· Trigonalmetric substitutions

Expression	Identity	Substitution
$\sqrt{a^2 - x^2}$	$1 - \sin^2\theta = \cos^2\theta$	$x = a\sin\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$\sqrt{a^2 + x^2}$	$1 + \tan^2\theta = \sec^2\theta$	$x = a\tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$\sqrt{x^2 - a^2}$	$\sec^2\theta - 1 = \tan^2\theta$	$x = a\sec\theta, \quad 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3}{2}\pi$

We want to get rid of the $\sqrt{}$.

Remarks:

Notice "=" when we have $\frac{1}{\sqrt{a^2 - x^2}}$, or $\frac{1}{\sqrt{x^2 - a^2}}$

T05 - Ex 7

$$\int \frac{1}{\sqrt{9x^2-36x+37}}\,dx$$

Solution.

$$9x^2-36x+37 = 9(x^2-4x+4)-4\times 9+37 = 9(x-2)^2+1 = \left(3(x-2)\right)^2+1$$

Let $3(x-2) = \tan\theta$, then $x = 2+\frac{1}{3}\tan\theta$,

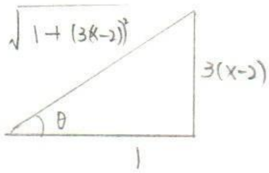
$-\frac{\pi}{2}<\theta<\frac{\pi}{2}$ $\qquad dx = \frac{1}{3}\sec^2\theta\,d\theta$

$$\int \frac{1}{\sqrt{9x^2-36x+37}}\,dx = \int \frac{1}{\sqrt{\tan^2\theta+1}}\cdot\frac{1}{3}\sec^2\theta\,d\theta$$

$$=\frac{1}{3}\int \frac{1}{|\sec\theta|}\sec^2\theta\,d\theta$$

$$=\frac{1}{3}\int \frac{1}{\sec\theta}\sec^2\theta\,d\theta.$$ since $\sec\theta>0$ when $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$

$$=\frac{1}{3}\int \sec\theta\,d\theta$$



$$=\frac{1}{3}\ln|\sec\theta+\tan\theta|+C$$

$$=\frac{1}{3}\ln\left|\sqrt{9x^2-36x+37}+3(x-2)\right|+C$$ $\square$

T05 - Ex 8

$$\int \frac{\sqrt{x^2+16}}{x^4}\,dx$$

Solution: $a=4$. Let $x = 4\tan\theta$, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$

then $dx = 4\sec^2\theta\, d\theta$.

$$\int \frac{\sqrt{x^2+16}}{x^4}\,dx = \int \frac{\sqrt{16\tan^2\theta+16}}{(4\tan\theta)^4}\cdot 4\sec^2\theta\, d\theta$$

$$= \frac{4\times 4}{4^4}\int \frac{|\sec\theta|}{\tan^4\theta}\sec^2\theta\, d\theta$$

$$= \frac{1}{16}\int \frac{\sec^3\theta}{\tan^4\theta}\,d\theta$$

save factor doesn't work ↓ rewrite it to $\sin\theta$ & $\cos\theta$

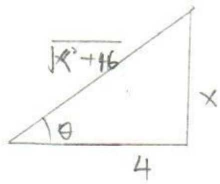
$$= \frac{1}{16}\int \frac{1}{\cos^3\theta}\cdot\frac{\cos^4\theta}{\sin^4\theta}\,d\theta$$

$$= \frac{1}{16}\int \frac{\cos\theta}{\sin^4\theta}\,d\theta$$

It works!

$$= \frac{1}{16}\int \frac{1}{\sin^4\theta}\,d\sin\theta$$

$\tan\theta = \frac{x}{4}$



$$= \frac{1}{16}\,\frac{1}{-4+1}(\sin\theta)^{-4+1} + C$$

$$= -\frac{1}{48}\,\frac{1}{\sin^3\theta} + C$$

$$= -\frac{1}{48}\left(\frac{\sqrt{x^2+16}}{x}\right)^3 + C$$

$$= -\frac{(x^2+16)^{\frac{3}{2}}}{48x^3} + C$$

Check the answer:

$$-\frac{\frac{3}{2}(x^2+16)^{\frac{1}{2}}\cdot(2x)\cdot 48x^3 - (x^2+16)^{\frac{3}{2}}\cdot 48\cdot 3x^2}{(48x^3)^2}$$

$$= -\frac{3x\sqrt{x^2+16}}{\underset{16}{\cancel{48}}x^{\cancel{4}\,2}} + \frac{\sqrt{x^2+16}\,(x^2+16)\cdot\cancel{48}\cdot 3\cancel{x^2}}{\underset{16}{\cancel{48^2}}x^{\cancel{6}\,4}} = -\frac{1}{16x^2}\sqrt{x^2+16} + \frac{x^2+16}{16x^4}\sqrt{x^2+16} = \frac{\sqrt{x^2+16}}{x^4}$$

