Models with Accelerated Failure Conditionals

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1 Introduction

As per Arnold and Arvanitis [1], the general accelerated failure conditionals model is of the form:

$$\bar{F}_X(x) = P(X > x) = \bar{F}_0(x), \quad x > 0, \tag{1}$$

for some survival function $\bar{F}_0(x) \in (0,1)\ \forall x > 0$, and for each $x > 0$:

$$P(Y > y \mid X > x) = \bar{F}_1\left(\beta(x)y\right), \quad x, y > 0, \tag{2}$$

for some survival function $\bar{F}_1(\beta(x)y) \in (0,1)\ \forall x, y > 0$ and a suitable acceleration function $\beta(x)$. The joint survival function is then:

$$P(X > x, Y > y) = \bar{F}_0(x)\bar{F}_1\left(\beta(x)y\right), \quad x, y > 0 \tag{3}$$

Assuming differentiability and X, Y are continuous, we obtain the marginal densities:

$$f_X(x) = -\frac{d}{dx}P(X > x) = -\frac{d}{dx}\bar{F}_0(x) = f_0(x),$$

and, since $\lim_{x\to 0^+} \bar{F}_0(x) = 1$, where $\beta(0) := \lim_{x\to 0^+} \beta(x)$:

$$P(Y > y) = \lim_{x\to 0^+} P(Y > y, X > x) = \bar{F}_1\big(\beta(0)y\big).$$

Hence the marginal distribution of Y belongs to the same family as $\bar{F}_1$, but with its argument scaled by $\beta(0)$. Now, the density of Y is:

$$f_Y(y) = -\frac{d}{dy}P(Y > y) = -\frac{d}{dy}\bar{F}_1(\beta(0)y) = \beta(0)\, f_1(\beta(0)y),$$

where $f_1(t) = -\bar{F}_1'(t)$ denotes the density associated with $\bar{F}_1$. For (3) to be a valid survival function, it must have a non-negative mixed partial derivative. Denoting f_0 and f_1 as the densities corresponding to the survival function $\bar{F}_0$ and $\bar{F}_1$, and differentiating, we obtain:

$$\begin{aligned}\frac{\partial}{\partial x}\frac{\partial}{\partial y}P(X > x, Y > y) &= f_0(x)f_1(\beta(x)y)\beta(x) - \bar{F}_0(x)f_1'(\beta(x)y)\beta(x)\beta'(x)y - \bar{F}_0(x)f_1(\beta(x)y)\beta'(x)\\ &= f_0(x)f_1(\beta(x)y)\beta(x) - \beta'(x)\bar{F}_0(x)\left[f_1(\beta(x)y) + f_1'(\beta(x)y)\beta(x)y\right].\end{aligned}$$

Denoting $t = \beta(x)y$, we have:

$$\begin{aligned}\frac{\partial}{\partial x}\frac{\partial}{\partial y}P(X > x, Y > y) &= f_0(x)f_1(t)\beta(x) - \beta'(x)\bar{F}_0(x)\left[f_1(t) + f_1'(t)t\right]\\ &= f_0(x)f_1(t)\beta(x) - \beta'(x)\bar{F}_0(x)\left[tf_1(t)\right]'.\end{aligned}$$

Thus a necessary and sufficient condition for $\frac{\partial}{\partial x}\frac{\partial}{\partial y}P(X > x, Y > y) \geq 0$ is:

$$\beta'(x)\begin{cases}\leq \dfrac{f_0(x)}{\bar{F}_0(x)}\,\dfrac{f_1(t)}{[tf_1(t)]'}\,\beta(x), & \text{if } [tf_1(t)]' > 0,\\[2ex] \geq \dfrac{f_0(x)}{\bar{F}_0(x)}\,\dfrac{f_1(t)}{[tf_1(t)]'}\,\beta(x), & \text{if } [tf_1(t)]' < 0,\\ \text{unconstrained}, & \text{if } [tf_1(t)]' = 0.\end{cases}$$

