1、(1) D

$A \in \mathbb{R}^{m\times n}$ $\forall b \in \mathbb{R}^m$, $Ax=b$ 有解 $\Rightarrow R(A)=m$

$R(A,b)=R(A)$

A: 如 $A=[I \mid \alpha_1\ \alpha_2]\ x=\alpha$

B: $A^T \in R^{n\times m}$

$\forall d$ $A^Tx=d$ 有解 $\Rightarrow R(A)=n$

C. $A=(\alpha_1 \cdots \alpha_n)\cdot\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}=\sum_{i=1}^{n}\alpha_i x_i$

D. $A \in \mathbb{R}^{m\times n}$

若 $B \in \mathbb{R}^{n\times m}$, $AB=I \Rightarrow B$ 为右逆

$A_{m\times n}$ 有右逆 $\Leftrightarrow R(A)=m$

$A=(\underbrace{\alpha_1 \cdots \alpha_n}_{m个线性无关})\cdot\begin{pmatrix} I_m \\ 0 \end{pmatrix}=A'_{m\times m}$ 可逆 $\Rightarrow A$ 的右逆为 $\begin{pmatrix} I_m \\ 0 \end{pmatrix}(A'_{m\times m})^{-1}$

只保留线性无关的列

(2) A.

A: $Ax=0 \Rightarrow EAx=0 = Bx$

B: 如 $A=\begin{bmatrix}1 & & \\ & 1 & \\ & & 0\end{bmatrix}$ $B=\begin{bmatrix} & 1 & \\ & & 1 \\ -1 & & 0\end{bmatrix}$

C: $A^Tx=0 \Leftrightarrow B^Tx=0 \Leftrightarrow A^TE^Tx=0$ ?

$x \neq E^Tx$

D: $B^TB = A^TE^TEA = A^TA$ ? $\times$

(3) B

A: A: not symmetric.

B: $Ax=0 \Leftrightarrow A^TAx=0$

C: 不能随意行变换

D: $n>m$.

(4) C

Hint: $C=(AB)^{-1}=B^{-1}A^{-1}$

(5) C

$A=A^2 \Rightarrow O=A(A-I)$

$\therefore R(A)+R(A-I)\leq n$

又 $R(A)+R(B)\geq R(A+B)$

取 $B=I-A$

$R(A)+R(I-A)\geq R(I)=n$

2、(1) $\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{6} \\ & \frac{1}{2} & -\frac{1}{6} \\ & & \frac{1}{3} \end{bmatrix}$

(2) $\begin{bmatrix} 1 & & \\ 2 & 1 & \\ 3 & 1 & 1 \end{bmatrix}$

(3) $\frac{1}{14}\begin{bmatrix} 4 & 6 & 2 \\ 6 & 9 & 3 \\ 2 & 3 & 1 \end{bmatrix}$

投影到 $V$, $V$ besis: $v_1, \dots, v_m$

$A = [v_1 \dots v_m]$

$P_V = A(A^TA)^{-1}A^T$

(4) 10

$A \in \mathbb{R}^{m \times n}$

$N(A) = n - R(A)$

$C(A) = R(A)$

$C(A^T) = R(A)$

$N(A^T) = m - R(A)$

$m = 2024$, $n = 2025$

3. (a) $I-A=\begin{bmatrix}1&-1&0\\-1&1&1\\0&-1&1\end{bmatrix}$

$I-A^2=\begin{bmatrix}0&0&1\\0&1&0\\-1&0&2\end{bmatrix}$

$X-XA^2-AX+AXA^2=I$

$(I-A)X(I-A^2)=I$

$X=(I-A)^{-1}(I-A^2)^{-1}=[(I-A^2)(I-A)]^{-1}=\begin{bmatrix}3&1&-2\\1&1&-1\\2&1&-1\end{bmatrix}$

4. (a) $\begin{bmatrix}1&0&2&0\\0&1&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}$ pivots 上方均为0!

(b) $C(A)=k_1\begin{bmatrix}1\\1\\1\\3\end{bmatrix}+k_2\begin{bmatrix}0\\1\\0\\2\end{bmatrix}+k_3\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$

$C(A^T)=k_1\begin{bmatrix}1\\0\\2\\0\end{bmatrix}+k_2\begin{bmatrix}0\\1\\1\\0\end{bmatrix}+k_3\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$

$N(A^T) = k_1 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

(c) $[A \mid b] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

特解：$x_r = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

通解：$x_p = k \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow x = x_r + x_p = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + k \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

5、(a) ① $v_1, v_2$ 线性无关

② $v_1, v_2 \in V \rightarrow$ 代入

③ $\mathrm{span}(v_1, v_2) \supseteq V$

$\left.\begin{array}{l} \text{②} \\ \text{③} \end{array}\right\} \mathrm{span}(v_1, v_2) = V$

解 $x_1 + x_2 + x_3 = 0 \Rightarrow k_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

