T2 — Area Between the Curve

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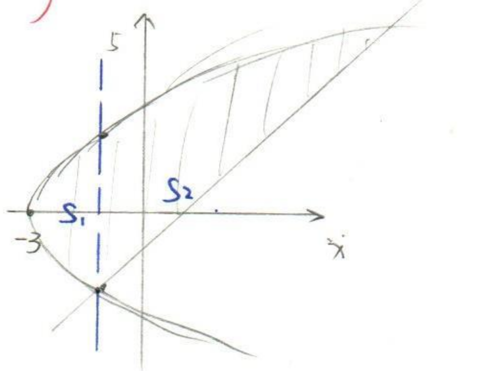
Find the area of the region enclosed by the curve $y=x-1$ and $x=\frac{1}{2}y^2-3$.

What will happen if we treat y as a function of x.



Solve $\begin{cases} y=x-1 \\ x=\frac{1}{2}y^2-3 \end{cases}$

we have $x=\frac{1}{2}(x-1)^2-3$

$x^2-4x-5=0$

$x_1=-1 \quad x_2=5$

Let $y_1(x)=\sqrt{2x+6}$, $y_2(x)=-\sqrt{2x+6}$ and $y_3(x)=x-1$

Let S_1 be the region enclosed by $y_1(x)$, $y_2(x)$ and $x=-1$

$$\text{area}(S_1)=\int_{-3}^{-1} y_1(x)-y_2(x)\,dx = 2\int_{-3}^{-1}\sqrt{2x+6}\,dx =: ①$$

Let $u=2x+6$ $\qquad x=-3 \Rightarrow u=2(-3)+6=0$

$\qquad x=-1 \Rightarrow u=2(-1)+6=4$

$du=2dx$

$$① = \int_0^4 u^{\frac{1}{2}}\,du = \left[\frac{1}{\frac{1}{2}+1}u^{\frac{3}{2}}\right]_0^4 = \frac{16}{3}$$

Let S_2 be the region enclosed by $y_1(x)$, $y_3(x)$ and $x=-1$

$$\text{area}(S_2)=\int_{-1}^{5} y_1(x)-y_3(x)\,dx = \int_{-1}^{5}\sqrt{2x+6}-x+1\,dx =: ②$$

Let $u=2x+6$ $\qquad x=-1 \Rightarrow u=2(-1)+6=4$

$du=2dx$ $\qquad x=5 \Rightarrow u=2(5)+6=16$

$$② = \frac{1}{2}\int_4^{16} u^{\frac{1}{2}}\,du - \int_{-1}^{5} x\,dx + 6$$

$$= \frac{1}{3}(2^6-2^3) - \left[\frac{1}{2}x^2\right]_{-1}^{5} + 6$$

$$= \frac{1}{3}(64-8) - \frac{1}{2}(25-(-1)^2) + 6$$

$$= \frac{1}{3}\times 56 - \frac{1}{2}\times 24 + 6 = \frac{38}{3}$$

$$\text{area}(S) = ① + ② = \frac{16}{3}+\frac{38}{3}=\frac{54}{3} = 18$$

Tutorial 2 - Volume.

Def: (Finding volume by using slices)

Let S be a solid that lies between $x = a$ and $x = b$.

If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis.

is $A(x)$, where A is a continuous function,

then. the volume of S is $V = \int_a^b A(x)\,dx$.

Step 1: Determine the solid S lies between $x = a$ and $x = b$ or $y = c$ and $y = b$.

a.b or c.d.

Step 2: Calculate $A(x)$ or $A(y)$

Cross-section is

Case I: Disk. $A = \pi \cdot (\text{radius})^2$

Case II: Washer $A = \pi \cdot (\text{outer radius})^2 - \pi \cdot (\text{inner radius})^2$

Case III: General ex. equilateral triangles. squares, ...

