南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

考试科目： 线 性 代 数 开课单位： 数 学 系
考试时长： 120 分钟 命题教师： 线性代数教学团队

题 号	1	2	3	4	5	6	7
分 值	15 分	20 分	10 分	24 分	20 分	5 分	6 分

本试卷共 （ 7 ）大题，满分 （ 100 ）分.（考试结束后请将试卷、答题本、草稿纸一起交给监考老师）
This exam paper contains 7 questions and the score is 100 in total. (Please hand in your exam paper, answer sheet, and your scrap paper to the proctor when the exam ends.)

Notation: $C(A)$ is the column space of matrix A, $C(A^T)$ is the row space of matrix A, $N(A)$ is the nullspace of matrix A, and $N(A^T)$ is the left nullspace of matrix A.

1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.
(共 15 分，每小题 3 分) 选择题，只有一个选项是正确的.

(1) Suppose A is an $m \times n$ real matrix with $m < n$, and the equation $A\mathbf{x} = \mathbf{b}$ has a solution for any m-dimensional real column vector $\mathbf{b}$. Which of the following assertions must be true?

(A) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^m$.
(B) $A^T\mathbf{x} = \mathbf{d}$ has a solution for any $\mathbf{d} \in \mathbb{R}^n$.
(C) $N(A) = \{\mathbf{0}\}$.
(D) A has a right inverse.

设 A 为一个 $m \times n$ 实矩阵 ($m < n$)，对于任意的 m 维实列向量 $\mathbf{b}$，$A\mathbf{x} = \mathbf{b}$ 都有解，以下说法一定正确的是

(A) 对于每个 $\mathbf{b} \in \mathbb{R}^m$，$A\mathbf{x} = \mathbf{b}$ 都有唯一解.
(B) 对于任意的 $\mathbf{d} \in \mathbb{R}^n$，$A^T\mathbf{x} = \mathbf{d}$ 都有解.
(C) $N(A) = \{\mathbf{0}\}$.
(D) A 有右逆.

(2) Suppose we have matrices A, B satisfying $EA = B$ for some invertible matrix E. Which of the following assertions must be true?

(A) $N(A) = N(B)$.
(B) $C(A) = C(B)$.
(C) $N(A^T) = N(B^T)$.
(D) $A^TA = B^TB$.

假定 A, B 满足 $EA = B$, E 为一个可逆矩阵, 以下结论一定正确的是

(A) $N(A) = N(B)$.

(B) $C(A) = C(B)$.

(C) $N(A^T) = N(B^T)$.

(D) $A^TA = B^TB$.

(3) For any $m \times n$ matrix A with reduced row echelon form U. Which of the following assertions must be true?

(A) $C(A) = C(A^TA)$.

(B) $\text{rank}(A) = \text{rank}(A^TA)$.

(C) $C(A) = C(U)$.

(D) If $C(A) = \mathbb{R}^m$, then A^TA is invertible.

对于任意 $m \times n$ 矩阵 A 以及它的简化阶梯型矩阵 U, 以下结论一定正确的是

(A) $C(A) = C(A^TA)$.

(B) $\text{rank}(A) = \text{rank}(A^TA)$.

(C) $C(A) = C(U)$.

(D) 如果 $C(A) = \mathbb{R}^m$, 则 A^TA 为可逆矩阵.

(4) Let A, B, C be $n \times n$ matrices such that $ABC = I$, where I is the identity matrix of order n, then

(A) $ACB = I$.

(B) $CBA = I$.

(C) $BCA = I$.

(D) $BAC = I$.

设 A, B, C 为满足 $ABC = I$ 的 n 阶矩阵, 其中 I 为 n 阶单位阵, 则

(A) $ACB = I$.

(B) $CBA = I$.

(C) $BCA = I$.

(D) $BAC = I$.

(5) Let A be an $n \times n$ matrix ($n > 1$) such that $A = A^2$, and I be the $n \times n$ identity matrix, then

(A) $\text{rank}(A) + \text{rank}(A - I) > n$.

(B) $\text{rank}(A) + \text{rank}(A - I) < n$.

(C) $\text{rank}(A) + \text{rank}(A - I) = n$.

(D) $\text{rank}(A) + \text{rank}(A - I) = n - 1$.

设 A 为一个满足 $A = A^2$ 的 n 阶矩阵 ($n > 1$), I 为一个 n 阶单位阵, 则

(A) $\text{rank}(A) + \text{rank}(A - I) > n$.

(B) $\text{rank}(A) + \text{rank}(A - I) < n$.

(C) $\text{rank}(A) + \text{rank}(A - I) = n$.

(D) $\text{rank}(A) + \text{rank}(A - I) = n - 1$.

2. (20 points, 5 points each) Fill in the blanks.
(共 20 分, 每小题 5 分) 填空题.

(1) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, then $A^{-1} =$__________.

设 $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, 则 $A^{-1} =$__________.

(2) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 3 \end{bmatrix}.$$

The LDU factorization of A has $L =$__________.
考虑矩阵

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 3 \end{bmatrix}.$$

矩阵 A 的 LDU 分解中的矩阵 $L =$__________.

(3) The matrix which projects every vector $\mathbf{b}$ in $\mathbb{R}^3$ onto the line in the direction of $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ through the origin is __________.

