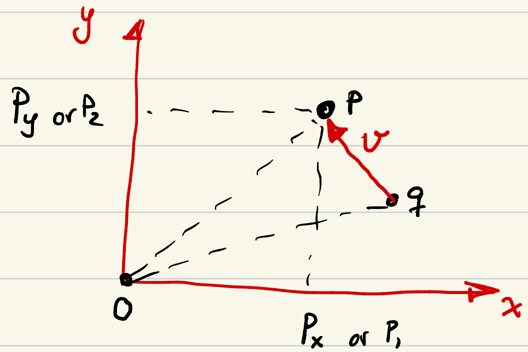
① 2 or 3 dimensional vector space

2-Dimensions



Points

$$p = \begin{bmatrix} p_x \\ p_y \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

$$q = \begin{bmatrix} q_x \\ q_y \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

vector : diference between two points

$$v = p - q = \begin{bmatrix} p_1 - q_1 \\ p_2 - q_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Magnitude - (norm) of a vector (or a point)

$$\| v \| = \sqrt{v_1^2 + v_2^2}$$

$$\| p \| = \sqrt{p_1^2 + p_2^2}$$

Matrix:

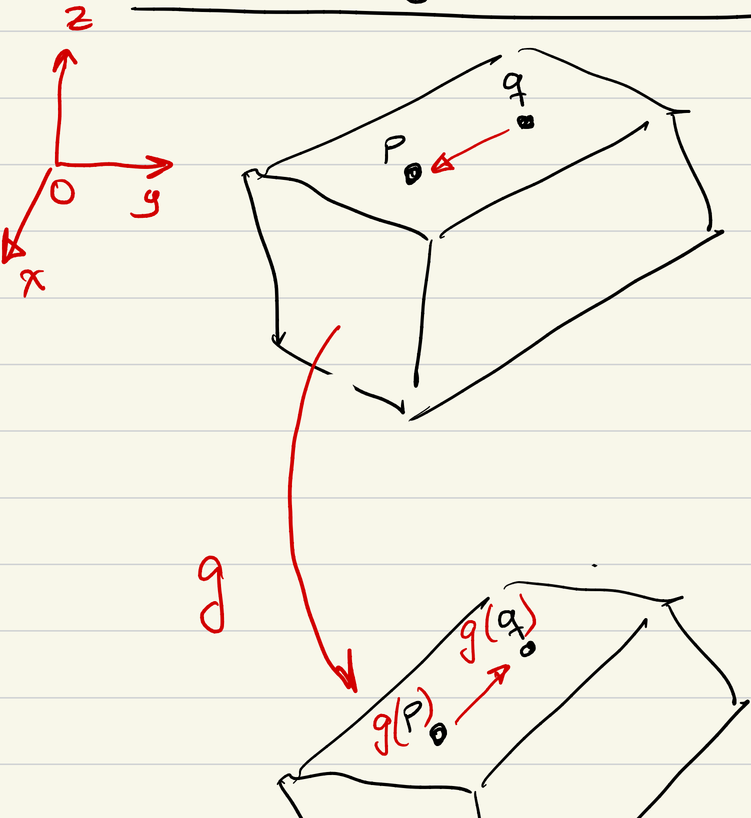
$$M \overset{\text{belongs}}{\in} \mathbb{R}^{n \times m}$$

$$M = \underbrace{\begin{bmatrix} m_{11} & \cdots & m_{1m} \\ \vdots & & \\ m_{n1} & \cdots & m_{nm} \end{bmatrix}}_{m \text{ columns}} \left.\vphantom{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}\right\} \begin{array}{l} n \\ \text{rows} \end{array}$$

$$f : \mathbb{R}_+ \longrightarrow \mathbb{R}^3$$

$$f(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix}$$

Rigid Body Transformation :



$g(p)$: position of point p after trans.

