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Testing Sample-Based Accounts of Probability Judgements Using a Ranking Task

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STATISTICAL MODELING in PSYCHOLOGY
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Introduction

- People's explicit probability judgements often appear to be probabilistically incoherent (i.e., Linda problem)

- Biases in probability judgements can arise from rational reasoning processes based on mental samples

- Sample-based probability judgement models assume that...

  ...samples generated from probability distributions in mind
  ...each sample takes time to generate
  ...infinite number of samples result in unbiased judgements

One sample-based model: PT+N model

Core assumptions in Probability Theory Plus Noise (PT+N) model (Costello & Watts, 2014):

- DM holds coherent sets of beliefs such that $P(a) + P(\neg a) = 1$
- When asked to explicitly estimate $P(A)$...
  ...DM requests N samples, according to **a fixed belief**, $P(a)$
  ...DM counts number of critical samples i that indicate A.
  ...sample has some chance d of being mistaken for its complement

  $\Rightarrow$ estimation based on relative frequencies of number of critical samples

One sample-based model: PT+N model

From these assumptions,

- Expected value of individual estimates (Costello & Watts, 2014, Eq. 3)
  $P_E(A) = (1 - 2d)P(a) + d$

- Estimate of number of critical samples (Howe & Costello, 2020, Eq. 4)
  $P(i|(N, p) = Bin(N, p) = \binom{N}{i} p^i (1 - p)^{N-i}$
  $p = P_E(A)$
  N: total sample size; i: the number of critical events

The limitation

- PT+N account only tested in estimation tasks
  - Mono-operation bias
  - Sampling parameters not identifiable without additional assumptions

- A novel ranking task is proposed
  - Ranking task
  - PT+N rank model
  - Derive qualitative predictions
  - Test qualitative predictions experimentally

The ranking Task

Please rank the four events below according to their likelihoods.

Assign **rank 1 to the event that you think is most likely to occur**, rank 2 to the event that you think is second most likely to occur, and so forth. Thus the event that you think is least likely to occur should receive rank 4. **Please specify the rank by selecting the corresponding number on the right.**

	1	2	3	4
On a random day in England, it will **not** be rainy.	○	○	○	○
On a random day in England, it will be cold.	○	○	○	○
On a random day in England, it will be rainy.	○	○	○	○
On a random day in England, it will **not** be cold.	○	○	○	○

Normative rules in ranking task

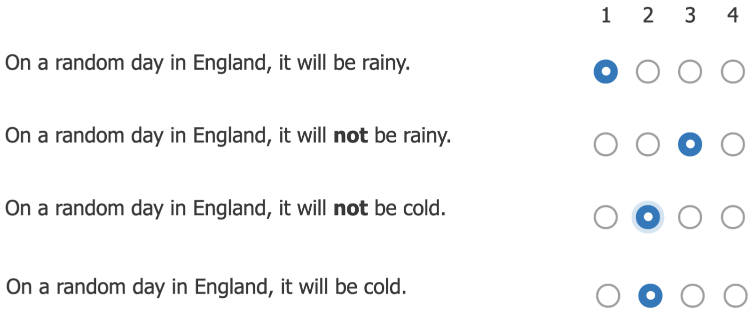
- The complement rule requires $P(a) + P(\neg a) = 1$
  - For logically possible rankings, $\text{Rank}(A) + \text{Rank}(\neg A) = \text{Rank}(B) + \text{Rank}(\neg B)$

- 75 possible ranks over four events
  - **17 of them are logically possible**
  - Rankings with vs. without ties
  - possibility to categorise logically impossible rankings

Normative rules in ranking task

Logically possible rankings. . .

