Extending the Accelerated Failure Conditionals Model to Location-Scale Families

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Abstract

Arnold and Arvanitis [1] introduced a novel class of bivariate conditionally specified distributions, in which dependence between two random variables is established by defining the distribution of one variable conditional on the other. This conditioning regime was formulated through survival functions and termed the accelerated failure conditional model. Subsequently, Lakhani [2] extended this conditioning framework to encompass distributional families whose marginal densities may exhibit unimodality and skewness, thereby moving beyond families with strictly monotonically decaying marginals. The present study builds on this line of work by proposing a conditional survival specification derived from a location-scale distributional family, where the dependence between X and Y arises not only through the acceleration function but also via a location function. An illustrative example of this new specification is developed using a Weibull marginal. The resulting models are fully characterized by closed-form expressions for their moments, and simulations are implemented using the Metropolis-Hastings algorithm. Finally, the model is applied to an dataset in which the empirical distribution of Y lies on the real line, demonstrating the model's capacity to accommodate Y marginals defined over $\mathbb{R}$.

1 Introduction

We extend the accelerated failure conditionals model, originally introduced by Arnold and Arvanitis [1], to the real line by re-specifying the conditional survival function to have support on $y \in (-\infty, \infty)$. Specifically, we consider a heterogeneous family specification in which X belongs to some positive support distributional family, while Y follows a location-scale family. Accordingly, the conditional survival function is redefined to conform to this specification, with $\mu(x)$ denoting the conditional location parameter and $\beta(x)$ the corresponding scale parameter. We note that Arnold and Arvanitis [1] originally introduced the model in which $\beta(x)$ acted as a rate parameter within the conditional survival function. The resulting general accelerated failure conditionals model is expressed as:

$$\bar{F}_X(x) = P(X > x) = \bar{F}_0(x), \quad x > 0, \tag{1}$$

for some survival function $\bar{F}_0(x) \in [0, 1]$ for $x > 0$, and for each $x > 0$:

$$P(Y > y \mid X > x) = \bar{F}_1\left(\frac{y - \mu(x)}{\beta(x)}\right), \quad x > 0,\ y \in \mathbb{R}, \tag{2}$$

for some survival function $\bar{F}_1\left(\frac{y-\nu}{\beta(x)}\right) \in [0, 1]$ for $x > 0$ and $y \in \mathbb{R}$, a suitable acceleration (or, perhaps more aptly, deceleration) function $\beta(x) \in \mathbb{R} \setminus \{0\}$ and a suitable location function $\mu(x) \in \mathbb{R}$. We call $\beta(x)$ an acceleration function because it scales the argument of the baseline survival $\bar{F}_1$, thereby accelerating or decelerating the rate of decay of the conditional survival $P(Y > y \mid X > x)$. By contrast, we call $\mu(x)$ the location function because it shifts the argument of the baseline survival $\bar{F}_1$. As discussed by Arnold and Arvanitis [1], in the analysis of dependent lifetimes of components in a system, it is more appropriate to consider the conditional density of Y given that the first component, with lifetime X, remains operational at time x. Hence why, the conditioning event is taken to be $\{X > x\}$, rather than conditioning on the exact value $X = x$.

The joint survival function is given by:

$$P(X > x, Y > y) = \bar{F}_0(x)\bar{F}_1\left(\frac{y - \mu(x)}{\beta(x)}\right), \quad x > 0,\ y \in \mathbb{R}. \tag{3}$$

Assuming differentiability and X, Y are continuous, we obtain the marginal densities:

$$f_X(x) = -\frac{d}{dx}P(X > x) = -\frac{d}{dx}\bar{F}_0(x) = f_0(x),$$

and, since $\lim_{x\to 0^+} \bar{F}_0(x) = 1$, where $\beta(0) := \lim_{x\to 0^+} \beta(x)$ and $\mu(0) := \lim_{x\to 0^+} \mu(x)$:

$$P(Y > y) = \lim_{x\to 0^+} P(Y > y, X > x) = \bar{F}_1\left(\frac{y-\mu(0)}{\beta(0)}\right).$$

Now, the density of Y is:

$$f_Y(y) = -\frac{d}{dy}P(Y > y) = -\frac{d}{dy}\bar{F}_1\left(\frac{y-\mu(0)}{\beta(0)}\right) = \frac{1}{\beta(0)}\, f_1\left(\frac{y-\mu(0)}{\beta(0)}\right),$$

where $f_1(t) = -\bar{F}_1'(t)$ denotes the density associated with $\bar{F}_1$. Hence the marginal distribution of Y belongs to the same family as $\bar{F}_1$, but with its argument scaled by $\frac{1}{\beta(0)}$. For (3) to be a valid survival function, it must have a non-negative mixed partial derivative. Denoting f_0 and f_1 as the densities corresponding to the survival function $\bar{F}_0$ and $\bar{F}_1$, and differentiating, we obtain:

$$\begin{aligned}\frac{\partial}{\partial x}\frac{\partial}{\partial y}P(X > x, Y > y) &= f_0(x)f_1\left(\frac{y-\mu(x)}{\beta(x)}\right)\frac{1}{\beta(x)} - \bar{F}_0(x)f_1'\left(\frac{y-\mu(x)}{\beta(x)}\right)\frac{1}{\beta(x)}\left[\frac{-\mu'(x)\beta(x)-\beta'(x)(y-\mu(x))}{\beta(x)^2}\right]\\ &+ \bar{F}_0(x)f_1\left(\frac{y-\mu(x)}{\beta(x)}\right)\frac{\beta'(x)}{\beta(x)^2}\\ &= f_0(x)f_1\left(\frac{y-\mu(x)}{\beta(x)}\right)\frac{1}{\beta(x)} - \frac{\bar{F}_0(x)}{\beta(x)^2}\Bigg[-f_1'\left(\frac{y-\mu(x)}{\beta(x)}\right)\mu'(x) - \beta'(x)\Bigg(\frac{y-\mu(x)}{\beta(x)}f_1'\left(\frac{y-\mu(x)}{\beta(x)}\right)\\ &+ f_1\left(\frac{y-\mu(x)}{\beta(x)}\right)\Bigg)\Bigg].\end{aligned}$$

Denoting $t = \frac{y-\mu(x)}{\beta(x)}$, we have:

$$\begin{aligned}\frac{\partial}{\partial x}\frac{\partial}{\partial y}P(X > x, Y > y) &= f_0(x)f_1(t)\frac{1}{\beta(x)} - \frac{\bar{F}_0(x)}{\beta(x)^2}\Big[-f_1'(t)\,\mu'(x) - \beta'(x)\big[tf_1'(t) + f_1(t)\big]\Big]\\ &= f_0(x)f_1(t)\frac{1}{\beta(x)} + \frac{\bar{F}_0(x)}{\beta(x)^2}\Big[f_1'(t)\,\mu'(x) + \beta'(x)\big[tf_1(t)\big]'\Big]\end{aligned} \tag{4}$$

To obtain interpretable bounds for $\beta'(x)$ and $\mu'(x)$ in (4), we proceed by isolating their respective effects: first by holding $\mu(x)$ constant while allowing $\beta(x)$ to vary with x, and then by holding $\beta(x)$ constant while allowing $\mu(x)$ to vary with x. With the first case, given a fixed location, $\mu(x) = \mu$, that is $\mu'(x) = 0$, a necessary and sufficient condition for $\frac{\partial}{\partial x}\frac{\partial}{\partial y}P(X > x, Y > y) \geq 0$, is:

$$\beta'(x)\begin{cases}\geq -\dfrac{f_0(x)}{\bar{F}_0(x)}\dfrac{f_1(t)}{[tf_1(t)]'}\,\beta(x), & \text{if } [tf_1(t)]' > 0,\\ \leq -\dfrac{f_0(x)}{\bar{F}_0(x)}\dfrac{f_1(t)}{[tf_1(t)]'}\,\beta(x), & \text{if } [tf_1(t)]' < 0,\\ \text{unconstrained}, & \text{if } [tf_1(t)]' = 0.\end{cases}$$

Denoting the hazard function $h_0(x) = \frac{f_0(x)}{\bar{F}_0(x)}$ and $S(t) = \frac{f_1(t)}{[tf_1(t)]'}$, we may write the bounds as such:

$$\sup_{t:[tf_1(t)]'>0} -S(t)h_0(x)\beta(x) \;\leq\; \beta'(x) \;\leq\; \inf_{t:[tf_1(t)]'<0} -S(t)h_0(x)\beta(x).$$

The role of the acceleration function, $\beta(x)$, in determining the sign of the correlation between X and Y is formalised in Theorem 1 (Appendix A). In particular, since several common location-scale distributions are symmetric about their location parameter μ, the theorem shows that such symmetry yields zero covariance between X and Y. Although this does not preclude nonlinear forms of dependence - such as scale or tail dependence - it implies the absence of linear association between the variables. Consequently, the study excludes the fixed location case, as such models would capture only scale-based dependence while exhibiting zero linear correlation, limiting their usefulness for characterising the direction of dependence.

