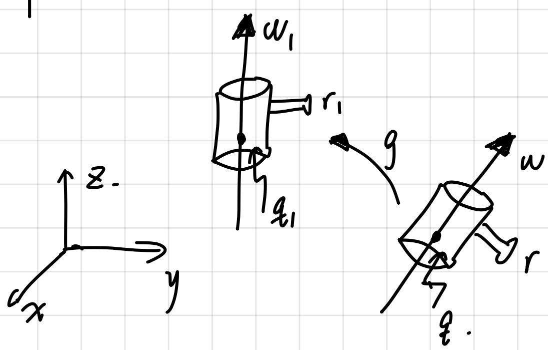
Velocity.

Recap:



$$\begin{cases} \bar{q}_1 = g\bar{q} \\ w_1 = Rw \end{cases} \Rightarrow \begin{bmatrix} v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}$$

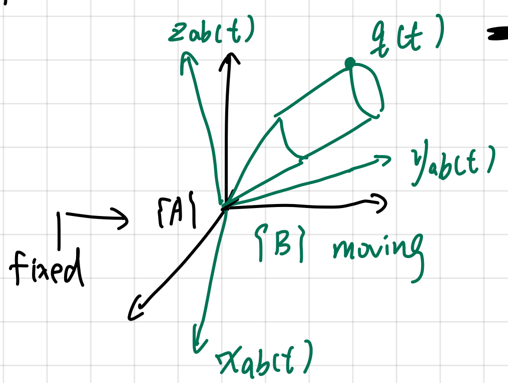
$$\Rightarrow \xi_1 = Ad_g \, \xi$$

(twist coordinate)

$$\Leftrightarrow \hat{\xi}_1 = g_{ab} \hat{\xi} g^{-1}_{ab}$$

$$\Leftrightarrow e^{\hat{\xi}_1 \theta} = g_{ab} e^{\hat{\xi}\theta} g^{-1}_{ab}$$

Pure Rotation



- For q_a:

$$q_a(t) = R_{ab}(t) \, q_b$$

Velocity: $\dot{q}_a(t) = \dot{R}_{ab}(t) \, \underbrace{q_b}_{R^{-1}_{ab}(t) q_a(t)}$

$$\Rightarrow \dot{q}_a(t) = \underbrace{\dot{R}_{ab}(t) R^{-1}_{ab}(t)}_{\hat{w}^a_{ab}} q_a(t)$$

$\Rightarrow$ skew-symmetric matrix.

$$\Rightarrow V_{q_a} = \hat{w}^a_{ab} \cdot q_a(t)$$

$$(\hat{w}^a_{ab})^T = -\hat{w}^a_{ab}$$

- For q_b:

☆ $V_{q_b} \neq \dot{q}_b = 0$

$$V_{q_b} = R^{-1}_{ab}(t) \, V_{q_a}$$

$$= R^{-1}_{ab}(t) \, \dot{R}_{ab}(t) \underbrace{R^{-1}_{ab}(t) q_a(t)}_{q_b}$$

$$= \underbrace{R^{-1}_{ab}(t) \, \dot{R}_{ab}(t)}_{\hat{w}^b_{ab}} q_b$$

☆ $V_{q_b} = \hat{w}^b_{ab} \cdot q_b$.

Rigid Body Motion.

For spatial frame velocity

$$\bar{q}_a(t) = g_{ab}\,\bar{q}_b$$

Velocity: $\dot{\bar{q}}_a(t) = \dot{g}_{ab}\,\bar{q}_b$

$$= \underbrace{\dot{g}_{ab}\, g_{ab}^{-1}}_{\hat{V}^s_{ab}}\, \bar{q}_a(t)$$

$\hat{V}^s_{ab}$: spatial velocity twist.

$$\hat{V}^s_{ab} = \left[\begin{array}{c|c} \hat{\omega}^s_{ab} & v^s_{ab} \\ \hline 0 & 0 \end{array}\right] \text{ where } v^s_{ab} = \dot{p}_{ab} - \hat{\omega}^s_{ab}\cancel{p_{ab}}$$

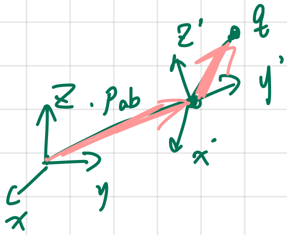
Plug into

$$V_{q_a} = \hat{V}^s_{ab}\, \bar{q}_a(t)$$

Write in matrix.

$$\begin{bmatrix} \dot{q}_a \\ 0 \end{bmatrix} = \left[\begin{array}{c|c} \hat{\omega}^s_{ab} & v^s_{ab} \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix} q_a \\ 1 \end{bmatrix} = \hat{\omega}_{ab} q_a + v^s_{ab}$$

$$= \dot{p}_{ab} + \hat{\omega}^s_{ab}(q_a - p_{ab})$$



For body frame velocity.

