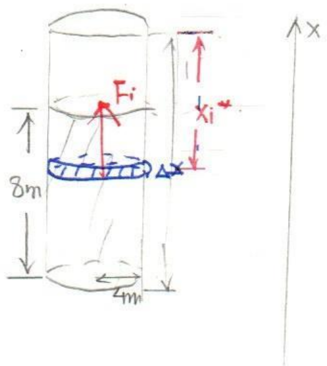
T03 - Ex 5.

Assume a cylindrical tank of radius 4m and height 10 m is filled to a depth of 8m. How much work does it take to pump all the water over the top edge of the tank?



$$F_i = \rho V_i g$$
$$= \rho \cdot \pi r^2 \Delta x \, g$$

$$W_i \approx F_i \cdot x_i^* = \pi r^2 \rho g \, x_i^* \Delta x \quad \text{with } 2 = 10-8 \leq x_i^* \leq 10$$

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} W_i = \pi r^2 \rho g \lim_{n\to\infty} \sum_{i=1}^{n} x_i^* \Delta x$$
$$= \pi r^2 \rho g \int_2^{10} x \, dx$$
$$= \pi r^2 \rho g \left[\tfrac{1}{2}x^2\right]_2^{10}$$
$$= 768 \pi \rho g$$

Tutorial 4

· Ave. value of a function:

Def: The ave. value of f on the interval $[a,b]$ is defined as

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

What you need: ① function f ② interval $[a,b]$

T04-1

Determine the ave. value of each of the following func.s on the given interval.

(a) $f(t) = t^2 - 5t + 6\cos(\pi t)$ on $[-1, \frac{5}{2}]$

Solution: $$f_{ave} = \frac{1}{b-a}\int_a^b f(t)\,dt$$

$$= \frac{1}{\frac{5}{2}-(-1)}\int_{-1}^{\frac{5}{2}} t^2 - 5t + 6\cos(\pi t)\,dt$$

$$= \frac{2}{7}\left[\frac{1}{3}t^3 - \frac{5}{2}t^2 + \frac{6}{\pi}\sin(\pi t)\right]_{-1}^{\frac{5}{2}}$$

$$= \frac{2}{7}\left[\frac{1}{3}\cdot\left(\frac{5}{2}\right)^3 - \frac{5}{2}\cdot\left(\frac{5}{2}\right)^2 + \frac{6}{\pi}\sin\left(\frac{5}{2}\pi\right) - \frac{1}{3}(-1)^3 + \frac{5}{2}(-1)^2 - \frac{6}{\pi}\sin(-\pi)\right]$$

$$= -\frac{13}{6} + \frac{12}{7\pi}$$

(b) $R(z) = \sin(2z)\,e^{1-\cos(2z)}$ on $[-\pi, \pi]$

Solution: $$R_{ave} = \frac{1}{\pi-(-\pi)}\int_{-\pi}^{\pi} \sin(2z)\,e^{1-\cos(2z)}\,dz = 0$$

since $R(z)$ is an odd function

The Mean Value Thm. for Integrals:

If f is continuous on $[a, b]$,

then, there exists a number c in $[a,b]$ s.t.

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

that is

$$\int_a^b f(x)\,dx = f(c)\,(b-a)$$

T04-2

Determine the number c that satisfies the Mean Value Thm. for Integrals for the function $f(x) = x^2+3x+2$ on the interval $[1, 4]$.

Solution: ① Calculate f_{ave}

$$\begin{aligned} f_{ave} &= \frac{1}{b-a}\int_a^b f(x)\,dx \\ &= \frac{1}{4-1}\int_1^4 (x^2+3x+2)\,dx \\ &= \frac{1}{3}\left[\frac{1}{3}x^3+\frac{3}{2}x^2+2x\right]_1^4 \\ &= \frac{1}{3}\left[\frac{1}{3}\cdot 4^3+\frac{3}{2}\cdot 4^2+2\times 4-\frac{1}{3}-\frac{3}{2}-2\right] \\ &= \frac{33}{2} \end{aligned}$$

② The Thm. guarantees the existence of $c \in [a,b]$ s.t. $f(c) = f_{ave}$.

Find it.

