Last Time

Chapter 1 Robotics History

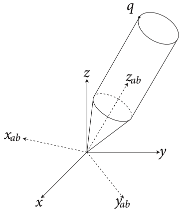
1. Robots and Robotics
2. Ancient History (3000 B.C.-1450 A.D.)
3. Early History (1451 A.D.-1960)
4. Modern History (1961-)
5. New Vistas

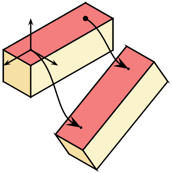
Today

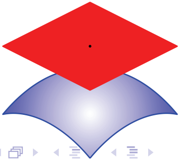
Chapter 2 Rigid Body Motion

1. Rigid Body Transformations
2. Rotational motion in $\mathbb{R}^3$

Chapter 2 Rigid Body Motion







Today

Chapter 2 Rigid Body Motion

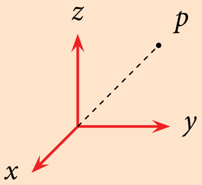
$\boxed{1}$ Rigid Body Transformations

- Length Preserving: $\left\|g(p) - g(q)\right\| = ||p - q||$
- Orientation Preserving: $g_*(v \times w) = g_*(v) \times g_*(w)$

2 Rotational motion in $\mathbb{R}^3$

2.1 Rigid Body Transformations

§ Notations:



$$p = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \text{ or } p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

For $p \in \mathbb{R}^n, n = 2, 3$(2 for planar, 3 for spatial)

Point: $p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}, \|p\| = \sqrt{p_1^2 + \cdots + p_n^2}$

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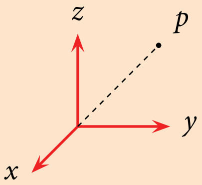

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Vector: $v = p - q = \begin{bmatrix} p_1 - q_1 \\ p_2 - q_2 \\ \vdots \\ p_n - q_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \ \|v\| = \sqrt{v_1^2 + \cdots + v_n^2}$

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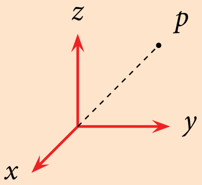

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Matrix: $A \in \mathbb{R}^{n \times m}, A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$

□ Description of point-mass motion:

$$p(0) = \begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix}$$: initial position

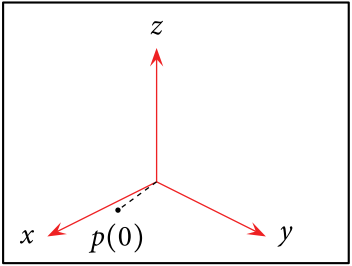

Figure 2.1

2.1 Rigid Body Transformations

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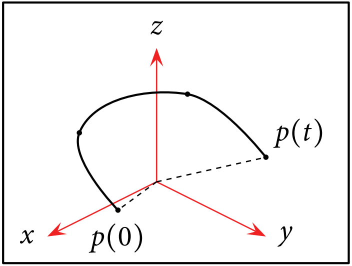

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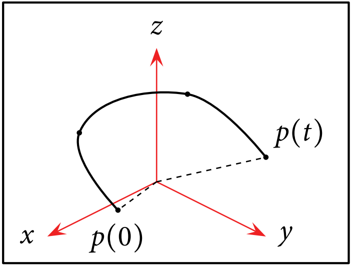

Figure 2.1

Definition: Trajectory

A **trajectory** is a curve $p : (-\varepsilon, \varepsilon) \mapsto \mathbb{R}^3, p(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$

Rigid Body Motion:

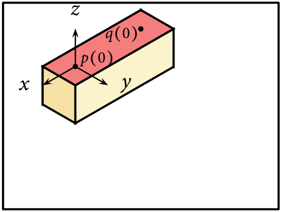

Figure 2.2

2.1 Rigid Body Transformations

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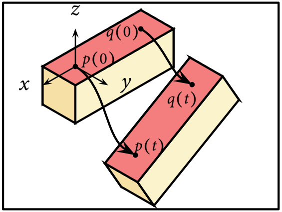

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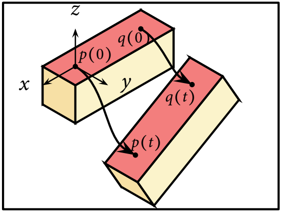

Figure 2.2

$$\|p(t) - q(t)\| = \|p(0) - q(0)\| = \text{constant}$$

Definition: Rigid body transformation

$$g : \mathbb{R}^3 \mapsto \mathbb{R}^3$$

s.t.