T06 - Ex 1

$$\int \frac{dx}{x^2\sqrt{4-x^2}}$$

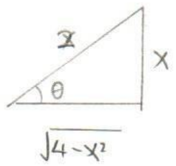
Solution:

Let $x = 2\sin\theta$, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, then $\sin\theta = \frac{x}{2}$ and $dx = 2\cos\theta\, d\theta$

$$\int \frac{1}{x^2\sqrt{4-x^2}}\,dx = \int \frac{1}{4\sin^2\theta\cdot 2\cos\theta}\cdot 2\cos\theta\, d\theta$$

$$= \frac{1}{4}\int \csc^2\theta\, d\theta$$

$$= -\frac{1}{4}\cot\theta + C$$



$$= -\frac{1}{4}\,\frac{\sqrt{4-x^2}}{x} + C$$

Check the answer $\left(-\frac{1}{4}\,\frac{\sqrt{4-x^2}}{x}\right)' = -\frac{1}{4}\,\frac{\frac{1}{2}(4-x^2)^{-\frac{1}{2}}\cdot(-2x)\cdot x^2 - \sqrt{4-x^2}}{x^2}$

$$= \frac{1}{4}\,\frac{x}{\sqrt{4-x^2}} + \frac{1}{4}\,\frac{\sqrt{4-x^2}}{x^2} = \frac{\frac{1}{4}x^2 + \frac{1}{4}(4-x^2)}{x^2\sqrt{4-x^2}} = \frac{1}{x^2\sqrt{4-x^2}}$$

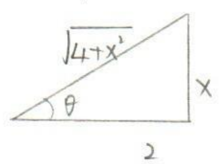
T06 - Ex 2

$$\int \frac{x^3}{\sqrt{x^2+4}}\,dx$$

Solution:

Let $x = 2\tan\theta$, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, then $\tan\theta = \frac{x}{2}$ and $dx = 2\sec^2\theta\, d\theta$

$$\int \frac{x^3}{\sqrt{x^2+4}}\,dx = \int \frac{8\tan^3\theta}{2\sec\theta}\cdot 2\sec^2\theta\, d\theta.$$



$$= 8\int \tan^3\theta\sec\theta\, d\theta$$

$$= 8\int \tan^2\theta\,(\tan\theta\sec\theta)\, d\theta$$

$$= 8\int (\sec^2\theta - 1)\, d\sec\theta$$

$$= \frac{8}{3}\sec^3\theta - 8\sec\theta + C$$

$$= \frac{8}{3}\left(\frac{\sqrt{4+x^2}}{2}\right)^3 - 8\cdot\frac{\sqrt{4+x^2}}{2} + C$$

$$= \frac{1}{3}(4+x^2)^{\frac{3}{2}} - 4(4+x^2)^{\frac{1}{2}} + C =: g(x)$$

Check the answer

$$g'(x) = \frac{1}{3}\cdot\frac{3}{2}\sqrt{4+x^2}\cdot(2x) - 4\cdot\frac{1}{2}\,\frac{1}{\sqrt{4+x^2}}(2x)$$

$$= x\sqrt{4+x^2} - 4\frac{x}{\sqrt{4+x^2}}$$

$$= \frac{x(4+x^2) - 4x}{\sqrt{4+x^2}} = \frac{x^3}{\sqrt{4+x^2}}$$

T06 - Ex 3

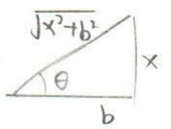
$$E(P) = \int_{-a}^{L-a} \frac{\lambda b}{4\pi\varepsilon_0 (x^2+b^2)^{\frac{3}{2}}}\,dx \qquad (b>0)$$

Solution 1 : (by indefinite integral)