By the Mean Value thm. for integral, $\exists c \in [1,4]$ s.t. $f(c) = \frac{33}{2}$

i.e. $c^2+3c+2 = \frac{33}{2}$

$$2c^2+6c-29 = 0$$

$$c = \frac{-6 \pm \sqrt{36+232}}{4} = \frac{-3 \pm \sqrt{67}}{2}$$

Since $\frac{-3-\sqrt{67}}{2} \notin [1,4]$, the number c is $\frac{\sqrt{67}-3}{2}$

Arc Length

$y = f(x)$ | $x = g(y)$

① Arc Length formula

If f' is continuous on $[a, b]$,

then the length of the curve $y = f(x)$, $a \le x \le b$

is $L = \int_a^b \sqrt{1+[f'(x)]^2}\, dx = \int_a^b \sqrt{1+(\frac{dy}{dx})^2}\, dx$

If g' is continuous on $[c, d]$,

then the length of the curve $x = g(y)$, $c \le y \le d$

$L = \int_c^d \sqrt{1+[g'(y)]^2}\, dy = \int_c^d \sqrt{1+(\frac{dx}{dy})^2}\, dy$

② Arc Length function

Given a smooth curve $C: y = f(x)$, $a \le x \le b$.

Let $s(x)$ be the distance along C

from the initial point $P_0(a, f(a))$

to the point $Q(x, f(x))$.

Then, s is a func., called the arc Length func.

it is given by

$$S(x) = \int_a^x \sqrt{1+[f'(t)]^2}\, dt$$

③ Relationships

$$\frac{ds}{dx} = \sqrt{1+[f'(x)]^2} = \sqrt{1+(\frac{dy}{dx})^2}$$

then $ds = \sqrt{(dx)^2 + (dy)^2}$

$$ds^2 = (dx)^2 + (dy)^2$$

We have $L = \int ds = \int \sqrt{1+(\frac{dy}{dx})^2}\, dx$

$$= \int \sqrt{1+(\frac{dx}{dy})^2}\, dy$$

T04 - Ex3

Find the length of the curve $f(x) = 2e^{x} + \frac{1}{8}e^{-x}$ on the interval $[0, \ln 2]$.

Solution: ① Find y is a function of x

② $f'(x) = 2e^{x} - \frac{1}{8}e^{-x}$

③ $L = \int_a^b \sqrt{1+[f'(x)]^2}\, dx$

$= \int_0^{\ln 2} \sqrt{1+[2e^{x}-\frac{1}{8}e^{-x}]^2}\, dx$

$= \int_0^{\ln 2} \sqrt{(2e^{x}+\frac{1}{8}e^{-x})^2}\, dx$

$= \int_0^{\ln 2} 2e^{x} + \frac{1}{8}e^{-x}\, dx$

$= \left[2e^{x} - \frac{1}{8}e^{-x}\right]_0^{\ln 2}$

$= 2\times 2 - \frac{1}{8}\times\frac{1}{2} - 2 + \frac{1}{8} = \frac{33}{16}$

Observe that

✓ $1+[2e^{x}-\frac{1}{8}e^{-x}]^2$

$= 1 + 4e^{2x} - \frac{1}{2} + \frac{1}{64}e^{-2x}$

$= 4e^{2x} + \frac{1}{2} + \frac{1}{64}e^{-2x}$

$= \left(2e^{x} + \frac{1}{8}e^{-x}\right)^2$

T04 - Ex4

Find the arc length of $g(y) = \frac{1}{8}y^2 - \ln y$ from $y=1$ to $y=2$.

Solution: ① Find x is a function of y

② $g'(y) = \frac{1}{4}y - \frac{1}{y}$

③ $L = \int_a^b \sqrt{1+[g'(y)]^2}\, dy$

$= \int_1^2 \sqrt{1+(\frac{1}{4}y-\frac{1}{y})^2}\, dy$

$= \int_1^2 \frac{1}{4}y + \frac{1}{y}\, dy$

$= \left[\frac{1}{8}y^2 + \ln|y|\right]_1^2$

$= \frac{1}{8}\cdot 4 + \ln 2 - \frac{1}{8} - \ln 1 = \frac{3}{8} + \ln 2.$

· Area of Surface of Revolution

· Rotating about x-axis : $S = \int 2\pi y \, ds$

$y = f(x),\ a \le x \le b$: $S = \int 2\pi f(x)\, ds = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2}\, dx$

$x = g(y),\ c \le y \le d$: $S = \int 2\pi y(x)\, ds = \int_c^d 2\pi y \sqrt{1 + [g'(y)]^2}\, dy$

· Rotating about y-axis : $S = \int 2\pi x \, ds$

$y = f(x),\ a \le x \le b$: $S = \int 2\pi x\, ds = \int_a^b 2\pi x \sqrt{1 + [f'(x)]^2}\, dx$

$x = g(y),\ c \le y \le d$: $S = \int 2\pi x(y)\, ds = \int_c^d 2\pi g(y) \sqrt{1 + [g'(y)]^2}\, dy$

