ & & y & = & -1. \end{cases}$$

该线性方程组的最小二乘解为______.

(5)设H为如下定义的一个R^3中的子空间

$$H=\left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\middle| x_1+2x_2+x_3=0\right\}.$$

一个和子空间H正交的**单位向量**为______.

3.(24 points)考虑以下这个4×5矩阵A以及他的简化阶梯形矩阵R:

$$A=\begin{bmatrix} 1 & 2 & * & 1 & * \\ 0 & 1 & * & 1 & * \\ -1 & 1 & * & 3 & * \\ 2 & 0 & * & 1 & * \end{bmatrix},\ R=\begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a)分别求矩阵 A 的四个基本子空间的一组基向量.

(b)求出矩阵 A 的第三个列向量.

4.(15 points) 设

$$A=\begin{bmatrix}1 & -1 & -1\\ 2 & a & 1\\ -1 & 1 & a\end{bmatrix},\ B=\begin{bmatrix}2 & 2\\ 1 & a\\ -a-1 & -2\end{bmatrix}.$$

当a为何值时，矩阵方程$AX=B$无解、有唯一解、有无穷多解？在有解时，求解此方程，这里的X为一个3×2矩阵

5.（15 points）设$M_{2\times 2}(\mathbb{R})$为所有$2\times 2$实矩阵构成的向量空间，设

$$A=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\ B=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\ C=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}.$$

考虑以下映射

$$T: M_{2\times 2}(\mathbb{R})\to\mathbb{R}^3,\ T(X)=\begin{bmatrix}tr(A^TX)\\ tr(B^TX)\\ tr(C^TX)\end{bmatrix},$$

对任意的 2×2实矩阵, 其中 $tr(D)$ 表示n阶矩阵D的迹.

方阵D的迹是指D的对角元之和，也即

$$tr(D)=d_{11}+d_{22}+\cdots+d_{nn}$$

(a)证明T是一个线性变换
(b)求T在$M_{2\times 2}(R)$的一组基

$$v_1=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, v_2=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, v_3=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, v_4=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

以及R^3的标准基

$$e_1=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}, e_2=\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}, e_3=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$$

下的矩阵表示.

(c)是否可以找到一个矩阵 X 使得$T(X)=\begin{bmatrix}1\\ -2\\ 1\end{bmatrix}$?如果可以,请求出一个符合要求的矩阵$X$. 如果不存在,请说明理由.

6.（6 points）设A为$m \times n$矩阵，B为$m \times p$矩阵,C为$q \times p$矩阵.证明:

$$\operatorname{rank}\begin{bmatrix} A & B \\ O & C \end{bmatrix} \geq \operatorname{rank} A + \operatorname{rank} C,$$

其中O为$q \times n$的零矩阵

1.(15 points, 3 points each) Multiple Choice. Only one choice is correct.

(1)Suppose that $\alpha_1, \alpha_2, \alpha_3$ are a basis for nullspace of a matrix $A, N(A)$. Which of the following lists of vectors is also a basis for $N(A)$?

(A)$\alpha_1 + \alpha_2 - \alpha_3, \alpha_1 + \alpha_2 + 5\alpha_3, 4\alpha_1 + \alpha_2 - 2\alpha_3.$

(B)$\alpha_1 + 2\alpha_2 + \alpha_3, 2\alpha_1 + \alpha_2 + 2\alpha_3, \alpha_3 + \alpha_1 + \alpha_2,$

(C)$\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3.$

(D)$\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \alpha_3 - \alpha_1.$

(2)Which of the following statements is correct?

(A) If the columns of A are linearly independent, then $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every b.

(B) Any 5×7 matrix has linearly dependent columns.

(C) If the columns of a matrix A are linearly dependent, so are the rows.

(D) The column space and row space of a 10×12 matrix may have different dimensions.