before

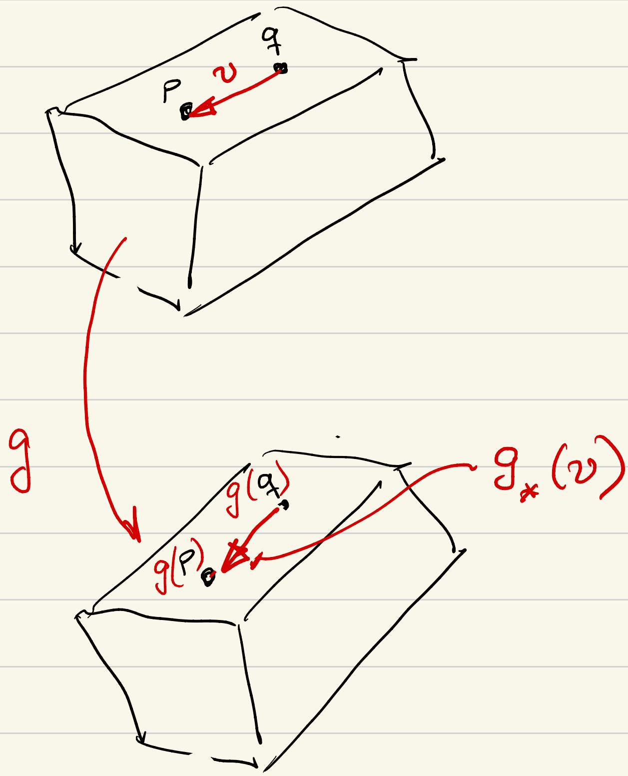
after

$$\boxed{\|p-q\| = \|g(p)-g(q)\|}$$

However :

$$p-q \neq g(p)-g(q)$$

Rigid body transformation



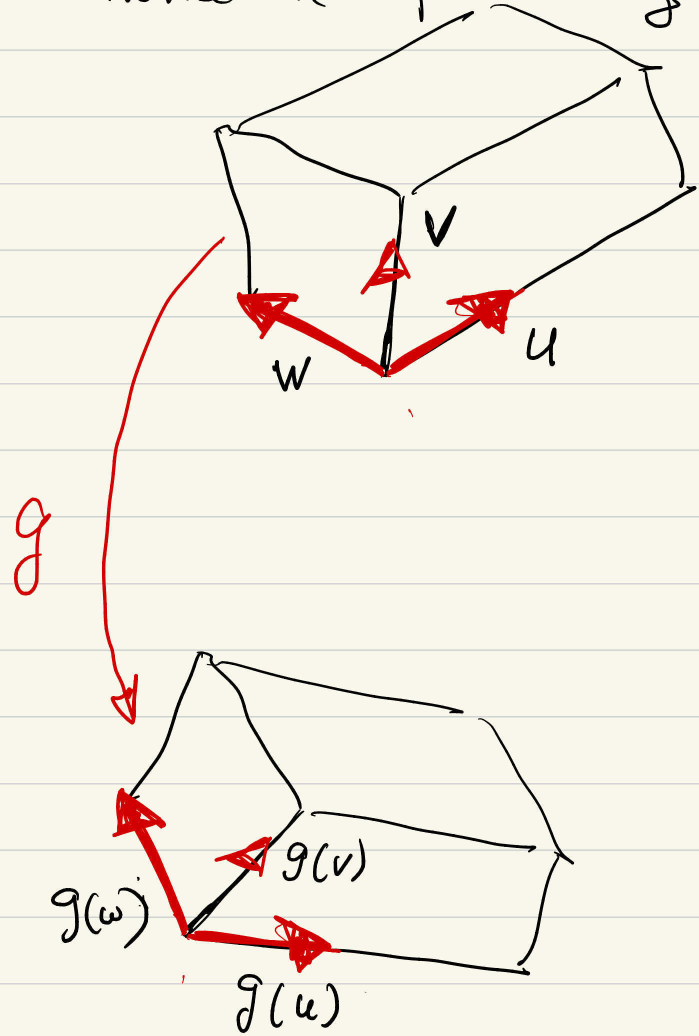
$$v = p - q$$

$$g_*(v) = g(p) - g(q)$$

I will drop the * subscript and simply use

$$g(v) = g(p-q) = g(p) - g(q)$$

Orientation - preserving



$$w = u \times v$$

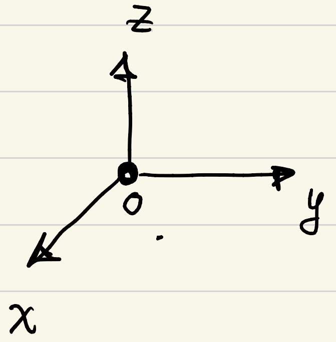
$$g(w) = g(u) \times g(v)$$

$$\Rightarrow \boxed{g(u \times v) = g(u) \times g(v)}$$

Frame:

