High-order accurate entropy stable discontinuous Galerkin schemes using artificial viscosity

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Introduction

Many governing equations in physics can be expressed in conservative form, which describes the (local) evolution of a (globally) conserved quantity according to its flux. A system of n conservation laws (CLs) over a 1D domain $\Omega \subset \mathbb{R}$ is described by

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} = 0, \tag{1}$$

where $\mathbf{u} : \Omega \times \mathbb{R}^+ \to \mathbb{R}^n$ are the conserved variables and $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ their fluxes. A weak solution of Eq. (1) must satisfy

$$\int_0^\infty \int_\Omega \mathbf{u}\partial_t\phi + \mathbf{f}(\mathbf{u})\partial_x\phi \, \mathrm{d}x\mathrm{d}t + \int_\Omega \mathbf{u}_0\phi(x,0)\mathrm{d}x = 0,$$

for some compactly supported $\phi : \Omega \times \mathbb{R}^+ \to \mathbb{R}$. The weak solutions are generally not unique and the entropy conditions are useful to single out the physically relevant one. The entropy $E : \mathbb{R}^n \to \mathbb{R}$ is a convex function. The corresponding entropy flux $Q : \mathbb{R}^n \to \mathbb{R}$ must satisfy $(\partial Q/\partial \mathbf{u})^T = \mathbf{v}^T \partial \mathbf{f}/\partial \mathbf{u}$, where $\mathbf{v} = \partial E/\partial \mathbf{u}$ is called the entropy variable. Entropy solutions must satisfy

$$W = \frac{\partial E}{\partial t} + \frac{\partial Q}{\partial x} \leq 0,$$

in the sense of distribution. This is called *entropy stable* (ES).

Numerical schemes

The domain is split in cells of size Δx containing $k+1$ quadrature points, allowing to interpolate the solution with a local polynomial of order k. Since the solution is smooth inside cells, the entropy is conserved with an entropy conservative flux $\widetilde{\mathbf{F}}$ which must satisfy

$$(\mathbf{v}_R - \mathbf{v}_L)^T \widetilde{\mathbf{F}}(\mathbf{u}_L, \mathbf{u}_R) = \psi_R - \psi_L,$$

where $\mathbf{u}_L$ and $\mathbf{u}_R$ are the left and right states and $\psi(\mathbf{u}) = \mathbf{v}^T(\mathbf{u})\mathbf{f}(\mathbf{u}) - Q(\mathbf{u})$ is called the entropy potential. At the cell interface, the entropy must be dissipated by an entropy stable numerical flux $\mathbf{F}$, such as the Lax-Friedrich flux, defined as:

$$\mathbf{F}(\mathbf{u}_L, \mathbf{u}_R) = \frac{1}{2}(\mathbf{f}(\mathbf{u}_L) + \mathbf{f}(\mathbf{u}_R)) - \frac{\alpha}{2}(\mathbf{u}_R - \mathbf{u}_L),$$

where $\alpha = \max(\lambda_{\max}(\mathbf{u}_L), \lambda_{\max}(\mathbf{u}_R))$, if $\lambda_{\max}(\mathbf{u})$ represents the maximal eigen value of the matrix $\partial \mathbf{f}/\partial \mathbf{u}$. On the i-th cell, the ES DG scheme is:

$$\frac{\Delta x}{2}\frac{\partial \mathbf{u}_l}{\partial t} + 2\sum_{r=0}^{k} D_{lr}\widetilde{\mathbf{F}}(\mathbf{u}_l, \mathbf{u}_r) = \frac{\tau_l}{\omega_l}(\mathbf{f}_l - \mathbf{f}_{*,l}),$$

where $\tau_j = -\delta_{0,j} + \delta_{k,j}$ and $\vec{\mathbf{f}}_* = (\mathbf{F}_{i-\frac{1}{2}}, 0, \ldots, 0, \mathbf{F}_{i+\frac{1}{2}})^T$. Artificial viscosity is introduced in the original equation with

