Spatial Velocity

$$V_{q_a} = \dot{q}_a(t) = \hat{V}^s_{ab}\, q_a = \dot{g}_{ab}(\cancel{t})\, g^{-1}_{ab}(t)\, q_a$$

$$\hat{V}^s_{ab} = \hat{\xi} = \begin{bmatrix} \hat{w} & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \dot{R}R^T & -\dot{R}R^T p + \dot{p} \\ 0 & 0 \end{bmatrix}$$

$$V^s_{ab} = \xi = \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} -\dot{R}R^T p + \dot{p} \\ (\dot{R}R^T)^\vee \end{bmatrix}$$

A point attached to the body moving around, determine the rate of position change respect to the spatial frame.

Body Velocity

$$V_{q_b} = g^{-1}_{ab}(t)\, V_{q_a} = g^{-1}_{ab}(t)\, \dot{g}_{ab}(\cancel{t})\, q_b$$

$$\hat{V}^b_{ab} = \hat{\xi} = \begin{bmatrix} \hat{w} & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R^T\dot{R} & R^T\dot{p} \\ 0 & 0 \end{bmatrix}$$

$$V^b_{ab} = \xi = \begin{bmatrix} R^T\dot{p} \\ (R^T\dot{R})^\vee \end{bmatrix}$$

Still the same time derivative (velocity) of point q relative to spatial frame, but expressed in body frame

Solution Tips

1. when determine spatial/body velocities,

① get $g_{ab}(t)$ and $g^{-1}_{ab}(t)$ first;

② then, ✱ derivative $\underline{\dot{g}_{ab}(t)}$;

note that $y = \sin\theta(t)$ $\quad \frac{dy}{dt} = \frac{dy}{d\theta}\frac{d\theta}{dt} = (\cos\theta)\underline{\dot{\theta}}$

2. $$\hat{w} = \widehat{\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}$$

3. Circle method to validate

That's how joints affect frames.

非 robot arm/joints, 直接看, 注意拆分 rotate & translate

Adjoint Transformation

$$Ad_{g_{ab}} = \begin{bmatrix} R_{ab} & \hat{p}_{ab} R_{ab} \\ 0 & R_{ab} \end{bmatrix}$$

$$Ad_{g^{-1}_{ab}} = \begin{bmatrix} R^T_{ab} & -R^T_{ab}\hat{p}_{ab} \\ 0 & R^T_{ab} \end{bmatrix}$$

$$V^a_{ab} = Ad_{g_{ab}} V^b_{ab} = \begin{bmatrix} R_{ab} & \hat{p}_{ab} R_{ab} \\ 0 & R_{ab} \end{bmatrix}\begin{bmatrix} v^b_{ab} \\ w^b_{ab} \end{bmatrix} = \begin{bmatrix} Rv + \hat{p}Rw \\ Rw \end{bmatrix}$$

$$\hat{V}^a_{ab} = g_{ab}\, \hat{V}^b_{ab}\, g^{-1}_{ab}$$

$$V^a_{ac} = V^a_{ab} + Ad_{g_{ab}} V^b_{bc}$$

$$V^c_{ac} = Ad_{g^{-1}_{bc}} V^b_{ab} + V^c_{bc}$$

$$(Ad_g)^{-1} = Ad_{g^{-1}}$$

$$Ad_{g_1 g_2} = Ad_{g_1} Ad_{g_2}$$

Useful formula

$$\hat{\xi}_a = g_{ab}\, \hat{\xi}_b\, g^{-1}_{ab}$$

$$\xi_a = Ad_{g_{ab}}\, \xi_b$$

$$g^{-1}_{ab} = \begin{bmatrix} R^T_{ab} & -R^T_{ab}\, p \\ 0 & 1 \end{bmatrix}$$

Spatial Jacobian

$$V_{ST}^{S} = J_{ST}^{S}(\theta)\,\dot{\theta}$$

$$J_{ST}^{S}(\theta) = \left[\left(\frac{\partial g_{ST}}{\partial \theta_1} \cdot g_{ST}^{-1}\right)^{\vee}, \ \cdots, \ \left(\frac{\partial g_{ST}}{\partial \theta_n} \cdot g_{ST}^{-1}\right)^{\vee}\right]$$

$$= \left[\xi_1, \ \xi_2', \ \cdots \ \xi_n'\right]$$

$$\xi_i' = Ad_{\left(e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_{i-1}\theta_{i-1}}\right)}\,\xi_i$$

Convert

$$J_{ST}^{S}(\theta) = Ad_{g_{ST}(\theta)}\, J_{ST}^{T}(\theta)$$

Tips

Tool frame is fixed to the tool, it will change when tool changes.