Origin : point

3 - orthonormal vectors



$$\|x\| = 1$$
$$\|z\| = 1$$
$$\|y\| = 1$$

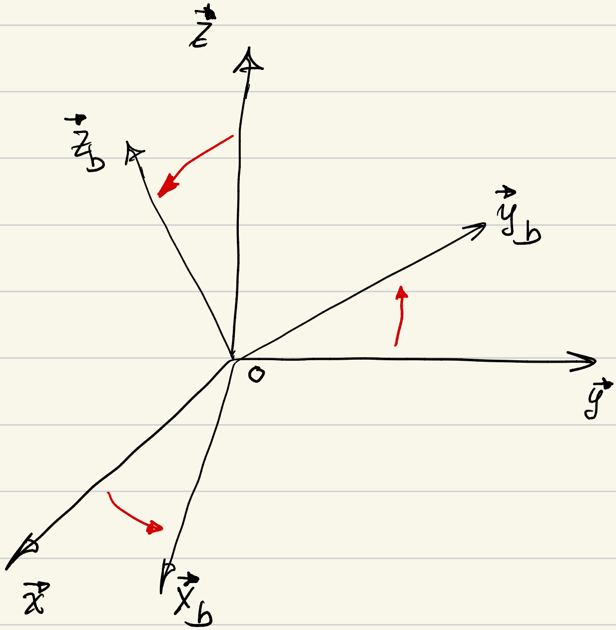
$$x^T y = 0$$
$$x^T z = 0$$
$$y^T z = 0$$

$$\left.\begin{aligned} x \times y &= z \\ y \times z &= x \\ z \times x &= y \end{aligned}\right\} \quad \text{Cross product}$$

Two Frames : A : $O - [x, y\ z]$

B : $O - [x_{ab}, y_{ab}, z_{ab}]$

Frame B is attached to a "rigid body" which is $\boxed{\text{rotated}}$



Express each of the axes of frame {B} relative to frame {A}

$$\vec{x}_b = x_{ab1}\vec{x}_a + x_{ab2}\vec{y}_a + x_{ab3}\vec{z}_a$$

$$X_{ab} = \begin{bmatrix} x_{ab1} \\ x_{ab2} \\ x_{ab3} \end{bmatrix}$$: Coordinates of $\vec{x}_b$ represented relative to frame $\{A\}$

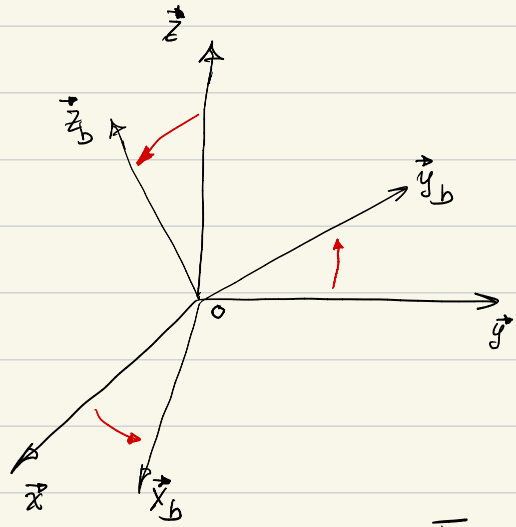
Similarly, we express $\vec{y}_b$ & $\vec{z}_b$ reative to frame $\{A\}$

$$R_{ab} = \left[X_{ab} \middle| y_{ab} \middle| z_{ab} \right] \in \mathbb{R}^{3\times3} \quad 3\times3 \text{ Matrix}$$

