Tutorial 1 LI Yixin

Thm (The fundamental thm. of calculus)

Sps f is continuous on $[a.b]$

① If $g(x) = \int_a^x f(t)\,dt$. then $g'(x) = f(x)$.

② $\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$. where F is the antiderivative of f

① Find the derivative of $y = \int_{-4}^{x^2} e^{2t}\cos^2(1-5t)\,dt$.

Step 1: Change the variable. to make it the same form as ①.

Let $u(x) = x^2$, then $y(u) = \int_{-4}^{u} e^{2t}\cos^2(1-5t)\,dt$

Step 2: Chain rule

$$y'(x) = y'(u)\,u'(x)$$

Apply ①

$$= e^{2u(x)}\cos^2(1-5u(x)) \cdot 2x$$

$$= e^{2x^2}\cos^2(1-5x^2) \cdot 2x$$ Get x back.

② Integrate each of the following

(a) $\int_{-2}^{2} (4x^4 - x^2 + 1)\, dx = ①$

Method 1: Let $g(x) = 4x^4 - x^2 + 1$

then $g(-x) = 4(-x)^4 - (-x)^2 + 1 = 4x^4 - x^2 + 1 = g(x)$

Thus, $g(x)$ is an even function.

$$① = 2\int_{0}^{2} (4x^4 - x^2 + 1)\, dx$$

$$= 2\left[\frac{4}{5}x^5 - \frac{1}{3}x^3 + x\right]_0^2$$

$$= 2\left[\left(\frac{4}{5}\cdot 2^5 - \frac{1}{3}\times 2^3 + 2\right) - (0 - 0 + 0)\right]$$

$$= \frac{2}{15}\left(3\times 4\times 2^5 - 5\times 2^3 + 2\times 15\right)$$

$$= \frac{2}{15}\left(384 - 40 + 30\right)$$

$$= \frac{2\times 374}{15} = \frac{748}{15}$$

Method 2: Use linearity

(More flexible for integrals like $\int_{-2}^{2} (4x^4 - 3x^3 + 2x^2)\, dx$)

$$① = 4\int_{-2}^{2} x^4 dx - \int_{-2}^{2} x^2 dx + \int_{-2}^{2} 1\, dx$$

(Observe that x^4, x^2, are even functions)

$$= 8\int_{0}^{2} x^4 dx - 2\int_{0}^{2} x^2 dx + 4$$

$$= 8\left[\frac{1}{5}x^5\right]_0^2 - 2\cdot\left[\frac{1}{3}x^3\right]_0^2 + 4$$

$$= 8\times\frac{2^5}{5} - 2\times\frac{8}{3} + 4 = \frac{748}{15}$$

(b) $\int_{-10}^{10} x^2 \sin(x^3)\, dx$

Let $g(x) = x^2 \sin(x^3)$

then $g(-x) = (-x)^2 \sin((-x)^3) = -x^2 \sin(x^3) = -g(x)$

Thus, $g(x)$ is an odd function.

We have $\int_{-10}^{10} x^2 \sin(x^3)\, dx = 0$.

Think about $\int_{-10}^{5} x^2 \sin(x^3)\, dx$

$$= \underbrace{\int_{-10}^{-5} x^2 \sin(x^3)\, dx}_{\text{by substitution rule}} + \underbrace{\int_{-5}^{5} x^2 \sin(x^3)\, dx}_{=0}$$

Let $u = x^3$

$du = 3x^2\, dx$

then $\int x^2 \sin(x^3)\, dx = \frac{1}{3}\int \sin(u)\, du = -\frac{1}{3}\cos(u) + C$

$= -\frac{1}{3}\cos(x^3) + C$

$\int_{-10}^{-5} x^2 \sin(x^3)\, dx = \left[-\frac{1}{3}\cos(x^3)\right]_{-10}^{-5} = -\frac{1}{3}\cos(5^3) + \frac{1}{3}\cos(10^3)$

seems does not help very much.

The substitution rule and the substitution rule for definite integrals:

(a) $\displaystyle\int_0^{\ln(1+\pi)} e^x \cos(1-e^x)\,dx = ①$

(Substitution rule for definite integrals)

Method 1 Let $u = 1-e^x$

$x=0 \Rightarrow u = 1-e^0 = 0$

then $du = -e^x dx$

$x = \ln(1+\pi) \Rightarrow u = 1-e^{\ln(1+\pi)} = 1-(1+\pi) = -\pi$

$$① = -\int_0^{-\pi} \cos(u)\,du$$

$$= -\Big[\sin(u)\Big]_0^{-\pi}$$

$$= -\Big[\sin(-\pi) - \sin(0)\Big]$$

$$= -[0-0] = 0$$

(Fundamental thm. of calculus & Substitution rule)