1. Length preserving: $\|g(p) - g(q)\| = \|p - q\|$
2. Orientation preserving: $g_*(v \times \omega) = g_*(v) \times g_*(\omega)$

† End of Section †

Today

Chapter 2 Rigid Body Motion

1 Rigid Body Transformations

- Length Preserving: $\left|\left|g(p) - g(q)\right|\right| = ||p - q||$
- Orientation Preserving: $g_*(v \times w) = g_*(v) \times g_*(w)$

2 Rotational motion in $\mathbb{R}^3$

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Chapter 2 Rigid Body Motion

1 Rigid Body Transformations

2 Rotational motion in $\mathbb{R}^3$

- Rotation Matrix
  - Represents *configuration*
  - Represents *(rotational) transformation*
- Rotation Matrices with matrix multiplication form a *Group*
- Rotational Transformation is a *Rigid Body Transformation*

□ Rotational Motion:

1 Choose a reference frame A (spatial frame)

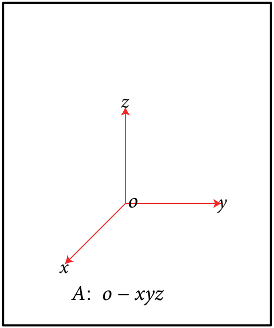

Figure 2.3

☐ Rotational Motion:

1 Choose a reference frame A (spatial frame)

2 Attach a frame B to the body (body frame)

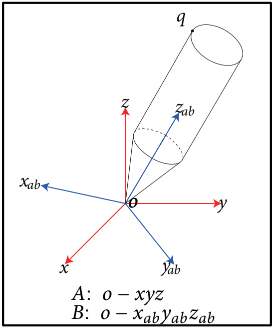

Figure 2.3

$$
\begin{aligned}
x_{ab} \in \mathbb{R}^3 &: \quad \text{coordinates of } x_b \text{ in frame } A \\
R_{ab} = [x_{ab}\ y_{ab}\ z_{ab}] \in \mathbb{R}^{3\times 3} &: \quad \text{Rotation (or orientation) matrix of } B \text{ w.r.t. } A
\end{aligned}
$$

□ Property of a Rotation Matrix:

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$$\det(R^T R) = \det R^T \cdot \det R = (\det R)^2 = 1, \det R = \pm 1$$

As $\det R = r_1^T(r_2 \times r_3) = 1 \Rightarrow \det R = 1$

2.2 Rotational Motion in $\mathbb{R}^3$

Definition:

$$SO(3) = \left\{ R \in \mathbb{R}^{3\times 3} \,\middle|\, R^T R = I, \det R = 1 \right\}$$

and

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4. $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$

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◇ Review: Examples of group

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$\square$

2.2 Rotational Motion in $\mathbb{R}^3$

Configuration and rigid transformation:

- $R_{ab} = [x_{ab}\ y_{ab}\ z_{ab}] \in SO(3)$
  Configuration Space

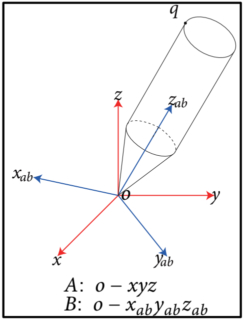

Figure 2.3

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- $R_{ab} = [x_{ab}\ y_{ab}\ z_{ab}] \in SO(3)$
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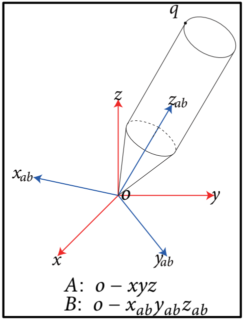
- Let $q_b = \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} \in \mathbb{R}^3$: coordinates of q in B.

$$q_a = x_{ab} \cdot x_b + y_{ab} \cdot y_b + z_{ab} \cdot z_b$$

$$= [x_{ab}\ y_{ab}\ z_{ab}] \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = R_{ab} \cdot q_b$$



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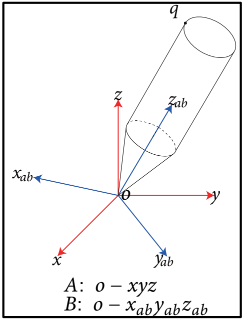

Figure 2.3

- A configuration $R_{ab} \in SO(3)$ is also a transformation:

$$R_{ab} : \mathbb{R}^3 \to \mathbb{R}^3, R_{ab}(q_b) = R_{ab} \cdot q_b = q_a$$

A config. $\Leftrightarrow$ A transformation in $SO(3)$

2.2 Rotational Motion in $\mathbb{R}^3$

Property 2: R_{ab} preserves distance between points and orientation.