Hence, we consider the second case from (4); given constant acceleration, $\beta(x) = \beta$, that is $\beta'(x) = 0$, a necessary and sufficient condition for $\frac{\partial}{\partial x}\frac{\partial}{\partial y}P(X > x, Y > y) \geq 0$, is:

$$\mu'(x)\begin{cases}\geq -\dfrac{f_0(x)}{\bar{F}_0(x)}\dfrac{f_1(t)}{f_1'(t)}\,\beta, & \text{if } f_1'(t) > 0,\\ \leq -\dfrac{f_0(x)}{\bar{F}_0(x)}\dfrac{f_1(t)}{f_1'(t)}\,\beta, & \text{if } f_1'(t) < 0,\\ \text{unconstrained}, & \text{if } f_1'(t) = 0.\end{cases}$$

Denoting the hazard function $h_0(x) = \frac{f_0(x)}{\bar{F}_0(x)}$ and $Q(t) = \frac{f_1(t)}{f_1'(t)}$, we may write the bounds as such:

$$\sup_{t:f_1'(t)>0} -Q(t)h_0(x)\beta \;\leq\; \mu'(x) \;\leq\; \inf_{t:f_1'(t)<0} -Q(t)h_0(x)\beta. \tag{5}$$

The role of the location function $\mu(x)$ in determining the sign of the correlation between X and Y is established in Theorem 2 (Appendix A). Accordingly, the present study uses $\mu^+(x)$ to denote the choice of location function that arises from the extreme case in which $\mu'(x)$ attains its theoretical upper bound, hence only allowing for modelling positively correlated X and Y. Conversely, $\mu^-(x)$ denotes the location function arising from the extreme case in which $\mu'(x)$ attains its theoretical lower bound, hence only allowing for modelling negatively correlated X and Y. Now given $\inf_{t:f_1'(t)<0}(-Q(t)) = s^* > 0$, from bounds in (5):

$$\begin{aligned}
\frac{d\left(\mu^+(x)\right)}{dx} &= s^* h_0(x)\beta, \\
\therefore \int d\left(\mu^+(x)\right) &= s^*\beta \int h_0(x)dx, \\
\therefore \mu^+(x) &= \gamma - s^*\beta \log\left(\bar{F}_0(x)\right),
\end{aligned} \tag{6}$$

for $\gamma \in \mathbb{R}$. This implies $\mu^+(x)$ is non-decreasing for $s^* > 0$. Furthermore, $\mu(0) := \lim_{x\to 0^+} \mu(x) = \gamma$ since $\lim_{x\to 0^+} \bar{F}_0(x) = 1$. We also note that since $\bar{F}_0(x)$ approaches zero only as $x \to \infty$, the expression in (6) remains non-zero for all finite x. Similarly, if we denote $\sup_{t:f_1'(t)>0}(-Q(t)) = c^* < 0$, we have:

$$\mu^-(x) = \gamma - c^*\beta \log\left(\bar{F}_0(x)\right), \tag{7}$$

for $\gamma \in \mathbb{R}$ and implies $\mu^-(x)$ is non-increasing for $c^* < 0$.

Furthermore, as done in Arnold and Arvanitis [1], we introduce a dependence parameter τ which controls the strength of the dependence between X and Y. Extending from (6) and (7), we define the location functions as such:

$$\mu^+(x) = \gamma - s^*\beta \log\left(\bar{F}_0(\tau x)\right), \tag{8}$$

$$\mu^-(x) = \gamma - c^*\beta \log\left(\bar{F}_0(\tau x)\right), \tag{9}$$

for $\gamma \in \mathbb{R}$, $s^* > 0$ and $c^* < 0$. Hence, the parameter τ modulates the rate at which $\mu(x)$ varies with x, thereby serving as a dependence-scaling parameter. When $\tau > 1$, the survival function $\bar{F}_0(\tau x)$ decays more rapidly in x, inducing a steeper change in the location function $\mu(x)$ and consequently amplifying the sensitivity of the conditional survival of Y to changes in X. Conversely, when $\tau < 1$, the dependence between X and Y weakens, while $\tau = 0$ implies independence as $\mu(x)$ becomes constant in x.

Given the bounds established in (5), and upon selecting the non-decreasing specification for $\mu(x)$ as defined in (8), and again assuming $\inf_{t:f_1'(t)<0}(-Q(t)) = s^* > 0$, we obtain bounds for τ as such:

$$\begin{aligned}
0 \leq \frac{d(\mu^+(x))}{dx} &\leq s^* h_0(x)\beta, \\
\therefore 0 \leq \tau s^*\beta h_0(\tau x) &\leq s^* h_0(x)\beta, \\
\therefore 0 \leq \tau \frac{h_0(\tau x)}{h_0(x)} &\leq 1.
\end{aligned}$$

We can express this inequality equivalently as:

$$0 \leq r(x,\tau) \leq 1, \tag{10}$$

where $r(x,\tau) = \tau\frac{h_0(\tau x)}{h_0(x)}$. The bounds in (10) also hold under the non-increasing specification of $\mu(x)$, as defined in (9).

2 Weibull Marginal

The following sections outline the range of models employed in this study. In particular, we consider a class of mixture models in which the marginal survival function of X follows a Weibull distribution (although any distributional family with positive support could, in principle, be used), while the conditional survival function of Y belongs to a location-scale distributional family. Specifically, we assume that $Y \sim Family(\mu(0), \beta)$, where the location parameter is given by $\mu(0) = \gamma \in \mathbb{R}$ and the scale parameter satisfies $\beta > 0$. Given the marginal Weibull specification, the survival function of X is expressed as

$$\bar{F}_0(x) = P(X > x) = e^{-(\alpha x)^\lambda}, \quad x > 0. \tag{11}$$

Consequently, the corresponding density function is $f_X(x) = \alpha\lambda(\alpha x)^{\lambda-1}e^{-(\alpha x)^\lambda}$ for $x > 0$ and hence, $X \sim Weibull(\alpha, \lambda)$ with:

$$E(X) = \frac{1}{\alpha}\Gamma\left(1 + \frac{1}{\lambda}\right), \tag{12}$$

$$Var(X) = \frac{1}{\alpha^2}\left(\Gamma\left(1 + \frac{2}{\lambda}\right) - \left(\Gamma\left(1 + \frac{1}{\lambda}\right)\right)^2\right). \tag{13}$$

Furthermore, assuming $\inf_{t:f_1'(t)<0}(-Q(t)) = s^* = 1$ and $\sup_{t:f_1'(t)>0}(-Q(t)) = c^* = -1$, we employ (8), (9) and (11) to obtain:

$$\begin{aligned} \mu^+(x) &= \gamma - s^*\beta \log\left(\bar{F}_0(\tau x)\right) \\ &= \gamma + \beta\left(\alpha\tau x\right)^\lambda, \end{aligned} \tag{14}$$

$$\mu^-(x) = \gamma - \beta\left(\alpha\tau x\right)^\lambda, \tag{15}$$

where $\gamma \in \mathbb{R}$ and $\alpha, \beta, \lambda > 0$ with $\tau \in [0, 1]$ for $x > 0$. Now with the aid of Theorem 3 in Appendix A, and expressing the location functions in (6) and (7) as $\mu(x) = \gamma + \text{sgn}\big(\mu'(x)\big)\beta\left(\alpha\tau x\right)^\lambda$ by remembering $\text{sgn}\big(\mu'(x)\big) = \begin{cases} +1, & \text{for } \mu^+(x) \text{ (non-decreasing case)}, \\ -1, & \text{for } \mu^-(x) \text{ (non-increasing case)}, \end{cases}$ we have:

$$\begin{aligned} \text{Cov}(X, Y) &= \int_0^\infty \bar{F}_0(x) \underbrace{\int_{-\infty}^\infty \left[\bar{F}_1\Big(\tfrac{y-\mu(x)}{\beta}\Big) - \bar{F}_1\Big(\tfrac{y-\mu(0)}{\beta}\Big)\right] dy}_{=\ \mu(x)-\mu(0)}\, dx \\ &= \int_0^\infty \bar{F}_0(x)\left[\mu(x) - \mu(0)\right] dx \\ &= \text{sgn}(\mu'(x))\,\beta \int_0^\infty e^{-(\alpha x)^\lambda}\left(\alpha\tau x\right)^\lambda dx \qquad \Big(\text{since } \mu(x) - \mu(0) = \text{sgn}(\mu'(x))\beta\left(\alpha\tau x\right)^\lambda\Big) \\ &= \text{sgn}(\mu'(x))\frac{\beta\tau^\lambda}{\alpha\lambda}\Gamma\left(1 + \frac{1}{\lambda}\right), \end{aligned} \tag{16}$$

where (16) illustrates that higher values of β and τ strengthen the dependence between X and Y, whereas larger values of α and λ weaken it. Furthermore, as shown by Lakhani [2], in order for condition (10) to hold, the parameter τ must satisfy $\tau \in [0, 1]$, given that the Weibull distribution has a hazard function of the form $h_0(x) = \lambda\alpha(\alpha x)^{\lambda-1}$.

Furthermore, the following sections present closed-form expressions of the models' moments, and additionally, theoretical correlation bounds are obtained (with the exception of the Weibull-Cauchy model since moments are undefined).

2.1 Weibull-Logistic

We restrict the conditional survival form to the logistic family. Hence for each $x > 0$:

$$\bar{F}_1\left(\frac{y - \mu(x)}{\beta}\right) = P(Y > y \mid X > x) = 1 - \frac{1}{1 + e^{-\frac{y-\mu(x)}{\beta}}} = \frac{1}{1 + e^{\frac{y-\mu(x)}{\beta}}}, \quad x > 0,\ y \in \mathbb{R},$$

where $\beta > 0$ for all $x > 0$ to ensure a valid survival function. The corresponding joint survival function will be:

$$\bar{F}_0(x)\bar{F}_1\left(\frac{y - \mu(x)}{\beta}\right) = P(X > x, Y > y) = \frac{e^{-(\alpha x)^\lambda}}{1 + e^{\frac{y-\mu(x)}{\beta}}}, \quad x > 0,\ y \in \mathbb{R}. \tag{17}$$

Accordingly, $f_Y(y) = \frac{e^{-\frac{y-\mu(0)}{\beta}}}{\beta\left(1+e^{-\frac{y-\mu(0)}{\beta}}\right)^2}$ for $y \in \mathbb{R}$. Hence, $Y \sim Logistic(\mu(0), \beta)$.