Step 3: Apply the definition:

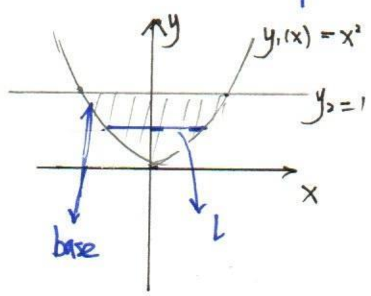
$$V = \int_a^b A(x)\,dx \quad \text{or} \quad V = \int_c^d A(y)\,dy$$

Step 4: Find the volume of the solid.

Exercise 1:

The solid whose base is the region bounded by $y=x^2$ and the line $y=1$, and whose cross-sections perpendicular to the base and parallel to the x-axis are squares.

Solution:



Step 1: The solid is bounded by $y=0$ and $y=1$.

Step 2: Case III: square.

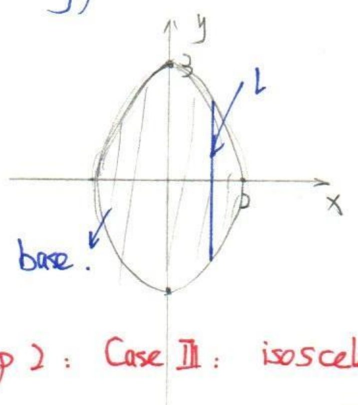
$$A(y) = l^2 = (2\,x(y))^2 = (2\sqrt{y})^2 = 4y.$$

Step 3: $V = \int_0^1 A(y)\,dy = \int_0^1 4y\,dy \overset{\text{Step 4.}}{=} 4\cdot\left[\frac{1}{2}y^2\right]_0^1 = 2$

Exercise 2:

The base of S is an elliptical region with boundary curve $9x^2+4y^2=36$, Cross-sections perpendicular to the x-axis are isosceles right triangulars with hypotenuse in the base.

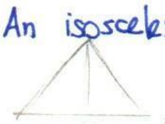
Solution:



Step 1: Let $y=0$, then $9x^2+4\cdot 0^2=36$, $x=\pm 2$.

The solid is bounded by $x=-2$ and $x=2$.

An isosceles with hypotenuse l



$$A(l) = \frac{1}{2}\cdot l\cdot\frac{l}{2} = \frac{l^2}{4}$$

Step 2: Case III: isosceles right triangulars.

$$l(x) = 2\,y(x) = 2\sqrt{\frac{36-9x^2}{4}} = \sqrt{36-9x^2}$$

$$A(x) = \frac{1}{4}\,l(x)^2 = \frac{36-9x^2}{4}$$

Step 3: $V = \int_{-2}^{2} A(x)\,dx = \int_{-2}^{2}\left(9-\frac{9}{4}x^2\right)dx$

$$= 2\int_0^2\left(9-\frac{9}{4}x^2\right)dx$$

$$= 2\left[18-\frac{9}{4}\cdot\left[\frac{1}{3}x^3\right]_0^2\right]$$

$$= 2\left(18-\frac{9}{4}\times\frac{8}{3}\right) = 2\times 12 = 24.$$

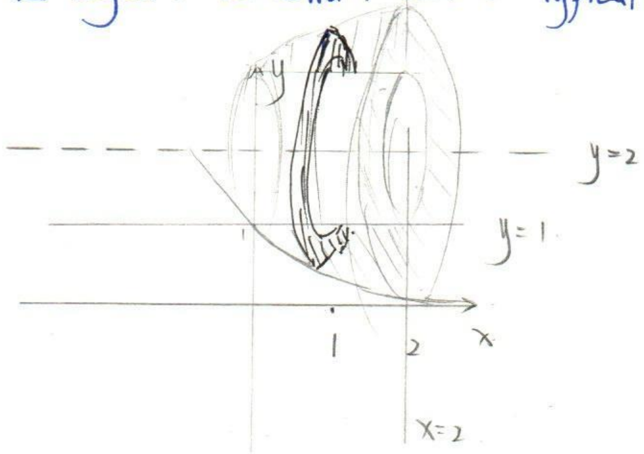
Exercise 3.

Find the volume of the solid obtained

by rotating the region bounded by the curves $y=e^{-x}$, $y=1$, $x=2$; about the line $y=2$.