Denoting the hazard function $h_0(x) = \frac{f_0(x)}{\bar{F}_0(x)}$ and $S(t) = \frac{f_1(t)}{[tf_1(t)]'}$, we may write the bounds as such:

$$\sup_{t:[tf_1(t)]'<0} h_0(x)S(t)\beta(x) \ \le\ \beta'(x) \ \le\ \inf_{t:[tf_1(t)]'>0} h_0(x)S(t)\beta(x). \tag{4}$$

A special case of the upper bound for $B'(x)$ arises for distribution families satisfying $f_1'(t) \le 0$. In this case, we note that $S(t) = \frac{f_1(t)}{[tf_1(t)]'} = \frac{f_1(t)}{f_1(t)+tf_1'(t)} \ge 1$ for $t \ge 0$, hence $\beta'(x) \ \le\ h_0(x)\beta(x)$.

It should be noted that multiple functional forms for $\beta(x)$ are possible. The question of selecting a form of $\beta(x)$ that provides the best model fit has already been investigated by Arnold and Arvanitis [1]. Accordingly, the present study restricts attention to those choices of $\beta(x)$ that arise from the extreme case in which $\beta'(x)$ attains its theoretical upper bound, that is, given $\inf_{t:[tf_1(t)]'>0} S(t) = c^*$:

$$\begin{aligned}
\beta'(x) &= c^* h_0(x)\beta(x) \\
\therefore \int \frac{1}{\beta(x)} d\beta(x) &= \int c^* h_0(x) dx \\
\therefore \log(\beta(x)) &= -c^* \log\left(\bar{F}_0(x)\right) + K \\
\therefore \beta(x) &= \gamma \frac{1}{\left[\bar{F}_0(x)\right]^{c^*}},
\end{aligned} \tag{5}$$

for $\gamma > 0$. This suggests $\beta(x) > 0$ for $x > 0$. Furthermore, $\beta(0) := \lim_{x\to 0^+} \beta(x) = \gamma$ since $\lim_{x\to 0^+} \bar{F}_0(x) = 1$.

The role of the acceleration function $\beta(x)$ in determining the sign of the correlation between X and Y is established in Theorem 1. In particular, since the present study considers only positive specifications of $\beta(x)$, the resulting framework is restricted to modelling data in which X and Y exhibit negative correlation.

Theorem 1. *Consider an accelerated conditional model of the form:*

$$P(X > x) = \bar{F}_0(x) \quad \textit{and} \quad P(Y > y \mid X > x) = \bar{F}_1\big(\beta(x)y\big),$$

where $\bar{F}_0$ and $\bar{F}_1$ are survival functions. Then:

$$\beta'(x) \ge 0 \quad\Longrightarrow\quad \mathrm{Cov}(X,Y) \le 0, \qquad \beta'(x) \le 0 \quad\Longrightarrow\quad \mathrm{Cov}(X,Y) \ge 0.$$

Proof. Since $P(Y > y, X > x) = \bar{F}_0(x)\bar{F}_1\big(\beta(x)y\big)$ then $P(Y > y) = \lim_{x\to 0^+} P(Y > y, X > x) = \bar{F}_0(0)\bar{F}_1\big(\beta(0)y\big) = \bar{F}_1\big(\beta(0)y\big)$ since $\lim_{x\to 0^+} \bar{F}_0(x) = 1$, we note:

$$\begin{aligned}
Cov(X,Y) &= \int_0^\infty \int_0^\infty \left(P(Y > y, X > x) - P(X > x)P(Y > y)\right) \, dxdy \\
&= \int_0^\infty \int_0^\infty \bar{F}_0(x)\left(\bar{F}_1\big(\beta(x)y\big) - \bar{F}_1\big(\beta(0)y\big)\right) \, dxdy.
\end{aligned}$$

