23 Fall Midterm Question

线代23秋期中试题 发布版 中英分离

1.(共 15 分，每小题 3 分) 选择题，只有一个选项是正确的.

(1)设$\alpha_1, \alpha_2, \alpha_3$为矩阵$A$的零空间$N(A)$的一组基. 下列哪一组向量也是矩阵A的零空间的一组基?

(A)$\alpha_1 + \alpha_2 - \alpha_3, \alpha_1 + \alpha_2 + 5\alpha_3, 4\alpha_1 + \alpha_2 - 2\alpha_3.$

(B)$\alpha_1 + 2\alpha_2 + \alpha_3, 2\alpha_1 + \alpha_2 + 2\alpha_3, \alpha_3 + \alpha_1 + \alpha_2,$

(C)$\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3.$

(D)$\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \alpha_3 - \alpha_1.$

(2)以下说法一定是正确的是?

(A)如果矩阵A的列向量线性无关，那么对任意的$\mathbf{b}$, $A\mathbf{x} = \mathbf{b}$有唯一的解.

(B)任意5×7矩阵的列向量一定是线性相关的.

(C)如果矩阵A 的列向量线性相关，该矩阵的行向量也线性相关.

(D)一个10×12矩阵的行空间和列空间可能具有不同的维数

(3)设 $\alpha_1 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 6 \\ 2 \\ -16 \end{bmatrix}, \beta = \begin{bmatrix} 2 \\ t \\ 3 \end{bmatrix}$

当$t = ()$时, β可用$\alpha_1, \alpha_2, \alpha_3$线性表示

(A)1.

(B)3.

(C)6.

(D)9.

(4) 以下说法一定是正确的事?

(A) 设E为一个可逆矩阵.如果A, B矩阵满足$EA=B$,则A和B的列空间相同

(B) 设A为秩为1的n阶的方阵, 则$A^n=cA$, 其 中 n 为 正 整 数 , c 为 实 数 .

(C) 如果A, B 为对称矩阵, 则AB为对称矩阵. 如果矩阵 A 为一个行满秩矩阵, 那么 $Ax=0$ 只 有 零 解 .

(D) 如果矩阵A为一个列满秩矩阵，那么$Ax=0$只有零解。

设A与B都为n阶矩阵，A为非零矩阵，且AB=0，则

(1)$BA=0$

(2)$B=0$

(3)$(A+B)(A-B)=A^2-B^2$

(4)$\text{rank}\ B<n$.

2.(共 25 分，每小题 5 分)填空题.

(1)记所有7×7实矩阵构成的向量空间为$M_{7\times 7}(\mathbb{R}), W$为$M_{7\times 7}(\mathbb{R})$中所有斜对称矩阵构成的子空间，则 $\dim W=$ ______.
如果$A^T=-A, A$就称之为斜对称的。

(2)设A, B为两个 可逆矩阵，假设的逆矩阵为，期中O为的零矩阵，则$D=$ ______.

(3)设 $A=\begin{bmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{bmatrix}$且 $rank(A)<4$，则 $a=$ ______.

(4)考虑一下线性方程组:

$$
A\mathbf{x} = \mathbf{b} : \begin{cases} x & + & 2y & = & 1 \\ x & - & y & = & 2 \\ & & y & = & -1. \end{cases}
$$

该线性方程组的最小二乘解为______.

(5)设H为如下定义的一个R^3中的子空间

$$
H = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_1 + 2x_2 + x_3 = 0 \right\}.
$$

一个和子空间H正交的**单位向量**为______.

3.(24 points)考虑以下这个4×5矩阵A以及他的简化阶梯形矩阵R:

$$A=\begin{bmatrix} 1 & 2 & * & 1 & * \\ 0 & 1 & * & 1 & * \\ -1 & 1 & * & 3 & * \\ 2 & 0 & * & 1 & * \end{bmatrix},\ R=\begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a)分别求矩阵 A 的四个基本子空间的一组基向量.

(b)求出矩阵 A 的第三个列向量.

