MATH 2411 TA Tutorial

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Basic Probability

Basic Concepts

- **Experiment/Trial**: A procedure that generates a uncertain outcome. Each experiment generates one and only one outcome.
- **Sample Space**: The set of all possible outcomes of an experiment, denoted by Ω.
- **Outcome**: An **element** of the sample space, denoted by $\omega \in \Omega$.
- **Event**: A **set** of outcomes. A **subset** of the sample space, denoted by $A \subset \Omega$. If the outcome of an experiment lies in event A, we say the event A **occurs**.

Operation rules of events

- **Union**: $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
- **Intersection**: $A \cap B = \{x : x \in A \text{ and } x \in B\}$.
- **Complement**: $A^c = \{x : x \notin A\}$.
- **Difference**: $A - B = \{x : x \in A \text{ and } x \notin B\}$.
- **Symmetric Difference**: $A \triangle B = (A - B) \cup (B - A)$.

Example

$\Omega = \{1, 2, 3, 4\}$,$A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$.

- $A \cup B = \{1, 2, 3, 4\}$.
- $A \cap B = \{2, 3\}$.
- $A^c = \{4\}$.
- $A - B = \{1\}$.
- $A \triangle B = \{1, 4\}$. Think about what will be changed for another Ω.

Operation rules of events

- **Commutative**: $A \cup B = B \cup A$, $A \cap B = B \cap A$.
- **Associative**: $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$.
- **Distributive**: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(For the above rules, note the similarity between $(\cup, \cap)$ and $(+, \times)$.)

- **De Morgan's Laws**: $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$.

Axioms of Probability

Axioms

A function P :Events in $\Omega \to \mathbb{R}$ is called a **probability** if it satisfies the following three axioms:

1. (Non-negativity) $0 \leq P(A) \leq 1$ for any event A.
2. (Normalization) $P(\Omega) = 1$.
3. (Countable Additivity) If $A_1, A_2, \cdots$ are mutually exclusive events, then $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$.

A sample space Ω together with a collection of events and a probability function P is called a **probability space**.

Properties of Probability

1. $P(A^c) = 1 - P(A)$.
2. $P(\emptyset) = 0$.
3. If $A \subset B$, then $P(A) \leq P(B)$.
4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
5. As $P(A \cap B) \geq 0$, we have $P(A \cup B) \leq P(A) + P(B)$.
6. Moreover, with De Morgan's Law, we have $P(A \cap B) \geq 1 - P(A^c) - P(B^c)$.
7. $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$.

Generally, for n events, we have the **Inclusion-Exclusion Principle**:

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \leq i < j \leq n} P(A_i \cap A_j) + \sum_{1 \leq i < j < k \leq n} P(A_i \cap A_j \cap A_k)$$
$$+ (-1)^{n-1} P(A_1 \cap A_2 \cap \cdots \cap A_n).$$

Example

Suppose that $P(A) = 0.4$, $P(B) = 0.3$, and $P(A \cap B) = 0.1$.

1. Find $P(A^c)$.
2. Find $P(A \cup B)$.
3. Find $P(A^c \cap B^c)$.

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Solution:

1. $P(A^c) = 1 - P(A) = 0.6$.
2. $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6$.
3. $P(A^c \cap B^c) = 1 - P(A \cup B) = 0.4$.

Sample Space with Equally Likely Outcomes

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If $\Omega = \{\omega_1, \omega_2, \cdots, \omega_n\}$ is a sample space with **finite** elements such that $P(\{\omega_i\}) = P(\{\omega_j\})$ for all i, j.

For any event E, we have $P(E) = \frac{|E|}{|\Omega|}$, where $|A|$ denotes the number of elements in set A.

Since we only need to calculate $|E|$ and $|\Omega|$, it becomes a counting problem.

Review of Counting

- **Multiplication Rule**: If there are n_1 ways to do the first task, and for each of these ways there are n_2 ways to do the second task, then there are $n_1 \times n_2$ ways to do the two tasks. (Tasks are done **sequentially**)
- Example: To choose a shirt and a pair of pants from 5 shirts and 4 pairs of pants, there are $5 \times 4 = 20$ ways.
- **Permutation**: The number of ways to arrange n distinct objects in a line is $n!$. (**With order**)
- Example: To put 5 different books on a shelf, there are $5! = 120$ ways.
- **Combination**: The number of ways to choose k objects from n distinct objects is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. (**Without order**)
- Example: To choose 3 books from 5 different books, there are $\binom{5}{3} = 10$ ways.

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Think that what will happen if we define Ω as the set of all results with order.

Conditional Probability

Definition

The **conditional probability** of event A given event B is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Note: If $P(B) > 0$, then $P(A|B)$ is a probability function on the sample space B.

Example

Suppose that a box contains 3 red balls and 5 blue balls. If we randomly choose 2 balls by order from the box, what is the probability that the two balls are of different colors given that the first ball is red?