Now since $\bar{F}_1'(\cdot) \le 0$ (as $\bar{F}_1$ is a survival function), if $\beta'(x) \ge 0$ then $\bar{F}_1\big(\beta(x)y\big) \le \bar{F}_1\big(\beta(0)y\big)$ and $Cov(X,Y) \le 0$ for $x, y \ge 0$. Conversely, if $\beta'(x) \le 0$ then $\bar{F}_1\big(\beta(x)y\big) \ge \bar{F}_1\big(\beta(0)y\big)$ and $Cov(X,Y) \ge 0$ for $x, y \ge 0$. Finally, if $\beta(x)$ is constant, say $\beta(x) \equiv c$, then $P(X > x, Y > y) = \bar{F}_0(x)\,\bar{F}_1(cy) = P(X > x)\,P(Y > y)$, so X and Y are independent and $Cov(X,Y) = 0$. □

2 Exponential

We restrict the survival function forms to the exponential family as such:

$$\bar{F}_0(x) = P(X > x) = e^{-\alpha x}, \quad x > 0,$$

where $\alpha > 0$ and, for each $x > 0$:

$$\bar{F}_1(\beta(x)y) = P(Y > y \mid X > x) = e^{-\beta(x)y}, \quad y > 0,$$

where $\beta(x) > 0$ for all $x > 0$ to ensure a valid survival function. The corresponding joint survival function will be:

$$\bar{F}_0(x)\bar{F}_1(\beta(x)y) = P(X > x, Y > y) = e^{-\alpha x - \beta(x)y}, \quad x, y > 0. \tag{6}$$

Accordingly, $f_X(x) = \alpha e^{-(\alpha x)}$ for $x > 0$ and $f_Y(y) = \beta(0)e^{-(\beta(0)y)}$ for $y > 0$. Hence, $X \sim Exp(\alpha)$ and $Y \sim Exp(\beta(0))$.

Now since $[tf_1(t)]' = (1-t)e^{-t} > 0$ for $t < 1$, and $S(t) = -\frac{1}{t-1}$ is increasing on this interval, $\inf_{t<1} S(t) = \lim_{t\to 0} S(t) = 1$. Additionally, since $[tf_1(t)]' = (1-t)e^{-t} < 0$ for $t > 1$, and $S(t) = -\frac{1}{t-1}$ is increasing on this interval, $\sup_{t>1} S(t) = \lim_{t\to\infty} S(t) = 0$. Consequently, applying (4) yields:

$$0 \le B'(x) \le h_0(x)\beta(x) = \alpha\beta(x),$$

corresponding to Proposition 3.1 in Arnold and Arvanitis [1]. Now, we select the form of the acceleration function $\beta(x)$ such that $\beta'(x) = \alpha\beta(x)$ (that is, equal to $\beta'(x)'s$ upper bound). Hence from (5) with $c^* = 1$, $\beta(x) = \gamma e^{\alpha x}$ with $\gamma > 0$. Furthermore, following Arnold and Arvanitis [1], we introduce a dependence parameter $\tau \in [0, 1]$ which governs the strength of the dependence between X and Y. The parameter τ is restricted to $[0, 1]$ since $\tau = 0$ implies $\beta(x) = \gamma$ is constant (so X and Y are independent with covariance 0), $\tau = 1$ corresponds to the case where $\beta'(x)$ attains its theoretical upper bound, where allowing $\tau > 1$ would imply $\beta'(x) > \alpha\beta(x)$, violating the theoretical upper bound. Hence we choose our acceleration function as:

$$\beta(x) = \gamma e^{\alpha\tau x}.$$

2.1 Moments

Noting $X \sim Exp(\alpha)$ and $Y \sim Exp(\gamma)$, we have:

$$E(X) = \frac{1}{\alpha}, \tag{7}$$
$$E(Y) = \frac{1}{\gamma}, \tag{8}$$
$$Var(X) = \frac{1}{\alpha^2}, \tag{9}$$
$$Var(Y) = \frac{1}{\gamma^2}. \tag{10}$$