Since $f_1'(t) = \frac{e^t(1-e^t)}{(e^t+1)^3} > 0$ for $t < 0$ and the function $-Q(t) = \frac{1}{\tanh\left(\frac{t}{2}\right)}$ is decreasing on this interval, it follows that $\sup_{t<0}(-Q(t)) = \lim_{t\to-\infty}(-Q(t)) = c^* = -1$. Additionally, since $f_1'(t) = \frac{e^t(1-e^t)}{(e^t+1)^3} < 0$ for $t > 0$ and the function $-Q(t) = \frac{1}{\tanh\left(\frac{t}{2}\right)}$ is decreasing on this interval, it follows that $\inf_{t>0}(-Q(t)) = \lim_{t\to\infty}(-Q(t)) = s^* = 1$. Hence, applying (5) yields:

$$-h_0(x)\beta \leq \mu'(x) \leq h_0(x)\beta.$$

Hence, given that $s^* = 1$ and $c^* = -1$, the location functions (14) and (15) may be employed.

2.1.1 Moments

We note that since $Y \sim \mathit{Logistic}(\gamma, \beta)$, we have:

$$E(Y) = \gamma, \tag{18}$$

$$Var(Y) = \frac{\beta^2\pi^2}{3}. \tag{19}$$

Now using (13), (16) and (19), we have:

$$\begin{aligned} \rho(X,Y) &= \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} \\ &= \operatorname{sgn}(\mu'(x))\frac{\sqrt{3}}{\pi}\frac{\tau^\lambda}{\lambda}\frac{\Gamma\left(1+\frac{1}{\lambda}\right)}{\sqrt{\Gamma\left(1+\frac{2}{\lambda}\right) - \Gamma^2\left(1+\frac{1}{\lambda}\right)}}. \end{aligned} \tag{20}$$

Consequently, (20) attains its maximum at $\tau = 1$ and its minimum (zero) at $\tau = 0$. Therefore,

$$0 \leq |\rho(X,Y)| \leq \rho_{\max(\lambda)} := \frac{\sqrt{3}}{\pi}\frac{1}{\lambda}\frac{\Gamma\left(1+\frac{1}{\lambda}\right)}{\sqrt{\Gamma\left(1+\frac{2}{\lambda}\right) - \Gamma^2\left(1+\frac{1}{\lambda}\right)}}.$$

Denoting $\lambda^* = \operatorname{argmax}_{\lambda>0}\rho_{\max}$, we have $\sup_{\lambda>0}\rho_{\max}(\lambda) = \rho_{\max}(\lambda^* = 1) = \frac{\sqrt{3}}{\pi}$. Hence, the Weibull-logistic model with location function $\mu^+(x)$ in (14) can accommodate correlations in the range $\rho(X,Y) \in \left[0, \frac{\sqrt{3}}{\pi} \approx 0.5513\right]$, whereas the specification with $\mu^-(x)$ in (15) can accomodate correlations in the range $\rho(X,Y) \in \left[-\frac{\sqrt{3}}{\pi} \approx -0.5513, 0\right]$.

2.2 Weibull-Gumbel

We restrict the conditional survival form to the Gumbel family. Hence for each $x > 0$:

$$\bar{F}_1\left(\frac{y-\mu(x)}{\beta}\right) = P(Y > y \mid X > x) = 1 - e^{-e^{-\frac{y-\mu(x)}{\beta}}}, \quad x > 0,\ y \in \mathbb{R},$$

where $\beta > 0$ with $\mu(x) \in \mathbb{R}$ for all $x > 0$ to ensure a valid survival function. The corresponding joint survival function will be:

$$\bar{F}_0(x)\bar{F}_1\left(\frac{y-\mu(x)}{\beta}\right) = P(X > x, Y > y) = e^{-(\alpha x)^\lambda}\left(1 - e^{-e^{-\frac{y-\mu(x)}{\beta}}}\right), \quad x > 0,\ y \in \mathbb{R}. \tag{21}$$

Accordingly, $f_Y(y) = \frac{e^{-\left(e^{-\frac{y-\mu(0)}{\beta}} + \frac{y-\mu(0)}{\beta}\right)}}{\beta}$ for $y \in \mathbb{R}$ and hence $Y \sim \mathit{Logistic}(\mu(0), \beta)$.

Since $f_1'(t) = (1-e^t)e^{-2t-e^{-t}} > 0$ for $t < 0$ and the function $-Q(t) = -\frac{e^t}{1-e^t}$ is decreasing on this interval, $\sup_{t<0}(-Q(t)) = \lim_{t\to-\infty}(-Q(t)) = c^* = 0$. Additionally, since $f_1'(t) = (1-e^t)e^{-2t-e^{-t}} < 0$ for $t > 0$ and $-Q(t) = -\frac{e^t}{1-e^t}$ is decreasing on this interval, $\inf_{t>0}(-Q(t)) = \lim_{t\to\infty}(-Q(t)) = s^* = 1$. Hence, applying (5) yields:

$$0 \leq \mu'(x) \leq h_0(x)\beta.$$

Hence, given that $s^* = 1$, the location function (14) is utilised. Moreover, since $c^* = 0$, the model is not capable of representing negative correlations.

2.2.1 Moments

We note that since $Y \sim Gumbel(\gamma, \beta)$, we have:

$$E(Y) = \gamma + \beta\gamma^*, \tag{22}$$

$$Var(Y) = \frac{\pi^2}{6}\beta^2, \tag{23}$$

where γ^* denotes the Euler-Mascheroni constant. Now using (13), (16) and (23), we have:

$$\begin{aligned}\rho(X,Y) &= \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} \\ &= \frac{\sqrt{6}}{\pi}\frac{\tau^\lambda}{\lambda}\frac{\Gamma\left(1+\frac{1}{\lambda}\right)}{\sqrt{\Gamma\left(1+\frac{2}{\lambda}\right)-\Gamma^2\left(1+\frac{1}{\lambda}\right)}}.\end{aligned} \tag{24}$$

Consequently, (24) attains its maximum magnitude at $\tau = 1$ and its minimum magnitude (zero) at $\tau = 0$. Therefore,

$$0 \leq |\rho(X,Y)| \leq \rho_{\max(\lambda)} := \frac{\sqrt{6}}{\pi}\frac{1}{\lambda}\frac{\Gamma\left(1+\frac{1}{\lambda}\right)}{\sqrt{\Gamma\left(1+\frac{2}{\lambda}\right)-\Gamma^2\left(1+\frac{1}{\lambda}\right)}}.$$

Denoting $\lambda^* = \operatorname{argmax}_{\lambda>0}\rho_{\max}$, we have $\sup_{\lambda>0}\rho_{\max}(\lambda) = \rho_{\max}(\lambda^* = 1) = \frac{\sqrt{6}}{\pi}$. Hence, the Weibull-Gumbel model with location function $\mu^+(x)$ in (14) can accommodate correlations in the range $\rho(X,Y) \in \left[0, \frac{\sqrt{6}}{\pi} \approx 0.7797\right]$.

2.3 Weibull-Laplace

We restrict the conditional survival form to the Laplace family. Hence for each $x > 0$:

$$\bar{F}_1\left(\frac{y-\mu(x)}{\beta}\right) = P(Y > y \mid X > x) = \frac{1}{2} - \frac{1}{2}\operatorname{sgn}(y-\mu(x))\left(1 - e^{-\frac{|y-\mu(x)|}{\beta}}\right), \quad x > 0,\ y \in \mathbb{R},$$

where $\beta > 0$ with $\mu(x) \in \mathbb{R}$ for all $x > 0$ to ensure a valid survival function. The corresponding joint survival function will be:

$$\bar{F}_0(x)\bar{F}_1\left(\frac{y-\mu(x)}{\beta}\right)P(X > x, Y > y) = e^{-(\alpha x)^\lambda}\left(\frac{1}{2} - \frac{1}{2}\operatorname{sgn}(y-\mu(x))\left(1 - e^{-\frac{|y-\mu(x)|}{\beta}}\right)\right), \quad x > 0,\ y \in \mathbb{R}. \tag{25}$$

Accordingly, $f_Y(y) = \frac{1}{2\beta}e^{-\frac{|y-\mu(0)|}{\beta}}$ for $y \in \mathbb{R}$ hence $Y \sim Laplace(\mu(0), \beta)$.

Since $f_1'(t) = -\frac{1}{2}\operatorname{sgn}(t)e^{-|t|} > 0$ for $t < 0$ and $-Q(t) = \operatorname{sgn}(t)$, $\sup_{t<0}(-Q(t)) = c^* = -1$. Additionally, since $f_1'(t) = -\frac{1}{2}\operatorname{sgn}(x)e^{-|x|} < 0$ for $t > 0$ and $-Q(t) = \operatorname{sgn}(t)$, $\inf_{t>0}(-Q(t)) = s^* = 1$. Hence, applying (5) yields:

$$-h_0(x)\beta \leq \mu'(x) \leq h_0(x)\beta.$$

Hence, given that $s^* = 1$ and $c^* = -1$, the location functions (14) and (15) may be employed.

