一、(1)~(5) B D B D C

(4) $f'(x)=\begin{cases} 2x\sin\frac{1}{x}-\cos\frac{1}{x} & x\neq 0 \\ 0 & x=0 \end{cases}$

$f''(x)=2\sin\frac{1}{x}-\frac{2\cos\frac{1}{x}}{x}-\frac{\sin\frac{1}{x}}{x^2}\quad(x\neq 0)$

$x\to 0$时$\Rightarrow$振荡$\Rightarrow$无法判断是否为拐点。

二、(1) $-\frac{3}{2}\pi$ (化成两个半圆的面积)

(2) 1 (注意是线性化而不是近似!)

(3) $-\frac{107}{333}$ (牛顿法公式: $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$)

(4) $-\sin 1$ (先得出 $f(0)=1$, 再对两边求导)

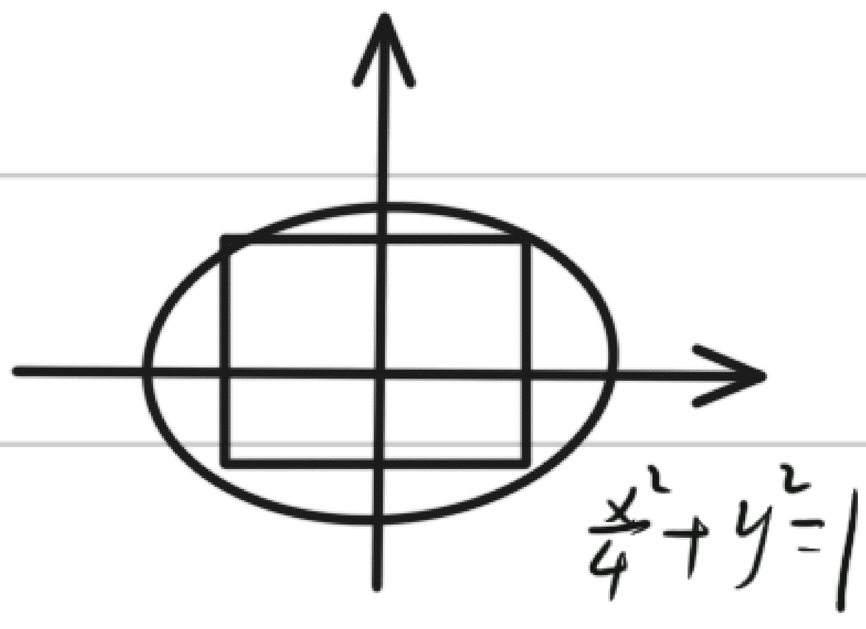
(5) 100

(5): $\lim\limits_{x\to\infty}(f(x+10)-f(x))=10\lim\limits_{x\to\infty}\frac{f(x+10)-f(x)}{10}$

中值定理$\exists\xi\in(x,x+10)$, s.t. $\frac{f(x+10)-f(x)}{10}=f'(\xi)$

原式 $=10\lim\limits_{x\to\infty}f'(\xi)=100$

三、



设长方形边长分别为x,y,则

$\frac{(\frac{x}{2})^2}{4}+(\frac{y}{2})^2=1$

即 $\frac{x^2}{4}+y^2=4$, $0\le y\le 2$

$A=xy=2y\sqrt{4-y^2}$

$\frac{dA}{dy}=\frac{4(2-y^2)}{\sqrt{4-y^2}}=0\Rightarrow y=\sqrt{2}\Rightarrow x=2\sqrt{2}$

$\Rightarrow A_{max}=4$

四. $\frac{d}{dx} f(\sin x) = f'(\sin x)\cos x$

$\frac{d}{dx} f^2(\sin x) = 2f(\sin x) f'(\sin x)\cos x$

Let $x=0$, $f'(0) = 2f(0)f'(0)$

$f'(0) \neq 0 \Rightarrow f(0) = \frac{1}{2}$

五. (1) $\lim\limits_{x\to 1} \frac{x^4-1}{\sqrt{3+x}-2} = \lim\limits_{x\to 1} \frac{x^4-1}{\sqrt{3+x}-2} \cdot \frac{\sqrt{3+x}+2}{\sqrt{3+x}+2}$

