MATH 1014 – Calculus II (Tutorial 1)

Department of Mathematics, HKUST

Arrangements of MATH1014-T8A

- **Teaching Assistant:** LI Yixin (yliqh@connect.ust.hk)
- **Time and Venue:** Th 06:00PM - 06:50PM, Rm 1034, LSK Bldg
- **Tutorial Notes** (Prepared by the TA Team of Professor WU)**:**
  - **Before Tutorials:** Homepage on Canvas (Available on Mondays)
  - **After Tutorials:** My homepage (Available on Fridays)
    - Handwritten notes and Photos during the tutorials
  - **NO need to download before or during the tutorials**
- **Procedure:** Review -> Do exercise by yourself (3 min) -> Share my methods -> …
- **Any questions or suggestions, feel free to contact!**

Know what you understand and what you do not.

Then, Practice!

Theorem (Fundamental Theorem of Calculus)

Suppose f is continuous on $[a, b]$.

1. *If $g(x) = \int_a^x f(t)\mathrm{d}t$, then $g'(x) = f(x)$.*
2. *$\int_a^b f(x)\mathrm{d}x = F(x)\Big|_a^b = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.*

Exercise Find the derivative of $y = \displaystyle\int_{-4}^{x^2} \mathrm{e}^{2t}\cos^2(1-5t)\,dt$.

Solution Let $u = x^2$. Then $y = g(u) = \displaystyle\int_{-4}^{u} \mathrm{e}^{2t}\cos^2(1-5t)\,dt$. According to first part of the Fundamental Theorem of Calculus and the Chain Rule we have

$$y' = \left(x^2\right)' \cdot \mathrm{e}^{2u}\cos^2(1-5u) = 2x\,\mathrm{e}^{2x^2}\cos^2\left(1-5x^2\right)$$

Table of the Indefinite Integrals

1. $\int cf(x)\mathrm{d}x = c\int f(x)\mathrm{d}x$
2. $\int (f(x)+g(x))\,\mathrm{d}x = \int f(x)\mathrm{d}x + \int g(x)\mathrm{d}x$
3. $\int k\mathrm{d}x = kx + C$
4. $\int x^n\mathrm{d}x = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$
5. $\int \frac{1}{x}\mathrm{d}x = \ln|x| + C$
6. $\int e^x\mathrm{d}x = e^x + C$
7. $\int a^x\mathrm{d}x = \frac{a^x}{\ln a} + C$
8. $\int \sin x\mathrm{d}x = -\cos x + C$
9. $\int \cos x\mathrm{d}x = \sin x + C$
10. $\int \sec^2 x\mathrm{d}x = \tan x + C$
11. $\int \csc^2 x\mathrm{d}x = -\cot x + C$
12. $\int \sec x\tan x\mathrm{d}x = \sec x + C$
13. $\int \csc x\cot x\mathrm{d}x = -\csc x + C$
14. $\int \frac{1}{1+x^2}\mathrm{d}x = \tan^{-1}x + C$
15. $\int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x = \sin^{-1}x + C$

Properties of the Definite Integral

1. $\int_b^a f(x)\mathrm{d}x = -\int_a^b f(x)\mathrm{d}x$
2. $\int_a^a f(x)\mathrm{d}x = 0$
3. $\int_a^b c\mathrm{d}x = c(b-a)$
4. $\int_a^b [f(x)+g(x)]\,\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x$
5. $\int_a^b cf(x)\mathrm{d}x = c\int_a^b f(x)\mathrm{d}x$
6. $\int_a^b [f(x)-g(x)]\,\mathrm{d}x = \int_a^b f(x)\mathrm{d}x - \int_a^b g(x)\mathrm{d}x$
7. $\int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x$

Theorem (Integrals of Symmetric Functions)

Suppose f is continuous on $[-a,a]$.

(a) *If f is even (i.e., $f(-x)=f(x)$), then* $\displaystyle\int_{-a}^{a} f(x)\mathrm{d}x = 2\int_0^a f(x)\mathrm{d}x.$

(b) *If f is odd (i.e., $f(-x)=-f(x)$), then* $\displaystyle\int_{-a}^{a} f(x)\mathrm{d}x = 0.$

Exercise Integrate each of the following.