2.3.1 Moments

We note that since $Y \sim Laplace(\gamma, \beta)$, we have:

$$E(Y) = \gamma, \tag{26}$$

$$Var(Y) = 2\beta^2. \tag{27}$$

Now using (13), (16) and (27), we have:

$$\begin{aligned}\rho(X,Y) &= \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} \\ &= \operatorname{sgn}(\mu'(x))\frac{1}{\sqrt{2}}\frac{\tau^\lambda}{\lambda}\frac{\Gamma\left(1+\frac{1}{\lambda}\right)}{\sqrt{\Gamma\left(1+\frac{2}{\lambda}\right)-\Gamma^2\left(1+\frac{1}{\lambda}\right)}}.\end{aligned} \tag{28}$$

Consequently, (28) attains its maximum magnitude at $\tau = 1$ and its minimum magnitude (zero) at $\tau = 0$. Therefore,

$$0 \le |\rho(X,Y)| \le \rho_{\max(\lambda)} := \frac{1}{\sqrt{2}}\frac{1}{\lambda}\frac{\Gamma\left(1+\frac{1}{\lambda}\right)}{\sqrt{\Gamma\left(1+\frac{2}{\lambda}\right) - \Gamma^2\left(1+\frac{1}{\lambda}\right)}}.$$

Denoting $\lambda^* = \text{argmax}_{\lambda>0}\, \rho_{\max}$, we have $\sup_{\lambda>0} \rho_{\max}(\lambda) = \rho_{\max}(\lambda^* = 1) = \frac{1}{\sqrt{2}}$. Hence, the Weibull-Laplace model with location function $\mu^+(x)$ in (14) can accommodate correlations in the range $\rho(X,Y) \in \left[0, \frac{1}{\sqrt{2}} \approx 0.7071\right]$, whereas the specification with $\mu^-(x)$ in (15) can accomodate correlations in the range $\rho(X,Y) \in \left[-\frac{1}{\sqrt{2}} \approx -0.7071, 0\right]$.

2.4 Weibull-Cauchy

We restrict the conditional survival form to the normal family. Hence for each $x > 0$:

$$\bar{F}_1\left(\frac{y-\mu(x)}{\beta}\right) = P(Y > y \mid X > x) = \frac{1}{2} - \frac{1}{\pi}\arctan\left(\frac{y-\mu(x)}{\beta}\right), \quad x > 0,\ y \in \mathbb{R},$$

where $\beta > 0$ with $\mu(x) \in \mathbb{R}$ for all $x > 0$ to ensure a valid survival function. The corresponding joint survival function will be:

$$\bar{F}_0(x)\bar{F}_1\left(\frac{y-\mu(x)}{\beta}\right) = P(X > x, Y > y) = e^{-(\alpha x)^\lambda}\left(\frac{1}{2} - \frac{1}{\pi}\arctan\left(\frac{y-\mu(x)}{\beta}\right)\right), \quad x > 0,\ y \in \mathbb{R}. \tag{29}$$

Accordingly, $f_Y(y) = \frac{1}{\pi\beta\left(1+\left(\frac{y-\mu(0)}{\beta}\right)^2\right)}$ for $y \in \mathbb{R}$. Hence, $Y \sim Cauchy(\mu(0), \beta)$.

Since $f_1'(t) = -\frac{2t}{\pi(t^2+1)^2} > 0$ for $t < 0$ it follows that the function $-Q(t) = \frac{t^2+1}{2x}$ is increasing on $(-\infty, -1)$ and decreasing on $(-1, 0)$. Hence, $Q(t)$ attains its maximum at $t = -1$, where $-Q(-1) = -1$. Consequently, $\sup_{t<0}(-Q(t)) = c^* = -1$. Additionally, since $f_1'(t) = -\frac{2t}{\pi(t^2+1)^2} < 0$ for $t > 0$ it follows that the function $-Q(t) = \frac{t^2+1}{2t}$ is decreasing on $(0, 1)$ and increasing on $(1, \infty)$. Hence, $Q(t)$ attains its minimum at $t = 1$, where $-Q(1) = 1$. Hence, $\inf_{t>0}(-Q(t)) = s^* = 1$. Now, applying (5) yields:

$$-h_0(x)\beta \le \mu'(x) \le h_0(x)\beta.$$

Hence, given that $s^* = 1$ and $c^* = -1$, the location functions (14) and (15) may be employed.

2.5 Weibull-Normal

We restrict the conditional survival form to the normal family. Hence for each $x > 0$:

$$\bar{F}_1\left(\frac{y-\mu(x)}{\beta}\right) = P(Y > y \mid X > x) = \frac{1}{2}\left(1 - \text{erf}\left(\frac{y-\mu(x)}{\beta\sqrt{2}}\right)\right), \quad x > 0,\ y \in \mathbb{R}$$

where $\beta > 0$ and $\mu(x) \in \mathbb{R}$ for all $x > 0$ to ensure a valid survival function. The corresponding joint survival function will be:

$$\bar{F}_0(x)\bar{F}_1\left(\frac{y-\mu(x)}{\beta}\right) = P(X > x, Y > y) = \frac{e^{-(\alpha x)^\lambda}}{2}\left(1 - \text{erf}\left(\frac{y-\mu(x)}{\beta\sqrt{2}}\right)\right), \quad x > 0,\ y \in \mathbb{R}.$$

Accordingly, $f_Y(y) = \frac{e^{-\frac{(x-\mu(0))^2}{2\beta^2}}}{\sqrt{2\pi\beta^2}}$ for $y \in \mathbb{R}$. Hence $Y \sim Normal(\mu(0), \beta)$.

Since $f_1'(t) = -\frac{te^{-\frac{t^2}{2}}}{\sqrt{2\pi}} > 0$ for $t < 0$ and the function $-Q(t) = \frac{1}{t}$ is decreasing on this interval, it follows that $\sup_{t<0}(-Q(t)) = \lim_{t\to-\infty}(-Q(t)) = c^* = 0$. Additionally, since $f_1'(t) = -\frac{te^{-\frac{t^2}{2}}}{\sqrt{2\pi}} < 0$ for $t > 0$ and the function $-Q(t) = \frac{1}{t}$ is decreasing on this interval, it follows that $\inf_{t>0}(-Q(t)) = \lim_{t\to\infty}(-Q(t)) = s^* = 0$. Consequently, applying (5) yields:

$$0 \le \mu'(x) \le 0.$$

Thus, the location function reduces to a constant form, $\mu(x) = \gamma$ for some $\gamma \in \mathbb{R}$, implying statistical independence between X and Y. As a result, employing the normal model with this location function is futile for representing dependence between the variables. Therefore, the study forgoes further utilisation of the normal model.

3 Simulation

We utilise the Metropolis-Hastings (MH) algorithm[1] to obtain X and Y (where Y is conditioned on $\{X > x\}$) draws by sampling from the joint density of the models considered in this study, where the target distribution is given by $\pi(x, y) = f_{X,Y}(x, y)$, noting that $X \sim Weibull(\alpha, \lambda)$ and $Y \sim Family(\mu(0), \beta)$, with $f_{X,Y}(x, y)$ given in Appendix B. Additionally, Appendix B presents the log-likelihoods for the study's models - m.l.e's are obtained numerically as $\text{argmax}_{\boldsymbol{\theta}}\, \ell(\boldsymbol{\theta})$ where $\boldsymbol{\theta} = [\alpha, \beta, \lambda, \gamma, \tau]'$. We employ $\mu^+(x)$ as defined in (14), although $\mu^-(x)$, given in (15), could equivalently have been used.

Furthermore, we conduct simulations based on 10000 data sets of size $n = 1000, 5000$ and 10000 for fixed paramter values: $\alpha = 1, \beta = 2, \lambda = 3, \gamma = -4$ and $\tau = 0.5$.

3.1 Simulation Data

The corresponding m.m.e and m.l.e results are reported in Tables 1 - 4, together with 95% confidence intervals computed as $\hat{\theta} \pm Z_{\alpha/2}\text{S.E.}(\hat{\theta})$, where $\hat{\theta}$ denotes the point estimator [2]. In addition, we report Pearson correlation coefficients denoted as PC.

As expected, the standard errors of both m.m.e's and m.l.e's [3] decrease as the sample size n increases, yielding narrower 95% confidence intervals. The simulation results indicate that no instability arises in the parameter estimation; however, as expected, more reliable estimation requires larger sample sizes. Furthermore, the population correlations converge to the PCs as sample size increases.

Figure 1[4] displays the bivariate densities of the models used in the study, given the fixed parameter set.

n	Parameter	m.m.e	m.l.e	SE(m.m.e)	SE(m.l.e)	95%CI (m.m.e)	95%CI (m.l.e)	PC
100	α	0.999	0.999	0.038	0.038	(0.925, 1.073)	(0.925, 1.073)	0.101
	β	1.977	1.989	0.186	0.177	(1.612, 2.342)	(1.642, 2.335)	
	λ	3.024	3.031	0.257	0.254	(2.520, 3.527)	(2.533, 3.529)	
	γ	-3.990	-3.991	0.389	0.372	$(-4.752, -3.227)$	$(-4.721, -3.261)$	
	τ	0.587	0.577	0.157	0.189	(0.280, 0.894)	(0.206, 0.948)	
	ρ	0.101	0.096	0.078	0.077	$(-0.052, 0.254)$	$(-0.056, 0.247)$	
500	α	1.000	1.000	0.017	0.017	(0.967, 1.033)	(0.967, 1.033)	0.068
	β	1.997	2.000	0.084	0.079	(1.832, 2.161)	(1.845, 2.155)	
	λ	3.004	3.006	0.113	0.112	(2.783, 3.225)	(2.787, 3.225)	
	γ	-3.999	-3.999	0.173	0.165	$(-4.338, -3.661)$	$(-4.321, -3.676)$	
	τ	0.514	0.512	0.115	0.122	(0.287, 0.740)	(0.272, 0.752)	
	ρ	0.068	0.067	0.041	0.039	$(-0.013, 0.149)$	$(-0.010, 0.145)$	
1000	α	1.000	1.000	0.012	0.012	(0.977, 1.023)	(0.977, 1.024)	0.064
	β	1.998	2.000	0.060	0.056	(1.881, 2.115)	(1.891, 2.109)	
	λ	3.003	3.004	0.080	0.080	(2.846, 3.161)	(2.847, 3.161)	
	γ	-3.997	-3.998	0.122	0.116	$(-4.236, -3.758)$	$(-4.225, -3.771)$	
	τ	0.503	0.503	0.096	0.094	(0.316, 0.691)	(0.318, 0.688)	
	ρ	0.064	0.064	0.032	0.030	(0.001, 0.128)	(0.004, 0.124)	

Table 1: Simulations for Weibull-Logistic model ($\alpha = 1$, $\beta = 2$, $\lambda = 3$, $\gamma = -4$ and $\tau = 0.5$ with $\rho = 0.063$).