把任意一个 $\mathbb{R}^3$ 中的向量 $\mathbf{b}$ 投影到过原点沿着方向 $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ 直线上的投影矩阵为 __________.

(4) Let A be a 2024×2025 real matrix with $\dim N(A) = 11$, then $\dim N(A^T) =$__________.
设 A 为一个 2024×2025 实矩阵, 且 $\dim N(A) = 11$, 则 $\dim N(A^T) =$__________.

3. (10 points) Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Consider

$$X - XA^2 - AX + AXA^2 = I,$$

where I is the 3×3 identity matrix, and X is a 3×3 matrix.

(a) Compute $I - A$ and $I - A^2$.

(b) Find all possible X.

设

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

考虑

$$X - XA^2 - AX + AXA^2 = I,$$

其中 I 为 3 阶单位阵, X 为一个 3 阶矩阵.

(a) 计算 $I - A$ 和 $I - A^2$.

(b) 求出所有可能的矩阵 X.

4. (24 points) Let

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 3 & 0 \\ 1 & 0 & 2 & 1 \\ 3 & 2 & 8 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 9 \end{bmatrix}.$$

(a) Find the reduced row echelon form of A.

(b) Find a basis for the row space $C(A^T)$, the column space $C(A)$, and the left nullspace $N(A^T)$.

(c) Find the complete solution to $A\mathbf{x} = \mathbf{b}$. In other words, find all the solutions to $A\mathbf{x} = \mathbf{b}$.

设

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 3 & 0 \\ 1 & 0 & 2 & 1 \\ 3 & 2 & 8 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 9 \end{bmatrix}$$

(a) 求矩阵 A 的简化行阶梯型矩阵.

(b) 分别求 $C(A^T)$, $C(A)$, 以及 $N(A^T)$ 的一组基向量.

(c) 求 $A\mathbf{x} = \mathbf{b}$ 的完全解, 也即, 求 $A\mathbf{x} = \mathbf{b}$ 的所有解.

5. (20 points) Consider the following subspace of $\mathbb{R}^3$:

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0 \right\}.$$

(a) Show that:

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

is a basis of V.

(b) Let T be the linear transformation from V to $\mathbb{R}^3$ defined as follows:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 \\ x_2 + x_3 \\ x_1 \end{bmatrix}.$$

Find the matrix representation of T with respect to the ordered basis $\mathbf{v}_1, \mathbf{v}_2$ of V and the ordered basis

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

of $\mathbb{R}^3$.

(c) Can we find a vector $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$? If so, find one such $\mathbf{v}$. Otherwise, give an explanation.

考虑以下 $\mathbb{R}^3$ 的一个子空间:

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0 \right\}.$$

(a) 证明:

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

为 V 的一组基.

(b) 设 T 为从 V 到 $\mathbb{R}^3$ 按照以下方式定义的线性变换:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 \\ x_2 + x_3 \\ x_1 \end{bmatrix}.$$

求 T 在 V 的有序基 $\mathbf{v}_1, \mathbf{v}_2$ 和 $\mathbb{R}^3$ 的有序基

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

下的矩阵表示.

(c) 是否可以找到 $\mathbf{v} \in V$, 使得 $T(\mathbf{v}) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$? 如果可以, 找出一个符合要求的 $\mathbf{v}$. 如若不然, 请阐明理由.

6. (5 points) Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \text{ and } U = \{B \in M_{3\times 3}(\mathbb{R}) \mid AB = BA\},$$

where $M_{3\times 3}(\mathbb{R})$ denotes the vector space of all 3×3 real matrices with the ordinary matrix addition and scalar multiplication.

(a) Show that U is a subspace of $M_{3\times 3}(\mathbb{R})$.

(b) Find a basis of U.

(c) Find the dimension of U.

设

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \ U = \{B \in M_{3\times 3}(\mathbb{R}) \mid AB = BA\}.$$

其中 $M_{3\times 3}(\mathbb{R})$ 为在正常矩阵加法和数乘定义下所有 3×3 实矩阵构成的向量空间.

(a) 证明: U 是 $M_{3\times 3}(\mathbb{R})$ 的一个子空间.

(b) 求 U 的一个基向量组.

(c) 求 U 的维数.

7. (6 points) Prove the following two independent statements.

(a) Let A be an $m \times n$ real matrix. Suppose $v_1, v_2, \ldots, v_s$ is a basis of $C(A^T)$ and $w_1, w_2, \ldots, w_t$ is a basis of $N(A)$. Show that: $s + t = n$ and

$$v_1, v_2, \ldots, v_s, w_1, w_2, \ldots, w_t$$

is a basis of $\mathbb{R}^n$.

(b) Let A be an $m \times n$ real matrix with rank m. Show that AA^T is invertible.

证明以下两个相互独立的结论.

(a) 设 A 为一个 $m \times n$ 实矩阵. 假定 $v_1, v_2, \ldots, v_s$ 为 $C(A^T)$ 的一组基, $w_1, w_2, \ldots, w_t$ 为 $N(A)$ 的一组基. 证明: $s + t = n$, 并且

$$v_1, v_2, \ldots, v_s, w_1, w_2, \ldots, w_t$$

为 $\mathbb{R}^n$ 的一组基.

(b) 设 A 为秩为 m 的 $m \times n$ 实矩阵. 证明: AA^T 为可逆矩阵.