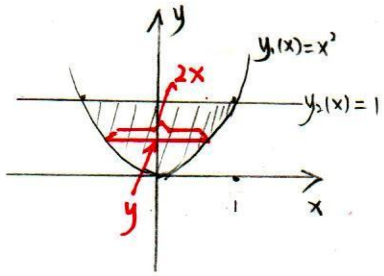
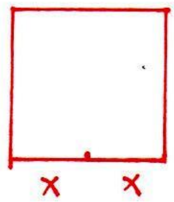
Graphs of T02 - Ex1 & 2

T02 - Exercise 1

xOy - plane



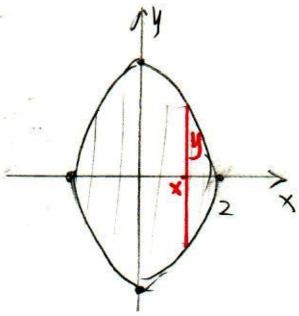
Cross - section



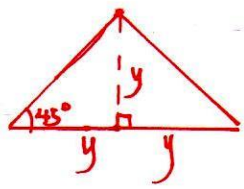
$$\text{Area} = (2x)^2 = 4x^2$$

T02 - Exercise 2

xOy - plane

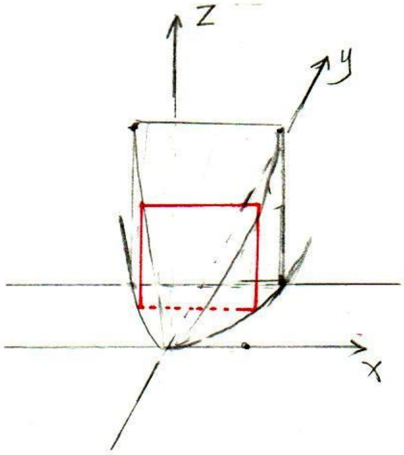


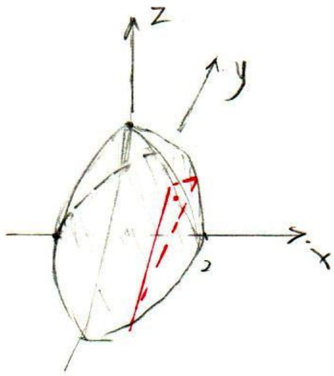
Cross - section



$$\text{Area} = \frac{1}{2} \cdot (2y) \cdot y = y^2$$

①





The Method of Cylindrical Shells

* For the solid obtained by rotation only

The volume of the solid obtained

by rotating about the y-axis $(x=0)$ the region under the curve $y=f(x)$ from a to b

is $V=\int_a^b \underbrace{2\pi x}_{\text{circumference}} \cdot \underbrace{f(x)}_{\text{height}} \cdot \underbrace{dx}_{\text{thickness}}$ where $0 \le a \le b$

Steps :

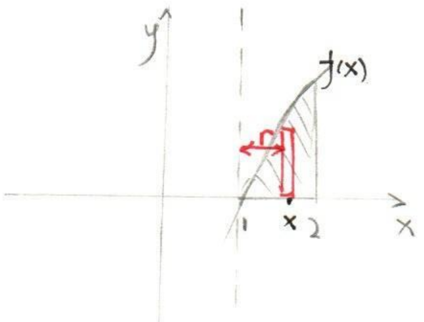
1. Determine ① the axis of rotation — $x=i$ $(i=0:$ y-axis$)$ — $y=j$ $(j=0:$ x-axis$)$

   ② the region — from $x=a$ to $x=b$ — from $y=c$ to $y=d$

2. Radius & Height (Positive !) — radius (x), height (x) — radius(y) height (y)

3. Apply the def — $V=\int_a^b 2\pi\,\text{radius}(x)\,\text{height}(x)\,dx$ — $V=\int_c^d 2\pi\, r(y)\, h(y)\, dy$

Draw a diagram may helps.



↖ axis of rotation

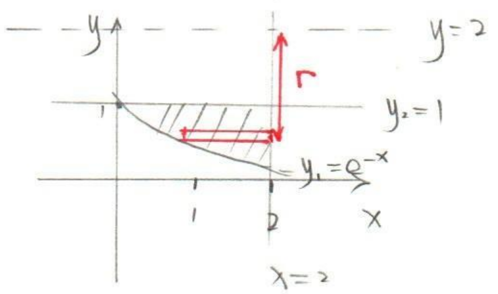
① rotating about $x=1$, from 1 to 2

② radius $(x) = x-1$

height $(x) = f(x)$

③ $V=\int_1^2 2\pi (x-1) f(x)\, dx.$

T02 - Ex3 (By the method of cylindrical shell)