Accordingly:

$$\begin{aligned}
Cov(X,Y) &= \int_0^\infty \int_0^\infty \left(P(Y > y, X > x) - P(X > x)P(Y > y)\right)\, dxdy \\
&= \int_0^\infty \int_0^\infty \bar{F}_0(x)\left(\bar{F}_1(\beta(x)y) - \bar{F}_1(\beta(0)y)\right)\, dxdy \\
&= \int_0^\infty \int_0^\infty e^{-(\alpha x)}\left(e^{-\beta(x)y} - e^{-\beta(0)y}\right)\, dydx \\
&= \int_0^\infty e^{-(\alpha x)}\left(\frac{1}{\beta(x)} - \frac{1}{\beta(0)}\right)\, dx \\
&= \frac{1}{\gamma}\int_0^\infty \left(e^{-\alpha(1+\tau)x} - e^{-(\alpha x)}\right)\, dx \\
&= -\frac{\tau}{\alpha\gamma(1+\tau)}.
\end{aligned} \tag{11}$$

Using (9), (10) and (11), we have the correlation between X and Y as:

$$\begin{aligned}
\rho(X,Y) &= \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} \\
&= -\frac{\tau}{1+\tau}.
\end{aligned}$$

Seeing as $\tau \in [0,1]$, this exponential model with particular acceleration acceleration funciton $\beta(x) = \gamma e^{\alpha\tau x}$, may only be used to model data for which $\rho(X,Y) \in [0, -\frac{1}{2}]$.

Additionally, to facilitate simulation, it is convenient to formally note the conditional cumulative distribution function $F_{Y|X}$ as

$$
\begin{aligned}
F_{Y|X=x}(y) &= \int_0^y \frac{f_{X,Y}(x,t)}{f_X(x)}\,dy \\
&= (-\gamma\tau y e^{\alpha\tau x} + e^{\gamma y e^{\alpha\tau x}} - 1)e^{-\gamma y e^{\alpha\tau x}},
\end{aligned}
$$

where $f_{X,Y}(x,y)$ is provided in Appendix **??**.

$$
-\frac{\tau W\left(\frac{(U-1)e^{-\frac{1}{\tau}}}{\tau}\right)+1}{\gamma\tau e^{\alpha\tau x}} \tag{12}
$$

where W is the lower branch of the Lambert W function.

3 Lomax

We restrict the survival function forms to the Lomax family as such:

$$
\bar{F}_0(x) = P(X > x) = (1+\alpha x)^{-\lambda}, \quad x > 0,
$$

where $\alpha, \lambda > 0$ and, for each $x > 0$:

$$
\bar{F}_1(\beta(x)y) = P(Y > y \mid X > x) = (1+\beta(x)y)^{-\nu}, \quad y > 0,
$$

where $\beta(x), \nu > 0$ for all $x > 0$ to ensure a valid survival function. The corresponding joint survival function will be:

$$
\bar{F}_0(x)\bar{F}_1(\beta(x)y) = P(X > x, Y > y) = (1+\alpha x)^{-\lambda}(1+\beta(x)y)^{-\nu}, \quad x, y > 0. \tag{13}
$$

Accordingly, $f_X(x) = \alpha\lambda(\alpha x+1)^{-(\alpha+1)}$ for $x > 0$ and $f_Y(y) = \beta(0)\nu(\beta(0)y+1)^{-(\nu+1)}$ for $y > 0$. Hence, $X \sim \mathit{Lomax}(\alpha,\lambda)$ and $Y \sim \mathit{Lomax}(\beta(0),\nu)$.

