Using Cartesian product to define a sample space.

Def (Cartesian product)

The cartesian product of two sets A and B, denoted $A \times B$.

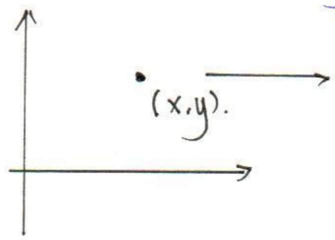
is the set of $\boxed{\text{all}}$ ordered pairs (a,b) where a is in A and b is in B.

same to Ω: the set of $\boxed{\text{all}}$ possible outcomes.

$$A \times B = \{(a,b): a \in A,\ b \in B\}$$

$|A \times B| = |A| \cdot |B|$ where $|\cdot|$ is the number of elements in set.

Ex 1: Cartesian coordinate system $\mathbb{R} \times \mathbb{R}$



Each point is an element in the set $\mathbb{R} \times \mathbb{R}$.

Ex 2: $A = \{A, B, C\}$ $B = \{1, 2\}$.

Then $A \times B = \{(a,b): a \in \{A, B, C\},\ b \in \{1, 2\}\}$

$$= \left\{ \begin{matrix} (A,1), & (A,2) \\ (B,1), & (B,2) \\ (C,1), & (C,2) \end{matrix} \right\}.$$

← 6 elements in total.

$= |A| \times |B| = 3 \times 2$

Ex 3: J is taking two books along on her holiday

W.p. 0.5. she will like the $\boxed{\text{first}}$ book. → order

⋮

Then, the sample space can be defined by the Cartesian product.

$A_i = \{0, 1\}$ with $1 =$ J will like book i

$0 =$ J will not like book i

Then $\Omega = A_1 \times A_2 = \{0,1\} \times \{0,1\}$

$= \{(0,0), (0,1), (1,0), (1,1)\}$ 4 elements.

like neither, like B only, like A only, like both

Sample space will equally likely outcomes:

$\Omega = \{w_1, \dots, w_n\}$ finite set $\quad |\Omega| = n$

with $\mathbb{P}(\{w_i\}) = \mathbb{P}(\{w_j\}) \quad \forall i, j \qquad *$

Then. by Axiom 2 $1 = \mathbb{P}(\Omega)$

$$= \mathbb{P}(\{w_1, \dots, w_n\})$$

$$= \mathbb{P}(\{w_1\} \cup \{w_2\} \cup \dots \cup \{w_n\}) \quad \text{with } \{w_i\} \cap \{w_j\} = \emptyset$$

$$= \sum_{i=1}^{n} \mathbb{P}(\{w_i\}) \quad \text{Axiom 3}$$

$$1 = n\,\mathbb{P}(\{w_i\}) \quad \forall i \qquad \text{by } *.$$

$$\Rightarrow \mathbb{P}(\{w_i\}) = \frac{1}{n}$$

To study the prob. of a set $E := \{w_{i_1}, \dots, w_{i_m}\}$ where $i_1, \dots, i_m \in \{1, \dots, n\}$

contains (m) outcomes $\qquad$ with $i_j \neq i_k \ \forall j \neq k$

$$\mathbb{P}(E) = \mathbb{P}(\{w_{i_1}, \dots, w_{i_m}\})$$

$$= \mathbb{P}(\{w_{i_1}\} \cup \dots \cup \{w_{i_m}\})$$

$$= \sum_{j=1}^{m} \mathbb{P}(\{w_{i_j}\}) = m \cdot \frac{1}{n}$$

we only need to calculate $m := |E| = \#$ elements in E

and $n = |\Omega| = \quad$ " $\quad \Omega$.

Then it becomes a problem of counting.

Sps that there are 3 red balls and 5 blue balls in a box

If we randomly choose 2 balls from the box,

what is the prob. that the two balls are of different colors?

Translation: "2 balls with different color" $\Leftrightarrow$ "1 red ball and 1 blue ball"

Denote the red balls R_1, R_2, R_3. blue balls $B_1, \cdots, B_5$

Solution ①: (Without order)

When we don't consider the order which means $\{R_1, R_2\} = \{R_2, R_1\}$

$$\Omega = \left\{ \begin{array}{llll} \{R_1, R_2\}, & \{R_1, R_3\}, & \{R_1, B_1\} \cdots & \{R_1, B_5\}, \\ & \{R_2, R_3\} & \{R_2, B_1\} \cdots & \{R_2, B_5\}, \\ & & \{R_3\ B_1\} \cdots & \{R_3\ B_5\}. \\ & & & \vdots \\ & & & \{B_4\ B_5\} \end{array} \right\} \begin{array}{l} 7 \\ 6 \\ \cdot \\ \cdot \\ 1 \end{array}$$

there are $\dfrac{(7+1)\times 7}{2} = 28$ (another way $\binom{8}{2} = 28$) outcomes

$|\Omega| = 28$

$$|E| = \underbrace{\binom{3}{1}}_{\text{\# ways choose 1 red from 3}} \overset{\text{multiplicative rule}}{\times} \underbrace{\binom{5}{1}}_{\text{\# ways choose 1 blue from 5}} = 3\times 5 = 15$$

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|} = \frac{15}{28}$$

Solution ② : With order

We consider the order of choosing the two balls.

We use (,) to denote the order

(R_1, R_2) : first choose R_1, then R_2

$\neq$

(R_2, R_1) : " R_2, R_1

$\Omega =$ $\qquad |\Omega| = 8 \times 7 = 56$

ways choose 1st ball

ways choose 2nd ball

E = # ways first choose a red ball × # ways then choose a blue ball.

Addition principle.

multiplicative rule

+ # ways first blue × # ways then red

Addition principle (Wikipedia)

A ways of doing something, B ways of doing another thing

* we cannot do both things at the same time

different with multiplicative which is sequentially

then there are A + B ways to choose one of the actions.

For disjoint set A and B, we have $|A \cup B| = |A| + |B|$.

$= 3 \times 5 + 5 \times 3 = 15 + 15 = 30$

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|} = \frac{30}{56} = \frac{15}{28}$$

$$\Omega = \left\{ \begin{array}{cccc} (R_1, R_2) & (R_1, R_3) & \cdots & (R_1, B_5) \\ (R_2, R_1) & (R_2, R_3) & \cdots & (R_2, B_5) \\ \vdots & & & \\ (B_5, R_1) & (B_5, R_2) & \cdots & (B_5, B_4) \end{array} \right\}$$