R_{ab} is the rotation matrix that "rotates"

a frame, initially alligned with frame $\{A\}$ to frame $\{B\}$

Notice



$$R_{ab} = \left[x_{ab} \;\vdots\; y_{ab} \;\vdots\; z_{ab} \right] = \begin{bmatrix} x_{ab1} & y_{ab1} & z_{ab1} \\ x_{ab2} & y_{ab2} & z_{ab2} \\ x_{ab3} & y_{ab3} & z_{ab3} \end{bmatrix}$$

x_{ab}, y_{ab} & z_{ab} are all orthonormal vectors

$$\left.\begin{aligned} x_{ab}^T x_{ab} &= 1 \\ y_{ab}^T y_{ab} &= 1 \\ z_{ab}^T z_{ab} &= 1 \end{aligned}\right\}$$

$$\left.\begin{aligned} x_{ab}^T y_{ab} &= 0 \\ x_{ab}^T z_{ab} &= 0 \\ y_{ab}^T z_{ab} &= 0 \end{aligned}\right\} \text{orthogonal}$$

$$\Rightarrow \quad R_{ab}^T R_{ab} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Moreover $R_{ab} R_{ab}^T = I$

R_{ab} is the rotation matrix

it rotates a frame, initially alligned with frame $\{A\}$ into frame $\{B\}$

Properties of Rotation Matrices

$$R = [\, r_1 \;\; r_2 \;\; r_3 \,] \in \mathbb{R}^{3\times 3}$$

① $R^T R = I$ & $R R^T = I$

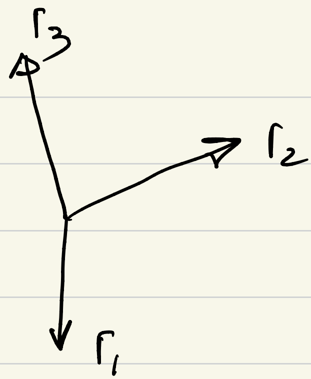
$$\det(R^T R) = \det(I) = 1$$

$$\Rightarrow \det(R^T) \cdot \det(R) = 1$$

but $\det(R^T) = \det(R)$

$$\Rightarrow \det(R)^2 = 1$$

$$\Rightarrow \boxed{\det(R) = +1 \text{ or } -1}$$



Notice that $r_2 \times r_3 = r_1$

& $r_1^T (r_2 \times r_3) = r_1^T r_1 = 1$

But $\det(R) = r_1^T (r_2 \times r_3)$

$\Rightarrow \boxed{\det(R) = 1}$

$SO(3)$: special orthogonal group of dimension (order) 3

aka the rotation $\boxed{\text{group.}}$

Properties of a group : G

1. There exists a binary operation $(\cdot)$ such that, if g_1 & $g_2 \in G$, then

$$g_1 \cdot g_2 \in G$$

(2) There exists an identity element (e) such that, for any $g \in G$

$$e \cdot g = g = g \circ e$$

(3) For any element $g \in G$

$\exists$ an inverse g^{-1} such that

$$g^{-1} \cdot g = e = g \circ g^{-1}$$

4 for any $g_1, g_2, g_3 \in G$

$$g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$$

$SO(3)$ satisfies all those properties with the binary operation being matrix multiplication

