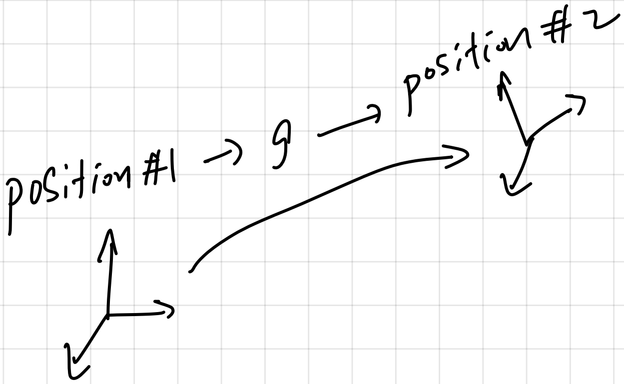
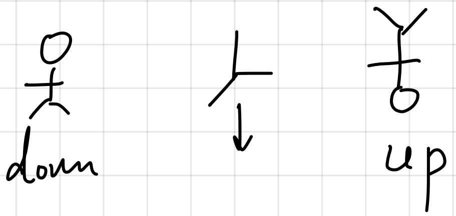


rigid body motion : transformation.

how fast it changes ? or, the rate of change : <u>velocity</u>

- body frame
- spatial frame

Reference frame is crucial

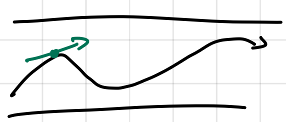


Set up:

Time derivative of position. ① transforming position into time derivative ② tagent vector.

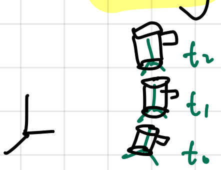
$$\dot{q} = \hat{V} q$$

matrix, pos.



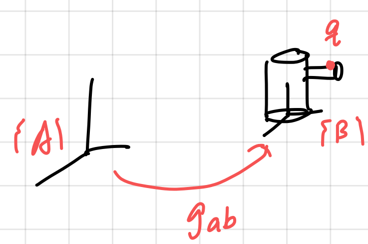
Typical transformation : $g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$

BUT, if frame is moving, orientation/translation will depend on time.



$$g(t) = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix}$$

$\downarrow$



Spatial Velocity. $q_a(t) = g_{ab}(t) \cdot \underline{\underline{q_b}}$

q is fixed to frame B.

$$V_{q_a} := \dot{q}_a(t) = \frac{d}{dt}\left(g_{ab}(t)\, q_b\right) = \dot{g}_{ab}(t)\, q_b \qquad \hat{V}^s_{ab}$$

$$= \underline{\underline{\dot{g}_{ab}(t)\, g^{-1}_{ab}(t)}}\, q_a(t)$$

Spatial Velocity

$$V_{q_a} = \dot{q}_a(t) = \hat{V}^s_{ab}\, q_a = \dot{g}_{ab}(\cancel{t})\, g^{-1}_{ab}(t)\, q_a$$

$$\hat{V}^s_{ab} = \hat{\xi} = \begin{bmatrix} \hat{w} & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \dot{R}R^T & -\dot{R}R^T p + \dot{p} \\ 0 & 0 \end{bmatrix}$$

$$V^s_{ab} = \xi = \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} -\dot{R}R^T p + \dot{p} \\ (\dot{R}R^T)^\vee \end{bmatrix}$$

A point attached to the body moving around, determine the rate of position change respect to the spatial frame.

Body Velocity

$$V_{q_b} = g^{-1}_{ab}(t)\, V_{q_a} = g^{-1}_{ab}(t)\, \dot{g}_{ab}(\cancel{t})\, q_b$$

$$\hat{V}^b_{ab} = \hat{\xi} = \begin{bmatrix} \hat{w} & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R^T\dot{R} & R^T\dot{p} \\ 0 & 0 \end{bmatrix}$$

$$V^b_{ab} = \xi = \begin{bmatrix} R^T\dot{p} \\ (R^T\dot{R})^\vee \end{bmatrix}$$

Still the same time derivative (velocity) of point q relative to spatial frame, but expressed in body frame

Adjoint Transformation

$$Ad_{g_{ab}} = \begin{bmatrix} R_{ab} & \hat{p}_{ab} R_{ab} \\ 0 & R_{ab} \end{bmatrix} \quad Ad_{g^{-1}_{ab}} = \begin{bmatrix} R^T_{ab} & -R^T_{ab}\hat{p}_{ab} \\ 0 & R^T_{ab} \end{bmatrix}$$

$$V^a_{ab} = Ad_{g_{ab}} V^b_{ab}$$
$$= \begin{bmatrix} R_{ab} & \hat{p}_{ab} R_{ab} \\ 0 & R_{ab} \end{bmatrix} \begin{bmatrix} V^b_{ab} \\ w^b_{ab} \end{bmatrix}$$
$$= \begin{bmatrix} Rv + \hat{p}Rw \\ Rw \end{bmatrix}$$

$$\hat{V}^a_{ab} = g_{ab}\, \hat{V}^b_{ab}\, g^{-1}_{ab}$$

$$V^a_{ac} = V^a_{ab} + Ad_{g_{ab}} V^b_{bc}$$

$$V^c_{ac} = Ad_{g^{-1}_{bc}} V^b_{ab} + V^c_{bc}$$

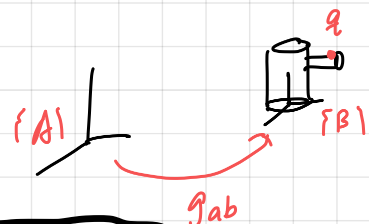
spatial velocity := twist.