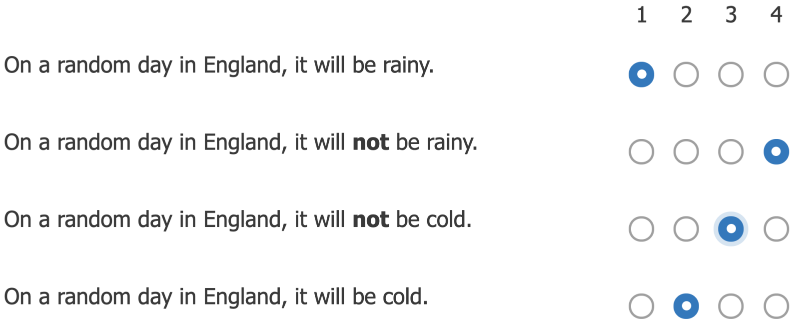
- With ties
  i.e., $\text{Rank}(A) + \text{Rank}(\neg A) = 4$; $\text{Rank}(B) + \text{Rank}(\neg B) = 4$



Normative rules in ranking task

Logically possible rankings. . .

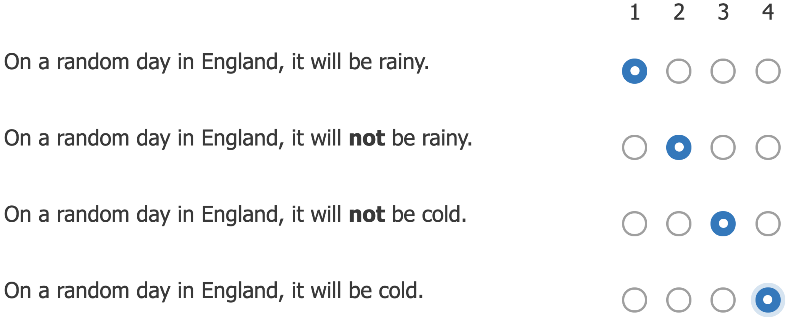
- Without ties
  i.e., $\text{Rank}(A) + \text{Rank}(\neg A) = 5$; $\text{Rank}(B) + \text{Rank}(\neg B) = 5$



Normative rules in ranking task

Categorise rankings without ties. . .

- class one illogical rankings:
  i.e., A & $\neg A$ simultaneously ranked above B & $\neg B$



Normative rules in ranking task

Categorise rankings without ties...

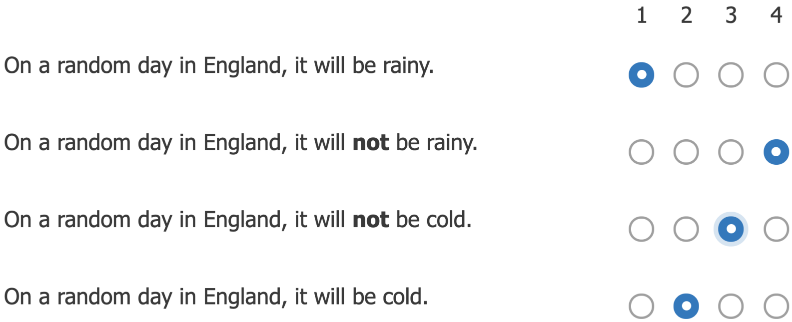
- class two illogical rankings:
  i.e., $A > B$ & $\neg A > \neg B$ simultaneously hold in one response

	1	2	3	4
On a random day in England, it will be rainy.	●	○	○	○
On a random day in England, it will **not** be rainy.	○	○	●	○
On a random day in England, it will **not** be cold.	○	○	○	●
On a random day in England, it will be cold.	○	●	○	○

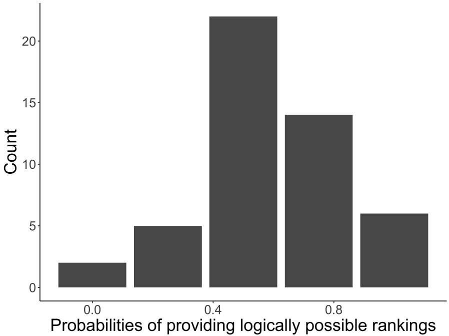
Normative rules in ranking task

Categorise rankings without ties. . .

- logically possible rankings



A new bias?



49 participants were asked to rank $\{\text{event } A, \text{ event} \neg A, \text{ event} B, \text{ event} \neg B\}$.

On average, around two out of the four rankings that participants provided were logically possible (mean $= 0.59$, SD $= 0.24$).

A new bias?

Is it a real bias, or merely the consequence of mental sampling processes?

