We have a state

describe how this point moves with its dynamics.

$\dot{x} = \frac{d}{dt} x$ = some function $= f(x)$ if we don't have any input.

$= f(x, u)$ if we have input, will be state+input.

$\Updownarrow$

$f(x(t), u(t))$

Note that, the form will not be arbitary.

but $\dot{x}(t) = \underline{A(t)}\, x(t) + \underline{B(t)}\, u(t) = [A\ B] \begin{bmatrix} x \\ u \end{bmatrix}$

matrix

if A, B not change over time → Linear Time-Invariant LTI

Global Asymptotic Stability.

A property of a dynamical system where, regardless of the initial condition, all solutions eventually converge to a single equillibrium points as $t \to \infty$

$\dot{x} = Ax \Rightarrow x = x(0)\, e^{At} \Rightarrow A = \Lambda V \Lambda^{-1}$

$e^{At} = \Lambda e^{Vt} \Lambda^{-1}$

Equilibrium point

$e^{Vt} \to 0$ as $t \to \infty$ when all λ of A is negative

$\dot{x}(t) = 0$ as next time point still be 0.

as $V = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \ddots \end{bmatrix}$

Controllability

Intuition:

If the system is not stable as written in last page,
or if we want to force it to follow a particular trajectory
Control Input u is used to change how dynamics behave.

Questions:

1. Whether we can use u (input) to drive x where we want in a finite time

not necessarily right away.

Use controllability matrix Q:

$$Q = [B \ AB \ A^2B \ \cdots \ A^{n-1}B] \in R^{n \times nm};$$

$$A \in R^{n \times n} \quad B \in R^{n \times m}$$

$\text{Rank}(Q) = n \Rightarrow$ completely controllable. $\Rightarrow$ any trajectory.

rank(B) = n is sufficient forall

why that?

$$x[t+1] = A(x[t]) + B(u[t])$$

$$x[t+2] = A(x[t+1]) + B(u[t+1])$$

$$= A(A\,x[t] + B\,u[t]) + B\,u[t+1]$$

$$= A^2 x[t] + AB\,u[t] + B\,u[t+1]$$

$$x[t+3] = A(x[t+2]) + B(u[t+2])$$

$$= A(A^2 x[t] + AB\,u[t] + B\,u[t+1]) + B\,u[t+2]$$

$$= A^3 x[t] + A^2 B\,u[t] + AB\,u[t+1] + B\,u[t+2]$$

$$x[t+n] = A^n x[t] + A^{n-1} B\,u[t] + \cdots + AB\,u[t+n-2] + B\,u[t+n-1]$$

$$= A^n x[t] + Q\,[u[t+n-1], \cdots u[t]]^T$$

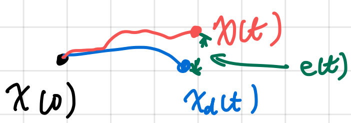
if $Q = [B \; AB \; A^2B \cdots A^{n-1}B]$ full rank,

it means all $u[t \cdots]$ actually do something

Choose actual control input.

error $e(t) = x(t) - x_d(t)$ how far off we are from our desired point.

$\dot{e} = \dot{x} - \dot{x}_d$



$= Ax + Bu - \dot{x}_d$

$= A(e + x_d(t)) + Bu - \dot{x}_d$

$= Ae + Bu + (Ax_d(t) - \dot{x}_d)$

$\Rightarrow \dot{e} = Ae(t) + Bu + (Ax_d(t) - \dot{x}_d)$

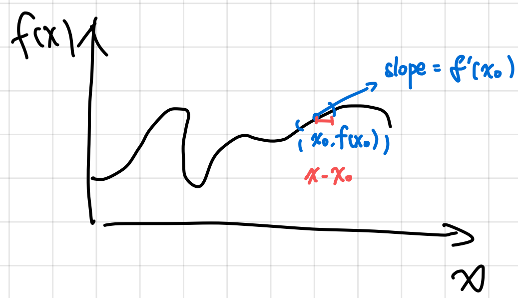
choose u so that $e(t)$ goes to 0, $u(t) = -K_p e(t)$

$\Rightarrow \dot{e} = Ae(t) + B(-K_p e(t)) + Ax_d(t) - \dot{x}_d$

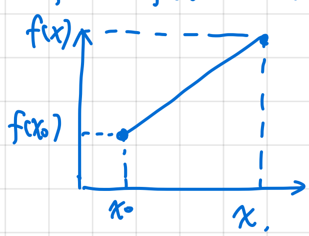
$= \underline{(A - BK_p)}e(t) + Ax_d(t) - \dot{x}_d$

eigenvalue $re(\lambda(A - BK_p)) < 0 \Rightarrow e(t) \to 0$ as $t \to \infty$

Linearization



$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$



$f'(x_0) = \frac{f(x) - f(x_0)}{x - x_0}$

where $x = x_0 + \Delta x$

Scale up to multiple dimension x

$f(x) = f(x_0) + \frac{\partial f}{\partial x_i}(x - x_0)$

$$\frac{\partial \vec{f}}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & & & \vdots \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Scale up to input & state

$f(x, u) = f(x_0, u_0) + \frac{\partial f}{\partial x_i}(x - x_0) + \frac{\partial f}{\partial u}(u - u_0)$

$\Rightarrow \dot{x} = f(x, u) = \begin{bmatrix} f_1(x, u) \\ f_2 \\ \vdots \end{bmatrix}$