[1] Presented in Appendix C

[2] The reported standard errors (S.E.) are calculated as the empirical standard deviation of the parameter estimates across n simulated datasets, representing the sampling variability of the estimator.

[3] m.m.e's (where applicable) were used as initial values when using R's `optim` function Furthermore, Appendix D provides a guide on how to obtain m.m.e's for the models (noting that moments were not provided for the Weibull-Cauchy model, we exclude m.m.e's for said model).

[4] Interactive content: this figure contains an embedded GIF. To view the animation, please open the PDF with **Adobe Acrobat Reader**.

n	Parameter	m.m.e	m.l.e	SE(m.m.e)	SE(m.l.e)	95%CI (m.m.e)	95%CI (m.l.e)	PC
100	α	0.999	1.000	0.039	0.039	(0.923, 1.077)	(0.922, 1.077)	0.116
	β	1.964	1.983	0.219	0.174	(1.536, 2.393)	(1.642, 2.325)	
	λ	3.029	3.034	0.265	0.263	(2.510, 3.548)	(2.519, 3.550)	
	γ	-3.987	-3.996	0.239	0.235	$(-4.455, -3.519)$	$(-4.457, -3.535)$	
	τ	0.549	0.541	0.141	0.144	(0.272, 0.826)	(0.260, 0.823)	
	ρ	0.116	0.112	0.085	0.072	$(-0.051, 0.283)$	$(-0.029, 0.254)$	
500	α	1.000	1.000	0.018	0.018	(0.966, 1.035)	(0.966, 1.035)	0.090
	β	1.994	1.997	0.101	0.077	(1.796, 2.192)	(1.847, 2.148)	
	λ	3.007	3.009	0.116	0.116	(2.779, 3.235)	(2.782, 3.237)	
	γ	-3.997	-3.999	0.106	0.105	$(-4.205, -3.789)$	$(-4.203, -3.794)$	
	τ	0.503	0.505	0.098	0.070	(0.311, 0.695)	(0.368, 0.642)	
	ρ	0.090	0.091	0.047	0.035	$(-0.002, 0.182)$	(0.022, 0.160)	
1000	α	1.000	1.000	0.012	0.012	(0.976, 1.025)	(0.976, 1.025)	0.089
	β	1.996	1.998	0.072	0.055	(1.856, 2.137)	(1.891, 2.105)	
	λ	3.004	3.004	0.083	0.083	(2.841, 3.166)	(2.841, 3.167)	
	γ	-3.998	-3.999	0.075	0.074	$(-4.145, -3.852)$	$(-4.143, -3.854)$	
	τ	0.500	0.501	0.075	0.049	(0.352, 0.647)	(0.405, 0.598)	
	ρ	0.089	0.090	0.036	0.026	(0.019, 0.159)	(0.039, 0.140)	

Table 2: Simulations for Weibull-Gumbel model ($\alpha = 1$, $\beta = 2$, $\lambda = 3$, $\gamma = -4$ and $\tau = 0.5$ with $\rho = 0.089$).

n	Parameter	m.m.e	m.l.e	SE(m.m.e)	SE(m.l.e)	95%CI (m.m.e)	95%CI (m.l.e)	PC
100	α	0.999	0.997	0.040	0.042	(0.921, 1.078)	(0.916, 1.079)	0.112
	β	1.963	1.971	0.235	0.222	(1.502, 2.423)	(1.535, 2.406)	
	λ	3.029	3.051	0.269	0.274	(2.502, 3.556)	(2.513, 3.588)	
	γ	-3.988	-4.062	0.307	0.259	$(-4.591, -3.385)$	$(-4.570, -3.555)$	
	τ	0.561	0.526	0.144	0.159	(0.279, 0.844)	(0.214, 0.839)	
	ρ	0.112	0.092	0.083	0.074	$(-0.050, 0.274)$	$(-0.053, 0.236)$	
500	α	1.000	0.999	0.018	0.019	(0.965, 1.036)	(0.962, 1.036)	0.083
	β	1.994	1.996	0.107	0.101	(1.785, 2.203)	(1.798, 2.194)	
	λ	3.008	3.016	0.119	0.122	(2.776, 3.241)	(2.778, 3.254)	
	γ	-3.997	-4.022	0.137	0.119	$(-4.267, -3.728)$	$(-4.256, -3.788)$	
	τ	0.506	0.492	0.102	0.093	(0.306, 0.706)	(0.310, 0.675)	
	ρ	0.083	0.077	0.045	0.040	$(-0.005, 0.171)$	$(-0.001, 0.155)$	
1000	α	1.000	1.000	0.012	0.012	(0.976, 1.025)	(0.976, 1.025)	0.089
	β	1.996	1.998	0.072	0.055	(1.856, 2.137)	(1.891, 2.105)	
	λ	3.004	3.004	0.083	0.083	(2.841, 3.166)	(2.841, 3.167)	
	γ	-3.998	-3.999	0.075	0.074	$(-4.145, -3.852)$	$(-4.143, -3.854)$	
	τ	0.500	0.501	0.075	0.049	(0.352, 0.647)	(0.405, 0.598)	
	ρ	0.089	0.090	0.036	0.026	(0.019, 0.159)	(0.039, 0.140)	

Table 3: Simulations for Weibull-Laplace model ($\alpha = 1$, $\beta = 2$, $\lambda = 3$, $\gamma = -4$ and $\tau = 0.5$ with $\rho = 0.081$).

n	Parameter	m.l.e	SE(m.l.e)	95%CI (m.l.e)	PC
100	α	1.000	0.037	(0.927, 1.073)	0.076
	β	1.969	0.288	(1.404, 2.535)	
	λ	3.034	0.247	(2.550, 3.517)	
	γ	-4.008	0.296	$(-4.587, -3.428)$	
	τ	0.504	0.256	(0.002, 1.005)	
500	α	1.000	0.017	(0.967, 1.033)	0.033
	β	1.991	0.128	(1.741, 2.242)	
	λ	3.008	0.112	(2.788, 3.227)	
	γ	-4.002	0.131	$(-4.259, -3.745)$	
	τ	0.482	0.208	(0.076, 0.889)	
1000	α	1.000	0.012	(0.977, 1.024)	0.024
	β	1.996	0.090	(1.820, 2.172)	
	λ	3.004	0.080	(2.847, 3.160)	
	γ	-4.002	0.091	$(-4.180, -3.823)$	
	τ	0.479	0.216	(0.056, 0.901)	

Table 4: Simulations for Weibull-Cauchy model ($\alpha = 1$, $\beta = 2$, $\lambda = 3$, $\gamma = -4$ and $\tau = 0.5$).

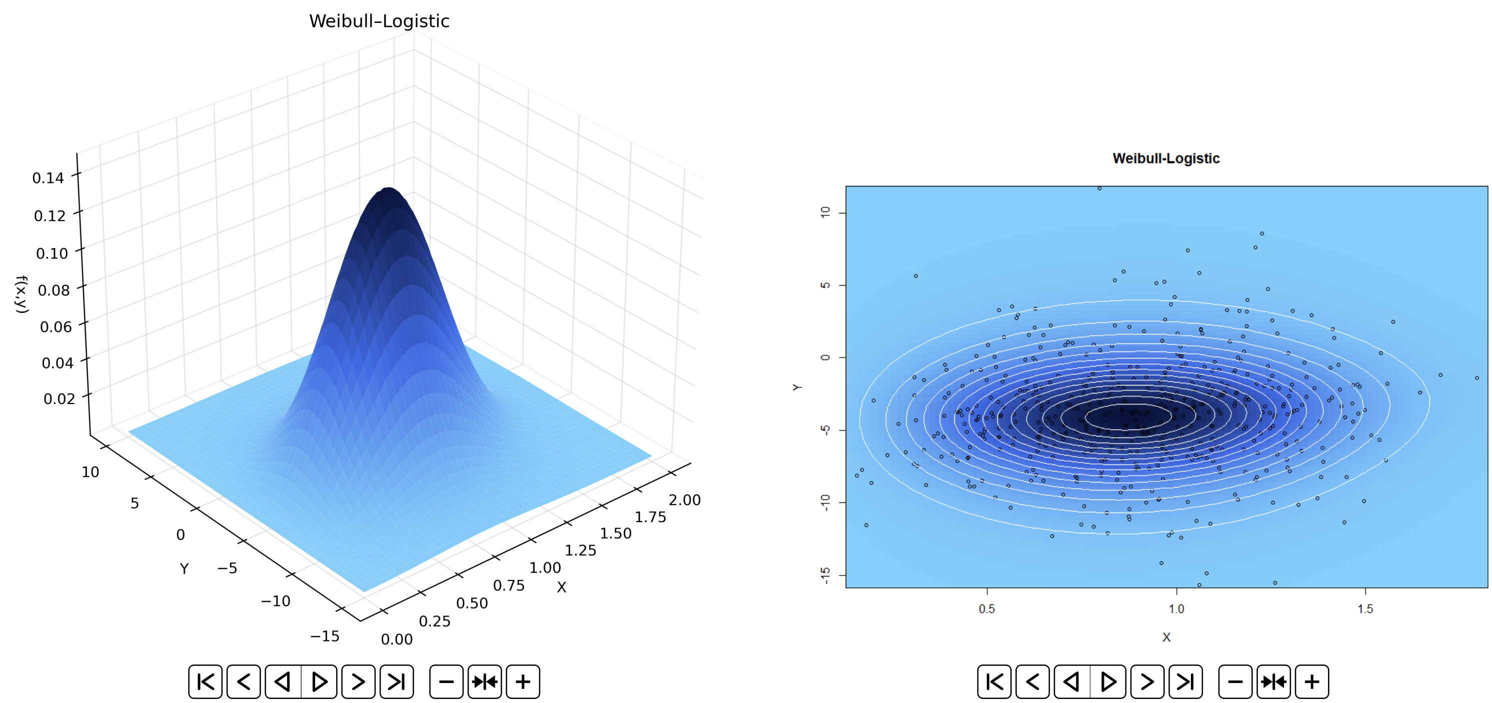

Figure 1: Bivariate densities of the models with parameter values: $\alpha = 1$, $\beta = 2$, $\lambda = 3$, $\gamma = -4$, and $\tau = 0.5$. The left panel displays the three-dimensional bivariate density surfaces, while the right panel presents the corresponding contour plots (a single dataset of $n = 500$ used; data in black circles).

