23 Spring Midterm Question

线性代数 23春季 期中试题 发布版

Q1.(20 points, 4 points each)
暂无选择题。

Q2.(25 points, 5 points each) Fill in the blanks

(1)Let $u, v \in \mathbb{R}^n$ with $\|u\| = 2$, $\|v\| = 4$ and $u^T v = 6$. Then $\|3u - v\| =$ ______.

(2) Let A be an $n \times n$ matrix with $A^2 = -A$ and let I be the $n \times n$ identity matrix. Then $(A - I)^{-1} =$ ______.

(3)Let$A = \begin{bmatrix} 1 & a & a & a \\ a & 1 & a & a \\ a & a & 1 & a \\ a & a & a & 1 \end{bmatrix}$with $rank\,(A) = 1$. Then $a =$ ______.

(4)Let α be a nonzero 3-dimensional real column vector in $\mathbb{R}^3$with $\alpha^T\alpha \neq 1$, and I_3 be the 3×3 identity matrix. Then rank $\left(I_3 - \alpha\alpha^T\right) =$ ______.

(5) Let$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

Then the least squares solution to $Ax = b$ is $\widehat{x} =$ ______.

Q3 (15 points) Let $\alpha \in R$, and

$$A_{\alpha} = \begin{bmatrix} 1 & -\alpha & 1+\alpha \\ \alpha & \alpha^2 & \alpha \\ -\alpha & 1 & -2 \end{bmatrix}.$$

(a) By applying row operations, determine for which values of α is the matrix A_{α} invertible?

(b) Find the values of α such that the nullspace of $A_{\alpha}, N(A_{\alpha})$, has dimension 1?

(c) Let $\alpha = 2$.Write down the matrix inverse of A_{α}.

Q4.(10points)
Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 9 & -3 & 1 \\ -1 & 2 & 2 \end{bmatrix}.$$

Find an LU factorization of A.

Q5.(10 points)

Consider the following system ofline are quations:

$$(I): \begin{cases} x_1 + x_2 = 0, \\ x_2 - x_4 = 0. \end{cases}$$

Note that the above system (I) has four variables x_1, x_2, x_3, x_4.
Suppose another homogeneous
system of linear equations (II) has special solutions

$$u = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, v = \begin{bmatrix} -1 \\ 2 \\ 2 \\ 1 \end{bmatrix}.$$

Find the common nonzero solutions of systems (I) and (II).

Q6.(8 points)

Let $R^{2\times 2}$ be the vector space consisting of all 2×2 real matrices.

Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$E=\left\{E_{11}=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21}=\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22}=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}.$$

(a) Show that E is a basis for $\mathbb{R}^{2\times 2}$.

(b) Show that T:$\mathbb{R}^{2\times 2}\to\mathbb{R}^{2\times 2}$, $X\mapsto XA$ is a linear transformation.

(c)Find the matrix representation of T with respect to the ordered basis $E_{11}, E_{12}, E_{21}, E_{22}$.

Q7.(6 points) Let A,B be two $m \times n$ matrices. Prove

(a)$rank\left(A+B\right) \leq rankA + rankB$

(b)$rank\left(A+B\right) \geq rankA - rankB$

Q8.(6 points)

Let A be an $m \times n$ matrix with rank r. Show that there exist an $m \times r$ matrix B and an $r \times n$ matrix C such that $A = BC$ and both B and C have rank r.

(共25分，每小题5分)填空题。

(1)设$u, v \in \mathbb{R}^n$且$\| u \| = 2, \| v \| = 4$以及$u^T v = 6$.则
$\|3u - v\| =$ ______.

(2)设A为一个n阶矩阵，且$A^2 = -A, I$表示n阶单位矩阵。则
$(A - I)^{-1} =$ ______.

(3)设$A = \begin{bmatrix} 1 & a & a & a \\ a & 1 & a & a \\ a & a & 1 & a \\ a & a & a & 1 \end{bmatrix}$且 $rank\,(A) = 1$. 则 $a =$ ______.

(4)设 $\alpha \in \mathbb{R}^3$为一个非零列向量且 $\alpha^T \alpha \neq 1$, I_3 为3×3 单位矩阵.则
$\operatorname{rank}\left(I_3 - \alpha\alpha^T\right) =$ ______

(5)
令$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.
则 $Ax = b$ 的最小二乘解 $\widehat{x} =$ ______.

Q3 (15 points)设α为实数，A_α为

$$A_\alpha = \begin{bmatrix} 1 & -\alpha & 1+\alpha \\ \alpha & \alpha^2 & \alpha \\ -\alpha & 1 & -2 \end{bmatrix}.$$

(a)对矩阵A_α做初等行变换，α为何时时，A_α为可逆矩阵?

(b)α取何值时，矩阵A_α的零空间的维数等于1?

(c)设 $\alpha = 2$ ，求矩阵A_α的逆矩阵.

Q4.(10 points)设

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 9 & -3 & 1 \\ -1 & 2 & 2 \end{bmatrix}.$$

求A的一个LU分解

Q5.(10 points) 考虑以下线性方程组:

$$(I):\begin{cases} x_1+x_2=0, \\ x_2-x_4=0. \end{cases}$$

注意上述方程组(I)有四个变量x_1, x_2, x_3, x_4。假设另一个齐次线性方程组(II)有特殊解

$$u=\begin{bmatrix}0\\1\\1\\0\end{bmatrix}, v=\begin{bmatrix}-1\\2\\2\\1\end{bmatrix}.$$

找出方程组(I)和(II)的共同非零解。

Q6.(8 points)

设$R^{2\times 2}$为所有2×2实矩阵构成的向量空间. 设$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, 且

$$E=\left\{E_{11}=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21}=\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22}=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}.$$

(a)证明：E为$R^{2\times 2}$的一组基。

(b)证明：$T:\mathbb{R}^{2\times 2}\to\mathbb{R}^{2\times 2}, X\mapsto XA$为线性变换

(c)求 T 在有序基$E_{11}, E_{12}, E_{21}, E_{22}$下的矩阵表示

Q7.(6 points)设A,B都为$m\times n$矩阵，证明：

(a)$rank\left(A+B\right)\leq rankA+rankB$

(b)$rank\left(A+B\right)\geq rankA-rankB$

Q8.(6 points)

设 A 为一个秩为r的$m \times n$矩阵. 证明：存在一个$m \times r$矩阵B和一个$\tau \times n$矩阵C,使得 $A = BC$,其中B, C的秩都为r.