Body Jacobian

$$V_{ST}^{T} = J_{ST}^{T}(\theta)\,\dot{\theta}$$

$$J_{ST}^{T}(\theta) = \left[\left(g_{ST}^{-1} \cdot \frac{\partial g_{ST}}{\partial \theta_1}\right)^{\vee}, \ \cdots \ \left(g_{ST}^{-1} \frac{\partial g_{ST}}{\partial \theta_n}\right)^{\vee}\right] = \left[\xi_1^{\dagger}, \ \xi_2^{\dagger}, \ \cdots \ \xi_n^{\dagger}\right]$$

$$\xi_i^{\dagger} = Ad_{\left(e^{\hat{\xi}_{i+1}\theta_{i+1}} \cdots e^{\hat{\xi}_n\theta_n} g_{ST}(0)\right)}\,\xi_i$$

Singularity

rank (J) < joint numbers

① 2 collinear revolute joints
② 3 parallel coplanar r joints
③ 4 intersected revolute joints

1. Rotation R is achieved by intrinsic rotations about the z, y, x. The spatial Jacobian for the rotation has a singularity when $\theta_y = \frac{\pi}{2}$

2. Any rotation about the linearly independent axes will also have a singularity.

   ※ when rotate about w_3, w_1 will not change, only rotate W_3.

3. When 4 revolute joint axes are coplanar, any six degree of freedom manipulator is at a singular configuration.

4. manipulability measure: how close we're encountering singularities.

   $\mu(\theta) = \prod_{i=1} \sigma_i(\theta)$ ← singular value of J^S

J^S singular $\Rightarrow$ non-trival $\underset{K\text{ dimensional}}{\underline{\text{null space}}}$ $\Rightarrow (J^S)^T J^S$ singular, null space k-d

$\Downarrow$

at least one eigenvale $= 0$

$\Downarrow$

$\sigma_i = 0$

计算 singular value.
① Calculate A^TA
② 求 A^TA eigenvalue
③ singular value = A^TA 的非负 eigenvalue
降序

Dynamics

Steps:

① Pick generalized coordinates q minimum info to describe system. only consider changing things
(position / angle)

② Find KE, PE.

③ Take Lagrangian $L = T - V$ and find its derivatives.

$$T = \frac{1}{2}[v^T \ w^T]\begin{bmatrix} mI_3 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} v \\ w \end{bmatrix} = \frac{1}{2}[v^T \ w^T]\begin{bmatrix} mv \\ Iw \end{bmatrix} = \underbrace{\frac{1}{2}mv^Tv}_{\text{trans KE}} + \underbrace{\frac{1}{2}w^TIw}_{\text{rotation KE}}$$

④ $$r = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) - \frac{\partial L}{\partial q}$$

$$V_g = mgh \qquad V_s = \frac{1}{2}kx^2$$

Frictions forces. Coriolis Matrix $C(q)$

$M(q)$ Inertial Matrix

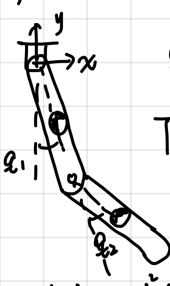
$G(q)$

$$m\ddot x - mg + kx + 0\dot x = 0$$

equation of motion of the system

$$\text{☆ } I_{total} = I_{rotation} + m \cdot l^2$$

⑥



$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$

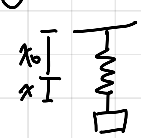
$$T = \frac{1}{2}mV_1^2 + \frac{1}{2}I_1\dot q_1^2 + \frac{1}{2}mV_2^2 + \frac{1}{2}I_2(\dot q_1 + \dot q_2)^2$$

$$V_1^2 = \left(-\frac{L}{2}\dot q_1 \sin q_1\right)^2 + \left(\frac{L}{2}\dot q_1\cos q_1\right)^2 = \frac{L^2}{4}\dot q_1^2$$

$$V_2^2 = \left(L\sin q_1 + \frac{L}{2}\sin(q_1+q_2)\right)^2 + \left(-L\cos q_1 - \frac{L}{2}\cos(q_1+q_2)\right)^2$$

$$V = -\frac{L}{2}mg\cdot\cos q_1 - mg\left(L\cos q_1 + \frac{L}{2}\cos(q_1+q_2)\right)$$

①



$q = [x]$

$L = \frac{1}{2}m\dot x^2$

$V = \frac{1}{2}kx^2 - mgx$

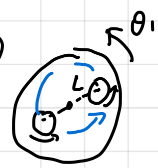
②

$q = [x]$

$L = \frac{1}{2}m\dot x^2$

$V = \frac{1}{2}kx^2 + mgx\cdot\sin\theta$

③

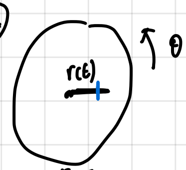


$q = [\theta_1 \theta_2 \theta_3]^T$

$$L = 0 + \frac{1}{2}I\dot\theta_1^2 + \frac{1}{2}m_2V_2^2 + \frac{1}{2}m_2\dot\theta_2^2 + \frac{1}{2}m_3V_3^2 + \frac{1}{2}m_3\dot\theta_3^2 \qquad V = 0$$

$$V_2^2 = \dot x_2^2 + \dot y_2^2 = (L\cos\theta_1)^2 + (L\sin\theta_1)^2 = (-L\sin\theta_1\cdot\dot\theta_1)^2 + (L\cos\theta_1\,\dot\theta_1)^2 = L^2\dot\theta_1^2$$

