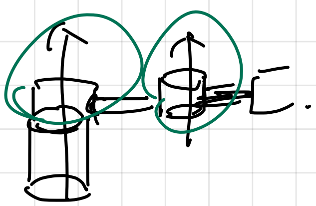
if we have a point q_a, $\curvearrowright$ velocity mapping: $\dot{q}_a = \hat{V} q_a$

find time derivative $\dot{q}_a$.

∅: instantaneous velocity of the B frame expressed in both spatial / body coord.

Velocity mapping: Matrix — points → time derivatives



only through joints, the end-effector moves.

Jocabian:

convert joints' differences (how fast they change) into actual spatial / body velocity

Adjoints:

Matrix: $\hat{\xi}^b = g^{-1} \hat{\xi}^a g \in \mathbb{R}^{4\times4}$

$\Updownarrow$

Twist: $\xi^b = Adg_{ab}^{-1} \xi^a \in \mathbb{R}^6$

Robot $\begin{cases} \text{joint angle: can control by motor} \\ \text{gripper frame velocity.} \end{cases}$

$\dot{q} = \hat{V} q$

Similarity: $\begin{bmatrix} v \\ w \end{bmatrix} = \underbrace{V^s}_{\mathbb{R}^6} = \underbrace{(\qquad)}_{\mathbb{R}^{6\times n}} \underbrace{\dot{\theta}}_{\mathbb{R}^n} \in \mathbb{R}^6$

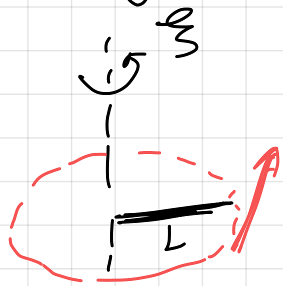
n joints

mapping. $\Rightarrow$ Jacobian. $J^s(\theta)$

current configuration

$\dot{\theta}$: how fast my joints are moving.

e.g.



① $\|V\| = L \cdot \dot{\theta}$

direction of v: tangent to the circle at the point.

② $\text{dir}(w): \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\|w\| = \dot{\theta}$

Find how my robot is moving?

$\frac{d\, g_{ST}(\theta)}{dt}$ pose where $g_{ST}(\theta) = e^{\hat{\xi}_1 \theta_1} \cdots e^{\hat{\xi}_n \theta_n} g_{ST}(0)$

$\Rightarrow \frac{d}{dt} g_{ST}(\theta) \cdot g^{-1} = \underbrace{\frac{\partial}{\partial \theta} g_{ST}(\theta)\, g^{-1}}_{\hat{J}} \dot{\theta}$

//

$$V_{ST}^{S} = J_{ST}^{S}(\theta)\,\dot{\theta}$$

$$J_{ST}^{S}(\theta) = \left[\left(\frac{\partial g_{ST}}{\partial \theta_1} \cdot g_{ST}^{-1} \right)^{\vee}, \; \cdots, \; \left(\frac{\partial g_{ST}}{\partial \theta_n} \cdot g_{ST}^{-1} \right)^{\vee} \right]$$

$$= \left[\xi_1, \; \xi_2', \; \cdots \; \xi_n' \right] \qquad \xi_i' : \text{transformed } \xi_i$$

$$\xi_i' = Ad_{(e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_{i-1}\theta_{i-1}})}\, \xi_i \qquad \text{e.g. } g(\theta) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} g(0)$$

$$\hat{\xi}_2' = \frac{\partial g_{ST}}{\partial \theta_2} \cdot g_{ST}^{-1} = e^{\hat{\xi}_1\theta_1} \hat{\xi}_2 \, e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} \cdot (e^{-\hat{\xi}_3\theta_3} e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1})$$

$$\dot{q}_a = \left[\underbrace{J_{ST}^{S}(\theta)\,\dot{\theta}}_{V_{ST}^{S}} \right]^{\wedge} q_a \qquad = \underbrace{e^{\hat{\xi}_1\theta_1} \hat{\xi}_2 \, e^{-\hat{\xi}_1\theta_1}}$$

$$\xi_2' = Ad_{(e^{\hat{\xi}_1\theta_1})}\, \xi_2$$

① map $\dot{\theta}$ to velocity

② map point to time derivative

Spatial Jacobian

$$V_{ST}^{S} = J_{ST}^{S}(\theta)\,\dot{\theta}$$

$$J_{ST}^{S}(\theta) = \left[\left(\frac{\partial g_{ST}}{\partial \theta_1} \cdot g_{ST}^{-1} \right)^{\vee}, \cdots, \left(\frac{\partial g_{ST}}{\partial \theta_n} \cdot g_{ST}^{-1} \right)^{\vee} \right]$$

$$= \left[\xi_1, \xi_2', \cdots \xi_n' \right]$$

$$\xi_i' = Ad_{(e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_{i-1}\theta_{i-1}})}\, \xi_i$$

Body Jacobian

$$V_{ST}^{T} = J_{ST}^{T}(\theta)\,\dot{\theta}$$

$$J_{ST}^{T}(\theta) = \left[\left(g_{ST}^{-1} \cdot \frac{\partial g_{ST}}{\partial \theta_1} \right)^{\vee}, \cdots \left(g_{ST}^{-1} \frac{\partial g_{ST}}{\partial \theta_n} \right)^{\vee} \right]$$

$$= \left[\xi_1^{+}, \xi_2^{+}, \cdots \xi_n^{+} \right]$$

$$\xi_i^{+} = Ad_{(e^{\hat{\xi}_{i+1}\theta_{i+1}} \cdots e^{\hat{\xi}_n\theta_n} g_{ST}(0))}\, \xi_i$$

Convert

$$J_{ST}^{S}(\theta) = Ad_{g_{ST}(\theta)}\, J_{ST}^{T}(\theta)$$

Tips

Tool frame is fixed to the tool, it will change when tool changes.

