Tutorial 2

Area Between Curves

Region $S = \{(x, y) : \underline{a} \le x \le \underline{b},\ g(x) \le y \le f(x)\}$

$$\text{area}(S) = \int_a^b \big(f(x) - g(x) \big)\, dx$$

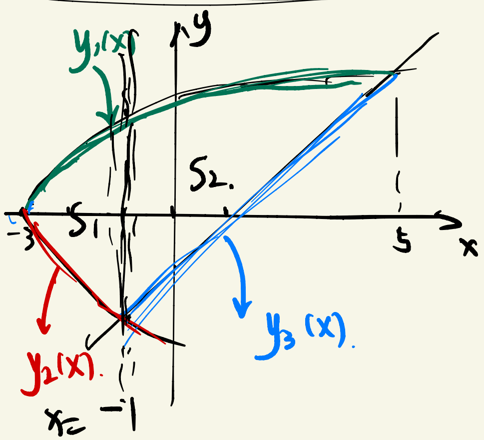
$$\text{area}(S) = \int_a^b \left| f(x) - g(x) \right| dx$$

② Find the area of the region enclosed by the curve $y=x-1$ and $x=\frac{1}{2}y^2-3$

Method 1: Consider x as a function of y

(Shown in the Lecture)

Method 2: Consider y as a function of x.



Rewrite $x=\frac{1}{2}y^2-3$

as $y^2=2x+6$

Denote $y_1(x)=\sqrt{2x+6}$

$y_2(x)=-\sqrt{2x+6}$

$y_3(x)=x-1$

$$\text{area}(S)=\text{area}(S_1)+\text{area}(S_2)$$

$$\text{area}(S_1)=\int_{-3}^{-1}\left(y_1(x)-y_2(x)\right)dx=\int_{-3}^{-1}\left(\sqrt{2x+6}-(-\sqrt{2x+6})\right)dx$$

$$\text{area}(S_2) = \int_{-1}^{5} (y_1(x) - y_3(x))\,dx = \int_{-1}^{5} \left(\sqrt{2x+6} - (x-1)\right)dx$$

③ Find the area of the region bounded by the curve

$y_1 = 4x+16$, $y_2 = 2x^2+10$, $x=-2$ and $x=5$,

① y is a function of x

② the region is $[a,b] = [-2, 5]$

③ $$\text{area}(S) = \int_{-2}^{5} |y_1(x) - y_2(x)|\, dx$$

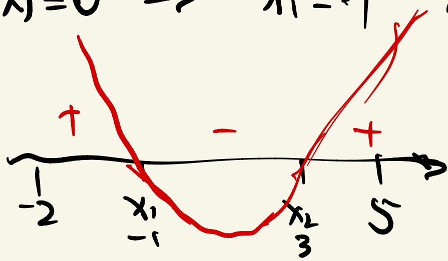
$$= \int_{-2}^{5} \left| (4x+16) - (2x^2+10) \right| dx$$

$$= \int_{-2}^{5} \left| -2x^2 + 4x + 6 \right| dx.$$

$$= |-2| \int_{-2}^{5} |x^2 - 2x - 3|\, dx = ①$$

Let $g(x) = x^2 - 2x - 3$

$g(x) = 0 \Rightarrow \quad x_1 = -1 \quad x_2 = 3$



$$① = 2\left(\int_{-2}^{-1} (x^2-2x-3)\,dx - \int_{-1}^{3} (x^2-2x-3)\,dx + \int_{3}^{5} (x^2-2x-3)\,dx \right)$$

Volumes

Definition

Step 1: Determine the solid S lies between $x=a$ and $x=b$ or $y=c$ and $y=d$.