Sketch the region, the solid, and a typical disk or washer.

Solution:



Step 1: Let $y_1(x) = e^{-x}$ and $y_2(x) = 1$, $y_3 = 2$

when $y_1(x) = y_2(x)$ we have $x = 0$

The region is bounded by $x=0$ and $x=2$.

Step 2: Case II: Washer

outer radius $= y_3(x) - y_1(x) = 2 - e^{-x}$

inner radius $= y_3(x) - y_2(x) = 2 - 1 = 1$

$$A(x) = \pi\,(\text{outer radius})^2 - \pi\,(\text{inner radius})^2$$

$$= \pi\left(2 - e^{-x}\right)^2 - \pi \cdot (1)^2$$

$$= \pi\left[\,e^{-2x} - 4e^{-x} + 4 - 1\right]$$

$$= \pi\left[e^{-2x} - 4e^{-x} + 3\right]$$

Step 3: $V = \int_0^2 A(x)\,dx = \pi\int_0^2 \left(e^{-2x} - 4e^{-x} + 3\right) dx$

Step 4: $V = \pi\left(\int_0^2 e^{-2x}\,dx - 4\int_0^2 e^{-x}\,dx + 6\right)$

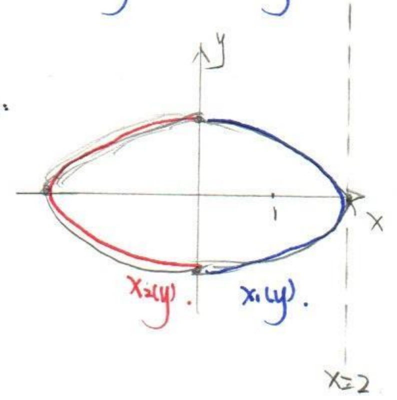
$$= \pi\left(\left[-\frac{1}{2}\,e^{-2x}\right]_0^2 - 4\left[\frac{1}{-1}\,e^{-x}\right]_0^2 + 6\right) = \pi\left[-\frac{1}{2}\left(e^{-4}-1\right) + 4\left(e^{-2}-1\right) + 6\right]$$

$$= \pi\left(-\frac{1}{2}e^{-4} + 4e^{-2} + \frac{5}{2}\right)$$

Exercise 4:

Set up an integral for the volume of the solid obtained by rotating the region bounded by the curves $x^2+4y^2=4$ about the line $x=2$.

Solution.



Step 1: Let $x=0$ for the curve $x^2+4y^2=4$, then $y=\pm 1$.

The region is bounded by $y=-1$ and $y=1$.

Step 2: Case II: Washer.

For the curve $x^2+4y^2=4$ $\qquad x_1(y)=\sqrt{4-4y^2}$

$$x_2(y) = -\sqrt{4-4y^2}$$

outer radius $= 2-x_2(y) = 2+\sqrt{4-4y^2}$

inner radius $= 2-x_1(y) = 2-\sqrt{4-4y^2}$

$$A(y) = \pi\,(\text{outer radius})^2 - \pi\,(\text{inner radius})^2$$

$$= \pi\left(\left(2+\sqrt{4-4y^2}\right)^2 - \left(2-\sqrt{4-4y^2}\right)^2\right) \qquad a^2-b^2=(a+b)(a-b).$$

$$= \pi\cdot(4)\cdot\left(2\sqrt{4-4y^2}\right)$$

$$= 16\pi\sqrt{1-y^2}$$

Step 3: $V=\int_{-1}^{1} A(y)\,dy = \int_{-1}^{1} 16\pi\sqrt{1-y^2}\,dy = 32\pi\int_0^1 \sqrt{1-y^2}\,dy$

To solve this integral, we need the knowledge of §7.3 trigonometric substitution.

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin x + C$$

$$V = 32\pi\cdot\frac{1}{2}\left(\arcsin(1) - \arcsin(0)\right) = 32\pi\cdot\frac{1}{2}\cdot\frac{\pi}{2} = 8\pi^2 \approx 78.95684.$$