4.(15 points) 设

$$A=\begin{bmatrix}1 & -1 & -1\\ 2 & a & 1\\ -1 & 1 & a\end{bmatrix},\ B=\begin{bmatrix}2 & 2\\ 1 & a\\ -a-1 & -2\end{bmatrix}.$$

当a为何值时，矩阵方程$AX=B$无解、有唯一解、有无穷多解？在有解时，求解此方程，这里的X为一个3×2矩阵

5.（15 points）设$M_{2\times 2}(\mathbb{R})$为所有$2\times 2$实矩阵构成的向量空间，设

$$A=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\ B=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\ C=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}.$$

考虑以下映射

$$T: M_{2\times 2}(\mathbb{R})\to\mathbb{R}^3,\ T(X)=\begin{bmatrix}tr(A^TX)\\ tr(B^TX)\\ tr(C^TX)\end{bmatrix},$$

对任意的 2×2实矩阵, 其中 $tr(D)$ 表示n阶矩阵D的迹.

方阵D的迹是指D的对角元之和，也即

$$tr(D)=d_{11}+d_{22}+\cdots+d_{nn}$$

(a)证明T是一个线性变换
(b)求T在$M_{2\times 2}(R)$的一组基

$$v_1=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, v_2=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, v_3=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, v_4=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

以及R^3的标准基

$$e_1=\begin{bmatrix}1\\0\\0\end{bmatrix}, e_2=\begin{bmatrix}0\\1\\0\end{bmatrix}, e_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

下的矩阵表示.

(c)是否可以找到一个矩阵 X 使得$T(X)=\begin{bmatrix}1\\-2\\1\end{bmatrix}$?如果可以,请求出一个符合要求的矩阵$X$. 如果不存在,请说明理由.

6.（6 points）设A为$m \times n$矩阵，B为$m \times p$矩阵,C为$q \times p$矩阵.证明:

$$\operatorname{rank}\begin{bmatrix} A & B \\ O & C \end{bmatrix} \geq \operatorname{rank} A + \operatorname{rank} C,$$

其中O为$q \times n$的零矩阵

1.(15 points, 3 points each) Multiple Choice. Only one choice is correct.

(1)Suppose that $\alpha_1, \alpha_2, \alpha_3$ are a basis for nullspace of a matrix $A, N(A)$. Which of the following lists of vectors is also a basis for $N(A)$?

(A)$\alpha_1+\alpha_2-\alpha_3, \alpha_1+\alpha_2+5\alpha_3, 4\alpha_1+\alpha_2-2\alpha_3$.

(B)$\alpha_1+2\alpha_2+\alpha_3, 2\alpha_1+\alpha_2+2\alpha_3, \alpha_3+\alpha_1+\alpha_2$,

(C)$\alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3$.

(D)$\alpha_1-\alpha_2, \alpha_2-\alpha_3, \alpha_3-\alpha_1$.

(2)Which of the following statements is correct?

(A) If the columns of A are linearly independent, then $A\mathbf{x}=\mathbf{b}$ has exactly one solution for every b.

(B) Any 5×7 matrix has linearly dependent columns.

(C) If the columns of a matrix A are linearly dependent, so are the rows.

(D) The column space and row space of a 10×12 matrix may have different dimensions.

(3)Let

$$\alpha_1=\begin{bmatrix}1\\4\\1\end{bmatrix},\ \alpha_2=\begin{bmatrix}2\\1\\-5\end{bmatrix}, \alpha_3=\begin{bmatrix}6\\2\\-16\end{bmatrix},\ \beta=\begin{bmatrix}2\\t\\3\end{bmatrix}.$$

β can be written as a linear combination of $\alpha_1, \alpha_2, \alpha_3$ if $t=$ ( )

(A)1.

(B)3.

(C)6.

(D)9.

(4) Which of the following statements is correct?

(A) Suppose that $EA = B$ and E is an invertible matrix, then the column space of $A$and the column space of B are the same.