① rotating about $y=2$ from e^{-2} to 1

② radius$(y) = 2-y$

height$(y) = 2-(-\log y)$ (Rewrite $y=e^{-x}$ as $x=-\log(y)$)

③ $$V = \int_{e^{-2}}^{1} 2\pi(2-y)(2+\log y)\,dy$$

$$= 8\pi(1-e^{-2}) - 4\pi\int_{e^{-2}}^{1} y\,dy + 4\pi\int_{e^{-2}}^{1} \log y\,dy - 2\pi\int_{e^{-2}}^{1} y\log y\,dy$$

$$\int \ln x\,dx = x(\ln x - 1) + C$$

$$\int x\ln x\,dx = \frac{1}{4}x^2(2\ln x - 1) + C$$

$$= 8\pi(1-e^{-2}) - 4\pi\left[\frac{1}{2}y^2\right]_{e^{-2}}^{1} + 4\pi\left[y(\log y - 1)\right]_{e^{-2}}^{1} - 2\pi\left[\frac{1}{4}y^2(2\ln y - 1)\right]_{e^{-2}}^{1}$$

$$= 8\pi(1-e^{-2}) - 4\pi\cdot\frac{1}{2}(1-e^{-4}) + 4\pi\left(1\cdot(0-1) - e^{-2}(-2-1)\right)$$

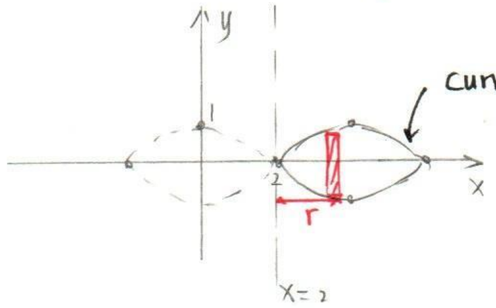
$$-2\pi\left(\frac{1}{4}(0-1) - \frac{1}{4}e^{-4}(2\times(-2)-1)\right)$$

$$= 8\pi(1-e^{-2}) - 2\pi(1-e^{-4}) + 4\pi(3e^{-2}-1) - 2\pi\left(-\frac{1}{4}+\frac{5}{4}e^{-4}\right)$$

$$= \left(8\pi - 2\pi - 4\pi + \frac{1}{2}\pi\right) + e^{-2}(-8\pi + 12\pi) + e^{-4}\left(2\pi - \frac{5}{2}\pi\right)$$

$$= \pi\left(-\frac{1}{2}e^{-4} + 4e^{-2} + \frac{5}{2}\right)$$

T02 - Exercise 4 (By the method of cylindrical shell)



Step 1: Rotating about $x=2$ from 2 to 6

Step 2: Radius $(x) = x-2$

height $(x) = 2 \cdot y(x) = 2\sqrt{\dfrac{4-(x-4)^2}{4}} = \sqrt{4-(x-4)^2}$

Step 3: $V = \displaystyle\int_2^6 2\pi (x-2)\sqrt{4-(x-4)^2}\, dx$ Let $u = x-4$

$= 2\pi \displaystyle\int_{-2}^{2} (u+2)\sqrt{4-u^2}\, du$

$= 2\pi \displaystyle\int_{-2}^{2} u\sqrt{4-u^2}\, du + 4\pi \int_{-2}^{2} \sqrt{4-u^2}\, du.$

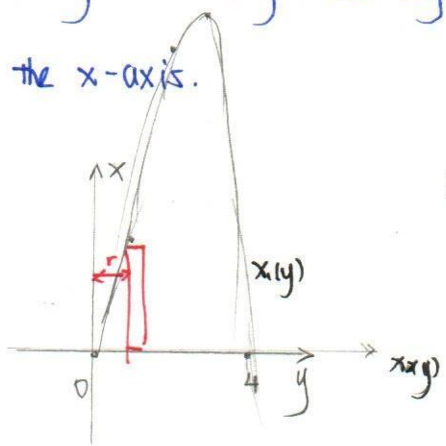
$= 0 + 8\pi \displaystyle\int_0^2 \sqrt{4-u^2}\, du$

$= 8\pi \cdot \frac{1}{4}\pi (2)^2 = 8\pi^2$

T02 - Ex 5

Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by the curves $x = 4y^2 - y^3$, $x' = 0$; about the x-axis.