(3)Let

$$\alpha_1 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \ \alpha_2 = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 6 \\ 2 \\ -16 \end{bmatrix}, \ \beta = \begin{bmatrix} 2 \\ t \\ 3 \end{bmatrix}.$$

β can be written as a linear combination of $\alpha_1, \alpha_2, \alpha_3$ if $t =$ ( )

(A)1.

(B)3.

(C)6.

(D)9.

(4) Which of the following statements is correct?

(A) Suppose that $EA = B$ and E is an invertible matrix, then the column space of $A$and the column space of B are the same.

(B) Let A be a $n \times n$ matrix with rank 1, then $A^n = cA$, where n is a positive integer and c is a real number.

(C) Let A, B be symmetric matrices, then AB is symmetric.

(D) If A is of full row rank, then $Ax = 0$ has only the zero solution.

(5)Let A and B be two $n \times n$ matrices. If A is a non-zero matrix and AB = 0, then

(1)$BA = 0$

(2)$B = 0$

(3)$(A + B)(A - B) = A^2 - B^2$

(4)$\text{rank } B < n$.

2.$(25 \text{ points, } 5 \text{ points each})$ Fill in the blanks.

(1)Denote the vector space of 7×7 real matrices by $M_{7\times 7}(\mathbb{R})$, and let W be the subspace of $M_{7\times 7}(\mathbb{R})$ consisting of skew-symmetric real matrices, then dim $W =$ ______.

A matrix A is called skew symmetric if $A^T = -A$.

(2)Let A, B be two $n \times n$ invertible matrices. Suppose the inverse of $\begin{bmatrix} A & C \\ O & B \end{bmatrix}$ is $\begin{bmatrix} A^{-1} & D \\ O & B^{-1} \end{bmatrix}$, where O is the $n \times n$ zero matrix. Then $D =$ ______.

(3)Let $A = \begin{bmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{bmatrix}$ with $rank(A) < 4$. Then $a =$ ______.

(4)Consider the system of linear equations

$$A\mathbf{x} = \mathbf{b} : \begin{cases} x & + & 2y & = & 1 \\ x & - & y & = & 2 \\ & & y & = & -1. \end{cases}$$

The least-squares solution for the system is______.

(5)

Let H be the subspace of R^3 be defined as follows:

$$H = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_1 + 2x_2 + x_3 = 0 \right\}.$$

A **unit** vector orthogonal to H is ______.

3.(24 points) Consider the following 4×5 matrix A with its reduced row echelon form R:

$$A = \begin{bmatrix} 1 & 2 & * & 1 & * \\ 0 & 1 & * & 1 & * \\ -1 & 1 & * & 3 & * \\ 2 & 0 & * & 1 & * \end{bmatrix}, \ R = \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for each of the four fundamental subspaces of A.

(b) Find the third column of matrix A.

4.(15points)Let

$$A=\begin{bmatrix}1 & -1 & -1\\ 2 & a & 1\\ -1 & 1 & a\end{bmatrix},\ B=\begin{bmatrix}2 & 2\\ 1 & a\\ -a-1 & -2\end{bmatrix}.$$

For which value(s) of a, the matrix equation $AX=B$ has no solution, a unique solution, or infinitely many solutions? Where X is a 3×2 matrix. Solve $AX=B$ if it has at least one solution.

5.(15 points) Let $M_2 \times 2(\mathbb{R})$ be the vector space of 2×2 real matrices. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Consider the map

$$T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^3, \; T(X) = \begin{bmatrix} tr(A^T X) \\ tr(B^T X) \\ tr(C^T X) \end{bmatrix},$$

for any 2×2 real matrix X, where $tr(D)$ denotes the trace of a matrix D.

The trace of an $n \times n$ matrix D is defined to be the sum of all the diagonal entries of D, in other words,

$$tr(D) = d_{11} + d_{22} + \cdots + d_{nn}.$$

(a) Show that T is a linear transformation.

(b) Find the matrix representation of T with respect to the ordered basis

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; v_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \; v_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; v_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

for $M_{2\times 2}$ $(\mathbb{R})$and the standard basis

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

for $\mathbb{R}^3$.