(a) $$\int_{-2}^{2} 4x^4 - x^2 + 1\,dx$$

(b) $$\int_{-10}^{10} x^2 \sin\left(x^3\right)\,dx$$

Solution

(a) In this case the integrand is even and the interval is correct so,

$$\begin{aligned}
\int_{-2}^{2} 4x^4 - x^2 + 1\,dx &= 2\int_{0}^{2} 4x^4 - x^2 + 1\,dx \\
&= 2\left(\frac{4}{5}x^5 - \frac{1}{3}x^3 + x\right)\Big|_0^2 \\
&= 2\left(\frac{4}{5}\cdot 2^5 - \frac{1}{3}\cdot 2^3 + 2\right) - 2\left(\frac{4}{5}\cdot 0^5 - \frac{1}{3}\cdot 0^3 + 0\right) \\
&= \frac{748}{15}
\end{aligned}$$

(b) The integrand in this case is odd and the interval is in the correct form and so we can directly conclude that

$$\int_{-10}^{10} x^2 \sin\left(x^3\right)\,dx = 0$$

Theorem (The Substitution Rule)

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f\left(g(x)\right) g'(x)\mathrm{d}x = \int f(u)\mathrm{d}u \ .$$

Theorem (The Substitution Rule for Definite Integrals)

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f\left(g(x)\right) g'(x)\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\mathrm{d}u \ .$$

Exercise Evaluate each of the following.

(a) $$\int_0^{\ln(1+\pi)} \mathrm{e}^x \cos\left(1-\mathrm{e}^x\right) dx$$

(b) $$\int_{-\pi}^{\frac{\pi}{2}} \cos(x)\cos(\sin(x))\; dx$$

Solution

(a) Let $u = 1-\mathrm{e}^x$. Then $du = -\mathrm{e}^x dx$. Furthermore, we have

$$x=0 \quad\Rightarrow\quad u = 1-\mathrm{e}^0 = 1-1 = 0$$
$$x=\ln(1+\pi) \quad\Rightarrow\quad u = 1-\mathrm{e}^{\ln(1+\pi)} = 1-(1+\pi) = -\pi$$

The integral is then,

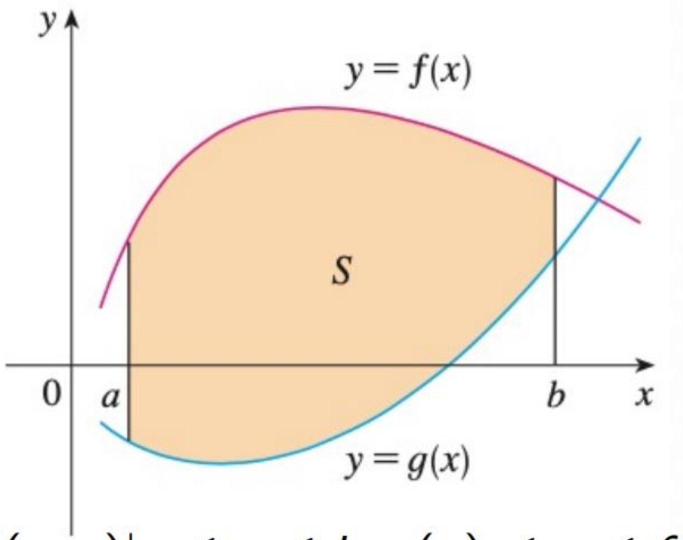
$$\begin{aligned}\int_0^{\ln(1+\pi)} \mathrm{e}^x \cos\left(1-\mathrm{e}^x\right) dx &= -\int_0^{-\pi} \cos u\, du \\ &= -\sin(u)\big|_0^{-\pi} \\ &= -\left(\sin(-\pi) - \sin 0\right) = 0\end{aligned}$$

(b) Let $u = \sin x$. Then $du = \cos x\, dx$. Furthermore, we have

$$x=\frac{\pi}{2} \quad\Rightarrow\quad u = \sin\frac{\pi}{2} = 1$$
$$x=-\pi \quad\Rightarrow\quad u = \sin(-\pi) = 0$$

The integral is then,

$$\begin{aligned}\int_{-\pi}^{\frac{\pi}{2}} \cos(x)\cos(\sin(x))\; dx &= \int_0^1 \cos u\, du \\ &= \sin(u)\big|_0^1 \\ &= \sin(1) - \sin(0) \\ &= \sin(1)\end{aligned}$$