$= \lim\limits_{x\to 1} \frac{(x^4-1)(\sqrt{3+x}+2)}{x-1} = 4\lim\limits_{x\to 1} \frac{(x-1)(x^3+x^2+x+1)}{x-1}$

$= 4\lim\limits_{x\to 1}(x^3+x^2+x+1) = 4\times 4 = 16$

(trick: $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$)

(2) $\lim\limits_{x\to -\infty} \frac{x^2 \sin\frac{1}{2x}}{\sqrt{x^2+2024x+1}} = \lim\limits_{x\to -\infty} \frac{x}{\sqrt{x^2+2024x+1}} x\sin\frac{1}{2x}$

$\lim\limits_{x\to -\infty} x\sin\frac{1}{2x} = \lim\limits_{x\to -\infty} \frac{\sin\frac{1}{2x}}{\frac{1}{2x}\cdot 2} = \frac{1}{2}$

$\lim\limits_{x\to -\infty} \frac{x}{\sqrt{x^2+2024x+1}} = \lim\limits_{x\to -\infty} \frac{x}{-x\sqrt{1+\frac{2024}{x}+\frac{1}{x^2}}} = \lim\limits_{x\to -\infty} \frac{1}{\sqrt{1+\frac{2024}{x}+\frac{1}{x^2}}} = -1$

六. (1) $f'(x) = \frac{2}{3}x^{-\frac{1}{3}}(6-x)^{\frac{1}{3}} - \frac{1}{3}x^{\frac{2}{3}}(6-x)^{-\frac{2}{3}}$

$= \frac{\frac{2}{3}(6-x) - x}{x^{1/3}(6-x)^{2/3}} = \frac{12-3x}{3x^{1/3}(6-x)^{2/3}} = \frac{4-x}{3x^{1/3}(6-x)^{2/3}}$

critical points: $x=0$, $x=4$, $x=6$

	$(-\infty, 0)$	$(0, 4)$	$(4, 6)$	$(6, +\infty)$
f'	$-$	$+$	$-$	$-$
f	$\searrow$	$\nearrow$	$\searrow$	$\searrow$

local minimum : $f(0)=0$

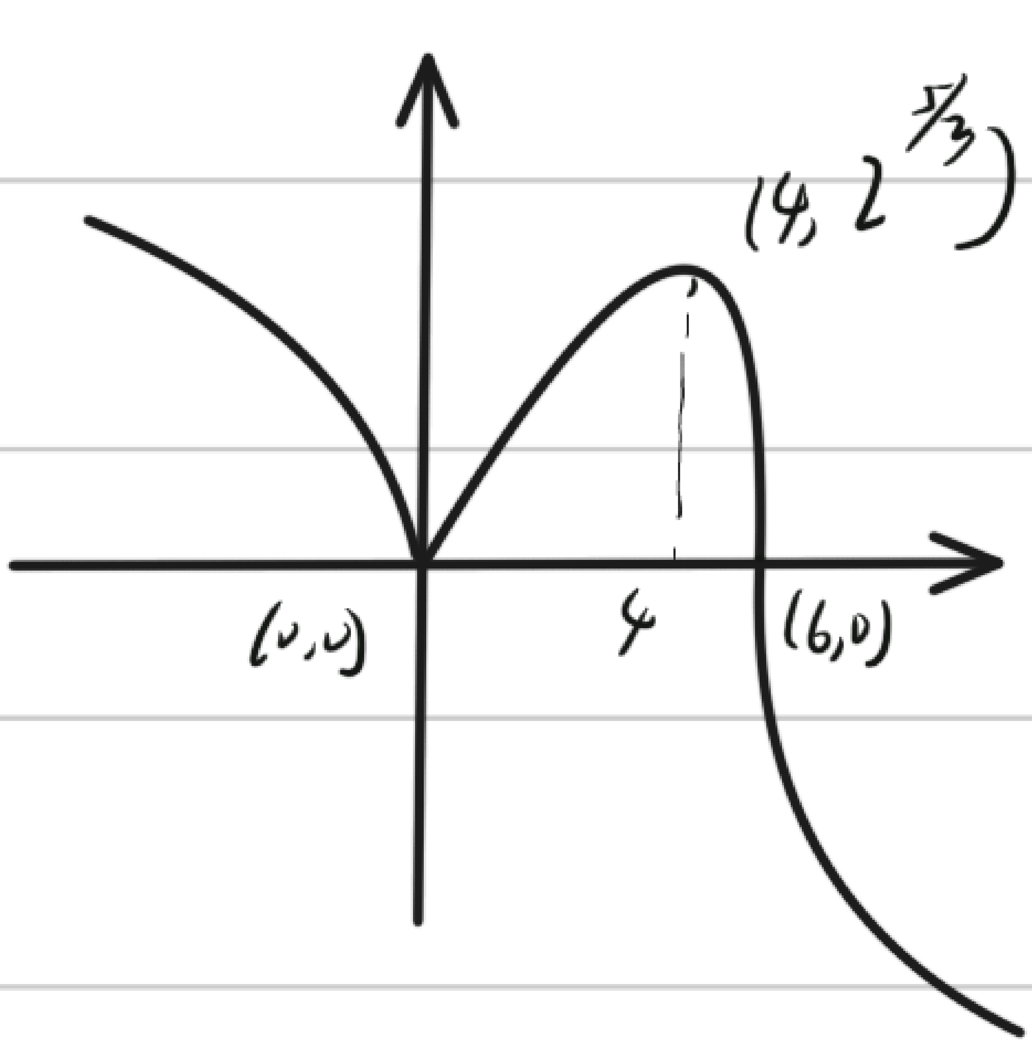
local maximum : $f(4)=2^{5/3}$ 这里化简方法同(1)，通分即可.