Rotating about / Curve described	x-axis $S = \int 2\pi y\, ds$	y-axis $S = \int 2\pi x\, ds$
$y = f(x),\ a \le x \le b$ $ds = \sqrt{1+[f'(x)]^2}\, dx$	$S = \int 2\pi f(x)\, ds$ $= \int_a^b 2\pi f(x) \sqrt{1+[f'(x)]^2}\, dx$	$S = \int 2\pi x\, ds$ $= \int_a^b 2\pi x \sqrt{1+[f'(x)]^2}\, dx$
$x = g(y)\quad c \le y \le d$ $ds = \sqrt{1+[g'(y)]^2}\, dy$	$S = \int 2\pi y\, ds$ $= \int_c^d 2\pi y \sqrt{1+[g'(y)]^2}\, dy$	$S = \int 2\pi g(y)\, ds$ $= \int_c^d 2\pi g(y) \sqrt{1+[g'(y)]^2}\, dy$

① Axis of rotation

② $y = f(x)$ or $x = g(y)$

$a \le x \le b$ $\quad$ $c \le y \le d$

T04-5

Consider the curve $y = \sin(x)$, $0 \le x \le \pi$

Find the area of the surface generated when the curve is revolved about the x-axis.

Solution :

① Rotating about x-axis

② $y = \sin(x)$, $0 \le x \le \pi$

③ $S = \int 2\pi\, y(x)\, ds$

$= \int_0^{\pi} 2\pi\, y(x) \sqrt{1+[y'(x)]^2}\, dx$

$= \int_0^{\pi} 2\pi \sin(x) \sqrt{1+\cos^2(x)}\, dx$

By substitution rule for definite integral

Let $u = \cos(x)$ $\quad du = -\sin(x)\, dx$

$x = 0 \Rightarrow u = \cos(0) = 1$

$x = \pi \Rightarrow u = \cos(\pi) = -1$

$= -2\pi \int_1^{-1} \sqrt{1+u^2}\, du$

$= 2\pi \int_{-1}^{1} \sqrt{1+u^2}\, du$

$= 4\pi \int_0^{1} \sqrt{1+u^2}\, du$

$= \cdots$

By trigonometric substitution.

$= 2\pi\left(\sqrt{2} + \ln(1+\sqrt{2})\right)$

T04-6

Consider the $y = \ln\left(\frac{x+\sqrt{x^2-1}}{2}\right)$.

Find the area of the surface generated when the part of the curve between the points $(\frac{5}{4}, 0)$ and $(\frac{17}{8}, \ln 2)$ is revolved about the y-axis.

Solution:

① Rotating about y-axis $\quad S = \int 2\pi x \, ds$

② If we treat y as a func. of x

$$S = \int_{\frac{5}{4}}^{\frac{17}{8}} 2\pi x \sqrt{1+(y'(x))^2} \, dx$$

which is difficult to solve.

Rewrite $\quad y = \ln\left(\frac{x+\sqrt{x^2-1}}{2}\right)$

$$e^y = \frac{x+\sqrt{x^2-1}}{2}$$

$$2e^y - x = \sqrt{x^2-1}$$

Take square on both sides $\quad 4e^{2y} - 4e^y x + x^2 = x^2 - 1 \qquad$ Since $x \geq 1$

$$x = g(y) = e^y + \frac{1}{4}e^{-y}, \quad 0 \leq y \leq \ln 2$$

Then. $g'(y) = e^y - \frac{1}{4}e^{-y}$

③ $$S = \int_0^{\ln 2} 2\pi g(y) \sqrt{1+[g'(x)]^2} \, dy$$

$$= \int_0^{\ln 2} 2\pi \left(e^y + \frac{1}{4}e^{-y}\right) \sqrt{1+\left(e^y - \frac{1}{4}e^{-y}\right)^2} \, dy$$

$$= \int_0^{\ln 2} 2\pi \left(e^y + \frac{1}{4}e^{-y}\right)^2 dy$$

$$= 2\pi \int_0^{\ln 2} e^{2y} + \frac{1}{2} + \frac{1}{16}e^{-2y} \, dy$$

$$= 2\pi \left[\frac{1}{2}e^{2y} + \frac{1}{2}y - \frac{1}{32}e^{-2y} \right]_0^{\ln 2}$$

$$= 2\pi \left[\frac{1}{2} \cdot 4 + \frac{1}{2}\ln 2 - \frac{1}{32} \times \frac{1}{4} - \frac{1}{2} + \frac{1}{32} \right]$$

$$= \left(\frac{195}{64} + \ln 2 \right) \pi$$