(B) Let A be a $n \times n$ matrix with rank 1, then $A^n = cA$, where n is a positive integer and c is a real number.

(C) Let A, B be symmetric matrices, then AB is symmetric.

(D) If A is of full row rank, then $Ax = 0$ has only the zero solution.

(5)Let A and B be two $n \times n$ matrices. If A is a non-zero matrix and AB = 0, then

(1)$BA = 0$

(2)$B = 0$

(3)$(A + B)(A - B) = A^2 - B^2$

(4)$\text{rank } B < n$.

2.$(25 \text{ points, } 5 \text{ points each})$ Fill in the blanks.

(1)Denote the vector space of 7×7 real matrices by $M_{7\times 7}(\mathbb{R})$, and let W be the subspace of $M_{7\times 7}(\mathbb{R})$ consisting of skew-symmetric real matrices, then dim $W =$ ______.

A matrix A is called skew symmetric if $A^T = -A$.

(2)Let A, B be two $n \times n$ invertible matrices. Suppose the inverse of $\begin{bmatrix} A & C \\ O & B \end{bmatrix}$ is $\begin{bmatrix} A^{-1} & D \\ O & B^{-1} \end{bmatrix}$, where O is the $n \times n$ zero matrix. Then $D =$ ______.

(3)Let $A = \begin{bmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{bmatrix}$ with $rank(A) < 4$. Then $a =$ ______.

(4)Consider the system of linear equations

$$A\mathbf{x} = \mathbf{b} : \begin{cases} x & + & 2y & = & 1 \\ x & - & y & = & 2 \\ & & y & = & -1. \end{cases}$$

The least-squares solution for the system is______.

(5)

Let H be the subspace of R^3 be defined as follows:

$$H = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_1 + 2x_2 + x_3 = 0 \right\}.$$

A **unit** vector orthogonal to H is ______.

3.(24 points) Consider the following 4×5 matrix A with its reduced row echelon form R:

$$A = \begin{bmatrix} 1 & 2 & * & 1 & * \\ 0 & 1 & * & 1 & * \\ -1 & 1 & * & 3 & * \\ 2 & 0 & * & 1 & * \end{bmatrix}, \; R = \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for each of the four fundamental subspaces of A.

(b) Find the third column of matrix A.

4.(15points)Let

$$A=\begin{bmatrix}1 & -1 & -1\\ 2 & a & 1\\ -1 & 1 & a\end{bmatrix},\ B=\begin{bmatrix}2 & 2\\ 1 & a\\ -a-1 & -2\end{bmatrix}.$$

For which value(s) of a, the matrix equation $AX=B$ has no solution, a unique solution, or infinitely many solutions? Where X is a 3×2 matrix. Solve $AX=B$ if it has at least one solution.

5.(15 points) Let $M_2 \times 2(\mathbb{R})$ be the vector space of 2×2 real matrices. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Consider the map

$$T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^3,\ T(X) = \begin{bmatrix} tr(A^T X) \\ tr(B^T X) \\ tr(C^T X) \end{bmatrix},$$

for any 2×2 real matrix X, where $tr(D)$ denotes the trace of a matrix D.

The trace of an $n \times n$ matrix D is defined to be the sum of all the diagonal entries of D, in other words,

$$tr(D) = d_{11} + d_{22} + \cdots + d_{nn}.$$

(a) Show that T is a linear transformation.

(b) Find the matrix representation of T with respect to the ordered basis

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ v_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ v_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ v_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

for $M_{2\times 2}$ $(\mathbb{R})$and the standard basis

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

for $\mathbb{R}^3$.

(c) Can we find a matrix X such that $T(X) = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$? If yes, please find one such matrix. Otherwise, give an explanation.

6.(6 points) Let A be an $m \times n$ matrix, B be an $m \times p$ matrix, and C be an $q \times p$ matrix. Show that

$$\operatorname{rank} \begin{bmatrix} A & B \\ O & C \end{bmatrix} \geq \operatorname{rank} A + \operatorname{rank} C,$$

where O is the $q \times n$ zero matrix.