Solution



Let $x_1(y) = 4y^2 - y^3$, $x_2(y) = 0$

$x_1(y) = x_2(y) \Rightarrow y = 0$, or $y = 4$

① Rotating about the x-axis from $y=0$ to $y=4$

② radius $(y) = y$

height $(y) = x_1(y) - x_2(y) = 4y^2 - y^3$

③ $V = \int_0^4 2\pi y (4y^2 - y^3)\, dy$

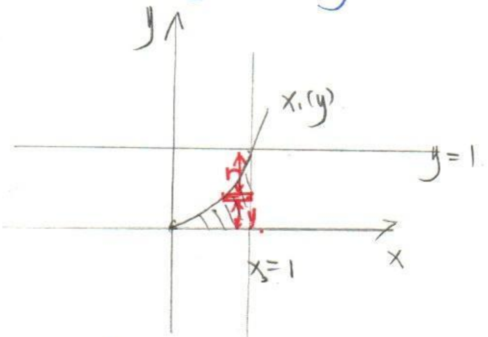
$= 8\pi \int_0^4 y^3\, dy - 2\pi \int_0^4 y^4\, dy = 8\pi \cdot \frac{1}{4} \left[y^4\right]_0^4 - 2\pi \cdot \frac{1}{5} \left[y^5\right]_0^4$

$= 512\pi - \frac{2048}{5}\pi = \frac{512}{5}\pi$

T02 - Exercise 6

Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the curves

$y = x^3$, $y = 0$, $x = 1$; about the axis $y = 1$.



Let $x_1(y) = x^{\frac{1}{3}}$, $x_2(y) = 1$

$x_1(y) = x_2(y)$ i.e. $\sqrt[3]{y} = 1 \Rightarrow y = 1$

Step 1: rotating about $y=1$, from $y=0$ to $y=1$

2: radius $(y) = (1-y)$

height $(y) = x_2(y) - x_1(y) = 1 - y^{\frac{1}{3}}$

3. $V = \int_0^1 2\pi(1-y)(1-y^{\frac{1}{3}})\,dy$

$= 2\pi \int_0^1 (1 - y - y^{\frac{1}{3}} + y^{\frac{4}{3}})\,dy$

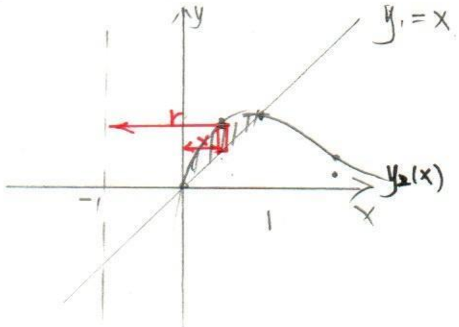
$= 2\pi \left[y - \frac{1}{2}y^2 - \frac{1}{\frac{1}{3}+1} y^{\frac{4}{3}} + \frac{1}{\frac{4}{3}+1} y^{\frac{7}{3}} \right]_0^1$

$= 2\pi \left[(1 - \frac{1}{2} - \frac{3}{4} + \frac{3}{7}) - 0 \right]$

$= \frac{5}{14}\pi$

Set up an integral for the volume of the solid obtained by rotating the region bounded by the curves $x = y$, $y = \frac{2x}{1+x^3}$; about the axis $x=-1$

Solution :



Let $y_1(x) = x$, $y_2(x) = \frac{2x}{1+x^3}$

$y_1(x) = y_2(x) \Rightarrow x = \frac{2x}{1+x^3}$

$x^4 - x = 0 \qquad x = 0$ or 1

① rotating about $x=-1$, from $x=0$ to $x=1$

② radius $(x) = x-(-1) = x+1$

height $(x) = y_2(x) - y_1(x) = \frac{2x}{1+x^3} - x$

③ $V = \int_0^1 2\pi (x+1)\left(\frac{2x}{1+x^3} - x\right) dx$

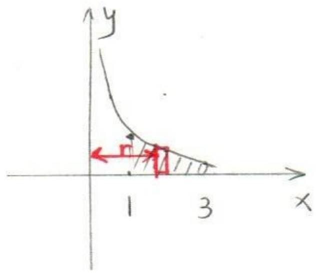
General method for solving this integral will be given in future lectures.

T03- Ex 1

Define R the region bounded above by the graph of $f(x) = \frac{1}{x}$ and below by the x-axis over the interval [1, 3].

Find the volume of the solid of revolution formed by revolving R around the y-axis.

Solution:



① rotating about y-axis. from 1 to 3

② radius(x) = x

height(x) = $\frac{1}{x}$

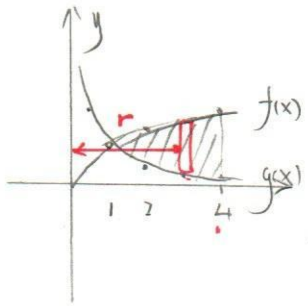
③ $V = \int_1^3 2\pi x \cdot \frac{1}{x}\, dx = 4\pi$

T03 - Ex 2

Define R the region bounded above by the graph of the func $f(x) = \sqrt{x}$ and below by the graph of the func. $g(x) = \frac{1}{x}$ over the interval [1, 4].

Find the volume of the solid of revolution generated by revolving R around the y-axis.

