# Tutorial 5

· Integration by parts

· Indefinite integrals $\int f(x)\, g'(x)\, dx = f(x)g(x) - \int f'(x)\, g(x)\, dx$

or $\int u\, dv = uv - \int v\, du$, where $u = f(x)$, $v = g(x)$.

$f(x) \xrightarrow{d} f'(x)$

$g'(x) \xrightarrow{\int} g(x)$

· Definite Integrals: $\int_a^b f(x)\, g'(x)\, dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)\, g(x)\, dx$

· Choose u and v

· inverse trigonalmetric func.s

· logarithmic func.s

· power func.s

· exponential func.s

· trigonalmetric func.s

Choose the upper one as u.

and the lower one as v.

TOS- Ex 1

$$\int t^7 \sin(2t^4)\, dt$$

Solution: let $w = 2t^4$, then $dw = 8t^3\, dt$

$$\int t^7 \sin(2t^4)\, dt = \frac{1}{16}\int w \sin(w)\, du$$

$u = w \longrightarrow du = dw$

$v' = \sin(w) \longrightarrow v = -\cos w$

$$= \frac{1}{16}\left[w(-\cos w) - \int -\cos(w)\, dw\right]$$

$$= \frac{1}{16}\left[-w\cos(w) + \int \cos(w)\, dw\right]$$

$$= \frac{1}{16}\left[-w\cos(w) + \sin(w) + C\right]$$

$$= -\frac{1}{16}\cdot 2t^4 \cos(2t^4) + \frac{1}{16}\sin(2t^4) + C$$

$$= -\frac{1}{8} t^4 \cos(2t^4) + \frac{1}{16}\sin(2t^4) + C$$

Solution on the slides:

$$\int t^7 \sin(2t^4)\, dt = \int t^4\, t^3 \sin(2t^4)\, dt =: A$$

$u = t^4 \longrightarrow du = 4t^3\, dt$

du = $t^3 \sin(2t^4)$ dt $\rightarrow v = -\frac{1}{8}\cos(2t^4)$

Let $w = 2t^4$, then $dw = 8t^3\, dt$

since $\int t^3 \sin(2t^4)\, dt = \frac{1}{8}\int \sin(w)\, dw = -\frac{1}{8}\cos(w) + C = -\frac{1}{8}\cos(2t^4) + C$

Then, $A = t^4\cdot\left(-\frac{1}{8}\cos(2t^4)\right) - \int -\frac{1}{8}\cos(2t^4)\cdot 4t^3\, dt$

$$= -\frac{1}{8}t^4\cdot\cos(2t^4) + \frac{1}{2}\int t^3 \cos(2t^4)\, dt$$

Let $s = 2t^4$, then $ds = 8t^3\, dt$

$$= -\frac{1}{8}t^4\cdot\cos(2t^4) + \frac{1}{2}\left[\frac{1}{8}\int \cos(s)\, ds\right]$$

$$= -\frac{1}{8}t^4\cdot\cos(2t^4) + \frac{1}{16}\sin(2t^4) + C$$

T05 - Ex 2

$$\int 6\tan^{-1}\left(\frac{8}{w}\right) dw$$

Solution: $u = \tan^{-1}\left(\frac{8}{w}\right) \longrightarrow du = \frac{1}{1+\left(\frac{8}{w}\right)^2} \cdot \left(-\frac{8}{w^2}\right) dw = -\frac{8}{w^2+64} dw$

$dv = dw \longrightarrow v = w$

$$A = 6\left[\tan^{-1}\left(\frac{8}{w}\right) w - \int w \cdot \left(-\frac{8}{w^2+64}\right) dw\right]$$

$$= 6w\tan^{-1}\left(\frac{8}{w}\right) + 48\int w \cdot \frac{1}{w^2+64} dw$$

Let $s = w^2+64$ , $ds = 2w\,dw$

$$\int \frac{w}{w^2+64} dw = \frac{1}{2}\int \frac{1}{s} ds = \frac{1}{2}\ln|s| + C = \frac{1}{2}\ln|w^2+64| + C$$

