Clifford Algebras and Tropical Geometry

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Summary

1 Introduction

1.1 Clifford Algebra

A Clifford algebra is the quotient of a tensor algebra generated by a vector space and a quadratic form. The Clifford algebra is denoted as $Cl(V, Q(v))$ where V is the vector space and Q(v) is the quadratic form and v is an element in V. So by the definition given

$$Cl(V, Q(v)) \cong \frac{TV}{(v \otimes v - Q(v))}$$

Complex Clifford algebras are Clifford algebras with the complex vector spaces and these are denoted as $C_n = Cl(\mathbb{C}^n, Q(v))$
The quadratic form for this Clifford algebra is given as an n by n matrix with 1s down the diagonal and 0s as all the other entries. So this would be the n by n identity matrix. With a vector from the vector space V. Let $v \in V$ then the quadratic form here would be given by $v^T I_{n\times n} v = \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2$ if the vector space is finite with n basis vectors.
There are some properties of Clifford algebras such as

$$C_n \cong C_{n-2} \otimes C_2 \tag{1}$$

The complex Clifford algebras will be our area of interest. An example of Clifford algebra isomorphisms would be $Cl(\mathbb{C}^2, Q(v)) \cong M_2(\mathbb{C})$. By the quadratic form given above. We know that

$$e_1^2 = 1, e_2^2 = 1, e_1e_2 = -e_2e_1$$

The anticommutative property comes from $v \otimes v - Q(v) = 0$. $e_1 \otimes e_2 = (e_1 + e_2)^2 = e_1^2 + e_1e_2 + e_2e_1 + e_2^2$ then we have that $Q(v) = e_1^2 + e_2^2$ leads to $e_1^2 + e_1e_2 + e_2e_1 + e_2^2 - e_1^2 - e_2^2 = e_1e_2 + e_2e_1 = 0$. To show this isomorphism we need a homomorphism say $\beta : Cl(\mathbb{C}^2, Q(v)) \to M_2(\mathbb{C})$. Because this is an isomorphism the properties of the Clifford algebra must be satisfied. Let the basis given to the Clifford algebra be $\{1, e_1, e_2, e_1e_2\}$. Then

$$\beta(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\beta(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\beta(e_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Note that the first two properties are satisfied but is anti commutativity?

$$\beta(e_1e_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\beta(e_2e_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Yes anti commutativity is satified. So the basis for $M_2(\mathbb{C})$ is given by

$$\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}$$

Clearly this homomorphism is surjective as it maps each basis from the Clifford Algebra to the 2 by 2 complex matrices. Next we also see that it is injective as there are four basis elements mapped to four basis elements. Hence the claim that the map is an isomorphism is correct.

Some Clifford algebra isomorphisms may need a non trivial change of basis. However as we will see later that tropicalising a Clifford algebra requires a good choice for a basis. So some non-degenerate quadratic forms may give isomorphic Clifford algebras but not necessarily the same tropicalisations.

1.2 Tropicalisations

Now to introduce tropicalisations. The tropical semiring is the semiring $\mathbb{T} = (\mathbb{R} \cup [+\infty], \oplus, \otimes)$ here we define the operations $\oplus = \min()$ and $\otimes = +$ This is also known as the min tropical semiring. There is also a max tropical semiring but we will not be covering that one. Notice that the min tropical semiring is idempotent i.e. $x \oplus x = x$ and this is a very useful property.

A map which maps a field F to the tropical semiring is called a valuation.[4] $v : F \to \mathbb{R} \cup [+\infty]$ The properties of this map are as follows

- $v(0) = \infty$
- $v(1) = 0$
- $v(a + b) \geq min(v(a), v(b))$
- $v(ab) = v(a) + v(b)$

The aim of this paper is to tropicalise clifford algebras and determine if there is a similar relation in the tropicalised version.

We will now introduce the Giansiracusa bend relations. These relations are defined as follows.
Let $X = \sum_{i\geq 1} \lambda_i x_i$ then the Giansiracusa bend relations are given by $\{X \sim \sum_{i\neq j} \lambda_i x_i\}$. They are equivalence relations of the element X but each relation has one term removed.[3] They can also be defined as in paper [2]

$$X = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3, \alpha_i \in \mathbb{C}$$

The bend relations will be as follows:

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \sim \alpha_1 x_1 + \alpha_2 x_2$$

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \sim \alpha_1 x_1 + \alpha_3 x_3$$

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \sim \alpha_2 x_2 + \alpha_3 x_3$$

They will be useful to use in equivalence relations and play an important role in the min tropical semiring. They are used in the first section.