We first solve $\int \frac{dx}{(x^2+b^2)^{\frac{3}{2}}}$

Let $x = b\tan\theta$, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, then $\tan\theta = \frac{x}{b}$ and $dx = b\sec^2\theta\,d\theta$

$$\int \frac{dx}{\left(\sqrt{x^2+b^2}\right)^3} = \int \frac{1}{(b\sec\theta)^3}\cdot b\sec^2\theta\,d\theta = \frac{1}{b^2}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{b^2}\int\cos\theta\,d\theta = \frac{1}{b^2}\sin\theta + C$$



Thus, $\int \frac{dx}{(x^2+b^2)^{\frac{3}{2}}} = \frac{1}{b^2}\frac{x}{\sqrt{x^2+b^2}} + C$

Check the answer:

$$\left(\frac{1}{b^2}\frac{x}{\sqrt{x^2+b^2}}\right)' = \frac{1}{b^2}\,\frac{\sqrt{x^2+b^2} - x\cdot\frac{1}{2}(x^2+b^2)^{-\frac{1}{2}}\cdot(2x)}{x^2+b^2}$$

$$= \frac{x^2+b^2-x^2}{b^2(x^2+b^2)^{\frac{3}{2}}} = \frac{1}{(x^2+b^2)^{\frac{3}{2}}}$$

$$E(P) = \frac{\lambda b}{4\pi\varepsilon_0}\int_{-a}^{L-a}\frac{1}{(x^2+b^2)^{\frac{3}{2}}}\,dx = \frac{\lambda b}{4\pi\varepsilon_0}\cdot\frac{1}{b^2}\left[\frac{x}{\sqrt{x^2+b^2}}\right]_{-a}^{L-a}$$

$$= \frac{\lambda}{4\pi\varepsilon_0 b}\left(\frac{L-a}{\sqrt{(L-a)^2+b^2}} + \frac{a}{\sqrt{a^2+b^2}}\right)$$

Solution 2 : (by definite integral)

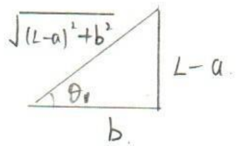
$$E(P) = \frac{\lambda b}{4\pi\varepsilon_0}\int_{-a}^{L-a}\frac{1}{(x^2+b^2)^{\frac{3}{2}}}\,dx$$

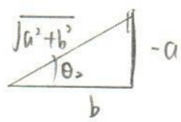
$x = L-a \rightarrow \theta = \arctan\left(\frac{L-a}{b}\right)$

$x = -a \rightarrow \theta = \arctan\left(\frac{-a}{b}\right)$

$$= \frac{\lambda b}{4\pi\varepsilon_0}\int_{\arctan(-\frac{a}{b})}^{\arctan(\frac{L-a}{b})}\frac{1}{(b\sec\theta)^3}\cdot b\sec^2\theta\,d\theta$$

$= \cdots$





$$= \frac{\lambda b}{4\pi\varepsilon_0}\cdot\frac{1}{b^2}\left[\sin\theta\right]_{\arctan(-\frac{a}{b})}^{\text{actan}(\frac{L-a}{b})}$$

$$= \frac{\lambda}{4\pi\varepsilon_0 b}\left[\sin\left(\arctan\left(\frac{L-a}{b}\right)\right) - \sin\left(\arctan\left(-\frac{a}{b}\right)\right)\right]$$

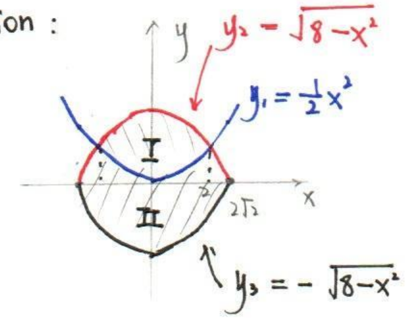
$$= \frac{\lambda}{4\pi\varepsilon_0 b}\left(\frac{L-a}{\sqrt{(L-a)^2+b^2}} + \frac{a}{\sqrt{a^2+b^2}}\right)$$

T06 – Ex4

The parabola $y = \frac{1}{2}x^2$ divides the disk $x^2+y^2 \le 8$ into two parts.

Find the areas of both parts.