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial}{\partial x}(\mathbf{f}(\mathbf{u}) - \mathbf{q}) = 0,$$

where $\mathbf{q} = \nu(\partial \mathbf{v}/\partial x)$ and ν is the viscosity. This new equation is discretized as follow:

$$\frac{\Delta x}{2}\frac{\partial \mathbf{u}_r}{\partial t} + \sum_{s=1}^{k} D_{rs}(2\widetilde{\mathbf{F}}(\mathbf{u}_r, \mathbf{u}_s) - \mathbf{q}_s) = \frac{\tau_r}{\omega_r}(\mathbf{f}_r - \mathbf{q}_r + \mathbf{q}_{*r} - \mathbf{f}_{*r})$$

$$\frac{\Delta x}{2}\mathbf{q}_r = \nu_r\left(\sum_{s=1}^{k} D_{rs}\mathbf{v}_s + \frac{\tau_r}{\omega_r}(\mathbf{v}_{*r} - \mathbf{v}_r)\right),$$

where $\vec{\mathbf{g}}_* = (\widehat{\mathbf{g}}_{i-\frac{1}{2}}, 0, \ldots, 0, \widehat{\mathbf{g}}_{i+\frac{1}{2}})^T$ with $\widehat{\mathbf{g}}_{i\pm\frac{1}{2}} = \pm(\mathbf{g}^-_{i\pm\frac{1}{2}} - \mathbf{g}^+_{i\pm\frac{1}{2}})$ (using $\mathbf{g} = \mathbf{v}, \mathbf{q}$). The 2D scheme is a tensor product of the 1D version.

Dilation based (DB) viscosity This model relies on the rapid variation of a variable s:

$$\nu_\beta = c_\beta \left|\frac{\partial s}{\partial x}\right| (h/k)^2,$$

For scalar CLs, the value s is the conserved variable and for Euler equations, it is the physical speed. This model limits the accuracy of the scheme to second order because of the h^2 term.

Entropy based (EB) viscosity This model relies on the entropy dissipation:

$$\nu_E = c_E \cdot (h/k) \cdot \max\left((h/k)\max_{x\in G_K}|W|\,,\ \max_{x\in\partial G_K}|[\![Q]\!]|\right),$$

where $[\![.]\!]$ is the jump across cells and G_K are the quadrature points.

Results

The convergence rate is tested on the advection equation with the smooth initial condition $u_0(x) = \sin(x)$, on the domain $\Omega = [0, 1]$. The numerical solution is compared with the analytical one ($u(x,t) = u_0(x-t)$) using various numbers N of cells. The convergence is shown in Table 1 for P^2 elements.

	No viscosity		DB viscosity		EB viscosity	
N	L^2 error	order	L^2 error	order	L^2 error	order
10	4.168e-03	-	6.834e-02	-	1.116e-02	-
20	5.337e-04	2.97	2.001e-02	1.77	1.058e-03	3.40
30	1.593e-04	2.98	9.244e-03	1.90	2.579e-04	3.48
40	6.741e-05	2.99	5.288e-03	1.94	9.592e-05	3.44
50	3.458e-05	2.99	3.415e-03	1.96	4.519e-05	3.37

Table 1: Convergence test on the advection equation using P^2 elements.

The scheme converges as expected including for DB viscosity where the accuracy is of order 2. Then the scheme is tested on the Euler equations, using Sod's shock tube test with P^4 elements and no viscosity (Fig. 1). This shows that the method is able to correctly capture shocks. Finally, the 2D scheme is tested on two Riemann problems of the Euler equations using P^2 elements and DB viscosity. The density and the viscosity at the final time are shown in Fig. 2 and Fig. 3, where the results are comparable to the reference solutions in the literature. From Fig. 3, it is clear that the viscosity is applied at the shocks.

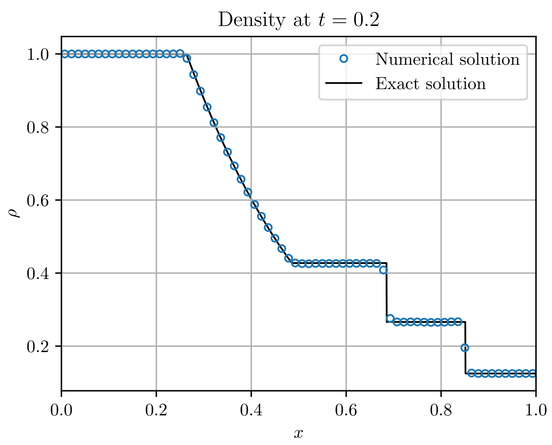

Figure 1: Density profile of the Sod's shock tube using P^4 elements without viscosity.