Definition 1.1. Strong Tropical Basis: polynomials which generate I and whose tropicalisations have bend relations which generate the bend relations of all the elements in trop(I)

This definition is from a paper written by Professor Jeffrey Giansiracusa.

1.3 Morita Equivalence

This paper does assume the reader has some knowledge about Category Theory. This book is a very good introduction to Category Theory [5] Recall that a Category is a collection of objects (for examples the sets A,B,C) and morphisms (f,g,h). Where each morphism has a domain and a codomain. So take for example $f : A \to B$. These morphisms must satisfy the associativity and identity axioms see [5] on pages 6-11.

Definition 1.2. Let us have a ring R with multiplicative identity 1. Then M is a right-R module if $(M, +, \cdot)$ with the operation $\cdot : M \times R \to M$ where $m, n \in M$ and $r, s \in R$

$$m \cdot 1 = m$$

$$(m + n) \cdot r = m \cdot r + n \cdot r$$

$$m \cdot (r + s) = m \cdot r + m \cdot s$$

$$m \cdot (r \cdot s) = (m \cdot r) \cdot s$$

Are satisfied.

There can also be a left module where the operation works on the left side instead.

Two rings say R and S are Morita equivalent if their right R-modules have a category equivalence [1].

$$M_R \approx M_S \tag{2}$$

In this case the equivalence is between the categories of the right modules of R and the right modules of S, of course one can also have a Morita equivalence of the left modules too. The category of right modules of both rings means that the objects are the right modules of R and the morphisms are the module homomorphisms between these modules.

Since we are working with the tropical semi ring we would like to determine a Morita equivalence. There are already some papers with Morita equivalence of semirings [6]. We know that

$$Cl(C_{n+2}) \approx Cl(C_n) \tag{3}$$

is a Morita equivalence so we would like to see if we have a similar relation for the tropical Clifford algebras. This will come up again in 3.4.

2 Useful tropical results

TV is the tensor algebra $TV = \bigoplus_{k=0}^{\infty} T^k V = F \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \cdots$. So $T^k V = V \otimes V \otimes V \otimes \cdots \otimes V$ k times. And F is a field usually $\mathbb{C}$ or $\mathbb{R}$ In this case we wanted to find a map α which sent elements of the tensor algebra (which is commutative only if n=1) to the real commutative polynomials. This map α can permute the

$$\alpha : TV \to \mathbb{R}[x_1, x_2, ..., x_n]$$

The initial tensor algebra may not necessarily be commutative.
We are interested in the kernel of this map because it will be a crucial step in proving an important result which will be used throughout this paper.
To take an example let the following element $X = \lambda_1 x_i x_j + \lambda_2 x_j x_i$ be in the kernel.

$$\alpha(\lambda_1 x_i x_j + \lambda_2 x_j x_i) = \lambda_1 x_i x_j + \lambda_2 x_j x_i = \lambda_1 x_i x_j + \lambda_2 x_i x_j = 0$$

$$\lambda_1 x_i x_j + \lambda_2 x_i x_j = 0$$

$$(\lambda_1 + \lambda_2) x_i x_j = 0$$

Hence this implies

$$\lambda_1 + \lambda_2 = 0$$

Another simple example would be

$$\lambda_1 x_i x_j + \lambda_2 x_j x_i + \lambda_3 x_i^2 x_j + \lambda_4 x_i x_j x_i + \lambda_5 x_j x_i^2$$

Take $\alpha(\lambda_1 x_i x_j + \lambda_2 x_j x_i + \lambda_3 x_i^2 x_j + \lambda_4 x_i x_j x_i + \lambda_5 x_j x_i^2) = \lambda_1 x_i x_j + \lambda_2 x_i x_j + \lambda_3 x_i^2 x_j + \lambda_4 x_i^2 x_j + \lambda_5 x_i^2 x_j$ Now this can only be zero if

$$(\lambda_1 + \lambda_2) x_i x_j + (\lambda_3 + \lambda_4 + \lambda_5) x_i^2 x_j = 0$$

$$(\lambda_1 + \lambda_2) x_i x_j = 0, (\lambda_3 + \lambda_4 + \lambda_5) x_i^2 x_j = 0$$

Hence we get that

$$\lambda_1 + \lambda_2 = 0$$

$$\lambda_3 + \lambda_4 + \lambda_5 = 0$$

We can see that for an element X to be in the kernel we need each $x_i^m x_j^n$ to have a sum of scalars to be zero.
Each polynomial is therefore in the kernel if the sum of scalars in each equivalence class is zero. Thus permuting two monomials to get the same monomial, such as $x_i^2 x_j$ and $x_i x_j x_i$, means that they are equivalent. This permutation arises from the symmetric group acting on the generators $x_1, x_2, ..., x_n$.

