1. (20 marks) Find the solution of the following first-order partial differential equation:

$$\begin{cases} \dfrac{\partial u}{\partial x} + t\dfrac{\partial u}{\partial t} = u + t, & t > 1,\ x \in \mathbb{R}, \\ u(x,1) = \phi(x), \end{cases}$$

where $\phi(x)$ is a smooth function.

2. (5 marks) Prove the uniqueness of the smooth solution under the Robin boundary condition. Let $B_1 = \{x \in \mathbb{R}^n : |x_1|^2 + \cdots + |x_n|^2 < 1\}$ be the unit open ball in $\mathbb{R}^n$ with boundary ∂B_1.

$$\begin{cases} u_t - a^2\Delta u + (t-1)u = f(x,t), & t > 0,\ x \in B_1, \\ u(x,0) = \phi(x), & x \in B_1, \\ \dfrac{\partial u}{\partial \mathbf{n}}(x,t) + u(x,t) = \mu(x,t), & x \in \partial B_1, \end{cases}$$

where $\mathbf{n}$ is the unit outer normal vector field on ∂B_1 and $f(x,t), \phi(x), \mu(x,t)$ are smooth functions.

3. (20 marks) Suppose that λ_m and λ_n are two different eigenvalues of the following general eigenvalue problem:

$$\begin{cases} X''(x) + \lambda X(x) = 0, & a < x < b, \\ \alpha_1 X'(a) + \alpha_2 X(a) = 0; \\ \beta_1 X'(b) + \beta_2 X(b) = 0, \end{cases}$$

where $\{\alpha_1, \alpha_2\}$ are not all zeros, $\{\beta_1, \beta_2\}$ are not all zeros, and $X_m(x), X_n(x)$ are eigenfunctions corresponding to λ_m, λ_n respectively.
Prove that the eigenfunctions $X_m(x)$ and $X_n(x)$ are orthogonal in the following sense:

$$\int_a^b X_m(x)X_n(x)dx = 0.$$

4. (30 marks)

(a) Solve the following eigenvalue problem:

$$\begin{cases} X''(x) + \lambda X(x) = 0, & 0 < x < l, \\ X(0) = 0, \quad X(l) = 0. \end{cases}$$

(b) Solve the non-homogeneous heat equation:

$$\begin{cases} u_t - a^2 u_{xx} = \dfrac{x}{\pi}, & t > 0,\ -\pi < x < \pi, \\ u(x,0) = \sin x, & \phi(x) \\ u(-\pi,t) = t,\ u(\pi,t) = 2t. \end{cases}$$

5. (25 marks) Consider the function $G(x,t;\xi)$ defined by

$$G(x,t;\xi) = \frac{1}{2a\sqrt{\pi t}} \exp\left(-\frac{(x-\xi)^2}{4a^2 t}\right), \tag{1}$$

where $x \in \mathbb{R}$ and $t > 0$ are variables, $\xi \in \mathbb{R}$ a parameter.

(a) Prove that the function $G(x,t;\xi)$ defined by (1) satisfies the homogeneous heat conduction equation for any ξ, i.e.

$$G_t = a^2 G_{xx}, \quad t > 0.$$

(b) Prove that

$$\lim_{t \to 0^+} \int_{-\infty}^{+\infty} f(\xi) G(x,t;\xi) d\xi = f(x),$$

where $f(x)$ is any bounded continuous function.