Singularity

rank (J) < joint numbers

Conclusions: (only keep statement in cheatsheet ∧ not proof)

1. Rotation R is achieved by intrinsic rotations about the z, y, x. The spatial Jacobian for the rotation has a singularity when $\theta_y = \frac{\pi}{2}$

$J^s = [w_1, w_2', w_3']$, where $w_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ rotate about z axis

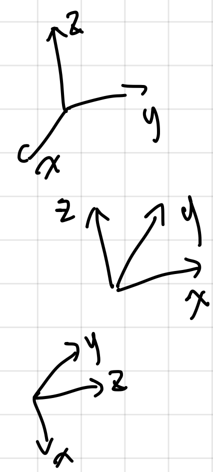
$$w_2' = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$ effect the w_1 has on y axis

$\times W_2$

$$w_3' = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

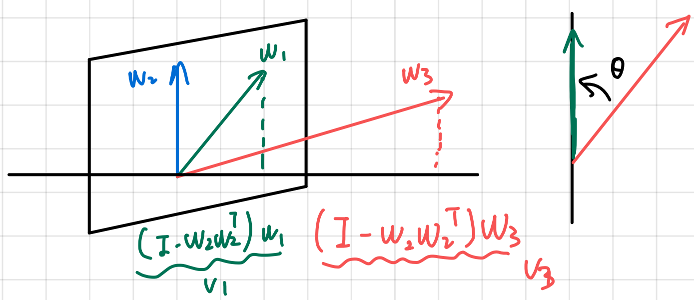
$$J^s = \begin{bmatrix} 0 & -\sin\theta_z & \cos\theta_z \cos\theta_y \\ 0 & \cos\theta_z & \sin\theta_z \cos\theta_y \\ 1 & 0 & -\sin\theta_y \end{bmatrix}$$ when $\theta_y = \frac{\pi}{2}$ $\cos\theta_y = 0$ $\sin\theta_y = 1$

$$= \begin{bmatrix} 0 & -s\theta_z & 0 \\ 0 & c\theta_z & 0 \\ 1 & 0 & -1 \end{bmatrix}$$ rank = 2 < 3 ⇒ singularity



2. Any rotation about the linearly independent axes will also have a singularity.

∵ when rotate about w_2, w_1 will not change, only rotate w_3.



$$\theta = \cos^{-1} \frac{v_1 \cdot v_3}{\|v_1\| \cdot \|v_3\|}$$

3. When 4 revolute joint axes are coplanar, any six degree of freedom manipulator is at a singular configuration.

∵ coplanar, let the span to be w_1, $n \times w_1$ where n is orthogonal to w_i s.t. $n^T w_i = 0$

∴ $w_2 = a_1 w_1 + a_2 (n \times w_1)$

$w_3 = b_1 w_1 + b_2 (n \times w_1)$

$w_4 = c_1 w_1 + C_2 (n \times w_1)$

let q_1 as origin

$$\Rightarrow J = \begin{bmatrix} 0 & (q_2-q_1)\times w_2 & (q_3-q_1)\times w_3 & (q_4-q_1)\times w_4 \\ 1 & a_1 & b_1 & c_1 \\ 0 & a_2 & b_2 & c_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$(q_i - q_1) \times w_i$ is parallel to n

$\Rightarrow (q_i - q_1) \times w_i = k_i n.$

$$\Rightarrow J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & k_2 & k_3 & k_4 \\ 1 & a_1 & b_1 & c_1 \\ 0 & a_2 & b_2 & c_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} < 6 \Rightarrow \text{singular.}$$

4. manipulability measure: how close we're encountering singularities.

$\mu(\theta) = \prod_{i=1} \sigma_i(\theta)$ ← singular value of J^s

J^s singular $\Rightarrow$ non-trival $\underbrace{\text{null space}}_{K\ \text{dimensional}}$ $\Rightarrow (J^s)^T J^s$ singular, null space k-d

$\Downarrow$

计算 singular value.

① Calculate A^TA

② 求 A^TA eigenvalue

③ singular value = A^TA 的非负 eigenvalue 降序

at least one eigenvale $= 0$

$\Downarrow$

$\sigma_i = 0$

HW

$$q_1 = [-L,\ 0,\ L]$$

$$-\begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ -L & -L & L \end{vmatrix} = [-L\ \ L\ \ 0]^T = \begin{bmatrix} -L \\ L \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$-\begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ 0 & -L & 0 \end{vmatrix} = [-L,\ 0,\ 0] = \begin{bmatrix} -L \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\xi_3^{\dagger} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xi_4^{1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$J^s = \begin{bmatrix} 0 & -\sin\theta_{\cancel{y}z} & \cos\theta_z\cos\theta_y \\ 0 & \cos\theta_{\cancel{y}z} & \sin\theta_z\cos\theta_y \\ 1 & 0 & -\sin\theta_y \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -s_z & 0 \\ 0 & c_z & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\theta_z = \theta_x = 0.$$

$$J^s = \begin{bmatrix} 0 & 0 & \cos\theta_y \\ 0 & 1 & 0 \\ 1 & 0 & -\sin\theta_y \end{bmatrix}$$

$$\theta_y = 0 \quad J^s = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \lambda = 1 \quad \sigma = 1$$

$$0$$

$$\theta_y = \frac{\pi}{2} \quad J^s = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\theta_y = \pi \quad J^s = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\theta_y = \frac{3}{2}\pi \quad J^s = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$