$S = \{(x, y) | a \leq x \leq b, g(x) \leq y \leq f(x)\}$

Formula for Area between Curves

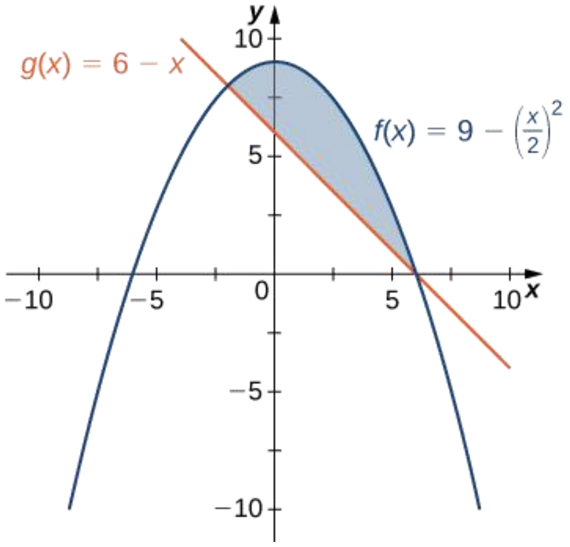
Let $a < b$ and $f(x) \geq g(x)$. Both $f(x)$ and $g(x)$ are continuous. Let $S = \{(x, y) | a \leq x \leq b, g(x) \leq y \leq f(x)\}$. Then

$$\text{area}(S) = \int_a^b [f(x) - g(x)]\,\mathrm{d}x\ .$$

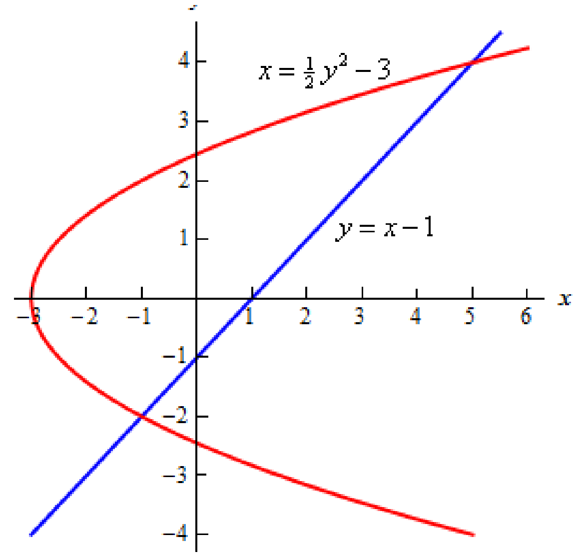
Exercise Find the area of the region enclosed by the curve $y = 6 - x$ and $y = 9 - (x/2)^2$.

Solution The coordinates of the two intersection points on the graph are $(-2, 8)$ and $(6, 0)$. The integrals for the area would then be,

$$\begin{aligned}
A &= \int_{-2}^{6} \left[9 - (x/2)^2\right] - (6 - x)\ dx \\
&= \int_{-2}^{6} -\frac{1}{4}x^2 + x + 3\, dx \\
&= \left(-\frac{1}{12}x^3 + \frac{1}{2}x^2 + 3x\right)\Big|_{-2}^{6} \\
&= \frac{64}{3}
\end{aligned}$$

