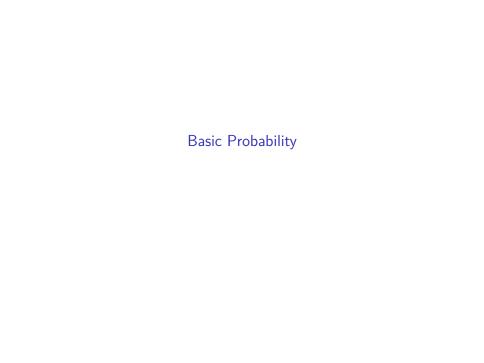
MATH 2411 TA Tutorial

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Basic Concepts

- Experiment/Trial: A procedure that generates a uncertain outcome. Each experiment generates one and only one outcome.
- **Sample Space**: The set of all possible outcomes of an experiment, denoted by Ω .
- ▶ **Outcome**: An **element** of the sample space, denoted by $\omega \in \Omega$.
- ▶ **Event**: A **set** of outcomes. A **subset** of the sample space, denoted by $A \subset \Omega$. If the outcome of an experiment lies in event A, we say the event A occurs.

Operation rules of events

- ▶ **Union**: $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
- ▶ Intersection: $A \cap B = \{x : x \in A \text{ and } x \in B\}.$
- **Complement**: $A^c = \{x : x \notin A\}$.
- ▶ **Difference**: $A B = \{x : x \in A \text{ and } x \notin B\}$.
- **Symmetric Difference**: $A \triangle B = (A B) \cup (B A)$.

Example

$$\Omega = \{1, 2, 3, 4\}, A = \{1, 2, 3\}, B = \{2, 3, 4\}.$$

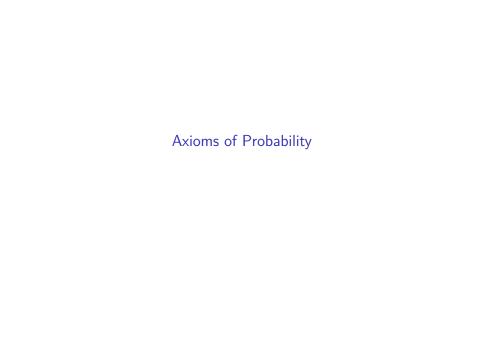
- $A \cup B = \{1, 2, 3, 4\}.$
- ► $A \cap B = \{2, 3\}.$
- $A^c = \{4\}.$
- ► $A B = \{1\}.$
- $A \triangle B = \{1,4\}$. Think about what will be changed for another Ω.

Operation rules of events

- ▶ Commutative: $A \cup B = B \cup A$, $A \cap B = B \cap A$.
- ► **Associative**: $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$.
- **Distributive**: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(For the above rules, note the similarity between (\cup, \cap) and $(+, \times)$.)

▶ De Morgan's Laws: $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$.



Axioms

A function P :Events in $\Omega \to \mathbb{R}$ is called a **probability** if it satisfies the following three axioms:

- 1. (Non-negativity) $0 \le P(A) \le 1$ for any event A.
- 2. (Normalization) $P(\Omega) = 1$.
- 3. (Countable Additivity) If A_1, A_2, \cdots are mutually exclusive events, then $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$

A sample space Ω together with a collection of events and a probability function P is called a **probability space**.

Properties of Probability

- 1. $P(A^c) = 1 P(A)$.
- 2. $P(\emptyset) = 0$.
- 3. If $A \subset B$, then $P(A) \leq P(B)$.
- 4. $P(A \cup B) = P(A) + P(B) P(A \cap B)$.
- 5. As $P(A \cap B) \ge 0$, we have $P(A \cup B) \le P(A) + P(B)$.
- 6. Moreover, with De Morgan's Law, we have $P(A \cap B) > 1 P(A^c) P(B^c)$.
- 7. $P(A \cup B \cup C) = P(A) + P(B) + P(C) P(A \cap B) P(A \cap C) P(B \cap C) + P(A \cap B \cap C)$.

Generally, for n events, we have the **Inclusion-Exclusion Principle**:

$$P\left(\bigcup_{i=1}^{n} A_{i}\right) = \sum_{i=1}^{n} P(A_{i}) - \sum_{1 \leq i < j \leq n} P(A_{i} \cap A_{j}) + \sum_{1 \leq i < j < k \leq n} P(A_{i} \cap A_{j} \cap A_{j}) + (-1)^{n-1} P(A_{1} \cap A_{2} \cap \cdots \cap A_{n}).$$

Example

Suppose that P(A) = 0.4, P(B) = 0.3, and $P(A \cap B) = 0.1$.

- 1. Find $P(A^c)$.
- 2. Find $P(A \cup B)$.
- 3. Find $P(A^c \cap B^c)$.

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Solution:

- 1. $P(A^c) = 1 P(A) = 0.6$.
- 2. $P(A \cup B) = P(A) + P(B) P(A \cap B) = 0.6$.
- 3. $P(A^c \cap B^c) = 1 P(A \cup B) = 0.4$.



Sample Space with Equally Likely Outcomes

If $\Omega = \{\omega_1, \omega_2, \cdots, \omega_n\}$ is a sample space with **finite** elements such that $P(\{\omega_i\}) = P(\{\omega_j\})$ for all i, j.

For any event E, we have $P(E) = \frac{|E|}{|\Omega|}$, where |A| denotes the number of elements in set A.

Since we only need to calculate |E| and $|\Omega|$, it becomes a counting problem.

Review of Counting

- ▶ Multiplication Rule: If there are n_1 ways to do the first task, and for each of these ways there are n_2 ways to do the second task, then there are $n_1 \times n_2$ ways to do the two tasks. (Tasks are done sequentially)
- Example: To choose a shirt and a pair of pants from 5 shirts and 4 pairs of pants, there are $5 \times 4 = 20$ ways.
- ▶ Permutation: The number of ways to arrange n distinct objects in a line is n!. (With order)
- Example: To put 5 different books on a shelf, there are 5! = 120 ways.
- ▶ **Combination**: The number of ways to choose k objects from n distinct objects is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. (**Without order**)
- Example: To choose 3 books from 5 different books, there are $\binom{5}{3} = 10$ ways.

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Solution: Let Ω is set of all results. With Combinations, we have $|\Omega|=\binom{8}{2}=28$, and the event E is to choose 1 red ball and 1 blue ball, so $|E|=\binom{3}{1}\binom{5}{1}=15$. Thus, $P(E)=\frac{15}{28}$.

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Wrong solution: $|\Omega| = 8 \times 7 = 56$, $|E| = 3 \times 5 = 15$, so $P(E) = \frac{15}{56}$.

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Think that what will happen if we define Ω as the set of all results with order.



Definition

The **conditional probability** of event A given event B is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Note: If P(B) > 0, then P(A|B) is a probability function on the sample space B.

Example

Suppose that a box contains 3 red balls and 5 blue balls. If we randomly choose 2 balls by order from the box, what is the probability that the two balls are of different colors given that the first ball is red?