Tutorial 6

- Trigonometric substitutions

Expression	Identity	Substitution
$\sqrt{a^2-x^2}$	$1-\sin^2\theta=\cos^2\theta$	$x=a\cdot\sin\theta \quad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$
1		$\sqrt{(b\cos\theta)^2}\ge 0$
		$\lvert b\cos\theta\rvert = b\cos\theta$
$\sqrt{a^2+x^2}$	$1+\tan$	
$1+\tan^2=\sec^2$		$x=a\cdot\tan x \quad -\frac{\pi}{2}<x<\frac{\pi}{2}$
$\sqrt{x^2-a^2}$		

$$\int \frac{dx}{x^2\sqrt{4-x^2}}$$

Let $x = 2\cdot\sin\theta \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

$dx = 2\cos\theta \, d\theta$

$$\sqrt{4-x^2} = \sqrt{4-4\cdot\sin^2\theta} = \sqrt{4(1-\sin^2\theta)} = \sqrt{4\cos^2\theta}$$

$$= |2\cdot\cos\theta| = 2\cos\theta$$

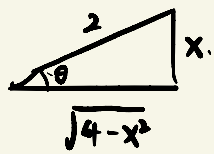
$$A = \int \frac{1}{4\cdot\sin^2\theta \cdot \cancel{2\cdot\cos\theta}} \cdot \cancel{2\cdot\cos\theta}\, d\theta.$$

$$= \frac{1}{4}\int \csc^2\theta \, d\theta$$

$$= -\frac{1}{4}\cot\theta + C$$

$$= -\frac{1}{4}\cdot\frac{\sqrt{4-x^2}}{x} + C.$$

$\sin\theta = \frac{x}{2}$



T05 - Ex 7

$$\int \frac{1}{\sqrt{9x^2-36x+37}}\,dx =: A$$

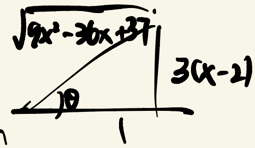
$$9x^2-36x+37 = 9(x^2-4x+4)-4\times 9+37$$
$$= 9(x-2)^2+1$$

$$3(x-2) = \tan\theta \qquad -\frac{\pi}{2}<\theta<\frac{\pi}{2}.$$

$$x = 2+\frac{1}{3}\tan\theta$$
$$dx = \frac{1}{3}\sec^2\theta\, d\theta$$

$$A = \int \frac{1}{\sqrt{9(x-2)^2+1}}\,dx$$
$$= \int \frac{1}{\sqrt{\tan^2\theta+1}}\cdot\frac{1}{3}\sec^2\theta\, d\theta$$
$$= \int \frac{1}{\cancel{\sec\theta}}\cdot\frac{1}{3}\sec^{\cancel{2}}\theta\, d\theta.$$
$$= \frac{1}{3}\int \sec\theta\, d\theta$$

$$= \frac{1}{3} \ln \left| \tan\theta + \sec\theta \right| + C$$



$$= \frac{1}{3} \ln \left| 3(x-2) + \sqrt{9x^2 - 36x + 37} \right| + C$$

T06 - Ex 3

$$E(P) = \int_{-a}^{L-a} \frac{\lambda b}{4\pi\varepsilon_0 (x^2+b^2)^{\frac{3}{2}}}\,dx \qquad (b>0)$$

$$= \frac{\lambda b}{4\pi\varepsilon_0} \int_{-a}^{L-a} \frac{1}{(x^2+b^2)^{\frac{3}{2}}}\,dx$$

Let $x = b\tan\theta \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$dx = b\sec^2\theta\, d\theta$$

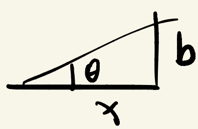
$$\theta = \arctan\left(\frac{x}{b}\right)$$

$$x = -a \longrightarrow \theta = \arctan\left(-\frac{a}{b}\right)$$

$$x = L-a \longrightarrow \theta = \arctan\left(\frac{L-a}{b}\right)$$

$$A = \frac{\lambda b}{4\pi\varepsilon_0} \int_{\arctan\left(-\frac{a}{b}\right)}^{\arctan\left(\frac{L-a}{b}\right)} \frac{1}{(b\sec\theta)^3} \cdot b\sec^2\theta\, d\theta$$

$$= \frac{\lambda}{4\pi\varepsilon_0 b} \int_{\arctan\left(-\frac{a}{b}\right)}^{\arctan\left(\frac{L-a}{b}\right)} \cos\theta\, d\theta$$