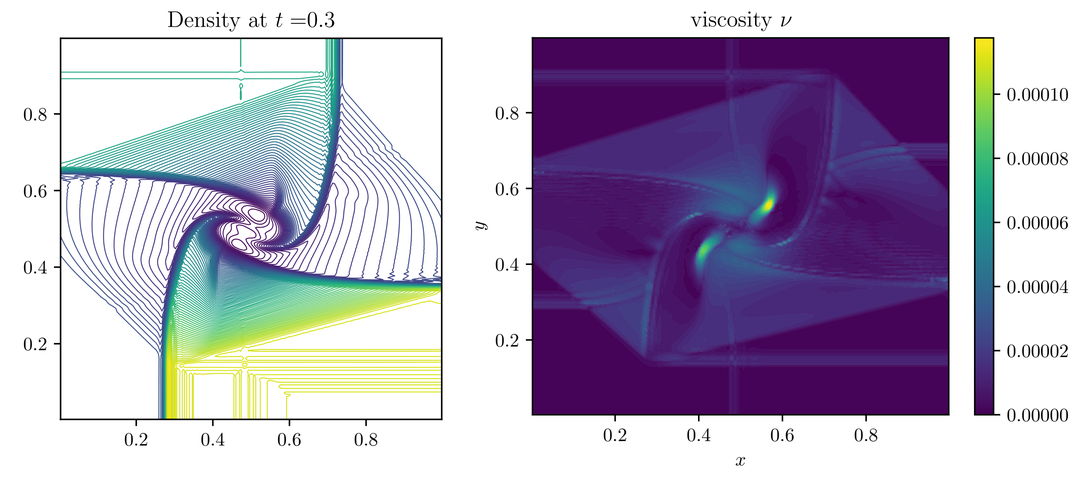

Figure 2: Results obtained with DB viscosity and P^2 elements for a 2D Riemann problem. Density and the corresponding artificial viscosity at the final time.

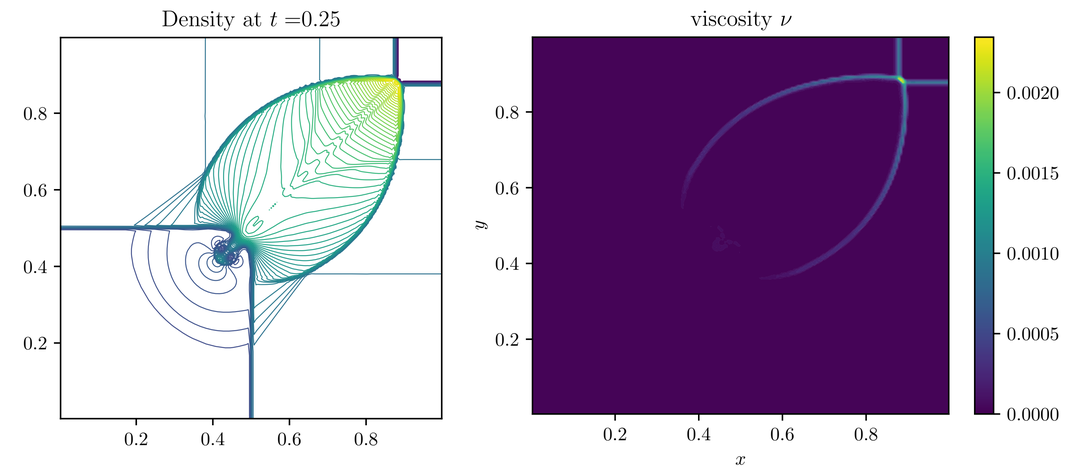

Figure 3: Results obtained with DB viscosity and P^2 elements for a 2D Riemann problem. Density and the corresponding artificial viscosity at the final time.

Conclusion

Entropy stable discontinuous Galerkin (ES DG) methods provide an elegant and parallel efficient way to get arbitrarily high-order numerical schemes for systems of conservation laws. This work further improves stability by introducing artificial viscosity. The method presented here provides an arbitrary convergence rate and is able to correctly capture shocks.

References

[1] Gregor J. Gassner. A skew-symmetric discontinuous galerkin spectral element discretization and its relation to sbp-sat finite difference methods. *SIAM Journal on Scientific Computing*, 35(3):A1233–A1253, 2013.

[2] Jian Yu and Jan S. Hesthaven. A study of several artificial viscosity models within the discontinuous galerkin framework. *Communications in Computational Physics*, 27(5):1309–1343, 2020.