Solution :



Let $y_1(x) = y_2(x)$,

we have $\frac{1}{2}x^2 = \sqrt{8-x^2}$

$$x^4 + 4x^2 - 32 = 0$$

$$(x-2)(x+2)(x^2+8) = 0$$

Recall Area under the curve:

then $x_1 = -2$, $x_2 = 2$

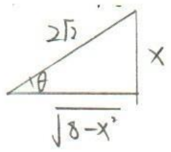
$$S_I = \int_{-2}^{2} y_2(x) - y_1(x)\, dx = \int_{-2}^{2} \sqrt{8-x^2} - \frac{1}{2}x^2\, dx = 2\int_0^2 \sqrt{8-x^2} - \frac{1}{2}x^2\, dx$$

We first find $\int \sqrt{8-x^2}\, dx$

Let $x = 2\sqrt{2}\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then $\sin\theta = \frac{x}{2\sqrt{2}}$ and $dx = 2\sqrt{2}\cos\theta\, d\theta$

$$\int \sqrt{8-x^2}\, dx = \int 2\sqrt{2}\cos\theta \cdot 2\sqrt{2}\cos\theta\, d\theta$$

$$= 8\int \cos^2\theta\, d\theta$$



$$= 8\int \frac{1}{2} + \frac{1}{2}\cos 2\theta\, d\theta$$

$$= 4\theta + 4\cdot\frac{1}{2}\sin 2\theta + C$$

$$= 4\theta + 4\sin\theta\cos\theta + C$$

$$= 4\sin^{-1}\left(\frac{x}{2\sqrt{2}}\right) + 4\cdot\frac{x}{2\sqrt{2}}\cdot\frac{\sqrt{8-x^2}}{2\sqrt{2}} + C$$

$$= 4\sin^{-1}\left(\frac{x}{2\sqrt{2}}\right) + \frac{1}{2}x\sqrt{8-x^2} + C$$

$$S_I = 2\int_0^2 \sqrt{8-x^2}\, dx - \int_0^2 x^2\, dx$$

$$= 2\left[4\sin^{-1}\left(\frac{x}{2\sqrt{2}}\right) + \frac{1}{2}x\sqrt{8-x^2}\right]_0^2 - \left[\frac{1}{3}x^3\right]_0^2$$

$$= 2\left(4\cdot\frac{\pi}{4} + \frac{1}{2}\cdot 2\cdot\sqrt{8-4} - 0 - 0\right) - \frac{1}{3}\cdot 8 = 2\pi + \frac{4}{3}$$

$$S_\odot = \int_{-2\sqrt{2}}^{2\sqrt{2}} y_2(x) - y_3(x)\, dx = \int_{-2\sqrt{2}}^{2\sqrt{2}} \sqrt{8-x^2} - (-\sqrt{8-x^2})\, dx = 4\int_0^{2\sqrt{2}} \sqrt{8-x^2}\, dx = 4\left[4\sin^{-1}\left(\frac{x}{2\sqrt{2}}\right) + \frac{1}{2}x\sqrt{8-x^2}\right]_0^{2\sqrt{2}}$$

$$= 4\times 4\times\frac{\pi}{2} = 8\pi$$

$$S_{II} = S_\odot - S_I = 8\pi - \left(2\pi + \frac{4}{3}\right) = 6\pi - \frac{4}{3}.$$

· Partial Fractions

$$f(x) = \frac{P(x)}{Q(x)}$$, where P and Q are polynomials.

Step 1: make f proper:

Long division $\longrightarrow$ $f(x) = S(x) + \frac{R(x)}{Q(x)}$

Step 2: Factor $Q(x) = c(x-a_1)^{\lambda_1} \cdots (x-a_s)^{\lambda_s}(x^2+p_1x+q_1)^{\mu_1} \cdots (x^2+p_tx+q_t)^{\mu_t}$

with $\lambda_i, \mu_j \in \mathbb{N}_+$, $\sum_{i=1}^{s}\lambda_i + 2\sum_{j=1}^{t}\mu_j = \deg(Q)$, $p_j^2 - 4q_j < 0$, $j=1,2,\cdots t$

Step 3: Express $\frac{R(x)}{Q(x)}$ as a sum of partial fractions

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^j} \quad (b^2-4ac<0)$$

linear factor irreducible quadratic factor

· 4 Cases

· Then, solve A, B, C, D ⋯ (May have tricks)

