23 Fall Midterm Answer

线代23秋期中试题答案 发布版

Q1 (1)A (2)B（存疑, 一说D） (3)D (4)B (5)D

Q2(1)21

(2)$-A^{-1}CB^{-1}$

(3)1or-3 （存疑，一说 1 or -2）

(4)$\begin{bmatrix} -\frac{19}{11} \\ -\frac{5}{11} \end{bmatrix}$

(5)$\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ or } -\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Q3

A basis for $C(A)$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

A basis for $C(A^T)$

$$\left\{\begin{bmatrix}1\\0\\1\\0\\3\end{bmatrix},\begin{bmatrix}0\\1\\2\\0\\1\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix}\right\}\text{or}\left\{\begin{bmatrix}1\\2\\5\\1\\6\end{bmatrix},\begin{bmatrix}0\\1\\2\\1\\2\end{bmatrix},\begin{bmatrix}-1\\1\\1\\3\\1\end{bmatrix}\right\}$$

A basis for $N(A)$

$$\left\{\begin{bmatrix}-1\\-2\\1\\0\end{bmatrix},\begin{bmatrix}-3\\-1\\0\\-1\end{bmatrix}\right\}.$$

A basis for $N(A^T)$

$$\left\{\begin{bmatrix}-5\\13\\-3\\1\end{bmatrix}\right\}.$$

$$(b)\begin{bmatrix}5\\2\\1\\2\end{bmatrix}$$

Q4

Gaussian Eliminations give:

$$\begin{bmatrix}1 & -1 & -1 & \vdots & 2 & 2\\ 2 & a & 1 & \vdots & 1 & a\\ -1 & 1 & a & \vdots & -a-1 & -2\end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & -1 & \vdots & 2 & 2 \\ 0 & a+2 & 3 & \vdots & -3 & a-4 \\ 0 & 0 & a-1 & \vdots & 0 & 0 \end{bmatrix}$$

If $a = -2$, then $rank A = 2 \neq 3 = rank(A \vdots B)$,$AX = B$ has no solution.

If $a \neq 1$ and $a \neq -1$, $AX = B$ has a unique solution.

$$\begin{bmatrix} 1 & -1 & -1 & 1 & 2 \\ 0 & a+2 & 3 & 1 & -3 \\ 0 & 0 & a-1 & 1 & 1-a \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & -1 & -1 & \vdots & 2 \\ 0 & a+2 & 3 & \vdots & -3 \\ 0 & 0 & a-1 & \vdots & 1-a \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & -1 & -1 & \vdots & z \\ 0 & a+2 & 3 & \vdots & a-4 \\ 0 & 0 & a-1 & \vdots & 0 \end{bmatrix} \Rightarrow X = \begin{bmatrix} \frac{3a}{a+2} \\ \frac{a-4}{a+2} \\ 0 \end{bmatrix}.$$

$$X = \begin{bmatrix} 1 & \frac{3a}{a+2} \\ 0 & \frac{a-4}{a+2} \\ -1 & 0 \end{bmatrix}$$

If $a = 1$, $Ax = B$ has infinitely many solutions

$$\begin{bmatrix} 1 & -1 & -1 & -2 & \\ 0 & 3 & 3 & -3 & \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 & 2 & \\ 0 & 3 & 3 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix} 1 & 1 \\ -k_1 - 1 & -k_2 - 1 \\ k_1 & k_2 \end{bmatrix}, \quad k_1, k_2 \quad \text{anbitrary constants.}$$

Q5

(a)Let $X, Y \in M_{2\times 2}(R)$ and $C \in R$, than we have

$$\begin{aligned} T\left(CX+Y\right) &= \begin{bmatrix} tr & A^T\left(CX+Y\right) \\ tr & B^T\left(CX+Y\right) \\ tr & C^T\left(CX+Y\right) \end{bmatrix} \\ &= c \begin{bmatrix} tr(A^TX) \\ tr(B^TX) \\ tr(C^TX) \end{bmatrix} + \begin{bmatrix} tr(A^TX) \\ tr(B^TY) \\ tr(C^TY) \end{bmatrix} \\ &= cT(X) + T(Y) \end{aligned}$$

(b)

$$T\left(V_1\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{=w_1} + 0 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{=w_2} + 0 \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{=w_3}$$

$$T\left(V_2\right) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T\left(V_3\right)=\begin{bmatrix}0\\1\\1\end{bmatrix}=0\begin{bmatrix}1\\0\\0\end{bmatrix}+1\cdot\begin{bmatrix}0\\1\\0\end{bmatrix}+1\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$T\left(V_4\right)=\begin{bmatrix}0\\1\\-1\end{bmatrix}=0\begin{bmatrix}1\\0\\0\end{bmatrix}+1\cdot\begin{bmatrix}0\\0\\0\end{bmatrix}+-1\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Therefore,the matix representation of T with respect to V_1, V_2, V_3, V_4, V_4, and W_1, V_2, W_3 , is:

$$\begin{bmatrix}1 & -1 & 0 & 0\\0 & 0 & 1 & 1\\0 & 0 & 1 & -1\end{bmatrix}.$$

(c)Since $T(A)=\begin{bmatrix}2\\0\\0\end{bmatrix}, T(B)=\begin{bmatrix}0\\1\\0\end{bmatrix}, T(C)=\begin{bmatrix}0\\0\\1\end{bmatrix}$,We can take X to be

$$\begin{aligned}&\frac{1}{2}A-2B+C\\&=\begin{bmatrix}y_2 & 0\\0 & -y_2\end{bmatrix}-\begin{bmatrix}0 & 2\\-0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\\&=\begin{bmatrix}y_2 & -2\\1 & -y_2\end{bmatrix}.\end{aligned}$$

Q6 Apply Elementary Row and Column Operations to A and C to obtain $D_1 = \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix}$ for A and $D_2 = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$for C.

Where $r = rankA, s = rankC$.
Let $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$. Then M can be converted to $M_1 = \begin{bmatrix} D_1 & C_1 \\ 0 & D_2 \end{bmatrix}$ via elementary row and column operations.

Furthermore, the pivots in D_1 and D_2 can be used to eliminate the nonzero entries in C_1, to obtain

$$M_2 = \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & C_2 \\ 0 & 0 & I_s & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In conclusion,

$$\begin{aligned} rankM &= rankM_1 = rankM_2 = r + s + rankC_2 \\ &\geq r + s = rankA + rankC \end{aligned}$$