Lemma 2.1. *An element X*

$$X = \lambda_1 x_i x_j + \lambda_2 x_j x_i + \lambda_3 x_i x_i x_j + \lambda_4 x_i x_j x_i + \lambda_5 x_j x_i x_i ... + \lambda_n x_i^{m_1} x_j^{m_2} x_i^{m_3}$$

with n terms. Then two monomials are equivalent if they can be permuted from one to the other. Furthermore X is in the kernel of α if and only if the sum of the coefficients across each equivalence class of terms is 0

$$\sum_{k=1}^{n} \lambda_k = 0$$

Proof. First claim:
Define an equivalence class $S_{mn} = x_i^m x_j^n$ so when an element X is mapped by α all terms with $x_i^m x_j^n$ are in this equivalence class.
Take say $x_i^{k_{i1}} x_j^{k_{j1}} x_i^{k_{i2}} x_j^{k_{j2}} ... x_i^{k_{im}} x_j^{k_{jn}}$ and map it with α. It is clear that both will be in the equivalence class S_{mn} if $k_{i1} + k_{i2} + \cdots + k_{im} = m$ and $k_{j1} + k_{j2} + \cdots + k_{jn} = n$ Therefore these two terms are equivalent.
The second claim:

$$\Rightarrow$$

Let an element of TV be X which is in the kernel.
This means it is given in the form

$$\lambda_1 x_i x_j + \lambda_2 x_j x_i + \lambda_3 x_i x_i x_j + \lambda_4 x_i x_j x_i + \lambda_5 x_j x_i x_i + +\lambda_6 x_i x_j x_j + \lambda_7 x_j x_i x_j + ...$$

so it is the summation of n monomial terms. Hence when mapped to the real commutative polynomials then

$$\alpha(\lambda_1 x_i x_j + \lambda_2 x_j x_i + \lambda_3 x_i x_i x_j + \lambda_4 x_i x_j x_i + \lambda_5 x_j x_i x_i + +\lambda_6 x_i x_j x_j + ...) = 0$$

because it is in the kernel.

$$(\lambda_1 + \lambda_2) x_i x_j + (\lambda_3 + \lambda_4 + \lambda_5) x_i^2 x_j + (\lambda_6 + \lambda_7 + \lambda_8) x_i x_j^2 + ... = 0$$

Then we can use the first claim which states that permuting monomials to the same term are equivalent. Take all monomials which are equivalent to the term $x_i^m x_j^n$ in the equivalence class S_{mn} Because the element X is in the kernel this means that each equivalence class must have a sum of monomials which is equal to zero.
So each term must satisfy

$$\sum_{k=1}^{n} \lambda_k x_i^m x_j^n = 0$$

hence this must have the sum of scalars to be equal to zero as $x_i \neq 0, x_j \neq 0$.

$$\sum_{k=1}^{n} \lambda_k = 0$$

$$\Leftarrow$$

Now let

$$\sum_{k=1}^{n} \lambda_k = 0$$

for each equivalence class S_{mn} Then when mapped by α to the commutative polynomials this will give zero overall.

$$(\lambda_1 + \lambda_2)x_i x_j + (\lambda_3 + \lambda_4 + \lambda_5)x_i^2 x_j + (\lambda_6 + \lambda_7 + \lambda_8)x_i x_j^2 + ... = 0$$

By assumption the summation of the scalars of each of the monomial terms is zero hence clearly this is in the kernel. $\square$

The Tropicalisation of this element of X

$$Trop(X) = v(\lambda_1)x_i x_j + v(\lambda_2)x_j x_i + v(\lambda_3)x_i x_i x_j + v(\lambda_4)x_i x_j x_i ...$$

Perhaps it is better to consider an example first.

$$Y = \lambda_1 x_i^2 x_j + \lambda_2 x_i x_j x_i + \lambda_3 x_j x_i^2$$

is an element in the kernel of α so this means that

$$\lambda_1 + \lambda_2 + \lambda_3 = 0$$

Now tropicalise this

$$v(\lambda_1)x_i^2 \otimes x_j \oplus v(\lambda_2)x_i^2 \otimes x_j \oplus v(\lambda_3)x_i^2 \otimes x_j$$

$$\infty = v(\lambda_1 + \lambda_2 + \lambda_3) > v(\lambda_1) \oplus v(\lambda_2) \oplus v(\lambda_3)$$

It is obvious that at least two scalars must be non zero as otherwise the equivalence class would not be zero and thus the element Y would not be in the Kernel. Using the valuation map property $v(-1) = v(1) = 0$ that means that there are two $\lambda_i, i = 1, 2, 3$ with the same valuation. For this particular case let $\lambda_1 = -\lambda_2$ and thus $\lambda_3 = 0$ Then by the Giansiracusa bend relations we have the following