Step 4: Take integrals

$$\int \frac{1}{x-a}\,dx = \ln|x-a| + C$$

$$\int \frac{1}{(x-a)^k}\,dx = \frac{1}{-k+1}(x-a)^{-k+1} + C \quad k \geq 2$$

completing the square

$$\int \frac{x-a}{[(x-a)^2+b^2]^k}\,dx \xlongequal{u=(x-a)^2+b^2} \frac{1}{2}\int \frac{1}{u^k}\,du + C$$

$$\int \frac{1}{[(x-a)^2+b^2]^k}\,dx \xlongequal{x-a = b\cdot\tan u\ ,\ -\frac{\pi}{2}<u<\frac{\pi}{2}} \frac{1}{b^{2k-1}}\int \cos^{2k-2}u\,du.$$

$k=1$, $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C$

$$\int \frac{1}{(x-1)(x+2)^2}\,dx$$

Solution :

Step 1: Proper ✓

Step 2: $Q(x) = (x-1)(x+2)^2$

Step 3:

$$\frac{1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

Solve A, B, C

Multiply $(x-1)(x-2)^2$ on both sides

$$1 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$$

$$1 = A(x^2+4x+4) + B(x^2+x-2) + C(x-1)$$

$$1 = (A+B)x^2 + (4A+B+C)x + 4A-2B-C$$

$$\Rightarrow \begin{cases} A+B=0 \\ 4A+B+C=0 \\ 4A-2B-C=1 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{9} \\ B = -\frac{1}{9} \\ C = -\frac{1}{3} \end{cases}$$

Step 4: $\int \frac{1}{(x-1)(x+2)^2}\,dx = A\int \frac{1}{x-1}\,dx + B\int \frac{1}{x+2}\,dx + C\int \frac{1}{(x+2)^2}\,dx$

$$= A\ln|x-1| + B\ln|x+2| - C\,\frac{1}{(x+2)} + \text{Const.}$$

$$= \frac{1}{9}\ln|x-1| - \frac{1}{9}\ln|x+2| + \frac{1}{3}\,\frac{1}{x+2} + C \qquad \square$$

$$\int \frac{x^2+1}{(x^2-2x+2)^2}\,dx$$

Solution :

Step 1: Proper ✓

Step 2 : $b^2-4ac = (-2)^2-4\times1\times2 = -4<0$

$Q(x) = (x^2-2x+2)^2$

Step 3: Method 1: solving the equations $\dfrac{x^2+1}{(x^2-2x+2)^2} = \dfrac{Ax+B}{x^2-2x+2} + \dfrac{Cx+D}{(x^2-2x+2)^2}$

Multiply $(x^2-2x+2)^2$ on both sides:

$$x^2+1 = (Ax+B)(x^2-2x+2) + Cx+D$$

$$= Ax^3 + (-2A+B)x^2 + (2A-2B+C)x + 2B+D$$

$$\Rightarrow \begin{cases} A=0 \\ -2A+B=1 \\ 2A-2B+C=0 \\ 2B+D=1 \end{cases} \Rightarrow \begin{cases} A=0 \\ B=1 \\ C=2 \\ D=-1 \end{cases}$$

we have $\dfrac{x^2+1}{(x^2-2x+2)^2} = \dfrac{1}{x^2-2x+2} + \dfrac{2x-1}{(x^2-2x+2)^2}$

Method 2: try to write as x^2-2x+2

$$\frac{x^2+1}{(x^2-2x+2)^2} = \frac{(x^2-2x+2)+2x-2+1}{(x^2-2x+2)^2} = \frac{1}{x^2-2x+2} + \frac{2x-1}{(x^2-2x+2)^2}$$

Step 4: Taking integrals.

$$\int \frac{dx}{x^2-2x+2} = \int \frac{1}{(x-1)^2+1^2}\,dx = \arctan(x-1) + C_1$$

$$\int \frac{2x-1}{(x^2-2x+2)^2}\,dx = 2\underbrace{\int \frac{x-1}{[(x-1)^2+1]^2}\,dx}_{①} + \underbrace{\int \frac{1}{[(x-1)^2+1]^2}\,dx}_{②}$$

$$① = \int \frac{x-1}{[(x-1)^2+1]^2}\,dx \ \xlongequal[du=2(x-1)dx]{u=(x-1)^2+1}\ \frac{1}{2}\int \frac{1}{[(x-1)^2+1]^2}\,2(x-1)\,dx$$