4 A Small Application

The dataset employs the logarithm of the Volatility Index (VIX) as the independent variable X, and the logarithm of daily returns of Nasdaq-100 E-mini futures (NQ) as the dependent variable Y conditioned on $\{X > x\}$, covering the period from 1 January 2020 to 17 October 2025 [5]. The data was obtained from Yahoo Finance [3]. The Pearson correlation between X and Y is -0.140; con-

[5]Observations containing missing values were removed. These omissions primarily arose from non-trading days (weekends and public holidays) and discrepancies in trading calendars between the VIX and NQ futures contracts.

sequently, the Weibull-Gumbel model is unsuitable for this dataset, as previously noted, such a specification cannot accommodate negative correlations.

The corresponding m.m.e's, m.l.e's, and AIC values for the VIX/NQ dataset is reported in Table 5. It is evident from the presented AIC values that the Weibull-Laplace model provides the best fit. This outcome is not entirely unexpected: as can be observed from the histogram of the logarithm of daily Nasdaq-100 returns, in Figure 2, the empirical distribution visually resembles that of a Laplace distribution. This finding underscores the importance of conducting formal goodness-of-fit tests for the marginal variables X and Y prior to estimating the joint models proposed in this study. In particular, given that the modeling framework assumes $X \sim Weibull(\alpha, \lambda)$ and $Y \sim Family(\mu(0), \beta)$, it is prudent to verify that the empirical distributions of X and Y from the data are consistent with the respective assumed distributional families.

Furthermore, the Weibull parameters α and λ reported in Table 5 are estimated almost identically across the three models. This indicates that all three models produce nearly identical Weibull fits for the logarithm of the VIX, as illustrated in Figure 2. In retrospect, it may have been more appropriate to construct the study's models using a Gamma marginal, which could perhaps offer a better fit for the logarithm of the VIX.

Estimated bivariate densities for each model, along with their corresponding contour plots, using m.l.e's from Table 5, are presented in Figure 3.

Model	Parameter	m.m.e	m.l.e	AIC
Weibull-Logistic	α	0.319	0.318	-6958.266
	β	0.009	0.008	
	λ	11.572	8.694	
	γ	0.001	0.001	
	τ	0.903	0.848	
	ρ	-0.140	-0.111	
Weibull-Laplace	α	0.319	0.318	$\mathbf{-6997.162}$
	β	0.011	0.011	
	λ	11.572	8.688	
	γ	0.001	0.001	
	τ	0.884	0.749	
	ρ	-0.140	-0.048	
Weibull-Cauchy	α	-	0.318	-6737.427
	β	-	0.008	
	λ	-	8.732	
	γ	-	0.001	
	τ	-	0.800	

Table 5: m.m.e's, m.l.e's and AIC's of models for the VIX/NQ dataset.

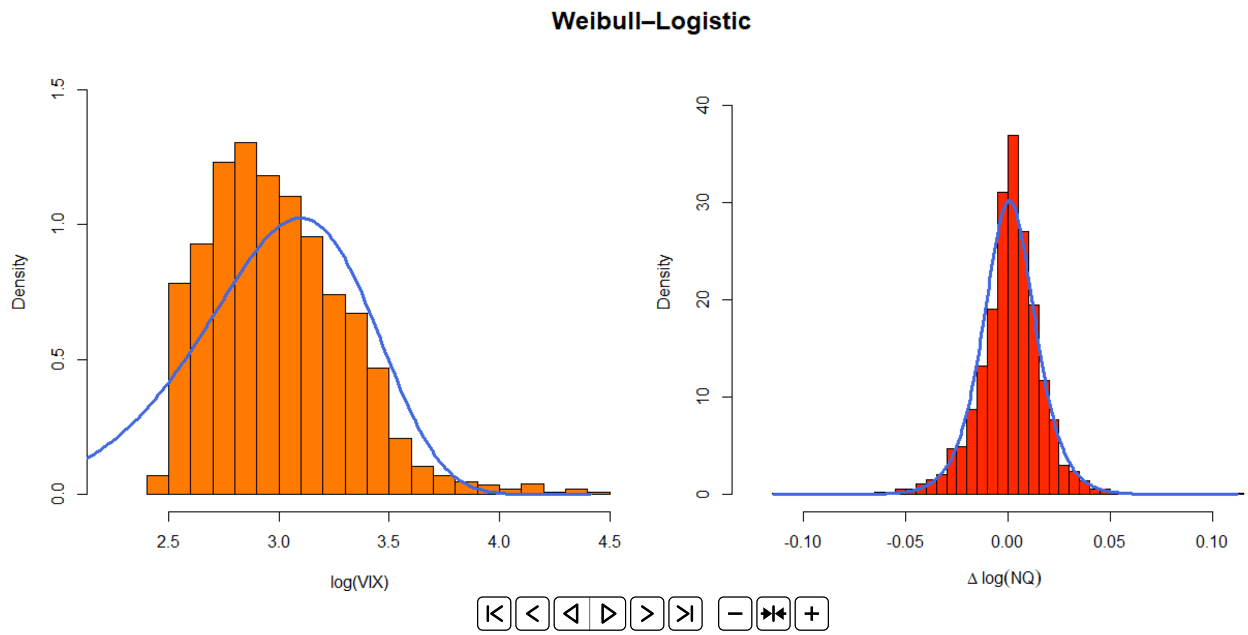

Figure 2: Histogram of the logarithm of the VIX (left) with estimated marginal Weibull densities based on m.l.e's reported in Table 5, and histogram of the logarithm of daily Nasdaq-100 (NQ) returns (right) with estimated marginal logistic, Laplace, and Cauchy densities, also based on the corresponding m.l.e's in Table 5.

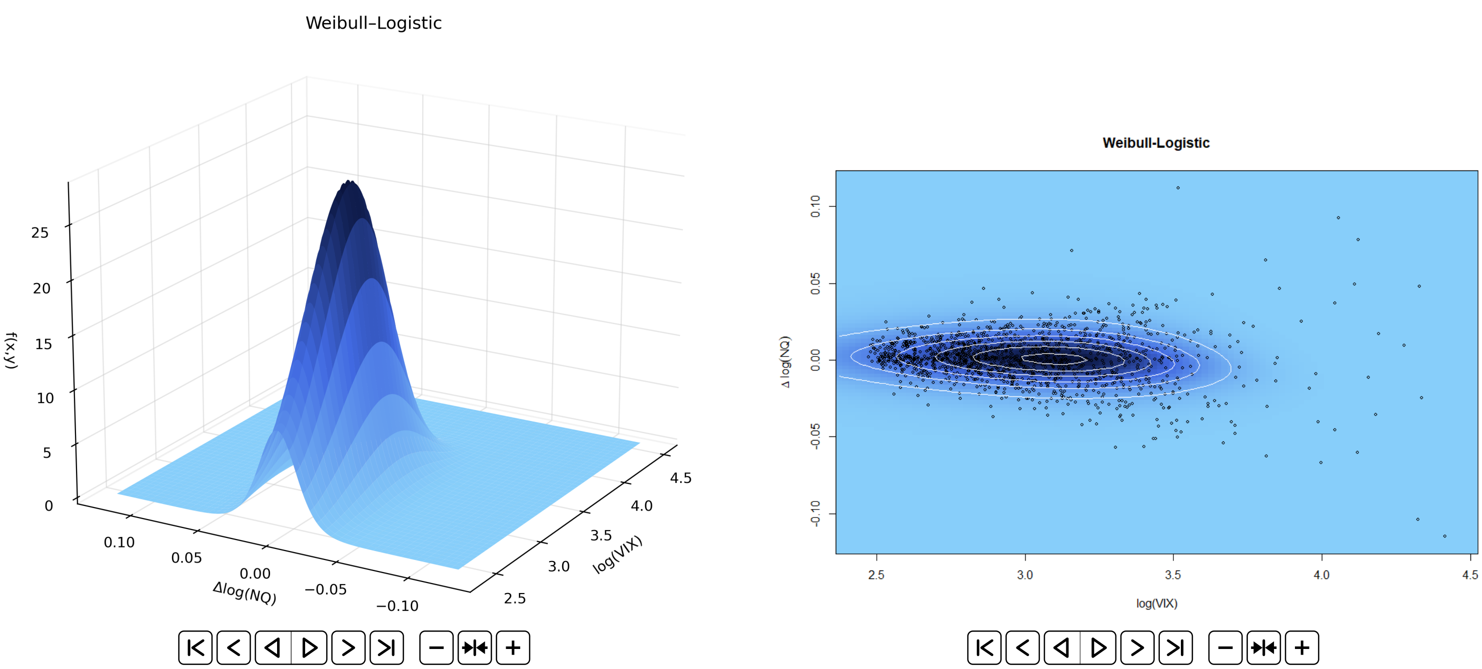

Figure 3: Estimated bivariate densities of the models using m.l.e's from Table 5. The left panel displays the estimated 3-D bivariate density surfaces, while the right panel presents the corresponding contour plots (observations in black circles).