Now since $[tf_1(t)]' = \frac{\nu(t+1)^{-\nu}(1-\nu t)}{t^2+2t+1} > 0$ for $t < \frac{1}{v}$, and $S(t) = -\frac{t+1}{vt-1}$ is increasing on this interval, $\inf_{t<\frac{1}{v}} S(t) = \lim_{t\to 0} S(t) = 1$. Additionally, since $[tf_1(t)]' = \frac{\nu(t+1)^{-\nu}(1-\nu t)}{t^2+2t+1} < 0$ for $t > \frac{1}{\nu}$, and $S(t) = -\frac{t+1}{vt-1}$ is increasing on this interval, $\sup_{t>\frac{1}{\nu}} S(t) = \lim_{t\to\infty} S(t) = -\frac{1}{\nu}$. Consequently, applying (4) yields:

$$
-\frac{1}{\nu}\frac{\alpha\lambda}{\alpha x+1}\beta(x) \leq B'(x) \leq \frac{\alpha\lambda}{\alpha x+1}\beta(x).
$$

Now, we select the form of the acceleration function $\beta(x)$ such that $\beta'(x) = \frac{\alpha\lambda}{\alpha x+1}\beta(x)$ (that is, equal to $\beta'(x)'s$ upper bound). Hence from (5) with $c^* = 1$, $\beta(x) = \gamma\,(1+\alpha x)^{\lambda}$ with $\gamma > 0$. Furthermore, following Arnold and Arvanitis [1], we introduce a dependence parameter $\tau \in [0,1]$ which governs the strength of the dependence between X and Y. Hence we choose our acceleration function as:

$$
\beta(x) = \gamma\,(1+\alpha\tau x)^{\lambda}.
$$

3.1 Moments

Noting $X \sim Lomax(\alpha, \lambda)$ and $Y \sim Lomax(\gamma, \nu)$, we have:

$$E(X) = \frac{1}{\alpha(\lambda-1)}, \quad \lambda > 1, \tag{14}$$

$$E(Y) = \frac{1}{\gamma(\nu-1)}, \quad \nu > 1, \tag{15}$$

$$Var(X) = \frac{\lambda}{\alpha^2(\lambda-1)^2(\lambda-2)}, \quad \lambda > 2, \tag{16}$$

$$Var(Y) = \frac{\nu}{\gamma^2(\nu-1)^2(\nu-2)}, \quad \nu > 2. \tag{17}$$

Accordingly:

$$\begin{aligned}
\mathrm{Cov}(X,Y) &= \int_0^\infty \int_0^\infty \left(P(Y>y, X>x) - P(X>x)P(Y>y)\right) dy\, dx \\
&= \int_0^\infty \int_0^\infty \bar{F}_0(x)\left(\bar{F}_1\big(\beta(x)y\big) - \bar{F}_1\big(\beta(0)y\big)\right) dy\, dx \\
&= \int_0^\infty (1+\alpha x)^{-\lambda} \left(\int_0^\infty \left((1+\beta(x)y)^{-\nu} - (1+\beta(0)y)^{-\nu}\right) dy\right) dx \\
&= \frac{1}{\nu-1}\int_0^\infty (1+\alpha x)^{-\lambda}\left(\frac{1}{\beta(x)} - \frac{1}{\beta(0)}\right) dx \\
&= \frac{1}{\gamma(\nu-1)}\left[\int_0^\infty \frac{dx}{(1+\alpha x)^\lambda (1+\alpha\tau x)^\lambda} - \int_0^\infty \frac{dx}{(1+\alpha x)^\lambda}\right] \\
&= \frac{1}{\alpha\gamma(\nu-1)}\left[\frac{1}{2\lambda-1}\,{}_2F_1(\lambda,1;2\lambda;1-\tau) - \frac{1}{\lambda-1}\right],
\end{aligned} \tag{18}$$

where the conditions $\nu > 1$ and $\lambda > 1$ ensure convergence of the inner and outer integrals, respectively. Now, using (16), (17) and (18), the correlation is:

$$\begin{aligned}
\rho(X,Y) &= \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} \\
&= (\lambda-1)\left(\frac{{}_2F_1(\lambda,1;2\lambda;1-\tau)}{2\lambda-1} - \frac{1}{\lambda-1}\right)\sqrt{\frac{(\lambda-2)(\nu-2)}{\lambda\,\nu}},
\end{aligned} \tag{19}$$