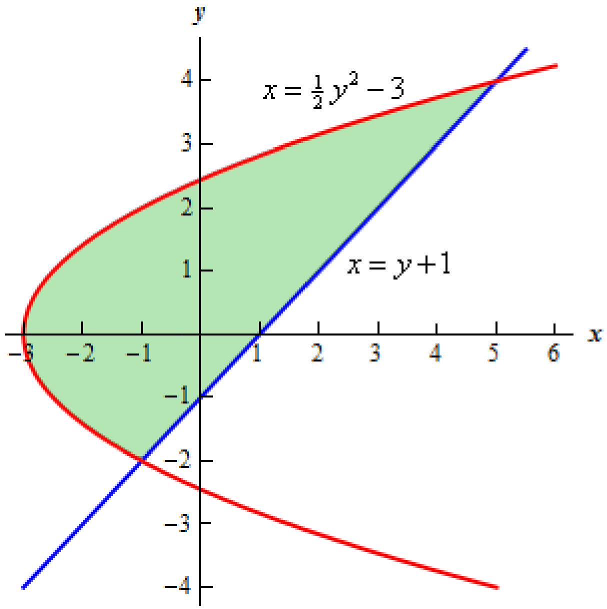


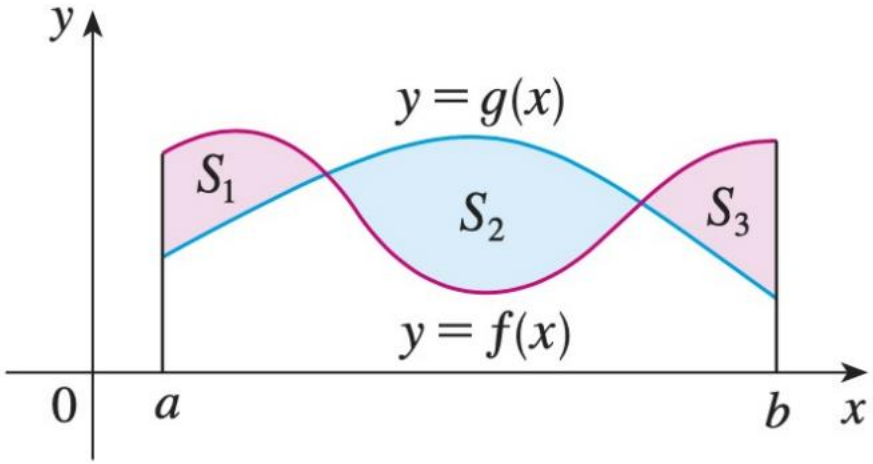
Exercise Find the area of the region enclosed by the curve $y = x - 1$ and $x = y^2/2 - 3$.



Solution Rewrite the equation of the line $y = x - 1$ as $x = y + 1$ then the curves are all in the form $x = g(y)$. The coordinates of the two intersection points on the graph are $(-1, -2)$ and $(5, 4)$. The integrals for the area would then be,

$$
\begin{aligned}
A &= \int_{-2}^{4} (y+1) - \left(\frac{1}{2}y^2 - 3\right) \, dy \\
&= \int_{-2}^{4} -\frac{1}{2}y^2 + y + 4 \, dy \\
&= \left(-\frac{1}{6}y^3 + \frac{1}{2}y^2 + 4y\right)\Bigg|_{-2}^{4} \\
&= 18
\end{aligned}
$$





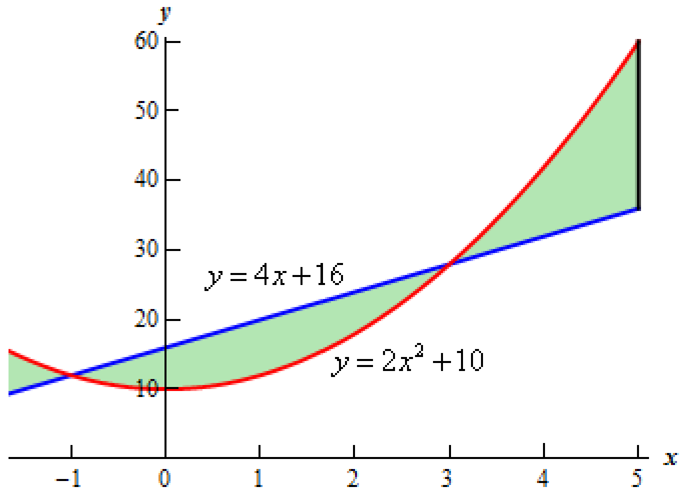
The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is

$$A = \int_a^b |f(x) - g(x)| \, \mathrm{d}x \, .$$

Exercise Find the area of the region bounded by the curve $y = 4x + 16$, $y = 2x^2 + 10$, $x = -2$ and $x = 5$.

Solution The coordinates of the two intersection points on the graph are $(-1, 12)$ and $(3, 28)$. The integrals for the area would then be,

$$\begin{aligned}
A &= \int_{-2}^{5} \left|2x^2 + 10 - (4x + 16)\right| \, dx \\
&= \int_{-2}^{-1} 2x^2 + 10 - (4x + 16) \, dx + \int_{-1}^{3} 4x + 16 - \left(2x^2 + 10\right) \, dx + \int_{3}^{5} 2x^2 + 10 - (4x + 16) \, dx \\
&= \int_{-2}^{-1} 2x^2 - 4x - 6 \, dx + \int_{-1}^{3} -2x^2 + 4x + 6 \, dx + \int_{3}^{5} 2x^2 - 4x - 6 \, dx \\
&= \left(\frac{2}{3}x^3 - 2x^2 - 6x\right)\Bigg|_{-2}^{-1} + \left(-\frac{2}{3}x^3 + 2x^2 + 6x\right)\Bigg|_{-1}^{3} + \left(\frac{2}{3}x^3 - 2x^2 - 6x\right)\Bigg|_{3}^{5} \\
&= \frac{14}{3} + \frac{64}{3} + \frac{64}{3} \\
&= \frac{142}{3}
\end{aligned}$$



Thanks!