A not so clear answer: PT+N can produce logically impossible rankings. But we need to test its sufficiency in explaining rankings.

PT+N rank model

What does PT+N model imply the rank-based probability judgements?

PT+N rank model:

- model assumptions:
  - DM holds coherent sets of beliefs
  - When asked to rank a set of events (i.e., $\{A, \neg A, B, \neg B\}$)...
    ...DM implements direct sampling for each event in set
    ...**for events in same set, one uses same sample size**
    ...**derives ranks by comparing number of critical samples**
  - sample has some chance d of being mistaken for its complement

PT+N rank model

Analytic formula for rankings without any ties:

$$P(A > B > C > D) = P(C_A > C_B > C_C > C_D)$$

$$= \sum_{i_d=0}^{N-3} f(i_d, N, d) \sum_{i_c=(i_d+1)}^{N-2} f(i_c, N, c) \sum_{i_b=(i_c+1)}^{N-1} f(i_b, N, b) \sum_{i_a=(i_b+1)}^{N} f(i_a, N, a)$$

$$f(i, N, p) = \binom{i}{N} p^i (1-p)^{N-i}$$

N: total sample size; i: the number of critical event; p: the expected value of P(A)

Assume N = 4:

i_A	i_B	i_C	i_D
3	2	1	0
4	2	1	0
4	3	1	0
4	3	2	0
4	3	2	1

- Substitute A, B, C, D with A, $\neg A$, B, $\neg B$ in 24 linear orderings

$\{A, \neg A, B, \neg B\}$
$\{A, \neg A, \neg B, B\}$
$\{A, B, \neg A, \neg B\}$
......

Predictions for rankings

Analytic formula for ranking with ties:

$$P(A > B > C = D)$$
$$= \sum_{i_{c,d}=0}^{N-2} f(i_{c,d}, N, d) f(i_{c,d}, N, c) \sum_{i_b=(i_{c,d}+1)}^{N-1} f(i_b, N, b) \sum_{i_a=(i_b+1)}^{N} f(i_a, N, a)$$
$$P(A > B = C > D) = \ldots$$
$$P(A = B > C > D) = \ldots$$

$$P(A = B > C = D)$$
$$= \sum_{i_{c,d}=0}^{N-1} f(i_{c,d}, d) f(i_{c,d}, n, c) \sum_{i_{a,b}=(i_{c,d}+1)}^{N} f(i_{a,b}, b) f(i_{a,b}, a)$$

$$P(A = B = C > D)$$
$$= \sum_{i_d=0}^{N-1} f(i_d, N, d) \sum_{i_{a,b,c}=(i_d+1)}^{N} f(i_{a,b,c}, N, c) f(i_{a,b,c}, N, b) f(i_{a,b,c}, N, a)$$
$$P(A > B = C = D) = \ldots$$

$$P(A = B = C = D)$$
$$= \sum_{i_{a,b,c,d}=0}^{N} f(i_{a,b,c,d}, N, d) f(i_{a,b,c,d}, N, c) f(i_{a,b,c,d}, N, b) f(i_{a,b,c,d}, N, a)$$

An empirical issue

Will tied rankings cause troubles on model testing?

We hypothesise that if ties are allowed, lazy participants will overuse ties.

The solution

- add additional assumptions:
  - ...
  - ...
  - ...
  - **When tied rankings not allowed, participants will produce a random linear ordering among tied events but still follow the ordering among the untied events**

The solution

	1	2	3	4
On a random day in England, it will be rainy.	●	○	○	○
On a random day in England, it will **not** be rainy.	○	○	●	○
On a random day in England, it will **not** be cold.	○	●	○	○
On a random day in England, it will be cold.	○	●	○	○

1	2	3	4
●	○	○	○
○	○	○	●
○	○	●	○
○	●	○	○

1	2	3	4
●	○	○	○
○	○	○	●
○	●	○	○
○	○	●	○

Derive qualitative predictions

- How do we derive the qualitative predictions?
  - Four parameters enter the model, N, d, A(or $\neg A$), B (or $\neg B\}$).
  - There categories of the responses
    - class one illogical rankings
    - class two illogical rankings
    - logically possible rankings