(2) $f''(x)=\dfrac{-x^{1/3}(6-x)^{2/3}-(4-x)\left[\frac{1}{3}x^{-2/3}(6-x)^{2/3}-\frac{2}{3}x^{1/3}(6-x)^{-1/3}\right]}{x^{2/3}(6-x)^{1/3}}=\dfrac{-8}{x^{4/3}(6-x)^{5/3}}$

	$(-\infty,0)$	$(0,6)$	$(6,+\infty)$
f''	$-$	$-$	$+$

concave up : $(6,+\infty)$. concave down $(-\infty,0),(0,6)$

(3)	$(-\infty,0)$	$(0,4)$	$(4,6)$	$(6,+\infty)$
f'	$-$	$+$	$-$	$-$
f''	$-$	$-$	$-$	$+$
f	╮	╭	╮	╰



七. $\dfrac{dy}{dx}=x$, $2y\dfrac{dy}{dx}+y+x\dfrac{dy}{dx}=0 \Rightarrow \dfrac{dy}{dx}=\dfrac{-y}{2y+x}$

切点处斜率相等：$x=\dfrac{-y}{2y+x}$ ①

切点处坐标相同：$y=\frac{1}{2}x^2$ ②

$y^2+xy=a$ ③

$\Rightarrow 3x^2+2x^3=0$

$$\begin{cases} x_1=0,\ y_1=0 \Rightarrow \text{①式不存在} \\ x_2=-\frac{3}{2},\ y_2=\frac{9}{8} \end{cases}$$

$\Rightarrow a=-\frac{27}{64}$

tangent line $y-\frac{9}{8}=-\frac{3}{2}(x+\frac{3}{2}) \Rightarrow y=-\frac{3}{2}x-\frac{9}{8}$

八. $\forall x\in(0,+\infty)$

$$\left(\frac{f(x)}{x}\right)'=\frac{xf'(x)-f(x)}{x^2}=\frac{xf'(x)-(f(x)-f(0))}{x^2}$$

中值定理 $\exists \xi\in(0,x)$, s.t $\frac{f(x)-f(0)}{x}=f'(\xi)$

$$\Rightarrow\left(\frac{f(x)}{x}\right)'=\frac{xf'(x)-xf'(\xi)}{x^2}=\frac{f'(x)-f'(\xi)}{x}$$

Since f' is increasing on $(0,+\infty)$, $0<\xi<x$

$f'(x)>f'(\xi)$

$\Rightarrow\left(\frac{f(x)}{x}\right)'=\frac{f'(x)-f'(\xi)}{x}>0 \Rightarrow \frac{f(x)}{x}$ is increasing on $(0,+\infty)$