(c) Can we find a matrix X such that $T(X) = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$? If yes, please find one such matrix. Otherwise, give an explanation.

6.(6 points) Let A be an $m \times n$ matrix, B be an $m \times p$ matrix, and C be an $q \times p$ matrix. Show that

$$\operatorname{rank} \begin{bmatrix} A & B \\ O & C \end{bmatrix} \geq \operatorname{rank} A + \operatorname{rank} C,$$

where O is the $q \times n$ zero matrix.

23 Fall Midterm Answer

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Q1 (1)A (2)B（存疑, 一说D） (3)D (4)B (5)D

Q2(1)21

(2)$-A^{-1}CB^{-1}$

(3)1or-3 （存疑，一说 1 or -2）

(4)$\begin{bmatrix} -\frac{19}{11} \\ -\frac{5}{11} \end{bmatrix}$

(5)$\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{or} -\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Q3
A basis for $C(A)$

$$\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}\right\}.$$

A basis for $C(A^T)$

$$\left\{\begin{bmatrix}1\\0\\1\\0\\3\end{bmatrix},\begin{bmatrix}0\\1\\2\\0\\1\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix}\right\}\text{or}\left\{\begin{bmatrix}1\\2\\5\\1\\6\end{bmatrix},\begin{bmatrix}0\\1\\2\\1\\2\end{bmatrix},\begin{bmatrix}-1\\1\\1\\3\\1\end{bmatrix}\right\}$$

A basis for $N(A)$

$$\left\{\begin{bmatrix}-1\\-2\\1\\0\end{bmatrix},\begin{bmatrix}-3\\-1\\0\\-1\end{bmatrix}\right\}.$$

A basis for $N(A^T)$

$$\left\{\begin{bmatrix}-5\\13\\-3\\1\end{bmatrix}\right\}.$$

$$(b)\begin{bmatrix}5\\2\\1\\2\end{bmatrix}$$

Q4

Gaussian Eliminations give:

$$\begin{bmatrix}1 & -1 & -1 & \vdots & 2 & 2\\ 2 & a & 1 & \vdots & 1 & a\\ -1 & 1 & a & \vdots & -a-1 & -2\end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & -1 & \vdots & 2 & 2 \\ 0 & a+2 & 3 & \vdots & -3 & a-4 \\ 0 & 0 & a-1 & \vdots & 0 & 0 \end{bmatrix}$$

If $a = -2$, then $rankA = 2 \neq 3 = rank(A \vdots B)$,$AX = B$ has no solution.

If $a \neq 1$ and $a \neq -1$, $AX = B$ has a unique solution.

$$\begin{bmatrix} 1 & -1 & -1 & 1 & 2 \\ 0 & a+2 & 3 & 1 & -3 \\ 0 & 0 & a-1 & 1 & 1-a \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & -1 & -1 & \vdots & 2 \\ 0 & a+2 & 3 & \vdots & -3 \\ 0 & 0 & a-1 & \vdots & 1-a \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & -1 & -1 & \vdots & z \\ 0 & a+2 & 3 & \vdots & a-4 \\ 0 & 0 & a-1 & \vdots & 0 \end{bmatrix} \Rightarrow X = \begin{bmatrix} \frac{3a}{a+2} \\ \frac{a-4}{a+2} \\ 0 \end{bmatrix}.$$

$$X = \begin{bmatrix} 1 & \frac{3a}{a+2} \\ 0 & \frac{a-4}{a+2} \\ -1 & 0 \end{bmatrix}$$

If $a = 1$, $Ax = B$ has infinitely many solutions

$$\begin{bmatrix} 1 & -1 & -1 & -2 & \\ 0 & 3 & 3 & -3 & \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 & 2 & \\ 0 & 3 & 3 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix} 1 & 1 \\ -k_1 - 1 & -k_2 - 1 \\ k_1 & k_2 \end{bmatrix}, \quad k_1, k_2 \quad \text{anbitrary constants.}$$