Step 2: Calculate $A(x)$ or $A(y)$

Case I: disk $= A=\pi(\text{radius})^2$

Case II: washer $= A=\pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$

Case III: e.g. squares, equilateral triangle

Step 3: Apply the definition

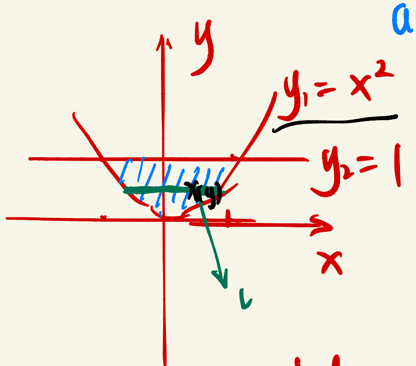
$$V = \int_a^b A(x)\,dx \quad \text{or} \quad V = \int_c^d A(y)\,dy$$

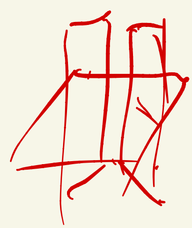
Step 4: Calculate the integral.

① The solid

whose base is the region bounded by $y=x^2$ and $y=1$, and

whose cross-sections perpendicular to the base plane and parallel to the x-axis are squares.





Step 1: the solid is bounded by $y=0$ and $y=1$

Step 2: Case Ⅲ : square with side length l

$$S=l^2$$

$$A(y)=l^2=(2\cdot x(y))^2=(2\sqrt{y})^2=4y$$

Step 3: $V=\int_0^1 A(y)\,dy=\int_0^1 4y\,dy=\underbrace{4\left[\frac{1}{2}y^2\right]_0^1}_{=2}$

② The base S is an elliptical region with boundary curve

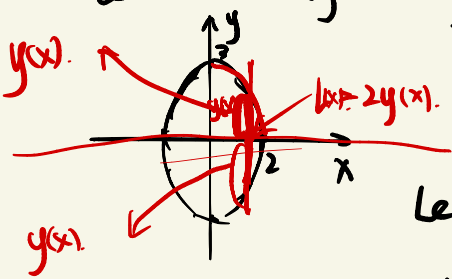
$9x^2+4y^2=36.$ $y(x)=\pm\sqrt{\frac{36-9x^2}{4}}$

Cross-sections perpendicular to the x-axis
are isosceles right triangles with hypotenuse in the base.

plane

line

Let $x=0$ $4y^2=36$ $y=\pm 3$



$l(x)=2y(x)$.

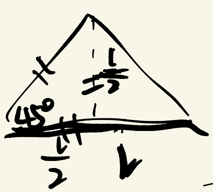
$l(x)=2y(x)=\sqrt{36-9x^2}$

Let $y=0$ in $9x^2+4y^2=36 \Rightarrow x=\pm 2$

Step 1: the solid is bounded by $x=-2$ and $x=2$.

Step 2: $A(x)$

An isosceles right triangle with the length of hypotenuse l



has the area $\frac{1}{2}\times\boxed{l}\times\boxed{\frac{l}{2}}=\frac{l^2}{4}$

$A(x)=\frac{1}{4}l^2=\frac{1}{4}(2y(x))^2.$

$=y^2(x)=\boxed{\frac{36-9x^2}{4}}$

Step 3:

$$V = \int_{-2}^{2} A(x)\, dx$$

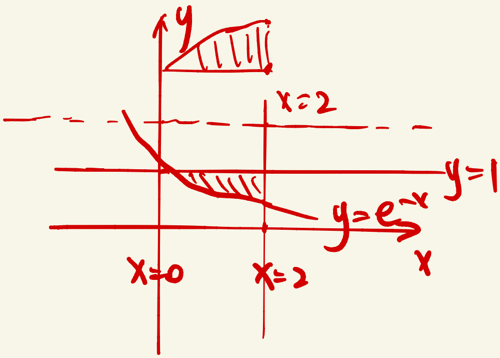
$$= \int_{-2}^{2} \frac{36-9x^2}{4}\, dx$$

$$= 2\int_{0}^{2} \frac{36-9x^2}{4}\, dx$$

$$=$$

③ Find the volume of the solid obtained by rotating the region bounded by the curves

$y = e^{-x}$, $y = 1$, $x = 2$; about the line $y = 2$.



④ Set up an integral for the volume of the solid obtained by rotating the region bounded by the curves $x^2+4y^2=4$ about the line $x=2$.