1.

$$v(\lambda_1)x_i^2 \otimes x_j \oplus v(\lambda_2)x_i^2 \otimes x_j \oplus v(\lambda_3)x_i^2 \otimes x_j = v(\lambda_1)x_i^2 \otimes x_j \oplus v(\lambda_2)x_i^2 \otimes x_j$$

2.

$$v(\lambda_1)x_i^2 \otimes x_j \oplus v(\lambda_2)x_i^2 \otimes x_j \oplus v(\lambda_3)x_i^2 \otimes x_j = v(\lambda_1)x_i^2 \otimes x_j \oplus v(\lambda_3)x_i^2 \otimes x_j$$

3.

$$v(\lambda_1)x_i^2 \otimes x_j \oplus v(\lambda_2)x_i^2 \otimes x_j \oplus v(\lambda_3)x_i^2 \otimes x_j = v(\lambda_2)x_i^2 \otimes x_j \oplus v(\lambda_3)x_i^2 \otimes x_j$$

For each case we can use the idempotency of the min tropical semiring. So for 1.

$$v(\lambda_1)x_i^2 \otimes x_j \oplus v(\lambda_2)x_i^2 \otimes x_j = \min\left(v(\lambda_1)x_i^2x_j, v(\lambda_2)x_i^2x_j, v(\lambda_3)x_i^2x_j\right)$$

Because in the min tropical semiring ∞ is the additive identity. For 2.

$$\min(v(\lambda_1)x_i^2x_j, v(\lambda_3)x_i^2x_j) = \min(v(\lambda_1)x_i^2x_j, v(\lambda_1)x_i^2x_j, v(\lambda_3)x_i^2x_j)$$

by idempotency and the by the relation given

$$\min(v(\lambda_1)x_i^2x_j, v(\lambda_3)x_i^2x_j) = \min(v(\lambda_1)x_i^2x_j, v(\lambda_2)x_i^2x_j, v(\lambda_3)x_i^2x_j)$$

For 3.

$$\min(v(\lambda_2)x_i^2x_j, v(\lambda_3)x_i^2x_j) = \min(v(\lambda_2)x_i^2x_j, v(\lambda_2)x_i^2x_j, v(\lambda_3)x_i^2x_j)$$

by idempotency

$$\min(v(\lambda_2)x_i^2x_j, v(\lambda_3)x_i^2x_j) = \min(v(\lambda_1)x_i^2x_j, v(\lambda_2)x_i^2x_j, v(\lambda_3)x_i^2x_j)$$

Then we can say that the bend relations can be generated by $x_i^2x_j \sim x_ix_jx_i$

What we would like to do now is introduce a claim. This claim was made by Professor Jeffrey Giansiracusa and it is one which will be useful later on. Let K be a binomial ideal generated by binomials of the form $x - y$. Where we have that $x \sim y \mod K$. The set of binomials in K is a strong tropical basis of K.
Next we tropicalise this and we would like to show that $\{x-y|x \sim y\}$ is a strong tropical basis. In other words we want to show that the equivalence relation $a \sim b$ generates the bend relations of the tropicalised K.
Here we say that I is a binomial ideal. Let $f \in I$ where

$$f = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n, \lambda_i \in \mathbb{C}$$

And allow $\{x_1, x_2, ..., x_{c_1}\}$ to be in the equivalence class S_1 where $c_1 < n$ $\{x_{c_1+1}, \ldots, x_{c_2}\}$ to be in the equivalence class S_2 and so on. Now because f is in the ideal I this means that by the previous lemma each equivalence class must have a summation of scalars equal to zero. Hence $\lambda_1 + \lambda_2 + \cdots + \lambda_{c_1} = 0$. Note that in the tropicalised version we have.

$$\infty = v(\lambda_1 + \lambda_2 + \cdots + \lambda_{c_1}) > \min\{(v(\lambda_1), v(\lambda_2), ...v(\lambda_{c_1})\}$$

By the definition of the valuation map. Notice that because the inequality is strict (it cannot be equal to infinity otherwise the scalars would all be zero) there are at least two scalars not equal to zero.
Without loss of generality let all scalars be zero and $\lambda_1 = -\lambda_2$.

$$v(\lambda_1)x_1 + v(\lambda_2)x_2 + \cdots + v(\lambda_{c_1})x_{c_1}$$

We can use the equivalence $x_1 \sim x_2$

$$v(\lambda_1)x_1 + v(\lambda_2)x_2 + \cdots + v(\lambda_{c_1})x_{c_1} \sim v(\lambda_1)x_2 + v(\lambda_2)x_2 + \cdots + v(\lambda_{c_1})x_{c_1}$$

Now we have the following

$$(v(\lambda_1) + v(\lambda_2))x_2 + \cdots + v(\lambda_{c_1})x_{c_1} \sim v(\lambda_2)x_2 + \cdots + v(\lambda_{c_1})x_{c_1}$$

The other bend relations are trivial as the scalars are all zero. This allows us to make use of the valuation map properties. Recall that $v(0) = \infty$ and recall that ∞ is the additive identity in the tropical min semiring. So this claim is true.