1. $\|R_{ab} \cdot (p_b - p_a)\| = \|p_b - p_a\|$
2. $R(v \times \omega) = (Rv) \times R\omega$

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Proof :

For $a \in \mathbb{R}^3$, let $\hat{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$

Note that $\hat{a} \cdot b = a \times b$

1 follows from
$$\|R_{ab}(p_b - p_a)\|^2 = (R_{ab}(p_b - p_a))^T R_{ab}(p_b - p_a) = (p_b - p_a)^T R_{ab}^T R_{ab}(p_b - p_a) = \|p_b - p_a\|^2$$

2 follows from $R\hat{v}R^T = (Rv)^\wedge$ (prove it yourself) $\square$

□ Parametrization of $SO(3)$ (the exponential coordinate):

♢ **Review:** $S^1 = \{z \in \mathbb{C} \| z| = 1\}$

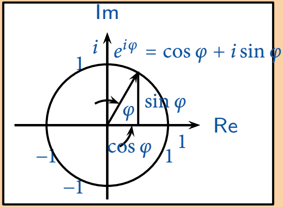

Figure 2.4

Euler's Formula

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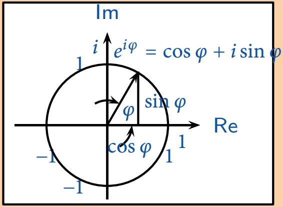

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◇ Review:

$$\begin{cases} \dot{x}(t) = ax(t) \\ x(0) = x_0 \end{cases} \Rightarrow x(t) = e^{at}x_0$$

(Continues next slide)

2.2 Rotational Motion in $\mathbb{R}^3$

$$R \in SO(3), R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$r_i \cdot r_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \quad \leftarrow \text{6 constraints}$$

$\Rightarrow$ 3 independent parameters!

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Consider motion of a point q on a rotating link

$$\begin{cases} \dot{q}(t) = \omega \times q(t) = \hat{\omega} q(t) \\ q(0)\text{: Initial coordinates} \end{cases}$$

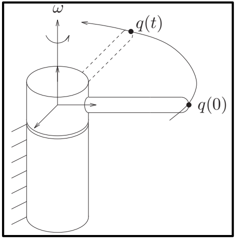

Figure 2.5

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$$\Rightarrow q(t) = e^{\hat{\omega} t} q_0 \text{ where } e^{\hat{\omega} t} = I + \hat{\omega} t + \frac{(\hat{\omega} t)^2}{2!} + \frac{(\hat{\omega} t)^3}{3!} + \cdots$$

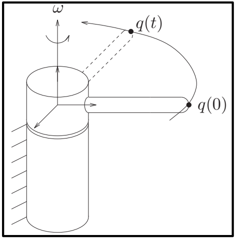


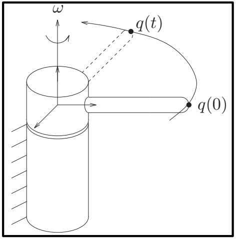
Figure 2.5

2.2 Rotational Motion in $\mathbb{R}^3$

$$R \in SO(3), R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$r_i \cdot r_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \quad \leftarrow \text{6 constraints}$$

$\Rightarrow$ 3 independent parameters!



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By the definition of rigid transformation, $R(\omega, \theta) = e^{\hat{\omega}\theta}$. Let $so(3) = \{\hat{\omega} | \omega \in \mathbb{R}^3\}$ or $so(n) = \{S \in \mathbb{R}^{n \times n} | S^T = -S\}$ where $\wedge : \mathbb{R}^3 \mapsto so(3) : \omega \mapsto \hat{\omega}$, we have:

Property 3: $\exp : so(3) \mapsto SO(3), \hat{\omega}\theta \mapsto e^{\hat{\omega}\theta}$

2.2 Rotational Motion in $\mathbb{R}^3$

Rodrigues' formula ($\|\omega\| = 1$):

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1 - \cos\theta)$$

Today

Chapter 2 Rigid Body Motion

1 Rigid Body Transformations

- Length Preserving: $||g(p) - g(q)|| = ||p - q||$
- Orientation Preserving: $g_*(v \times w) = g_*(v) \times g_*(w)$

2 Rotational motion in $\mathbb{R}^3$

- Rotation Matrix
  - Represents *configuration*
  - Represents *(rotational) transformation*
- Rotation Matrices with matrix multiplication form a *Group*
- Rotational Transformation is a *Rigid Body Transformation*