(b) $T(v_1, v_2) = (e_1, e_2, e_3) A_{3\times 2}$

$T(v_1) = \begin{bmatrix} -3 \\ 1 \\ -1 \end{bmatrix} = [e_1\ e_2\ e_3] \begin{bmatrix} -3 \\ 1 \\ -1 \end{bmatrix}$

$T(v_2) = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = [e_1\ e_2\ e_3] \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$

$A = \begin{bmatrix} -3 & -2 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}$

线性变换：$T: V^n \to W^m$ （回顾）

$v_1, v_2, \ldots, v_n \to w_1, \ldots, w_m$

$Tv_1 = ?$ $Tv_2 = ?$

$T(v_1, v_2 \ldots v_n) = (Tv_1, Tv_2, \ldots Tv_n)$

$= [w_1 \cdots w_m] A_{m\times n}$

(c) $Tv = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \Rightarrow T(v_1, v_2) = \underbrace{[e_1\ e_2\ e_3]}_{I_{3\times 3}} \begin{bmatrix} -3 & 2 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -3 & 2 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$\left[\begin{array}{cc|c} -3 & 2 & 3 \\ 1 & 1 & 2 \\ -1 & -1 & 1 \end{array}\right] \longrightarrow \cdots \longrightarrow \left[\begin{array}{cc|c} -3 & 2 & 3 \\ 0 & \frac{5}{3} & 3 \\ 0 & -\frac{5}{3} & 0 \end{array}\right] \Rightarrow$ 无解

或 $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 \\ x_2 + x_3 \\ x_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ 也无解

6. (a) 不妨设 $B_1, B_2 \in U$

$$AB_1 = B_1A, \quad AB_2 = B_2A$$

$$\Rightarrow A(B_1+B_2) = AB_1 + AB_2$$ 验证加法、数乘封闭

$$A(kB_1) = k(AB_1)$$

(b) $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$

$$AB = \begin{bmatrix} b_{11}+b_{21}+b_{31} & b_{12}+b_{22}+b_{32} & b_{13}+b_{23}+b_{33} \\ b_{31} & b_{32} & b_{33} \\ -b_{31} & -b_{32} & -b_{33} \end{bmatrix}$$

$$BA = \begin{bmatrix} b_{11} & b_{11} & b_{11}+b_{12}-b_{13} \\ b_{21} & b_{21} & b_{21}+b_{22}-b_{23} \\ b_{31} & b_{31} & b_{31}+b_{32}-b_{33} \end{bmatrix}$$

$$AB = BA \Rightarrow (AB)_{ij} = (BA)_{ij}$$

$\Rightarrow$ $b_{31}=0$, $b_{32}=0$, $b_{21}=0$ ✓

$b_{22}-b_{23}=b_{33}$ ✓

$b_{11}=b_{12}+b_{22}$ ✓

$b_{11}+b_{12}-b_{13}=b_{13}+b_{23}+b_{33}$ $\Rightarrow$ $b_{12}=b_{13}$

$$B=\begin{bmatrix} b_{12}+b_{22} & b_{12} & b_{12} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{22}-b_{23} \end{bmatrix}$$

basis : $\left\{ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \right\}$

(c) $\dim(U)=3$

7、(a) $C(A^T): v_1, \dots, v_s$

$N(A): w_1, \dots, w_t$

$t = R(N(A)) = n - R(A) = n - \dim(C(A^T)) = n - s$

$\Rightarrow s + t = n$

$k_1 v_1 + \dots + k_s v_s + \dots + k_n w_t = 0 \iff k_i = 0,\ i = 1, 2, \dots, n.$

$A = [\alpha_1\ \alpha_2\ \dots\ \alpha_m]$ $\quad A^T = \begin{bmatrix} \alpha_1^T \\ \vdots \\ \alpha_m^T \end{bmatrix}$

$C(A^T) = \mathrm{span}(\alpha_1 \dots \alpha_m)$

$N(A) = \{x : Ax = 0\}$

$= \{x : \alpha_i^T x = 0,\ i = 1, 2, \dots, m\}$

$\iff v_i^T w_j = 0,\ i = 1, \dots, s,\ j = 1, \dots, t$

$N(A) \perp C(A^T)$

结论：

$C(A^T) \perp N(A)$

$\alpha = k_1 v_1 + \cdots + k_s v_s + \cdots + k_n w_t = 0$

$\Rightarrow v + w = 0$

$v^T(v+w) = v^T v = 0 \Rightarrow v = 0$

$\Rightarrow k_1, \ldots, k_s = 0$

同理 $k_{s+1}, \ldots k_n = 0$.

$\Rightarrow$ 线性无关 $\Rightarrow$ 为一组基.

(b) $A \in \mathbb{R}^{m \times n}$

$R(A) = m \Rightarrow R(A^T) = m$

$R(AA^T) = m - N(AA^T) = m - N(A^T) = m - (m - R(A)) = R(A)$