Q5

(a)Let $X, Y \in M_{2\times 2}(R)$ and $C \in R$, than we have

$$T(CX+Y) = \begin{bmatrix} tr & A^T(CX+Y) \\ tr & B^T(CX+Y) \\ tr & C^T(CX+Y) \end{bmatrix}$$
$$= c \begin{bmatrix} tr(A^TX) \\ tr(B^TX) \\ tr(C^TX) \end{bmatrix} + \begin{bmatrix} tr(A^TX) \\ tr(B^TY) \\ tr(C^TY) \end{bmatrix}$$
$$= cT(X) + T(Y)$$

(b)

$$T(V_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{=w_1} + 0 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{=w_2} + 0 \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{=w_3}$$

$$T(V_2) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T\left(V_3\right)=\begin{bmatrix}0\\1\\1\end{bmatrix}=0\begin{bmatrix}1\\0\\0\end{bmatrix}+1\cdot\begin{bmatrix}0\\1\\0\end{bmatrix}+1\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$T\left(V_4\right)=\begin{bmatrix}0\\1\\-1\end{bmatrix}=0\begin{bmatrix}1\\0\\0\end{bmatrix}+1\cdot\begin{bmatrix}0\\0\\0\end{bmatrix}+-1\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Therefore,the matix representation of T with respect to V_1, V_2, V_3, V_4, V_4, and W_1, V_2, W_3 , is:

$$\begin{bmatrix}1 & -1 & 0 & 0\\0 & 0 & 1 & 1\\0 & 0 & 1 & -1\end{bmatrix}.$$

(c)Since $T(A)=\begin{bmatrix}2\\0\\0\end{bmatrix}, T(B)=\begin{bmatrix}0\\1\\0\end{bmatrix}, T(C)=\begin{bmatrix}0\\0\\1\end{bmatrix}$,We can take X to be

$$\begin{aligned}&\frac{1}{2}A-2B+C\\&=\begin{bmatrix}y_2 & 0\\0 & -y_2\end{bmatrix}-\begin{bmatrix}0 & 2\\-0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\\&=\begin{bmatrix}y_2 & -2\\1 & -y_2\end{bmatrix}.\end{aligned}$$

Q6 Apply Elementary Row and Column Operations to A and C to obtain $D_1 = \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix}$ for A and $D_2 = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$ for C.

Where $r = rankA, s = rankC$.
Let $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$. Then M can be converted to $M_1 = \begin{bmatrix} D_1 & C_1 \\ 0 & D_2 \end{bmatrix}$ via elementary row and column operations.

Furthermore, the pivots in D_1 and D_2 can be used to eliminate the nonzero entries in C_1, to obtain

$$M_2 = \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & C_2 \\ 0 & 0 & I_s & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In conclusion,
$$\begin{aligned} rankM &= rankM_1 = rankM_2 = r + s + rankC_2 \\ &\geq r + s = rankA + rankC \end{aligned}$$

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Q1.(20 points, 4 points each)

暂无选择题。

Q2.(25 points, 5 points each) Fill in the blanks

(1)Let $u, v \in \mathbb{R}^n$ with $\|u\| = 2$, $\|v\| = 4$ and $u^T v = 6$. Then $\|3u - v\| =$ ______.

(2) Let A be an $n \times n$ matrix with $A^2 = -A$ and let I be the $n \times n$ identity matrix. Then $(A - I)^{-1} =$ ______.

(3)Let $A = \begin{bmatrix} 1 & a & a & a \\ a & 1 & a & a \\ a & a & 1 & a \\ a & a & a & 1 \end{bmatrix}$ with $rank\,(A) = 1$. Then $a =$ ______.

(4)Let α be a nonzero 3-dimensional real column vector in $\mathbb{R}^3$with $\alpha^T \alpha \neq 1$, and I_3 be the 3×3 identity matrix. Then rank $\left(I_3 - \alpha\alpha^T\right) =$ ______.