Method 2: We first find $\displaystyle\int e^x\cos(1-e^x)\,dx$

Let $u = 1-e^x$, then $du = -e^x dx$

$$\int e^x \cos(1-e^x)\,dx = -\int \cos(u)\,du$$

$$= -\sin(u) + C$$

$$= -\sin(1-e^x) + C$$

By the fundamental thm of calculus,

$$\int_0^{\ln(1+\pi)} e^x\cos(1-e^x)\,dx = \Big[-\sin(1-e^x)\Big]_0^{\ln(1+\pi)}$$

$$= -\sin(1-e^{\ln(1+\pi)}) + \sin(1-e^0)$$

$$= -\sin(-\pi) + \sin(0) = 0$$

(b) $\int_{-\pi}^{\frac{\pi}{2}} \cos(x) \cos(\sin(x))\, dx = ②$

Method 1 : let $u = \sin(x)$ $\quad x = -\pi \Rightarrow u = \sin(-\pi) = 0$

$du = \cos(x)\, dx \quad x = \frac{\pi}{2} \Rightarrow u = \sin(\frac{\pi}{2}) = 1$

$$② = \int_0^1 \cos(u)\, du = \Big[\sin(u)\Big]_0^1 = \sin(1) - \sin(0) = \sin(1)$$

Method 2 : We first solve $\int \cos(x) \cos(\sin(x))\, dx$

Let $u = \sin(x)$ then $du = \cos(x)\, dx$

$$\int \cos(x) \cos(\sin(x))\, dx = \int \cos(u)\, du = \sin(u) + C$$
$$= \sin(\sin(x)) + C$$

$$② = \Big[\sin(\sin(x))\Big]_{-\pi}^{\frac{\pi}{2}} = \sin(\sin(\tfrac{\pi}{2})) - \sin(\sin(-\pi))$$
$$= \sin(1) - \sin(0) = \sin(1).$$

Discussion: Let $g(x) = \cos(x) \cos(\sin(x))$

$$g(-x) = \cos(-x) \cos(\sin(-x)) = \cos(x) \cos(-\sin(x)) = \cos(x) \cos(\sin(x))$$
$$= g(x).$$

We observe that g is an even function

$$① = \int_{-\pi}^{-\frac{\pi}{2}} \cos(x) \cos(\sin x)\, dx + 2\int_0^{\frac{\pi}{2}} \cos(x) \cos(\sin(x))\, dx.$$

Seems Notes not help very much.

· Area between the curves

Let $a<b$, both $f(x)$ and $g(x)$ are continuous

Let $S=\{(x,y) \mid a\le x\le b,\ g(x)\le y\le f(x)\}$

Then. $\text{area}(S)=\int_a^b |f(x)-g(x)|\,dx$

① Find the area of region enclosed by $y_1=6-x$ and $y_2=9-\left(\frac{x}{2}\right)^2$

Step 1 : Observe $y=f(x)$ or $x=g(y)$.

y is a function of x for both $y=6-x$ and $y=9-\left(\frac{x}{2}\right)^2$

Step 2 : Calculate the intersections

$$6-x=9-\left(\frac{x}{2}\right)^2 \iff \frac{1}{4}x^2-x-3=0$$

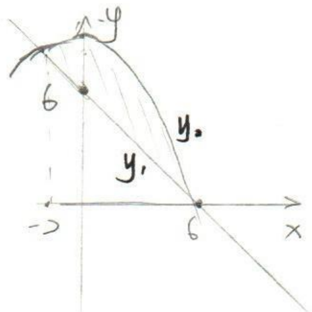
$$x^2-4x-12=0$$

$$(x+2)(x-6)=0$$

$$x_1=-2 \quad x_2=6.$$

Step 3 : Decide the sign of $f(x)-g(x)$.

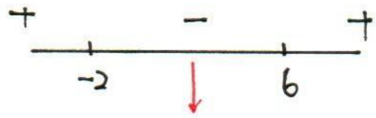
Method 1 : Draw a graph.



