Calcus & Algebra

Question 1

For point A, $\frac{dy}{dx} = 0$ because it's a <u>local minimum</u> where the slope of the tangent at A is 0

$\frac{d^2y}{d^2x} > 0$ because the curve is <u>concave up</u> at A.

For point B, $\frac{dy}{dx} = 0$ because it's a <u>local maximum</u> where the slope of the tangent at B is 0

$\frac{d^2y}{d^2x} < 0$ because the curve is <u>concave down</u> at B

For point C, $\frac{dy}{dx} > 0$ because the function is <u>increasing</u> at C, and the slope of the tangent is positive

$\frac{d^2y}{d^2x} > 0$ because the curve is <u>concave up</u> at C.

Question 2

First, we need to find the derivative of $f(x)$

$$f'(x) = x^2 - 2x - 3$$

To find minimum/maximum points, let $f'(x) = 0$

$$x^2 - 2x - 3 = 0$$

then $(x-3)(x+1) = 0$

$$x_1 = 3 \quad x_2 = -1$$

To determine whether they're max or min, find the second derivative.

$$f''(x) = 2x - 2$$

plug $x_1 = 3$ into $f''(x)$, get $f''(x) = 4 > 0 \mapsto$ local minimum

$x_2 = -1$ into $f''(x)$. get $f''(x) = -4 < 0 \mapsto$ local maximum

Then, plug x_1, x_2 into $f(x)$ to get y_1, y_2,

$$f(x_1) = f(3) = -2$$

$$f(x_2) = f(-1) = \frac{26}{3}$$

So, the local minimum is $(3, -2)$, the local maximum is $(-1, \frac{26}{3})$

Question 3

(a)

$$\frac{\partial h(x,y,z)}{\partial x} = -\frac{1}{(y^7-4z)x} + \frac{6xz}{y^4} - 2ye^{2xy}\ln(z)$$

(b)

$$\frac{\partial h(x,y,z)}{\partial y} = -(\ln(x)-z)\cdot\left(-\frac{1}{(y^7-4z)^2}\cdot 7y^6\right) + \left(-\frac{12x^2z}{y^5}\right)$$

$$- 2xe^{2xy}\ln(z) + 20yz$$

$$= \frac{7y^6(\ln(x)-z)}{(y^7-4z)^2} - \frac{12x^2z}{y^5} - 2xe^{2xy}\ln(z) + 20yz$$

Probability and Statistics

Question 4

(a) $P = \frac{9}{32} \cdot \frac{23}{31} = \frac{207}{992} \approx 0.209$

(b) $P = \frac{7}{32} \cdot \frac{7}{32} + \frac{7}{32} \cdot \frac{25}{32} \cdot 2$

$= \frac{49}{1024} + \frac{350}{1024} = \frac{399}{1024}$

(c) $P = \frac{8}{32} + \frac{8+9}{32} = \frac{25}{32}$

(d) $P = \frac{8}{32} + \frac{9}{32} = \frac{17}{32}$

Linear Algebra

Question 5

(a) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{Rank}(A) = 2$

As A has 2 columns and $\text{rank}(A) = 2$, thus Full Rank

(b) $B = \begin{bmatrix} -4 & 4 \\ -3 & 3 \end{bmatrix} \xrightarrow{R_2 - \frac{3}{4}R_1} \begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{Rank}(B) = 1$

As B has 2 colums and $\text{rank}(B) = 1 < 2$, thus not full rank

$\vec{V_2} = -\vec{V_1}$

(c) $C = \begin{bmatrix} 0 & 5 & 15 \\ 1 & 0 & 15 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 15 \\ 0 & 5 & 15 \end{bmatrix} \Rightarrow \text{Rank}(C) = 2$

As C has 3 columns and $\text{rank}(C) = 2 < 3$, thus not full rank

$\vec{V_3} = 15\vec{V_1} + 3\vec{V_2}$

(d) $D = \begin{bmatrix} -2 & 2 & 0 \\ -2 & 4 & 2 \\ 5 & -2 & 3 \end{bmatrix} \xrightarrow[R_3 + \frac{5}{2}R_1]{R_2 - R_1} \begin{bmatrix} -2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 3 & 3 \end{bmatrix} \xrightarrow{R_3 - \frac{3}{2}R_2} \begin{bmatrix} -2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \text{rank}(D) = 2 < 3 \Rightarrow$ not full rank

$\vec{V_1} + \vec{V_2} = \vec{V_3}$