④

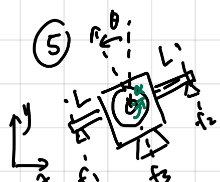


$q = \begin{bmatrix} r \\ \theta \end{bmatrix}$

$$L = 0 + \frac{1}{2}I\dot\theta_1^2 + \frac{1}{2}mv^2$$

$$V = \dot x^2 + \dot y^2 = (r(t)\cos\theta)^2 + (r(t)\sin\theta)^2 = (\dot r\cos\theta - r(t)\sin\theta\,\dot\theta)^2 + (\dot r\sin\theta + r(t)\cos\theta\,\dot\theta)^2 = \dot r^2 + \dot r(t)\dot\theta$$

⑤



$q = \begin{bmatrix} x \\ y \\ \theta \\ \alpha \end{bmatrix}$

$$L = \frac{1}{2}m_b(\dot x^2 + \dot y^2) + \frac{1}{2}I_b\dot\theta^2 + \frac{1}{2}m_r(\dot x^2 + \dot y^2) + \frac{1}{2}I_r(\dot\theta + \dot\alpha)^2$$

$\tau_x = -(f_1 + f_2 + f_3)\sin\theta$

$\tau_y = (f_1 + f_2 + f_3)\cos\theta$

$\tau_\theta = (f_2 - f_1)L - \tau$

$\tau_\alpha = \tau$

A point of mass

$$T = \frac{1}{2}m\|v\|^2$$

2D

$$T = \frac{1}{2}m\|v\|^2 + \frac{1}{2}\underbrace{I\dot\theta^2}_{\text{scalar}}$$

3D

$$T = \frac{1}{2}m\|v^2\| + \frac{1}{2}w^T I w$$

vector

matrix.

$$M = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

$$M_{box} = \begin{bmatrix} \frac{m}{12}(w^2+h^2) & 0 & 0 \\ 0 & \frac{m}{12}(l^2+h^2) & 0 \\ 0 & 0 & \frac{m}{12}(l^2+w^2) \end{bmatrix}$$

Stability:

$\dot{x} = Ax(t)$

if $re(\lambda(A)) < 0$, then $x(t) \to 0$ as $t \to \infty$

equilibrium point: $\dot{x} = 0$

Controllability:

$\dot{x}(t) = Ax(t) + Bu(t)$

$Q = [B \;\; AB \;\; A^2B \; \cdots \; A^{n-1}B]$

$rank(Q) \geq n \Rightarrow$ completely controlable

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error $e(t) = x(t) - x_d(t)$ how far off we are from our desired point.

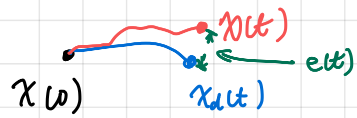
$\underline{\dot{e}} = A\underline{e(t)} + Bu + (Ax_d(t) - \dot{x}_d)$

choose control input

choose u so that $e(t)$ goes to 0, $u(t) = -K_p e(t)$

$\Rightarrow \dot{e} = Ae(t) + B(-K_p e(t)) + Ax_d(t) - \dot{x}_d$

$= \underline{(A - BK_p)} e(t) + Ax_d(t) - \dot{x}_d$



eigenvalue $re(\lambda(A - BK_p)) < 0 \Rightarrow e(t) \to 0$ as $t \to \infty$

Linearization

① $x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

② $\dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ \text{formula} \end{bmatrix} = \begin{bmatrix} f_1(x,u) \\ f_2(x,u) \end{bmatrix}$

③ $f(x,u) = f(x_0, u_0) + \frac{\partial f}{\partial x_i}(x - x_0)\Big|_{x=x_0, u=u_0} + \frac{\partial f}{\partial u}\Big|(u - u_0)$

$x = \;, \; u =$

$\dot{x} = f(x,u)$

④ $\underbrace{\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix}}_{\dot{x}} = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix} \underbrace{\begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}}_{x} + \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} u$

$$\frac{\partial \vec{f}}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & & & \vdots \\ \vdots & & \ddots & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

$$\begin{bmatrix} x_p \\ y_p \\ 1 \end{bmatrix} = \begin{bmatrix} f_x & 0 & u \\ 0 & f_y & v \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_c/Z_c \\ Y_c/Z_c \\ 1 \end{bmatrix}$$

$x_{pixel} = f_x \cdot \frac{X_c}{Z_c} + u$

estimated g_{21} must satisfy $\underline{\text{epipolar constraints}}$

$g_{21} = \begin{bmatrix} R_{21} & T_{21} \\ 0 & 1 \end{bmatrix}$

$x_2^T E x_1 = 0$ where

$E = \hat{T}_{21} R_{21}$