$y_2\ge y_1$

Method 2 : Sign chart

$$y_1(x)-y_2(x)=6-x-9+\left(\frac{x}{2}\right)^2$$
$$=\frac{1}{4}x^2-x-3$$



means $y_1\le y_2$ on $[-2,6]$

Step 4 : Take the integral

$$\int_{-2}^{6} -\frac{1}{4}x^2+x+3 = -\frac{1}{4}\int_{-2}^{6}x^2\,dx+\int_{-2}^{6}x\,dx+\int_{-2}^{6}3\,dx$$

$$=-\frac{1}{4}\left[\frac{1}{3}x^3\right]_{-2}^{6}+\left[\frac{1}{2}x^2\right]_{-2}^{6}+3\times(6-(-2))$$

$$= -\frac{1}{4\times3}\left(6^3 - (-2)^3\right) + \frac{1}{2}\left[6^2 - (-2)^2\right] + 3\times 8$$

$$= -\frac{1}{12}(216+8) + \frac{1}{2}(36-4) + 24$$

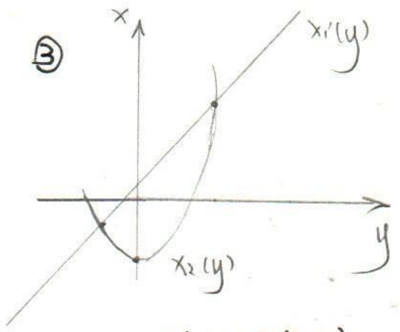
$$= \frac{64}{3}$$

② Find the area of region enclosed by the curve $y=x-1$ and $x=\frac{1}{2}y^2-3$.

① Notice that x is a function of y $x_1(y) = y+1$, $x_2(y) = \frac{1}{2}y^2-3$

② $y+1 = \frac{1}{2}y^2-3 \iff y^2-2y-8=0$ we have $y_1 = 4$, $y_2 = -2$

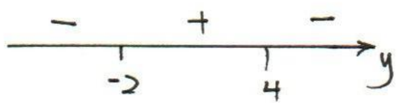
③



$x_1(y) \geq x_2(y)$

Sign chart

$x_1(y) - x_2(y) = -y^2 + 2y + 8$



$x_1(y) \geq x_2(y)$

④ $$\int_{-2}^{4} (y+1) - \left(\frac{1}{2}y^2-3\right) dy$$

$$= \int_{-2}^{4} -\frac{1}{2}y^2 + y + 4 \, dy$$

$$= -\frac{1}{2}\times\frac{1}{3}\left[y^3\right]_{-2}^{4} + \frac{1}{2}\left[y^2\right]_{-2}^{4} + (4-(-2))\times 4$$

$$= -\frac{1}{6}\left(4^3 - (-2)^3\right) + \frac{1}{2}\left(4^2 - (-2)^2\right) + (4+2)\times 4$$

$$= -\frac{1}{6}(64+8) + \frac{1}{2}(16-4) + 6\times 4$$

$$= -12 + 6 + 24 = 18$$

③ Find the area of the region bounded by the curve

$y = 4x + 16$, $y = 2x^2 + 10$, $x = -2$ and $x = 5$.

① y is a function of x

② We already know the region.

$$\text{Area}(S) = \int_{-2}^{5} \left| (4x+16) - (2x^2+10) \right| dx$$

$$= \int_{-2}^{5} \left| -2x^2 + 4x + 6 \right| dx$$

Let $g(x) = -2x^2 + 4x + 6$

$g(x) = 0 \Rightarrow x_1 = 3 \quad x_2 = -1$

sign chart

We divide $[-2, 5]$ into 3 intervals $[-2, -1]$, $[-1, 3]$, $[3, 5]$

$$\text{Area}(S) = \int_{-2}^{-1} (2x^2 - 4x - 6)\, dx + \int_{-1}^{3} (-2x^2 + 4x + 6)\, dx$$

$$+ \int_{3}^{5} (2x^2 - 4x - 6)\, dx$$

$$= \left[\frac{2}{3}x^3 - 2x^2 - 6x \right]_{-2}^{-1} + \left[-\frac{2}{3}x^3 + 2x^2 + 6x \right]_{-1}^{3}$$

$$+ \left[\frac{2}{3}x^3 - 2x^2 - 6x \right]_{3}^{5}$$

$$= \frac{2}{3}\left[(-1)^3 - (-2)^3\right] - 2\left[(-1)^2 - (-2)^2\right] - 6\left[(-1) - (-2)\right]$$

$$- \frac{2}{3}\left[3^3 - (-1)^3\right] + 2\left[3^2 - (-1)^2\right] + 6\left[3 - (-1)\right]$$

$$+ \frac{2}{3}\left[5^3 - 3^3\right] - 2\left[5^2 - 3^2\right] - 6\left[5 - 3\right]$$

$$= \frac{2}{3}\left(-1 + 8 - 27 - 1 + 125 - 27\right)$$

$$+ 2\left(-1 + 4 + 9 - 1 - 25 + 9\right)$$

$$+ 6\left(1 - 2 + 3 + 1 - 5 + 3\right)$$

$$= \frac{2}{3} \times 77 + 2(-5) + 6 \times 1 = \frac{142}{3}.$$