Then,

$$A = 6w\tan^{-1}\left(\frac{8}{w}\right) + 24\ln(w^2+64) + C$$

T05 - Ex 3

$$\int (\ln x)^2 dx$$

Solution: $u = (\ln x)^2 \longrightarrow du = 2\ln x \cdot \frac{1}{x} dx$

$dv = dx \longrightarrow v = x$

$$\int (\ln x)^2 dx = (\ln x)^2 \cdot x - \int x \cdot \left(2\ln x \cdot \frac{1}{x}\right) dx$$

$$= x(\ln x)^2 - 2\int \ln x\, dx$$

$s = \ln x \longrightarrow ds = \frac{1}{x} dx$

$dt = dx \longrightarrow t = x$

$$= x(\ln x)^2 - 2\left[\ln x \cdot x - \int x \cdot \frac{1}{x} dx\right]$$

$$= x(\ln x)^2 - 2x\ln x + 2x + C$$

· Trigonalmetric Integrals

Type I : $\int \sin^m x \cos^n x \, dx$

Possible factors to be saved : $\sin(x)$ / $\cos(x)$

Formulas : $\sin x \, dx = -d\cos x$ , $\sin^2 x = 1 - \cos^2 x$

$\cos x \, dx = d\sin x$ , $\cos^2 x = 1 - \sin^2 x$

When m, n are even.

$\sin^2 x = \frac{1-\cos 2x}{2}$ , $\cos^2 x = \frac{1+\cos 2x}{2}$ , $\sin x \cos x = \frac{1}{2}\sin 2x$

Example : Consider $\sin^2 x \cos^3 x$ , $\sin^2 x \cos^2 x$ , $\sin^3 x \cos^2 x$ , $\sin^3 x \cos^3 x$

Save $\cos x$ ; $= \frac{1}{4}\sin^2 2x$ ; save $\sin x$ ; save $\sin x$ or $\cos x$.

$= \frac{1}{4}\left(\frac{1-\cos 4x}{2}\right)$

$= \frac{1}{8} - \frac{1}{8}\cos 4x$

Type II : $\int \tan^m x \sec^n x \, dx$

Possible factors to be saved : $\sec^2 x$ / $\tan x \sec x$

Formulas : $\sec^2 x \, dx = d\tan x$. $\sec^2 x = 1 + \tan^2 x$

$\tan x \sec x \, dx = d\sec x$ , $\tan^2 x = \sec^2 x - 1$

$\int \tan x \, dx = \ln|\sec x| + C$

$\int \sec x \, dx = \ln|\tan x + \sec x| + C$

Example : Consider $\tan^2 x \sec^3 x$ . $\tan^2 x \sec^2 x$ . $\tan^3 x \sec^2 x$ , $\tan^3 x \sec^3 x$.

No general strategy ; Save $\sec^2 x$ ; save $\sec^2 x$ or save $\tan x \sec x$ ; save $\tan x \sec x$

Type III : $\int \sin mx \cos nx \, dx$ , $\int \sin mx \sin nx \, dx$, $\int \cos mx \cos nx \, dx$

Identities :

$\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$

$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$ .

E05 - Ex 4

$$\int \tan^3(6x)\sec^{10}(6x)\,dx$$

Substitution rule to make it simpler

Let $u = 6x$. then $du = 6\,dx$

$$\int \tan^3(6x)\sec^{10}(6x)\,dx = \frac{1}{6}\int \tan^3(u)\sec^{10}(u)\,du =: A \qquad m=3 \quad n=10.$$

Method a:

$$A = \frac{1}{6}\int \tan^3(u)\sec^8 u\,\sec^2 u\,du$$ 1. save the factor $\sec^2 u$

$$= \frac{1}{6}\int \tan^3(u)\,(\tan^2 u + 1)^4\,d\tan u$$ 2. use $\sec^2 u = \tan^2 u + 1$

Let $t = \tan u = \tan(6x)$ $$= \frac{1}{6}\int t^3 (t^2+1)^4\,dt$$ NOT very easy to solve.