Solution:



① rotating about y-axis, from 1 to 4

② radius (x) = x

height (x) = $f(x) - g(x) = \sqrt{x} - \frac{1}{x}$

③ $V = \int_1^4 2\pi x \left(\sqrt{x} - \frac{1}{x}\right) dx$

$= 2\pi \int_1^4 x^{\frac{3}{2}}\, dx - 2\pi(4-1)$

$= 2\pi \left[\frac{1}{\frac{5}{2}} x^{\frac{5}{2}}\right]_1^4 - 6\pi$

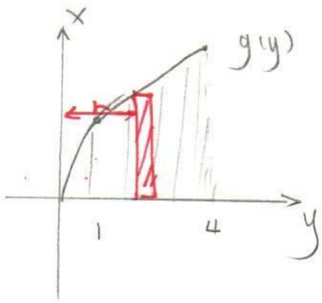
$= 2\pi \cdot \frac{2}{5}(32-1) - 6\pi = \frac{94}{5}\pi$

T03 - Ex3

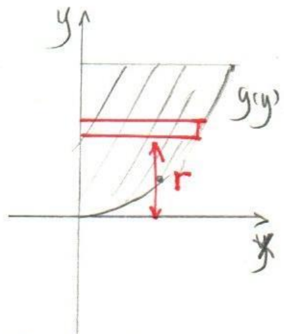
Define Q as the region bounded on the right by the graph of $g(y) = 2\sqrt{y}$ and on the left by the y-axis for $y \in [0, 4]$.

Find the volume of the solid of revolution formed by revolving Q around the x-axis.

Solution:



or



① rotating about x-axis from 0 to 4

② radius $(y) = y$

height $(y) = g(y) = 2\sqrt{y}$

③ $V = \int_0^4 2\pi y \cdot 2\sqrt{y}\, dy = 4\pi \int_0^4 y^{\frac{3}{2}}\, dy = 4\pi \cdot \frac{2}{5} \left[y^{\frac{5}{2}}\right]_0^4 = \frac{256}{5}\pi.$

T03- Ex 4

Work : $W = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \int_a^b f(x)\,dx$

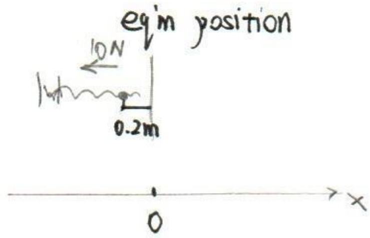
Sps it takes a force of 10N (in the negative direction)

to compress a spring 0.2m from the eq'm position.

How much work is done to stretch the spring 0.5m from the eq'm position?

Solution:

Find the spring constant $k>0$

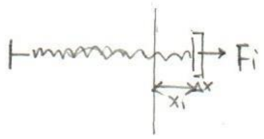


by Hooke's Law $\quad f(x) = kx$

$$-10N = k\cdot(-0.2m)$$

we have $k = \dfrac{-10N}{-0.2m} = 50(N/m)$

$W_i \approx f(x_i^*)\Delta x = k\cdot x_i^*\,\Delta x$

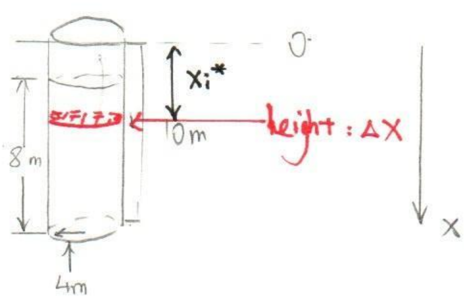


$$W = \lim_{n\to\infty}\sum_{i=1}^{n} W_i = \lim_{n\to\infty}\sum_{i=1}^{n} k x_i^*\,\Delta x = \int_0^{0.5} kx\,dx = k\cdot\left[\frac{1}{2}x^2\right]_0^{0.5} = \frac{k}{8} = 6.25(J)$$

Exercise 5:

Assume that a cylindrical tank of radius 4m and height 10m is filled to a depth of 8m.

How much work does it take to pump all the water over the top edge of the tank?



Solution: $F_i = \rho V_i g$

$V_i = \pi\cdot r^2\cdot\Delta x$

$W_i \approx F_i\cdot x_i^* = \rho g\pi r^2 x_i^*\,\Delta x$

$$W = \lim_{n\to\infty}\sum_{i=1}^{n} W_i = \lim_{n\to\infty}\sum_{i=1}^{n}\rho g\pi r^2 x_i^*\Delta x$$

$$= \int_2^{10}\rho g\pi r^2 x\,dx$$

$$= \rho g\pi r^2\left[\frac{1}{2}x^2\right]_2^{10}$$

$$= \rho g\pi r^2\cdot\frac{1}{2}(100-4)$$

$$= 768\pi\rho g$$