A General Theorems

Theorem 1. *Consider an accelerated conditional model of the form:*

$$P(X > x) = \bar{F}_0(x) \quad \textit{and} \quad P(Y > y \mid X > x) = \bar{F}_1\left(\frac{y-\mu}{\beta(x)}\right),$$

where $\bar{F}_0$ and $\bar{F}_1$ are survival functions. If the marginal of Y is symmetric about μ, then $Cov(X,Y) = 0$.

Proof. Since $P(Y > y, X > x) = \bar{F}_0(x)\bar{F}_1\left(\frac{y-\mu}{\beta(x)}\right)$ then $P(Y > y) = \lim_{x\to 0^+} P(Y > y, X > x) = \bar{F}_0(0)\bar{F}_1\left(\frac{y-\mu}{\beta(0)}\right) = \bar{F}_1\left(\frac{y-\mu}{\beta(0)}\right)$ since $\lim_{x\to 0^+} \bar{F}_0(x) = \bar{F}_0(0) = 1$ and $\beta(0) := \lim_{x\to 0^+} \beta(x)$. We have from Hoeffding's covariance identity:

$$\begin{aligned}
Cov(X,Y) &= \int_0^\infty \int_{-\infty}^\infty \left(P(Y > y, X > x) - P(X > x)P(Y > y)\right) dy\, dx \\
&= \int_0^\infty \int_{-\infty}^\infty \bar{F}_0(x) \left\{ \bar{F}_1\left(\frac{y-\mu}{\beta(x)}\right) - \bar{F}_1\left(\frac{y-\mu}{\beta(0)}\right) \right\} dy\, dx.
\end{aligned}$$

We split the inner integral at the symmetry point $y = \mu$:

$$\int_{-\infty}^\infty [\cdots]\, dy = \int_{-\infty}^\mu [\cdots]\, dy + \int_\mu^\infty [\cdots]\, dy.$$

By symmetry of $\bar{F}_1$, for any $\beta(x) > 0$ and all $y < \mu$,

$$\bar{F}_1\left(\frac{y-\mu}{\beta(x)}\right) = 1 - \bar{F}_1\left(\frac{\mu-y}{\beta(x)}\right).$$

Substituting into the lower-half integral, it becomes:

$$\int_{-\infty}^\mu [\cdots]\, dy = -\int_{-\infty}^\mu \left\{ \bar{F}_1\left(\frac{\mu-y}{\beta(x)}\right) - \bar{F}_1\left(\frac{\mu-y}{\beta(0)}\right) \right\} dy.$$

Changing the integration limits by the reflection identity:

$$\int_{-\infty}^\mu \phi(\mu-y)\, dy = \int_\mu^\infty \phi(y-\mu)\, dy,$$

we rewrite the lower-half integral as:

$$\int_{-\infty}^\mu [\cdots]\, dy = -\int_\mu^\infty \left\{ \bar{F}_1\left(\frac{y-\mu}{\beta(x)}\right) - \bar{F}_1\left(\frac{y-\mu}{\beta(0)}\right) \right\} dy.$$

Hence, for each fixed x,

$$\int_{-\infty}^\infty [\cdots]\, dy = \int_{-\infty}^\mu [\cdots]\, dy + \int_\mu^\infty [\cdots]\, dy = -\int_\mu^\infty \left\{ \bar{F}_1\left(\frac{y-\mu}{\beta(x)}\right) - \bar{F}_1\left(\frac{y-\mu}{\beta(0)}\right) \right\} dy + \int_\mu^\infty \left\{ \bar{F}_1\left(\frac{y-\mu}{\beta(x)}\right) - \bar{F}_1\left(\frac{y-\mu}{\beta(0)}\right) \right\} dy = 0.$$

Hence, $Cov(X,Y) = \int_0^\infty \bar{F}_0(x) \cdot 0\, dx = 0$. □

Theorem 2. *Consider an accelerated conditional model of the form:*

$$P(X > x) = \bar{F}_0(x) \quad \textit{and} \quad P(Y > y \mid X > x) = \bar{F}_1\left(\frac{y-\mu(x)}{\beta}\right), \quad x > 0,\ y \in \mathbb{R},$$

where $\bar{F}_0$ and $\bar{F}_1$ are survival functions and $\beta > 0$ and $\mu(x) \in \mathbb{R}$. Then:

$$\mu'(x) \leq 0 \quad \Longrightarrow \quad Cov(X,Y) \leq 0, \qquad \mu'(x) \geq 0 \quad \Longrightarrow \quad Cov(X,Y) \geq 0.$$

Proof. Since $P(Y > y, X > x) = \bar{F}_0(x)\bar{F}_1\left(\frac{y-\mu(x)}{\beta}\right)$ then $P(Y > y) = \lim_{x\to 0^+} P(Y > y, X > x) = \bar{F}_0(0)\bar{F}_1\left(\frac{y-\mu(0)}{\beta}\right) = \bar{F}_1\left(\frac{y-\mu(0)}{\beta}\right)$ since $\lim_{x\to 0^+} \bar{F}_0(x) = \bar{F}_0(0) = 1$ and $\mu(0) := \lim_{x\to 0^+} \mu(x)$, we note:

$$\begin{aligned} Cov(X,Y) &= \int_{-\infty}^{\infty}\int_0^{\infty} \left(P(Y > y, X > x) - P(X > x)P(Y > y)\right)\, dx\, dy \\ &= \int_{-\infty}^{\infty}\int_0^{\infty} \bar{F}_0(x)\left(\bar{F}_1\left(\frac{y-\mu(x)}{\beta}\right) - \bar{F}_1\left(\frac{y-\mu(0)}{\beta}\right)\right)\, dx\, dy. \end{aligned}$$

Now since $\bar{F}_1$ is non-increasing (as $\bar{F}_1$ is a survival function), if $\mu'(x) \leq 0$ then $\bar{F}_1\left(\frac{y-\mu(x)}{\beta}\right) \leq \bar{F}_1\left(\frac{y-\mu(0)}{\beta}\right)$ and $Cov(X,Y) \leq 0$ for $x, y \in \mathbb{R}$. Conversely, if $\mu'(x) \geq 0$ then $\bar{F}_1\left(\frac{y-\mu(x)}{\beta}\right) \geq \bar{F}_1\left(\frac{y-\mu(0)}{\beta}\right)$ and $Cov(X,Y) \geq 0$ for $x, y \in \mathbb{R}$. Finally, if $\mu(x)$ is constant, say $\mu(x) \equiv c$, then $P(X > x, Y > y) = \bar{F}_0(x)\,\bar{F}_1\left(\frac{y-c}{\beta}\right) = P(X > x)\,P(Y > y)$, so X and Y are independent and $Cov(X,Y) = 0$. $\square$

Theorem 3. *Let F_1 be any (not necessarily symmetric) distribution function on $\mathbb{R}$ with survival $\bar{F}_1(u) = 1 - F_1(u)$, and let $\beta > 0$, $\mu, \gamma \in \mathbb{R}$. Then*

$$\int_{-\infty}^{\infty} \left(\bar{F}_1\left(\tfrac{y-\mu}{\beta}\right) - \bar{F}_1\left(\tfrac{y-\gamma}{\beta}\right)\right) dy \;=\; \mu - \gamma. \tag{30}$$

Proof. Assume in addition that F_1 has a density f_1 so that $\bar{F}_1'(u) = -f_1(u)$. Define

$$H(a) := \int_{-\infty}^{\infty} \bar{F}_1\left(\tfrac{y-a}{\beta}\right) dy, \qquad a \in \mathbb{R}.$$

Formally differentiating under the integral sign (justified by dominated convergence since $0 \leq \bar{F}_1 \leq 1$ and f_1 integrates to 1), we obtain

$$H'(a) = \int_{-\infty}^{\infty} \frac{\partial}{\partial a}\bar{F}_1\left(\tfrac{y-a}{\beta}\right) dy = \int_{-\infty}^{\infty} \frac{1}{\beta} f_1\left(\tfrac{y-a}{\beta}\right) dy = \int_{-\infty}^{\infty} f_1(t)\, dt = 1,$$

where we used the substitution $t = (y-a)/\beta$. Hence $H(a) = a + C$ for some constant C. Therefore,

$$\int_{-\infty}^{\infty} \left(\bar{F}_1\left(\tfrac{y-\mu}{\beta}\right) - \bar{F}_1\left(\tfrac{y-\gamma}{\beta}\right)\right) dy = H(\mu) - H(\gamma) = (\mu + C) - (\gamma + C) = \mu - \gamma,$$

which is (30). $\square$

B Joint Densities and Log-likelihoods

B.1 Weibull-Logistic

We obtain the joint density function by differentiating the joint survival function (17):

$$f_{X,Y}(x,y) = \frac{\left(\alpha^\lambda\beta\lambda x^\lambda e^{\frac{y}{\beta}} + \alpha^\lambda\beta\lambda x^\lambda e^{\frac{\mu(x)}{\beta}} - xe^{\frac{y}{\beta}}\frac{d}{dx}\mu(x) + xe^{\frac{\mu(x)}{\beta}}\frac{d}{dx}\mu(x)\right)e^{-\alpha^\lambda x^\lambda + \frac{y}{\beta} + \frac{\mu(x)}{\beta}}}{\beta^2 x\left(e^{\frac{3y}{\beta}} + 3e^{\frac{y+2\mu(x)}{\beta}} + 3e^{\frac{2y+\mu(x)}{\beta}} + e^{\frac{3\mu(x)}{\beta}}\right)}$$

where $\lambda \in \mathbb{R}$ and $\alpha, \beta, \lambda > 0$ with $\tau \in [0,1]$. The log-likelihood is thus:

$$\begin{aligned} \ell(\alpha,\beta,\gamma,\lambda,\tau) = \sum_{i=1}^{n} \Bigg[&\log\Big(\alpha^\lambda\beta\lambda x_i^\lambda\left(e^{y_i/\beta} + e^{\mu(x_i)/\beta}\right) + x_i\frac{d\mu(x_i)}{dx}\left(-e^{y_i/\beta} + e^{\mu(x_i)/\beta}\right)\Big) - \alpha^\lambda x_i^\lambda + \frac{y_i}{\beta} + \frac{\mu(x_i)}{\beta} \\ &- 2\log\beta - \log x_i - 3\log\left(e^{y_i/\beta} + e^{\mu(x_i)/\beta}\right)\Bigg]. \end{aligned}$$