Method b)

$$A = \frac{1}{6}\int \tan^2 u\,\sec^9 u\,(\tan u\sec u)\,du$$ 1. save the factor $\tan u \sec u$

$$= \frac{1}{6}\int (\sec^2 u - 1)\sec^9 u\,d\sec u$$ 2. use $\tan^2 u = \sec^2 u - 1$

Let $t = \sec u = \sec(6x)$ $$= \frac{1}{6}\int (t^2-1)\,t^9\,dt$$

$$= \frac{1}{6}\int (t^{11} - t^9)\,dt$$

$$= \frac{1}{6}\left(\frac{1}{12}t^{12} - \frac{1}{10}t^{10}\right) + C$$

$$= \frac{1}{72}t^{12} - \frac{1}{60}t^{10} + C$$

$$= \frac{1}{72}\sec^{12}(6x) - \frac{1}{60}\sec^{10}(6x) + C$$

Comment: Try to rewrite the term whose order is small!

T05 - Ex 5.

$$\int_1^3 \sin(8x)\sin x\,dx:$$

Use the identity: $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

$$A = \frac{1}{2}\int_1^3 \cos(8x-x) - \cos(8x+x)\,dx$$

$$= \frac{1}{2}\int_1^3 \cos(7x)\,dx - \frac{1}{2}\int_1^3 \cos(9x)\,dx$$

$$= \frac{1}{2\times7}\int_1^3 d\sin(7x) - \frac{1}{2\times9}\int_1^3 d\sin(9x)$$

$$= \frac{1}{14}\left[\sin(7x)\right]_1^3 - \frac{1}{18}\left[\sin(9x)\right]_1^3$$

$$= \frac{1}{14}\sin(21) - \frac{1}{14}\sin(7) - \frac{1}{18}\sin(27) + \frac{1}{18}\sin(9)$$

T05 - Ex 6

$$\int \frac{2+7\sin^3(z)}{\cos^2(z)}\,dz$$

$$= 2\int \frac{1}{\cos^2(z)}\,dz + 7\int \frac{\sin^2 z}{\cos^2(z)}\sin z\,dz$$

$$= 2\int d\tan(z) - 7\int \frac{1-\cos^2(z)}{\cos^2(z)}\,d\cos(z)$$

Let $u = \cos z$

$$= 2\tan(z) - 7\int \frac{1-u^2}{u^2}\,du$$

$$= 2\tan(z) - 7\left(-\frac{1}{u} - u\right) + C$$

$$= 2\tan(z) + 7\cdot\frac{1}{\cos(z)} + 7\cos(z) + C$$

$$= 2\tan(z) + 7\sec(z) + 7\cos(z) + C$$

· Trigonalmetric substitutions

| Expression | Identity | Substitution |
| --- | --- | --- |
| $\sqrt{a^2-x^2}$ | $1-\sin^2\theta = \cos^2\theta$ | $x = a\sin\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ |
| $\sqrt{a^2+x^2}$ | $1+\tan^2\theta = \sec^2\theta$ | $x = a\tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$ |
| $\sqrt{x^2-a^2}$ | $\sec^2\theta - 1 = \tan^2\theta$ | $x = a\sec\theta, \quad 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3}{2}\pi$ |

We want to get rid of the $\sqrt{\quad}$.