(5) Let$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

Then the least squares solution to $Ax = b$ is $\widehat{x} =$ ______.

Q3 (15 points) Let $\alpha \in R$, and

$$A_\alpha = \begin{bmatrix} 1 & -\alpha & 1+\alpha \\ \alpha & \alpha^2 & \alpha \\ -\alpha & 1 & -2 \end{bmatrix}.$$

(a) By applying row operations, determine for which values of α is the matrix A_α invertible?

(b) Find the values of α such that the nullspace of $A_\alpha, N(A_\alpha)$, has dimension 1?

(c) Let $\alpha = 2$.Write down the matrix inverse of A_α.

Q4.(10points)

Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 9 & -3 & 1 \\ -1 & 2 & 2 \end{bmatrix}.$$

Find an LU factorization of A.

Q5.(10 points)

Consider the following system ofline are quations:

$$(I):\begin{cases} x_1+x_2=0,\\ x_2-x_4=0.\end{cases}$$

Note that the above system (I) has four variables x_1, x_2, x_3, x_4.

Suppose another homogeneous

system of linear equations (II) has special solutions

$$u=\begin{bmatrix}0\\1\\1\\0\end{bmatrix}, v=\begin{bmatrix}-1\\2\\2\\1\end{bmatrix}.$$

Find the common nonzero solutions of systems (I) and (II).

Q6.(8 points)

Let $R^{2\times 2}$ be the vector space consisting of all 2×2 real matrices.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$E = \left\{ E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

(a) Show that E is a basis for $\mathbb{R}^{2\times 2}$.

(b) Show that T:$\mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$, $X \mapsto XA$ is a linear transformation.

(c)Find the matrix representation of T with respect to the ordered basis $E_{11}, E_{12}, E_{21}, E_{22}$.

Q7.(6 points) Let A,B be two $m \times n$ matrices. Prove

(a)$rank\left(A+B\right) \leq rankA + rankB$

(b)$rank\left(A+B\right) \geq rankA - rankB$

Q8.(6 points)

Let A be an $m \times n$ matrix with rank r. Show that there exist an $m \times r$ matrix B and an $r \times n$ matrix C such that $A = BC$ and both B and C have rank r.

(共25分，每小题5分)填空题。

(1)设$u,v\in\mathbb{R}^n$且$\|u\|=2,\|v\|=4$以及$u^Tv=6$.则$\|3u-v\|=$______.

(2)设A为一个n阶矩阵，且$A^2=-A,I$表示n阶单位矩阵。则$(A-I)^{-1}=$______.

(3)设$A=\begin{bmatrix}1&a&a&a\\a&1&a&a\\a&a&1&a\\a&a&a&1\end{bmatrix}$且$rank\,(A)=1$. 则 $a=$______.

(4)设 $\alpha\in\mathbb{R}^3$为一个非零列向量且 $\alpha^T\alpha\neq 1$, I_3 为3×3 单位矩阵.则 $\operatorname{rank}\left(I_3-\alpha\alpha^T\right)=$______

(5)

令$A=\begin{bmatrix}1&1\\1&0\\0&-1\end{bmatrix},b=\begin{bmatrix}2\\1\\1\end{bmatrix}$.

则 $Ax=b$ 的最小二乘解 $\widehat{x}=$______.

Q3 (15 points)设α为实数，A_α为

$$A_\alpha = \begin{bmatrix} 1 & -\alpha & 1+\alpha \\ \alpha & \alpha^2 & \alpha \\ -\alpha & 1 & -2 \end{bmatrix}.$$

(a)对矩阵A_α做初等行变换，α为何时时，A_α为可逆矩阵?

(b)α取何值时，矩阵A_α的零空间的维数等于1?

(c)设 $\alpha = 2$ ，求矩阵A_α的逆矩阵.