B.2 Weibull-Gumbel

We obtain the joint density function by differentiating the joint survival function (21):

$$f_{XY}(x,y) = \frac{\left(\alpha^\lambda \beta\lambda x^\lambda e^{\frac{y}{\beta}} - x e^{\frac{y}{\beta}} \frac{d\mu(x)}{dx} + x e^{\frac{\mu(x)}{\beta}} \frac{d\mu(x)}{dx}\right) e^{-\alpha^\lambda x^\lambda - e^{\frac{-y+\mu(x)}{\beta}} - \frac{2y}{\beta} + \frac{\mu(x)}{\beta}}}{\beta^2 x}.$$

where $\lambda \in \mathbb{R}$ and $\alpha, \beta, \lambda > 0$ with $\tau \in [0,1]$. The log-likelihood is thus:

$$\ell(\alpha,\beta,\gamma,\lambda,\tau) = \sum_{i=1}^{n} \left[\log\left(\alpha^\lambda \beta\lambda x_i^\lambda e^{\frac{y_i}{\beta}} - x_i e^{\frac{y_i}{\beta}} \frac{d\mu(x_i)}{dx} + x_i e^{\frac{\mu(x_i)}{\beta}} \frac{d\mu(x_i)}{dx}\right) - \alpha^\lambda x_i^\lambda - e^{\frac{-y_i+\mu(x_i)}{\beta}} - \frac{2y_i}{\beta} + \frac{\mu(x_i)}{\beta} - 2\log\beta - \log x_i\right].$$

B.3 Weibull-Laplace

We obtain the joint density function by differentiating the joint survival function (25):

$$f_{XY}(x,y) = \frac{\left[\alpha^\lambda \beta\lambda x^\lambda - \operatorname{sgn}(y-\mu(x))\, x\, \mu'(x)\right] \exp\left(-\alpha^\lambda x^\lambda - \frac{|y-\mu(x)|}{\beta}\right)}{2\beta^2 x}$$

where $\lambda \in \mathbb{R}$ and $\alpha, \beta, \lambda > 0$ with $\tau \in [0,1]$. The log-likelihood is thus:

$$\ell(\alpha,\beta,\gamma,\lambda,\tau) = \sum_{i=1}^{n} \left[\log\left(\alpha^\lambda \beta\lambda x_i^\lambda - \operatorname{sgn}\left(y_i-\mu(x_i)\right) x_i\, \mu'(x_i)\right) - \alpha^\lambda x_i^\lambda - \frac{|y_i-\mu(x_i)|}{\beta} - \log 2 - 2\log\beta - \log x_i\right].$$

B.4 Weibull-Cauchy

We obtain the joint density function by differentiating the joint survival function (29):

$$f_{XY}(x,y) = \frac{\beta\left[\lambda(\alpha x)^\lambda\left(\beta^2 + (y-\mu(x))^2\right) - 2x(y-\mu(x))\frac{d\mu(x)}{dx}\right] e^{-(\alpha x)^\lambda}}{\pi x\left(\beta^2+(y-\mu(x))^2\right)^2}.$$

where $\lambda \in \mathbb{R}$ and $\alpha, \beta, \lambda > 0$ with $\tau \in [0,1]$. The log-likelihood is thus:

$$\ell(\alpha,\beta,\gamma,\lambda,\tau) = \sum_{i=1}^{n} \left[\log\left(\beta\left[\lambda(\alpha x_i)^\lambda\left(\beta^2 + (y_i-\mu(x_i))^2\right) - 2x_i(y_i-\mu(x_i))\frac{d\mu(x_i)}{dx}\right]\right) - (\alpha x_i)^\lambda - \log\pi - \log x_i - 2\log\left(\beta^2 + (y_i-\mu(x_i))^2\right)\right].$$

C Metropolis-Hastings

We detail the Metropolis-Hastings algorithm used in the study, to generate dependent pairs (X,Y), where $f_{X,Y}$ can be obtained in Appendix B.

Step 1 (Initialization): Choose any starting point $(x^{(0)}, y^{(0)})$ in the support of $f_{X,Y}$. Set $j=0$.

Step 2 (Proposal): Given the current state $(x^{(j)}, y^{(j)})$, draw a candidate

$$(x^*, y^*) \sim q(\cdot \mid x^{(j)}, y^{(j)}).$$

The study utilises an independence proposal:

$$q(x^*, y^* \mid x^{(j)}, y^{(j)}) = q(x^*, y^*) = f_X(x^*)\, f_Y(y^*).$$

Step 3 (Acceptance ratio α_a): Compute:

$$\begin{aligned}\alpha_a &= \min\left(\frac{f_{X,Y}(x^*,y^*)\, q(x^{(j)}, y^{(j)} \mid x^*, y^*)}{f_{X,Y}(x^{(j)}, y^{(j)})\, q(x^*, y^* \mid x^{(j)}, y^{(j)})}, 1\right) \\ &= \min\left(\frac{f_{X,Y}(x^*,y^*)}{f_X(x^*)\, f_Y(y^*)} \Big/ \frac{f_{X,Y}(x^{(j)}, y^{(j)})}{f_X(x^{(j)})\, f_Y(y^{(j)})}, 1\right).\end{aligned}$$

Equivalently, using logs for numerical stability,

$$\log(\alpha_a) = \min\Bigg\{\Big[\log\big(f_{X,Y}(x^*,y^*)\big) - \log\big(f_X(x^*)\big) - \log\big(f_Y(y^*)\big)\Big] - \Big[\log\big(f_{X,Y}(x^{(j)},y^{(j)})\big) - \log\big(f_X(x^{(j)})\big) - \log\big(f_Y(y^{(j)})\big)\Big], 0\Bigg\}.$$

Subsequently, (x^*, y^*) is accepted as the new pair in the Markov chain under the following acceptance criterion:

$$(x^{(j+1)}, y^{(j+1)}) = \begin{cases} (x^*, y^*), & \text{if } U < \alpha_a \\ (x^{(j)}, y^{(j)}), & \text{otherwise.} \end{cases}$$

where $U \sim Uniform(0,1)$.

D Method of Moment Estimators

Suppose data in the form of $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, ..., \mathbf{X}^{(n)}$ (where for each $i = 1, \ldots, n$, $\mathbf{X}^{(i)} = (X_{1i}, X_{2i})^T$) are obtained, which are independent and identically distributed according to 1 and 2. If $M_1, M_2 > 0$, consistent asymptotically normal method of moment estimates (m.m.e's) are acquired.
Define:

$$\begin{aligned} M_1 &= \frac{1}{n}\sum_{i=1}^{n} x_{1i} \\ M_2 &= \frac{1}{n}\sum_{i=1}^{n} x_{2i} \\ S_{12} &= \frac{1}{n}\sum_{i=1}^{n} (x_{1i} - M_1)(x_{2i} - M_2) \\ S_1 &= \frac{1}{n}\sum_{i=1}^{n} (x_{2i} - M_2)^2 \\ S_2 &= \frac{1}{n}\sum_{i=1}^{n} (x_{2i} - M_2)^2. \end{aligned}$$

We denote the method of moment estimators for $\alpha, \beta, \lambda, \gamma$ and τ as $\tilde{\alpha}, \tilde{\beta}, \tilde{\lambda}, \tilde{\gamma}$ and $\tilde{\tau}$ respectively.

Furthermore, $\tilde{\alpha}$ and $\tilde{\lambda}$ may be solved for numerically by equating (12) to M_1 and (13) to S_1. After solving for $\tilde{\alpha}$, $\tilde{\lambda}$ and $\tilde{\beta}$, use (16) to obtain:

$$\tilde{\tau} = \left(\operatorname{sgn}\left(\mu'(x)\right) \frac{\tilde{\alpha}\tilde{\lambda}S_{12}}{\tilde{\beta}\Gamma\left(1 + \frac{1}{\tilde{\lambda}}\right)}\right)^{1/\tilde{\lambda}}.$$

D.1 Weibull-Logistic

Using (18) and (19), we obtain:

$$\begin{aligned} \tilde{\gamma} &= M_2, \\ \tilde{\beta} &= \sqrt{3S_2/\pi^2}. \end{aligned}$$

D.2 Weibull-Gumbel

Using (22) and (23), we obtain:

$$\begin{aligned} \tilde{\beta} &= \sqrt{6S_2/\pi^2} \\ \tilde{\gamma} &= M_2 - \tilde{\beta}\gamma^*, \end{aligned}$$

D.3 Weibull-Laplace

Using (26) and (27), we obtain:

$$\tilde{\gamma} = M_2,$$
$$\tilde{\beta} = \sqrt{S_2/2}.$$

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