Remarks:

Notice "=" when we have $\dfrac{1}{\sqrt{a^2-x^2}}$, or $\dfrac{1}{\sqrt{x^2-a^2}}$

T05 - Ex 7

$$\int \frac{1}{\sqrt{9x^2-36x+37}}\,dx$$

Solution.

$$9x^2-36x+37 = 9(x^2-4x+4)-4\times 9+37 = 9(x-2)^2+1 = \left(3(x-2)\right)^2+1$$

Let $3(x-2) = \tan\theta$ , then $x = 2+\frac{1}{3}\tan\theta$,

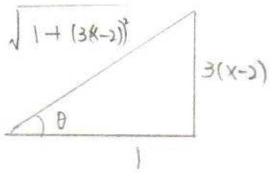
$-\frac{\pi}{2}<\theta<\frac{\pi}{2}$ $\quad dx = \frac{1}{3}\sec^2\theta\,d\theta$

$$\int \frac{1}{\sqrt{9x^2-36x+37}}\,dx = \int \frac{1}{\sqrt{\tan^2\theta+1}}\cdot\frac{1}{3}\sec^2\theta\,d\theta$$

$$= \frac{1}{3}\int \frac{1}{|\sec\theta|}\sec^2\theta\,d\theta$$

$$= \frac{1}{3}\int \frac{1}{\sec\theta}\sec^2\theta\,d\theta.$$ since $\sec\theta>0$ when $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$

$$= \frac{1}{3}\int \sec\theta\,d\theta$$




$$= \frac{1}{3}\ln|\sec\theta+\tan\theta| + C$$

$$= \frac{1}{3}\ln\left|\sqrt{9x^2-36x+37}+3(x-2)\right| + C$$

□

T05 - Ex 8

$$\int \frac{\sqrt{x^2+16}}{x^4}\,dx$$

Solution: $a=4$. Let $x = 4\tan\theta$, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$

then $dx = 4\sec^2\theta\, d\theta$.

$$\int \frac{\sqrt{x^2+16}}{x^4}\,dx = \int \frac{\sqrt{16\tan^2\theta+16}}{(4\tan\theta)^4}\cdot 4\sec^2\theta\, d\theta$$

$$= \frac{4\times4}{4^4}\int \frac{|\sec\theta|}{\tan^4\theta}\sec^2\theta\, d\theta$$

$$= \frac{1}{16}\int \frac{\sec^3\theta}{\tan^4\theta}\, d\theta$$

save factor doesn't work
↓
rewrite it to $\sin\theta$ & $\cos\theta$

$$= \frac{1}{16}\int \frac{1}{\cos^3\theta}\cdot\frac{\cos^4\theta}{\sin^4\theta}\, d\theta$$

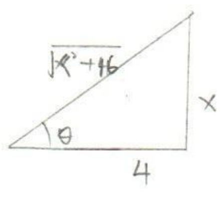
$$= \frac{1}{16}\int \frac{\cos\theta}{\sin^4\theta}\, d\theta$$

It works!

$$= \frac{1}{16}\int \frac{1}{\sin^4\theta}\, d\sin\theta$$

$$= \frac{1}{16}\,\frac{1}{-4+1}(\sin\theta)^{-4+1} + C$$

$\tan\theta = \frac{x}{4}$




$$= -\frac{1}{48}\,\frac{1}{\sin^3\theta} + C$$

$$= -\frac{1}{48}\left(\frac{\sqrt{x^2+16}}{x}\right)^3 + C$$

$$= -\frac{(x^2+16)^{\frac{3}{2}}}{48x^3} + C$$

Check the answer:

$$-\frac{\frac{3}{2}(x^2+16)^{\frac{1}{2}}\cdot(2x)\cdot 48x^3 - (x^2+16)^{\frac{3}{2}}\cdot 48\cdot 3x^2}{(48x^3)^2}$$

$$= -\frac{3x\sqrt{x^2+16}}{48x^2} + \frac{\sqrt{x^2+16}\,(x^2+16)\cdot 48\cdot 3x^2}{48^2x^6} = -\frac{1}{16x^2}\sqrt{x^2+16} + \frac{x^2+16}{16x^4}\sqrt{x^2+16} = \frac{\sqrt{x^2+16}}{x^4}$$