Twists can be interpreted as velocities and velocities can be interpreted as twists.

Find $\xi = \begin{bmatrix} v \\ w \end{bmatrix} \in \mathbb{R}^6$ s.t.

$$\hat{\xi} = \hat{V}^s_{ab} = \dot{g}_{ab}(t)\, g^{-1}_{ab}(t)$$
$$= \begin{bmatrix} \dot{R}_{ab}(t) & \dot{p}_{ab}(t) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T_{ab}(t) & -R^T_{ab}(t) p_{ab} \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \underbrace{\dot{R}R^T}_{\hat{w}} & \underbrace{-\dot{R}R^T p + \dot{p}}_{v} \\ 0 & 0 \end{bmatrix}$$

$$V^s_{ab} = \xi = \begin{pmatrix} v \\ w \end{pmatrix} = \begin{bmatrix} -\dot{R}R^T p + \dot{p} \\ (\dot{R}R^T) \end{bmatrix} ✓$$



the velocity of q with respect to {B} is just 0. 'cause it doesn't move from itself.

Thus, body velocity := the velocity of the point q relative to A but expressed in terms of B.

$V_{q_b}(t) = g^{-1}_{ab} V_{q_a}(t)$ why? Still the same time derivative, but expressed in frame {B}.

$= g^{-1}_{ab}\, \hat{V}^a_{ab}\, q_a$ Like $q_b = g^{-1}_{ab} q_a$

$= g^{-1}_{ab}\, \dot{g}_{ab}\, g^{-1}_{ab}\, q_a$ still the same point, but expressed in diff frame.

$= \underbrace{g^{-1}_{ab}\, \dot{g}_{ab}}_{\text{Body velocity.}}\, q_b$

Another way to get body velocity. change basis

↓

Useful formula

$$\hat{\xi}_a = g_{ab}\,\hat{\xi}_b\, g^{-1}_{ab}$$

$$\xi_a = Ad_{g_{ab}}\,\xi_b$$

$$g^{-1}_{ab} = \begin{bmatrix} R^T_{ab} & -R^T_{ab}\,p \\ 0 & 1 \end{bmatrix}$$

Solution Tips

1. when determine spatial / body velocities,

① get $g_{ab}(t)$ and $g^{-1}_{ab}(t)$ first;

② then, ✰ derivative $\underline{\dot{g}_{ab}(t)}$;

note that $y = \sin\theta(t)$ $\quad \frac{dy}{dt} = \frac{dy}{d\theta}\frac{d\theta}{dt} = (\cos\theta)\underline{\dot{\theta}}$

2. $$\hat{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}^{\wedge} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}$$

3. Circle method to validate

That's how joints affect frames.

As $\hat{\xi}_a = g_{ab}\,\hat{\xi}_b\, g^{-1}_{ab}$

$$\hat{V}^b_{ab} = g^{-1}_{ab}\,\hat{V}^a_{ab}\, g_{ab}$$

$$= g^{-1}\dot{g}\,\cancel{g^{-1}g}$$

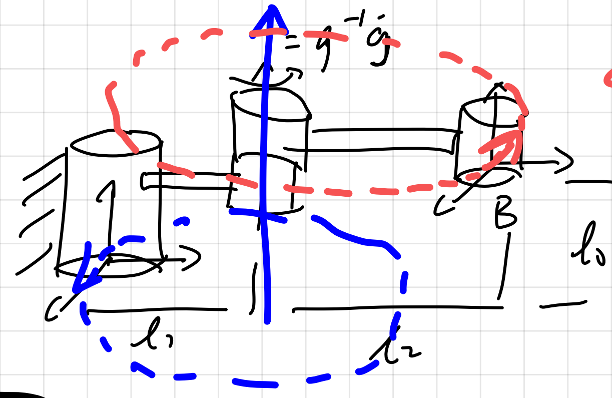
$$= g^{-1}\dot{g}$$

dir: x

mag: $l_1\dot{\theta}$

direction: $-x$

magnitude $l_2\dot{\theta}$



$$V^s_{ab} = \begin{bmatrix} l_1\dot{\theta} \\ 0 \\ 0 \\ 0 \\ 0 \\ \dot{\theta} \end{bmatrix}$$

velocity of B dependon

- how fast motor (joint) move around
- length of l_2.

$$V^b_{ab} = \begin{bmatrix} -l_2\dot{\theta} \\ 0 \\ 0 \\ 0 \\ 0 \\ \dot{\theta} \end{bmatrix}$$

Velocity ξ (Body / Spatial) = how the joint affect the frame

Adjoint is used to transform

$\underline{\text{twists}}$ velocity

$$V^a_{ab} = Ad_{g_{ab}}\,V^b_{ab}$$

$$Ad_{g_{ab}} = \begin{bmatrix} R_{ab} & \hat{p}R_{ab} \\ 0 & R_{ab} \end{bmatrix}$$