3 Tropicalisation of Clifford Algebras

3.1 Tropicalising our first Clifford Algebra

In the next couple of sections we discuss the tropicalisation of Clifford algebras. As mentioned in the introduction Clifford algebras have relations. We try to find a relation between tropical Clifford algebras.

$$Cl(\mathbb{C}(e_1, e_2)) \cong M_2(\mathbb{C})$$

The Clifford algebra is four dimensional and has the following basis elements $(1, e_1, e_2, e_1e_2)$. In this case the Clifford algebra has the following properties which must be satisfied. $e_1^2 = 1, e_2^2 = 1, e_1e_2 = -e_2e_1$ so the matrix basis which does satisfy these properties is the following

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The aim of the research is to determine if the tropical Clifford algebras follow a similar result. The tropical version of the Clifford algebra would be

$$Trop(Cl(\mathbb{C}(e_1, e_2), Q(v)))$$

An interesting question would be if the tropical clifford algebra is the same as

$$M_2(\mathbb{T})$$

The basis matrices would be given by

$$\begin{pmatrix} 0 & \infty \\ \infty & \infty \end{pmatrix}, \begin{pmatrix} \infty & 0 \\ \infty & \infty \end{pmatrix}, \begin{pmatrix} \infty & \infty \\ 0 & \infty \end{pmatrix}, \begin{pmatrix} \infty & \infty \\ \infty & 0 \end{pmatrix}$$

One problem we run into is that we would like the tropical Clifford algebra to satisfy the tropicalised Clifford relations.

$$e_1^2 = 0, e_2^2 = 0, e_1e_2 = e_2e_1$$

The answer to the question is no. There are only three matrices which could satisfy the first two relations

$$\begin{pmatrix} 0 & \infty \\ \infty & 0 \end{pmatrix}, \begin{pmatrix} \infty & 0 \\ 0 & \infty \end{pmatrix}, \begin{pmatrix} \infty & 1 \\ -1 & \infty \end{pmatrix}$$

The identity would map to the tropical identity matrix, e_1 is mapped to the second matrix and e_2 is mapped to the thrid matrix. The problem is that these matrices do not satisfy the commutativity property.
So the next possible isomorphism could be: $Trop(C_1) \otimes Trop(C_1)$ Where

$$Trop(C_1) \cong \frac{\mathbb{T}(x)}{x^2 = 0}$$

We know that this tropicalisation has a basis of two $(0, x)$ We want to show that

$$Trop(Cl(\mathbb{C}(x_1, x_2), Q(v))) \cong \frac{\mathbb{T}(x_1)}{x_1^2 = 0} \otimes \frac{\mathbb{T}(x_2)}{x_2^2 = 0}$$

We know that $Trop(Cl(\mathbb{C}(e_1, e_2), Q(v)))$ has basis $(0, e_1, e_2, e_1e_2)$. Now to deal with $\frac{\mathbb{T}(x_1)}{x_1^2=0} \otimes \frac{\mathbb{T}(x_2)}{x_2^2=0}$ we see that because both have the tropical numbers both have the same identity of 0. There are two elements x_1, x_2 with $x_1^2 = 0, x_2^2 = 0$. The final element would be $x_1 \otimes x_2$. But we do not yet know if it satisfies commutativity. So we need to see if

$$\frac{\mathbb{T}(x_1)}{x_1^2 = 0} \otimes \frac{\mathbb{T}(x_2)}{x_2^2 = 0} \cong \frac{\mathbb{T}(x_2)}{x_2^2 = 0} \otimes \frac{\mathbb{T}(x_1)}{x_1^2 = 0}$$

This can be done if we take a morphism γ and map
$\gamma(0) = 0, \gamma(x_1) = x_2, \gamma(x_2) = x_1$ then we get that
$\gamma(x_1x_2) = \gamma(x_1)\gamma(x_2) = x_2x_1$ The map γ is surjective and it easy to see it is injective, futhermore the dimesion of both tensor products is the same so they are indeed isomorphic.
Now we want to see if

$$Trop(Cl(\mathbb{C}(x_1, x_2), Q(v))) \cong \frac{\mathbb{T}(x_1)}{x_1^2 = 0} \otimes \frac{\mathbb{T}(x_2)}{x_2^2 = 0}$$

is true. We can take a morphism say

$$\delta : \frac{\mathbb{T}(x_1)}{x_1^2 = 0} \otimes \frac{\mathbb{T}(x_2)}{x_2^2 = 0} \to Trop(Cl(\mathbb{C}(x_1, x_2), Q(v)))$$

Where $\delta(0) = 0, \delta(x_1) = e_1, \delta(x_2) = e_2, \delta(x_1 \otimes x_2) = e_1e_2$. So it is clearly a surjective morphism. It is also clear that it is injective because four elements are mapped to four elements and the dimension is four. Hence this must be an isomorphism.