① $R_1^T R_1 = I$ & $R_2^T R_2 = I$

$$\Rightarrow (R_1 \cdot R_2)^T (R_1 \cdot R_2) =$$

$$\underbrace{R_2^T \cdot \underbrace{R_1^T \cdot R_1}_{I} \cdot R_2}_{I} = I$$

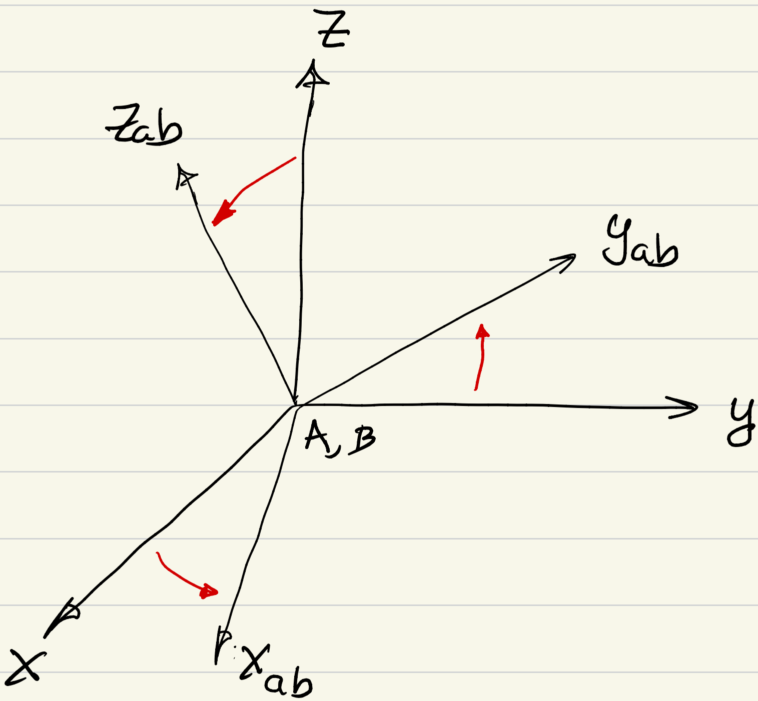
② $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in SO(3)$

③ For any $R \in SO(3)$, $\exists$

$R^{-1} = R^T$ such that $R^T R = I$

④ Verify by yourself.

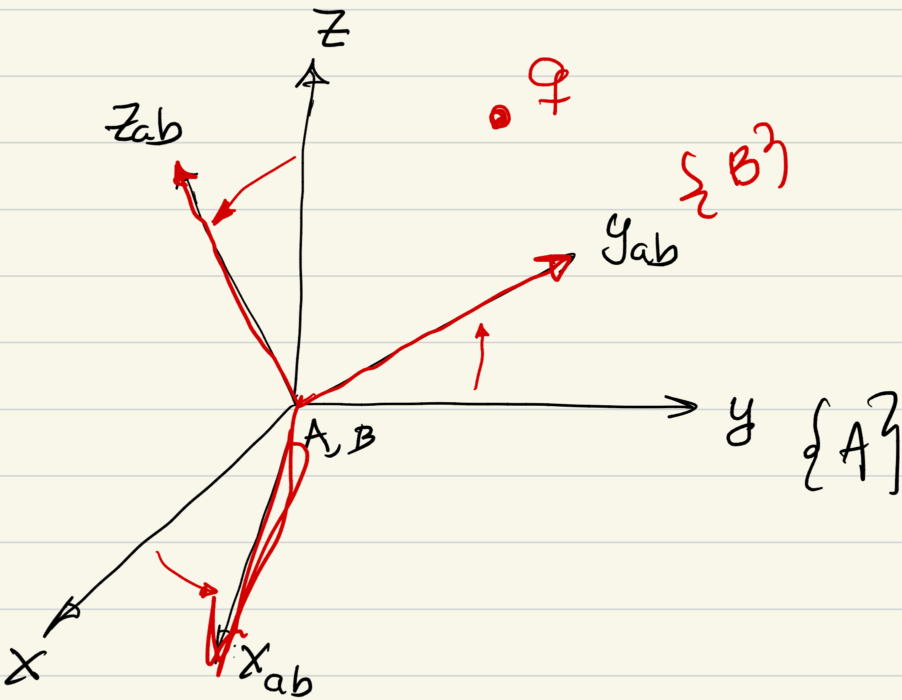
Rotation Matrices are also coordinate transformation matrices.



$$R_{ab} = \left[X_{ab} \,\vdots\, y_{ab} \,\vdots\, Z_{ab} \right]$$

$$X_{ab} = \begin{bmatrix} X_{ab1} \\ X_{ab2} \\ X_{ab3} \end{bmatrix}$$: components of X_b in terms of frame $\{A\}$