$$V_{q_b} = g_{ab}^{-1}\, V_{q_a}$$

$$= g_{ab}^{-1}\, \dot{g}_{ab}\, g_{ab}^{-1}\, \bar{q}_a(t)$$

$$= \underbrace{g_{ab}^{-1}\, \dot{g}_{ab}}_{\hat{V}^b_{ab}}\, q_b$$

$$V_{q_b} = \hat{V}^b_{ab}\, q_b$$

$$\hat{V}^b_{ab} = \left[\begin{array}{c|c} R_{ab}^T \dot{R}_{ab} & R_{ab}^T \dot{p}_{ab} \\ \hline 0 & 0 \end{array}\right]$$

(i.e. for the velocity of origin of {B})

1. $\hat{V}_{ab}^{a}$, $\hat{V}_{ab}^{b}$ are twists.

2. For twists, $\hat{\xi}_1 = g\hat{\xi}g^{-1} \Rightarrow \hat{V}_{ab}^{a} = g_{ab}\hat{V}_{ab}^{b}g_{ab}^{-1}$

3. For twist coordinates. $\xi_1 = Adg\,\xi \Rightarrow V_{ab}^{a} = Adg_{ab}\,V_{ab}^{b}$

4. For $Adg \Rightarrow \begin{cases} Adg_1 \cdot Adg_2 = Adg_1 \cdot g_2 \\ Adg^{-1} = (Adg)^{-1} \end{cases}$

Multiple frame velocity.

$$\hat{V}_{ac}^{a} = \dot{g}_{ac}\,g_{ac}^{-1} \qquad \hat{V}_{ac}^{c} = g_{ac}^{-1}\dot{g}_{ac} \qquad g_{ac} = g_{ab}\cdot g_{bc}$$

- Spatial Velocity

$$\hat{V}_{ac}^{a} = \frac{d}{dt}(g_{ab}\cdot g_{bc})\cdot(g_{ab}\cdot g_{bc})^{-1}$$

$$= (\dot{g}_{ab}g_{bc} + g_{ab}\dot{g}_{bc})\cdot(g_{bc}^{-1}g_{ab}^{-1})$$

$$= \dot{g}_{ab}\underbrace{g_{bc}g_{bc}^{-1}}_{I}g_{ab}^{-1} + g_{ab}\dot{g}_{bc}g_{bc}^{-1}g_{ab}^{-1}$$

$$= \hat{V}_{ab}^{a} + g_{ab}\hat{V}_{bc}^{b}g_{ab}^{-1}$$

$$V_{ac}^{a} = V_{ab}^{a} + Adg\,V_{bc}^{b}$$

- Body velocity

$$\hat{V}_{ac}^{c} = g_{bc}^{-1}V_{ab}^{b}g_{bc} + V_{bc}^{c}$$

$$V_{ac}^{c} = Adg_{bc}^{-1}V_{ab}^{b} + V_{bc}^{c}$$

If the motion is a Screw

$$g_{ab}(\theta) = e^{\hat{\xi}\theta} g_{ab}(0)$$

chain rule $\dfrac{d e^{\hat{\xi}\theta}}{dt} = \dfrac{d e^{\hat{\xi}\theta}}{d\theta} \cdot \dfrac{d\theta}{dt}$.

$$\dot{g}_{ab}(\theta) = \frac{d}{dt}\left(e^{\hat{\xi}\theta}\right) g_{ab}(0)$$

$$= \hat{\xi} e^{\hat{\xi}\theta}\, \dot{\theta}\; g_{ab}(0)$$

$$= \hat{\xi}\,\dot{\theta}\; e^{\hat{\xi}\theta} g_{ab}(0)$$

$$= \hat{\xi}\,\dot{\theta}\; g_{ab}(\theta)$$

The velocity will be :

- Spatial frame.

$$\hat{V}^{a}_{ab} = \dot{g}_{ab}\, g^{-1}_{ab} = \hat{\xi}\,\dot{\theta}$$

- Body frame.

$$\hat{V}^{b}_{ab} = g^{-1}_{ab}\, \dot{g}_{ab} = g^{-1}_{ab}(0)\,\hat{\xi}\,\dot{\theta}\, g_{ab}(0)$$

Recap: $g_{ab}(\theta) = e^{\hat{\xi}\theta} g_{ab}(0)$

$g^{-1}_{ab}(\theta) = g^{-1}_{ab}(0)\, e^{-\hat{\xi}\theta}$

Plug: $\hat{V}^{b}_{ab} = g^{-1}_{ab}(0)\, e^{-\hat{\xi}\theta}\, \hat{\xi}\,\dot{\theta}\, e^{\hat{\xi}\theta} g_{ab}(0)$

$\circledast$ $e^{At}A = Ae^{At}$

$$= g^{-1}_{ab}(0)\, \hat{\xi}\,\dot{\theta}\, g_{ab}(0)$$

$$= g^{-1}_{ab}(0)\, \hat{V}^{a}_{ab}\, g_{ab}(0)$$