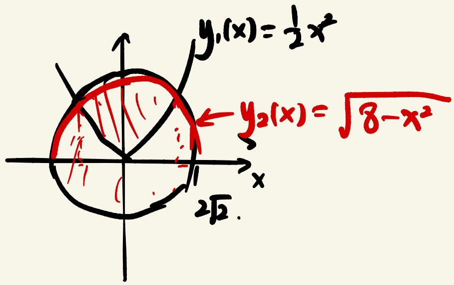
$$= \frac{\lambda}{4\pi\varepsilon_0 b} \left[\sin\left(\arctan\left(\frac{L-a}{b}\right)\right) - \sin\left(\arctan\left(\frac{-a}{b}\right)\right) \right]$$

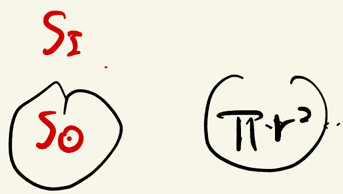
$$= \frac{\lambda}{4\pi\varepsilon_0 b}\left[\frac{L-a}{\sqrt{(L-a)^2+b^2}} + \frac{+a}{\sqrt{a^2+b^2}}\right]$$

T06 - Ex 4

The parabola $y=\frac{1}{2}x^2$ divides the disk $x^2+y^2 \le 8$ into two parts

Find the area of both parts.





$S_{II} = S_0 - S_I$.

$y_1(x) = y_2(x)$

$-\frac{1}{2}x^2 = \sqrt{8-x^2}$

$(x-2)(x+2)(x^2+8)=0$

$x=\pm 2$.

$$A = \int_{-2}^{2} y_2(x) - y_1(x)\, dx$$

$$= \int_{-2}^{2} \sqrt{8-x^2} - \frac{1}{2}x^2\, dx$$

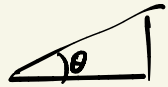
$$= 2\int_{0}^{2} \sqrt{8-x^2} - \frac{1}{2}x^2\, dx$$

$\int_0^2 \sqrt{8-x^2}\, dx$. Let $x = \sqrt{8}\cdot \sin\theta$ $\quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

$dx = \sqrt{8}\cdot \cos\theta\, d\theta$.

$$\int \sqrt{8}\cdot\cos\theta \cdot \underbrace{\sqrt{8}\cdot\cos\theta\, d\theta}_{dx}$$

$$= 8 \cdot \int \cos^2\theta \, d\theta .$$



$$= 8 \cdot \left[\frac{1}{2} \sin\theta \cos\theta + \frac{1}{2}\theta \right] + C .$$

Integration of rational func.s by partial fractions $\frac{Q(x)}{P(x)}$.

Step: 1 $\deg(Q) < \deg(P)$

Step 2 Factor Q(x

$$\frac{A}{(ax+b)^i} \qquad \frac{Ax+B}{(ax^2+bx+c)^j} \qquad b^2-4ac$$

$$\int \frac{1}{x-a}\,dx = \ln|x-a| + C.$$

$$\int \frac{1}{(x-a)^k}\,dx \qquad u = x-a.$$

$$\int \frac{1}{u^k}\,du.$$

$k \geq 2$

$t = x-a.$

$$\underbrace{\int \frac{\boxed{x-a}}{[(x-a)^2+b^2]^k}\,dx}_{A}$$

$$\int \frac{t}{[t^2+b^2]^k}\,dt$$

$$\frac{1}{2}\int \frac{1}{[t^2+b^2]^k}\,dt^2$$

$$\underbrace{C\int \frac{\boxed{\;}\,dx}{[(x-a)^2+b^2]}}_{B}$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C,$$

$$\int \frac{dx}{\left[(x-a)^2+b^2\right]^k}$$

$$x-a = b\tan u.$$

$$\int \frac{1}{(x-1)(x+2)^2}\,dx$$

$$\frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$\int \frac{x^2+1}{(x^2-2x+2)^2}\,dx$$

$b^2-4ac = 4-4\times2 = -4<0.$

$$\frac{Ax+B}{x^2-2x+2} + \frac{Cx+D}{(x^2-2x+2)^2}$$

$$\frac{(x^2-2x+2)+2x-2+1}{(x^2-2x+2)^2} = \frac{1}{x^2-2x+2} + \frac{2x-1}{(x^2-2x+2)^2}$$

$(x-1)^2+1$

$$\int \frac{1}{(x-1)^2+1}\,dx = \arctan(x-1) + C_1.$$

$$\frac{2(x-1)+2-1}{((x-1)^2+1)^2} = 2\frac{x-1}{((x-1)^2+1)^2} - \frac{1}{((x-1)^2+1)^2}$$

$t = x-1$

$$= 2\int \frac{t}{(t^2+1)^2}\,dt - \int \frac{1}{(t^2+1)^2}\,dt$$

$t = \tan\theta$