Consider a point q



q_b : coordinates of point q relative to frame $\{B\}$

$$q_b = \begin{bmatrix} q_{bx} \\ q_{by} \\ q_{bz} \end{bmatrix}_{\{B\}}$$

Now:

$$q_a = q_{bx} \cdot \vec{x}_{ab} + q_{by} \cdot \vec{y}_{ab} + q_{bz} \cdot \vec{z}_{ab}$$

or

$$\underline{q_a} = \underbrace{\begin{bmatrix} x_{ab} & y_{ab} & z_{ab} \end{bmatrix}}_{R_{ab}} \begin{bmatrix} q_{bx} \\ q_{by} \\ q_{bz} \end{bmatrix}$$

$$\begin{bmatrix} q_{ax} \\ q_{ay} \\ q_{az} \end{bmatrix} = \begin{bmatrix} & & \\ & R_{ab} & \\ & & \end{bmatrix} \begin{bmatrix} q_{bx} \\ q_{by} \\ q_{bz} \end{bmatrix}$$

$$\Rightarrow \quad \boxed{q_a = R_{ab} \cdot q_b}$$

coordinates of q relative to frame {B}

Coordinate transformation matrix from frame {B} to frame {A}

Coordinates of q relative to frame {A}

Properties of R

① $\| R(p-q) \| = \| p-q \|$

Proof :

$$\left(R(p-q)\right)^T \left(R(p-q)\right) = (p-q)^T \underbrace{R^T R}_{I} (p-q)$$

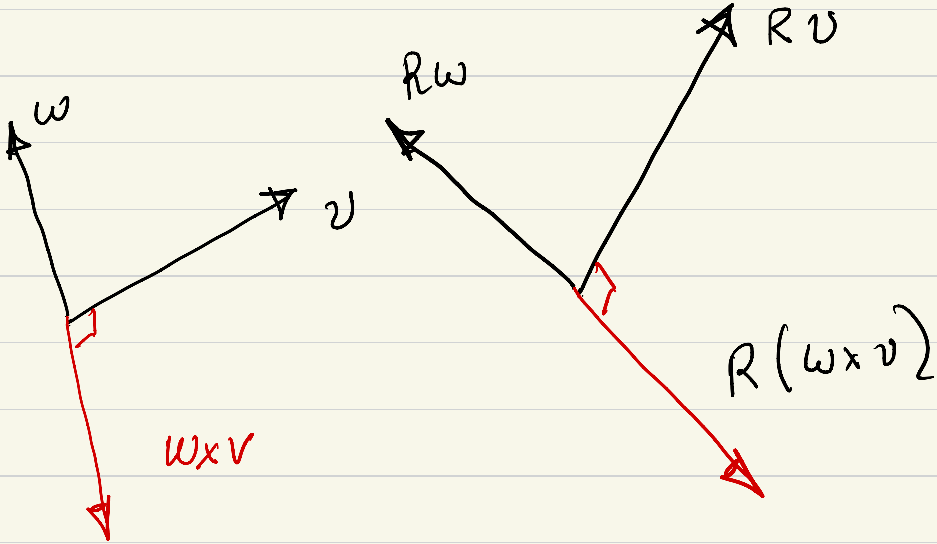
$$= (p-q)^T (p-q)$$

$$\Rightarrow \quad \| R(p-q) \|^2 = \| p-q \|^2$$

$$\Rightarrow \quad \| R(p-q) \| = \| p-q \|$$

(2)

$$R(\omega \times v) = R(\omega) \times R(v)$$



lets talk about the cross product

$$\omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \qquad v = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

$$\omega \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \begin{array}{cc} \begin{bmatrix} \omega_y v_z - \omega_z v_y \\ \omega_z v_x - \omega_x v_z \\ \omega_x v_y - \omega_y v_x \end{bmatrix} & \begin{array}{c} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \end{array}$$

$$\omega \times v = \underbrace{\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}}_{\hat{\omega}\ :\ \text{cross product matrix}} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

if $\omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & -\omega_x & 0 \end{bmatrix}$$

Now:

$$\boxed{\hat{\omega}^T = -\hat{\omega}}$$

skew symmetric matrix

$$\hat{\omega} \in \underbrace{so}(3)$$

"little" so(3)

"lower case"

$\wedge$: hat function
Converts a vector in $\mathbb{R}^3$
into a matrix in $so(3)$

$\vee$: vee function
Converts a matrix in
$so(3)$ into a vector in $\mathbb{R}^3$