Thus it is true that

$$Trop(Cl(\mathbb{C}(x_1, x_2), Q(v))) \cong \frac{\mathbb{T}(x_1)}{x_1^2 = 0} \otimes \frac{\mathbb{T}(x_2)}{x_2^2 = 0}$$

.

Another clifford algebra result is

$$Cl(\mathbb{C}^3, Q(v)) \cong \frac{T(\mathbb{C}^3)}{(v \otimes v - Q(v))}$$

$$Cl(\mathbb{C}^3, Q(v)) \cong M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$$

We want to see if a similar relation holds here as it did for the previous Clifford algebra. Namely if

$$Trop(C_3) \cong Trop(C_2) \otimes Trop(C_1)$$

$$Trop(Cl(\mathbb{C}(e_1, e_2, e_3), Q(v))) \cong \frac{\mathbb{T}(x_1, x_2)}{x_1^2 = 0, x_2^2 = 0} \otimes \frac{\mathbb{T}(x_3)}{x_3^2 = 0}$$

The basis for $\frac{\mathbb{T}(x_1,x_2)}{x_1^2=0,x_2^2=0}$ is $(0, x_1, x_2, x_1x_2)$ and the basis for $\frac{\mathbb{T}(x_3)}{x_3^2=0}$ is $(0, x_3)$ So the tensor product would be $(0, x_1, x_2, x_3, x_1x_2, x_1 \otimes x_3, x_2 \otimes x_3, x_1x_2 \otimes x_3)$ And the tropical Clifford algebra has a basis of $(0, e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3, e_1e_2e_3)$ so both have a dimension of eight. Take a morphism

$$\beta : \frac{\mathbb{T}(x_1, x_2)}{x_1^2 = 0, x_2^2 = 0} \otimes \frac{\mathbb{T}(x_3)}{x_3^2 = 0} \to Trop(Cl(\mathbb{C}^3, Q(v)))$$

$\beta(0) = 0, \beta(x_1) = e_1, \beta(x_2) = e_2, \beta(x_3) = x_3, \beta(x_1x_2) = e_1e_2, \beta(x_1x_3) = e_1e_3, \beta(x_2x_3) = e_2e_3, \beta(x_1x_2x_3) = e_1e_2e_3$ Once again we see this is a surjective map and it is also injective. We see this because eight elements are mapped to eight elements and thus it is an isomorphism.

3.2 The Clifford Algebra Tropicalised

The next question would be: does this mean that

$$Trop(C_{n+1}) \cong Trop(C_n) \otimes Trop(C_1)$$

We know that $Trop(C_{n+1})$ has 2^{n+1} elements

$$(0, e_1, e_2, ..., e_{n+1}, ..., e_1e_2 \cdots e_{n+1})$$

We also know that $Trop(C_n) \otimes Trop(C_1) = \frac{\mathbb{T}(x_1,x_2,...,x_n)}{x_1^2=0,x_2^2=0,...,x_n^2=0} \otimes \frac{\mathbb{T}(x_{n+1})}{x_{n+1}^2=0}$ will also have $2^n \times 2$ elements because the $Trop(C_n)$ has dimension 2^n and $Trop(C_1)$ has a dimension of two. $\frac{\mathbb{T}(x_1,x_2,...,x_n)}{x_1^2=0,x_2^2=0,...,x_n^2=0} \otimes \frac{\mathbb{T}(x_{n+1})}{x_{n+1}^2=0}$ has a basis

$$(0, x_1, x_2, ...x_n, ..., x_1x_2 \cdots x_n, x_{n+1}, ..., x_1x_2 \cdots x_nx_{n+1})$$

and we see that, similar to the previous two cases, we can define a morphism

$$\Lambda : \frac{\mathbb{T}(x_1, x_2, ..., x_n)}{x_1^2 = 0, x_2^2 = 0, ..., x_n^2 = 0} \otimes \frac{\mathbb{T}(x_{n+1})}{x_{n+1}^2 = 0} \to \frac{\mathbb{T}(e_1, e_2, ..., e_{n+1})}{e_1^2 = 0, e_2^2 = 0, ..., e_{n+1}^2 = 0}$$

Take $\Lambda(0) = 0, \Lambda(x_1) = e_1, ..., \Lambda(x_{n+1}) = e_{n+1},$. This is surjective and we can see it is injective too. Hence we see that this is indeed true.