Q4.(10 points)设

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 9 & -3 & 1 \\ -1 & 2 & 2 \end{bmatrix}.$$

求A的一个LU分解

Q5.(10 points) 考虑以下线性方程组：

$$(I):\begin{cases}x_1+x_2=0,\\ x_2-x_4=0.\end{cases}$$

注意上述方程组(I)有四个变量x_1,x_2,x_3,x_4。假设另一个齐次线性方程组(II)有特殊解

$$u=\begin{bmatrix}0\\1\\1\\0\end{bmatrix},v=\begin{bmatrix}-1\\2\\2\\1\end{bmatrix}.$$

找出方程组(I)和(II)的共同非零解。

Q6.(8 points)

设$R^{2\times 2}$为所有2×2实矩阵构成的向量空间. 设$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, 且

$$E=\left\{E_{11}=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21}=\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22}=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}.$$

(a)证明：E为$R^{2\times 2}$的一组基。

(b)证明：$T:\mathbb{R}^{2\times 2}\to\mathbb{R}^{2\times 2}, X\mapsto XA$为线性变换

(c)求 T 在有序基$E_{11}, E_{12}, E_{21}, E_{22}$下的矩阵表示

Q7.(6 points)设A, B都为$m \times n$矩阵，证明：

(a)$rank\left(A+B\right) \leq rankA + rankB$

(b)$rank\left(A+B\right) \geq rankA - rankB$

Q8.(6 points)

设 A 为一个秩为r的$m \times n$矩阵. 证明：存在一个$m \times r$矩阵B和一个$\tau \times n$矩阵C,使得 $A = BC$,其中B, C的秩都为r.

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Q1 (1)A (2)D (3)C (4)B (5)B

Q2

(1)4

(2)$-\dfrac{1}{2}A - I$

(3)1

(4)3

(5)$\begin{bmatrix} \dfrac{5}{3} \\ -\dfrac{1}{3} \end{bmatrix}$

Q3

(a) $\alpha \neq 0, 1, -3$

(b) $\alpha = 0, 1, -3$

(c) $\alpha = 2$

$$A_\alpha^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{20} & -\frac{4}{5} \\ 0 & \frac{1}{5} & \frac{1}{5} \\ -\frac{1}{2} & \frac{3}{20} & \frac{2}{5} \end{bmatrix}$$

Q4

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 9 & 1 & 0 \\ -1 & -\frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -12 & -8 \\ 0 & 0 & 1 \end{bmatrix}$$

Q5

$$k \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, k \neq 0.$$

Q6 (a)

- linear independent
- E spans $\mathbf{R}^{2\times 2}$.

(b)

$$\begin{aligned} &\cdot T(X+Y) = T(X) + T(Y), \\ &\cdot T(\lambda X) = \lambda T(X). \end{aligned}$$

(c)

$$M = \begin{bmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & 0 & d \end{bmatrix}.$$

Q7(a)

Pivot columns of A: $a_1, a_2, \cdots, a_r$;

Pivot columns of B : $b_1, b_2, \cdots, b_s$;

$rankA = r, rankB = s$.

$$
\begin{aligned}
V &= span\,(a_1, \cdots, a_s, b_1, \cdots, b_s).\, dimV \leq r+s \\
&= span\,(a_1, \cdots, a_s, b_1, \cdots, b_s) \supseteq C\,(A+B) \\
&\implies dimC(A+B) \leq dimV \\
&\implies rank(A+B) \leq rank(A) + rank(B)
\end{aligned}
$$

(b)

$A + B - B = A$

$rank(A+B-B) \leq rank(A+B) + rank(-B) \ldots \text{by (a)}$

$rank(A+B) + rank(-B) = rank(B)$

$\implies rankA - rankB \leq rank(A+B)$

Q8 P_1, Q_1 invertible.

$$
\begin{aligned}
A &= P_1 \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q_1 \\
&= \underbrace{P_1 \begin{bmatrix} I_r \\ 0 \end{bmatrix}}_{B} \underbrace{[I_r \quad 0] Q_1}_{C}
\end{aligned}
$$