Derive qualitative predictions

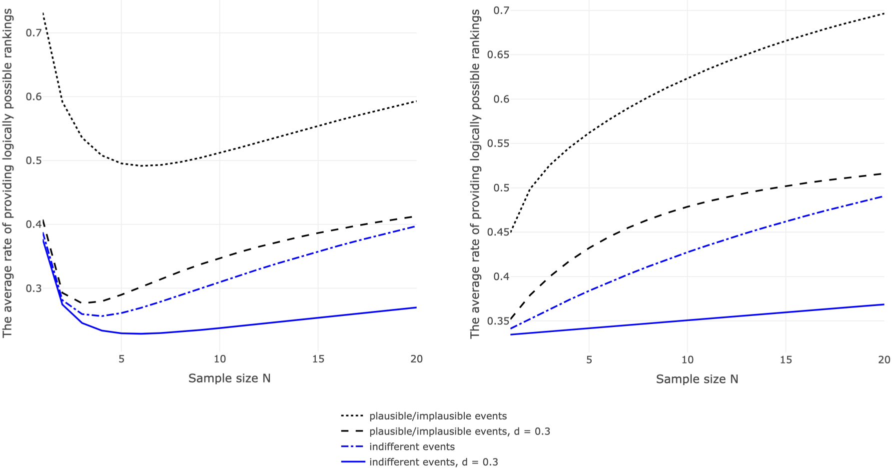
Simulation

- Inputs: different parameter values
  - Inherent beliefs (i.e., $P(a)$)
    - beliefs about plausible/implausible events, drawn from Beta(1, 10)
    - beliefs about indifferent events, drawn from Beta(10, 10)
  - reading error d
    - $d = 0$ or 0.3
  - sample size N
    - range from 1 to 20 in a step of 1
- Repeat 100000 times
- Results
  - simulate ranking data for two types of event sets
    - plausible/implausible vs. indifferent event sets
  - calculate prob. of providing three categories of rankings

Derive qualitative predictions

Average rate of producing logically possible rankings

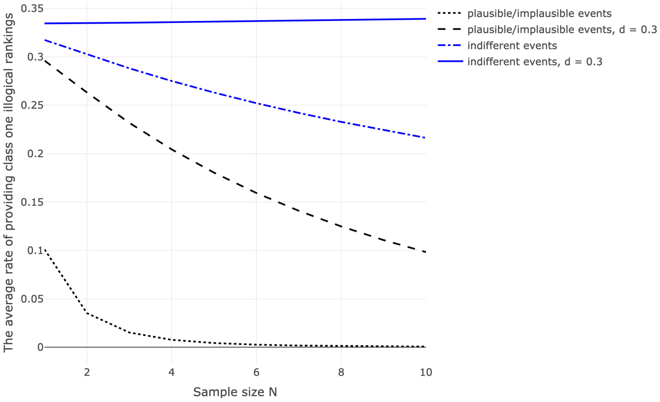
left: allow ties vs. right: not allow ties



Derive qualitative predictions

Average rate of producing class one illogical rankings

ties not allowed



Testing the qualitative predictions experimentally

Design

- manipulate inherent beliefs within subjects
  - ask participants to rank two types of events
    plausible/implausible vs. indifferent event set
- manipulate whether or not ties are allowed between subjects

Hypothesis

If we compare plausible/implausible event sets to indifferent event sets,

- Average rate of providing class one illogical rankings will be lower
- Average rate of providing logically possible rankingswill be higher

Summary

- Ranking task to investigate coherence of beliefs in a novel way
  - Biases in rank-based probability judgements
    - can be explained by existing sample-based models

- Need to understand and test the consequences of sampling
  - Ranking tasks allow us to do so
    - testing qualitative predictions of rankings

Discussion

- Future research ideas
  - Model is identifiable
    $\Rightarrow$ parameters can be estimated on individual level
  - Manipulate sample size N by...
    ...manipulating response time
    ...manipulating the structure of sentences
    - A randomly selected resident is living in southern Germany.
    - There are more people living in southern Germany than in northern Germany.

Thank you!

This research is a joint work by

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and Arndt Bröder (University of Mannheim).

Thank you!

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