We can now make the claim that

$$Trop(C_n \otimes C_m) \cong Trop(C_n) \otimes Trop(C_m) \tag{4}$$

We may use a similar method to the one for the n+1 case. Let F be a morphism so

$$F : \frac{\mathbb{T}(y_1, ..., y_n)}{y_1^2 = 0, ..., y_n^2 = 0} \otimes \frac{\mathbb{T}(x_1, ..., x_m)}{x_1^2 = 0, ..., x_m^2 = 0} \to \frac{\mathbb{T}(e_1, ..., e_{n+m})}{e_1^2 = 0, ..., e_{n+m}^2 = 0}$$

Take

$$F(0) = 0, F(y_1) = e_1, ..., F(y_n) = e_n, F(x_1) = e_{n+1}, ...F(x_m) = e_{n+m}$$

The basis on the left hand side is

$$(y_1, y_2, ..., y_n, x_1, x_2, ..., x_m, y_1y_2, ..., y_1x_1, ..., y_1...y_nx_1...x_m)$$

By using the morphism F we see that this maps to

$$(e_1, e_2, ...e_n, e_{n+1}, e_{n+2}, ..., e_{n+m}, e_1e_2, ..., e_1e_{n+1}, ..., e_1, ...e_ne_{n+1}...e_{n+m})$$

So the map is surjective as each element from the left hand side is mapped to an element on the right hand side. It is also clearly injective as there are 2^{n+m} elements in both $Trop(C_n \otimes C_m)$ and $Trop(C_n) \otimes Trop(C_m)$. Hence this is an isomorphism as needed.

3.3 General Tropicalisation

Now a more general result. Let α, β, ζ and γ be morphisms such that

$$\alpha : T\mathbb{C}^n \to A$$

$$\beta : T\mathbb{C}^m \to B$$

Then let

$$\zeta : T\mathbb{C}^{n+m} \to T\mathbb{C}^n \otimes T\mathbb{C}^m$$

Followed by

$$\gamma : T\mathbb{C}^n \otimes T\mathbb{C}^m \to A \otimes B$$

We would like to see if the following is true

$$Trop(A \otimes B) \cong Trop(A) \otimes Trop(B) \tag{5}$$

Clearly it worked before for the Clifford algebra by equation (1) so now we would like to see if it holds more generally. We can also write A and B in terms of the tensor algebra in quotient form with ideals I,J respectively.

$$A = \frac{T\mathbb{C}^n}{I} \tag{6}$$

$$B = \frac{T\mathbb{C}^m}{J} \tag{7}$$

First we define the map ζ is defined as

$$\zeta(e_1) = x_1, ..., \zeta(e_n) = x_n, \zeta(e_{n+1} = y_1, ..., \zeta(e_{n+m}) = y_m)$$

and we can see that this indeed maps the left hand side to the right hand side. Then notice that this is an equivalence given by $e_ie_j \sim e_je_i$ and this is a binomial ideal. Where $1 < i < n$ and $n+1 < j < n+m$.

Then we can say

$$\frac{T\mathbb{C}^n}{I} \otimes \frac{T\mathbb{C}^m}{J} = A \otimes B \cong \frac{T\mathbb{C}^{n+m}}{S} \tag{8}$$

Where the ideal S is the ideal generated by the two surjections ζ and γ

Proof. The result we would like to prove is 5

$$Trop(A \otimes B) = Trop(\frac{T\mathbb{C}^{n+m}}{S}) = Trop(\frac{\frac{T\mathbb{C}^{n+m}}{e_ie_j \sim e_je_i}}{I \otimes 1 + 1 \otimes J})$$

We will write the right hand side as

$$Trop(\frac{\frac{T\mathbb{C}^{n+m}}{e_ie_j \sim e_je_i}}{I \otimes 1 + 1 \otimes J}) \cong Trop(\frac{T\mathbb{C}^n \otimes T\mathbb{C}^m)}{I \otimes 1 + 1 \otimes J} = \frac{T\mathbb{T}^n \otimes T\mathbb{T}^m}{Trop(I \otimes 1 + 1 \otimes J)}$$

and we want to determine if this is isomorphic to

$$\frac{T\mathbb{T}^n}{Trop(I)} \otimes \frac{T\mathbb{T}^m}{Trop(J)}$$

Let ψ be a morphism given by

$$\psi : \frac{T\mathbb{T}^n}{Trop(I)} \otimes \frac{T\mathbb{T}^m}{Trop(J)} \to \frac{T\mathbb{T}^n \otimes T\mathbb{T}^m}{Trop(I \otimes 1 + 1 \otimes J)}$$