$$= \frac{1}{2}\int \frac{1}{u^2}\,du$$

$$= -\frac{1}{2}\,\frac{1}{u} + C_2$$

$$= -\frac{1}{2}\,\frac{1}{x^2-2x+2} + C_2$$

$$② = \int \frac{1}{[(x-1)^2+1]^2}\,dx$$

Let $x-1 = \tan u \quad -\frac{\pi}{2}<u<\frac{\pi}{2}$

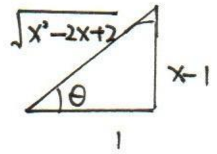
$dx = \sec^2 u\,du$

$$= \int \frac{1}{\sec^4 u}\cdot \sec^2 u\,du$$

$$= \int \cos^2 u\,du$$

by $\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$

$$= \frac{1}{2}\cos u\sin u + \frac{2-1}{2}\int \cos^0 u\,du$$



$$= \frac{1}{2}\cos u\sin u + \frac{1}{2}u + C_3$$

$$= \frac{1}{2}\,\frac{1}{\sqrt{x^2-2x+2}}\cdot\frac{x-1}{\sqrt{x^2-2x+2}} + \frac{1}{2}\tan^{-1}(x-1) + C_3$$

$$= \frac{1}{2}\,\frac{x-1}{x^2-2x+2} + \frac{1}{2}\tan^{-1}(x-1) + C_3.$$

Thus,

$$\int \frac{x^2+1}{(x^2-2x+2)^2} = \tan^{-1}(x-1) - \frac{1}{x^2-2x+2} + \frac{1}{2}\,\frac{x-1}{x^2-2x+2} + \frac{1}{2}\tan^{-1}(x-1) + C$$

$$= \frac{x-3}{2(x^2-2x+2)} + \frac{3}{2}\tan^{-1}(x-1) + C \qquad \square$$

Method in the slides:

$$\int \frac{2x-1}{(x^2-2x+2)^2}\,dx = \underbrace{\int \frac{2x-2}{(x^2-2x+2)^2}\,dx}_{=:①} + \underbrace{\int \frac{1}{[(x-1)^2+1]^2}\,dx}_{=:②} \qquad d(x^2-2x+2) = (2x-2)\,dx$$

$$① = \int \frac{1}{(x^2-2x+2)^2}\,d(x^2-2x+2) = -\frac{1}{x^2-2x+2} + C_1$$

② Let $t = x-1$, $dt = dx$

$$② = \int \frac{1}{(t^2+1)^2}\,dt$$

$$= \int \frac{t^2+1}{(t^2+1)^2}\,dt - \int \frac{t^2}{(t^2+1)^2}\,dt \qquad d\frac{1}{t^2+1} = \frac{-2t}{(t^2+1)^2}$$

$$= \int \frac{1}{t^2+1}\,dt + \frac{1}{2}\int t\, d(t^2+1)^{-1}$$

$$= \arctan(t) + \frac{1}{2}\left[\frac{t}{t^2+1} - \int \frac{1}{t^2+1}\,dt\right] \qquad \text{integration by parts.}$$

$$= \frac{t}{2(t^2+1)} + \frac{3}{2}\arctan(t) + C_2$$

$$= \frac{x-1}{2(x^2-2x+2)} + \frac{3}{2}\arctan(x-1) + C_2$$

$$\int \frac{2x-1}{(x^2-2x+2)^2}\,dx = -\frac{1}{x^2-2x+2} + \frac{x-1}{2}\,\frac{1}{x^2-2x+2} + \frac{3}{2}\arctan(x-1) + C$$

$$= \frac{x-3}{2(x^2-2x+2)} + \frac{3}{2}\arctan(x-1) + C \qquad \square$$

· Rationalizing substitutions for $\sqrt[n]{g(x)}$ Let $u = \sqrt[n]{g(x)}$ then $u^n = g(x)$

$$\int \frac{\sqrt{x}}{x+1}\, dx$$

Solution: Let $u = \sqrt{x}$, then $x = u^2$, $dx = 2u\,du$

$$\int \frac{\sqrt{x}}{x+1}\, dx = \int \frac{u}{u^2+1} \cdot 2u\, du$$

$$= 2\int \frac{u^2}{u^2+1}\, du \qquad \frac{u^2}{u^2+1} = \frac{(u^2+1)-1}{u^2+1} = 1 - \frac{1}{u^2+1}$$

$$= 2\int du - 2\int \frac{1}{u^2+1}\, du$$

$$= 2u - 2\arctan(u) + C$$

$$= 2\sqrt{x} - 2\arctan(\sqrt{x}) + C \qquad \square$$