So

$$\psi(\bar{x} \otimes \bar{y}) = x \mathbin{\bar{\otimes}} y$$

If we let say

$$x \sim x' \mod I \tag{9}$$

then we can we can use this to determine if this is well defined.

$$\psi(\bar{x} \otimes \bar{y}) = x \mathbin{\bar{\otimes}} y = x' \mathbin{\bar{\otimes}} y$$

This is because the right hand side is given by $x \otimes y \mod (I \otimes 1 + 1 \otimes J)$ and then the relation (6) was used. Hence this morphism is well defined.

This map is surjective and well defined but injectivity is a little tricker to determine.
Instead we take the inverse map

$$\phi : \frac{T\mathbb{T}^n \otimes T\mathbb{T}^m}{Trop(I \otimes 1 + 1 \otimes J)} \to \frac{T\mathbb{T}^n}{Trop(I)} \otimes \frac{T\mathbb{T}^m}{Trop(J)}$$

Notice that $x \bar{\otimes} y$ is the same as $x \otimes 1 + 1 \otimes y$ as defined by the ideal. So if $x \bar{\otimes} y \sim x' \bar{\otimes} y$. Then we have

$$x \otimes 1 + 1 \otimes y \sim x' \otimes 1 + 1 \otimes y \mod (I \otimes 1 + 1 \otimes J)$$

And therefore

$$x \otimes 1 \sim x' \otimes 1 \mod (I \otimes 1 + 1 \otimes J) \tag{10}$$

And we would now like to show that this implies

$$x \sim x' \mod I$$

$$\phi(x \bar{\otimes} 1) = \bar{x} \otimes \bar{1}$$
$$\phi(x' \bar{\otimes} 1) = \bar{x'} \otimes \bar{1}$$

And because of the relation (7)

$$\phi(x' \bar{\otimes} 1) \sim \phi(x \bar{\otimes} 1)$$

This implies that

$$\bar{x} \otimes \bar{1} \sim \bar{x'} \otimes \bar{1}$$

As a result it must mean that

$$x \sim x' \mod I$$

As needed. □

3.4 Morita equivalence

We now would like to look at finding endomorphisms of the The following definition of Morita equivalence for semi rings comes from a paper referenced on semirings [6].

Definition 3.1. For semirings R and S

1. R and S are Morita equivalent
2. $R \cong eM_2(S)e$ for some e idempotent matrix in the matrix semiring $M_2(S)$.

Are equivalent statements

This can be found on page 17 and proposition 5.2 of the paper [6]. Applying this to $Trop(C_2)$ we have that

$$Trop(C_2) \cong eM_2(Trop(C_2))e \tag{11}$$

where e is an idempotent matrix which also satisfies

$$M_2(Trop(C_2))eM_2(Trop(C_2)) = M_2(Trop(C_2))$$

This can be done by setting

$$e = \begin{pmatrix} 0 & 0 \\ \infty & \infty \end{pmatrix}$$

Then by 3.4

$$eM_2(Trop(C_2))e \cong Trop(C_2)$$

We leave it as an exercise to the reader to check this.
And this brings us neatly along to the next point. By the definition of Morita equivalence of semirings from the paper this means that $Trop(C_2)$ is Morita equivalent to $Trop(C_2)$. So now the aim would be to find a module of $Trop(C_2)$ such that this relation is satisfied.

4 What we would like to show

Much like the Clifford algebra having the relation 1 this means that C_{n+2} and C_n are Mortia equivalent as given by 3.
What we would like to show next would be if there is a similar "Tropical" Morita equivalence. As we saw before in 3.4 there is a Morita equivalence for semi rings so we would like to see if there is a Morita equivalence for the tropical Clifford Algebras as well.
A question we would like to be answered is if

$$Trop(C_{n+2}) \approx Trop(C_n) \tag{12}$$

is true. But before we can answer this we might want to first find any endomorphisms of the tropical Clifford algebra

$$Trop(C_2) \cong End_{\mathbb{T}}(M) \tag{13}$$

where M is some module similar to how

$$C_2 \cong M_2(\mathbb{C}) = End_{\mathbb{C}}(\mathbb{C}^2)$$

As mentioned in 3

$$M_2(\mathbb{T}) \ncong Trop(C_2)$$

So first one must mind an endomorphism first as we believe this would help to answer 12. So finding the endomorphisms of $Trop(C_2)$, if there are any, would be a good start to solve this.
This brings us to the end of this part of the paper.

5 References

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