valid for $\lambda, \nu > 2$. Since ${}_2F_1(\lambda,1;2\lambda;z)$ has a power series with strictly positive coefficients, it is strictly increasing in $z \in [0,1]$; hence ${}_2F_1(\lambda,1;2\lambda;1-\tau)$ is strictly decreasing in $\tau \in [0,1]$. Consequently, (19) attains its minimum at $\tau = 1$ and its maximum (zero) at $\tau = 0$. Therefore,

$$-\frac{\lambda}{2\lambda-1}\sqrt{\frac{\lambda-2}{\lambda}}\sqrt{\frac{\nu-2}{\nu}} \le \rho(X,Y) \le 0.$$

Moreover, $\rho_{\min}(\lambda,\nu) := -\frac{\lambda}{2\lambda-1}\sqrt{\frac{\lambda-2}{\lambda}}\sqrt{\frac{\nu-2}{\nu}}$ is strictly decreasing in each of λ and ν, so $\inf_{\lambda>2,\,\nu>2}\rho_{\min}(\lambda,\nu) = \lim_{\lambda,\nu\to\infty}\rho_{\min}(\lambda,\nu)$ $= -\frac{1}{2}$. Thus, as in the exponential case, this Lomax model with acceleration $\beta(x) = \gamma\,(1+\alpha\tau x)^\lambda$ can only accommodate correlations in the range $\rho(X,Y) \in \left[-\frac{1}{2},\, 0\right]$.

4 Weibull

We restrict the survival function forms to the Weibull family as such:

$$\bar{F}_0(x) = P(X > x) = e^{-(\alpha x)^\lambda}, \quad x > 0,$$

where $\alpha, \lambda > 0$ and, for each $x > 0$:

$$\bar{F}_1(\beta(x)y) = P(Y > y \mid X > x) = e^{-(\beta(x)y)^{\nu}}, \quad y > 0,$$

where $\beta(x), \nu > 0$ for all $x > 0$ to ensure a valid survival function. The corresponding joint survival function will be:

$$\bar{F}_0(x)\bar{F}_1(\beta(x)y) = P(X > x, Y > y) = e^{-(\alpha x)^{\lambda} - (y\beta(x))^{\nu}}, \quad x, y > 0. \tag{20}$$

Accordingly, $f_X(x) = \alpha\lambda(\alpha x)^{\lambda-1}e^{-(\alpha x)^{\lambda}}$ for $x > 0$ and $f_Y(y) = \beta(0)\nu(\beta(0)y)^{\nu-1}e^{-(\beta(0)y)^{\nu}}$ for $y > 0$. Hence, $X \sim Weibull(\alpha, \lambda)$ and $Y \sim Weibull(\beta(0), \nu)$.

Now since $[tf_1(t)]' = \nu^2 t^{\nu-1}(1 - t^{\nu})e^{-t^{\nu}} > 0$ for $t < 1$, and $S(t) = \frac{1}{\nu(1-t^{\nu})}$ is increasing on this interval, $\inf_{t<1} S(t) = \lim_{t\to 0} S(t) = \frac{1}{\nu}$. Additionally, since $[tf_1(t)]' = \nu^2 t^{\nu-1}(1-t^{\nu})e^{-t^{\nu}} < 0$ for $t > 1$, and $S(t) = \frac{1}{\nu(1-t^{\nu})}$ is increasing on this interval, $\sup_{t>1} S(t) = \lim_{t\to\infty} S(t) = 0$. Consequently, applying (4) yields:

$$0 \leq B'(x) \leq \frac{1}{\nu}\lambda\alpha(\alpha x)^{\lambda-1}\beta(x).$$

Now, we select the form of the acceleration function $\beta(x)$ such that $\beta'(x) = \frac{1}{\nu}\lambda\alpha(\alpha x)^{\lambda-1}\beta(x)$ (that is, equal to $\beta'(x)'s$ upper bound). Hence from (5) with $c^* = \frac{1}{\nu}$, $\beta(x) = \gamma e^{\frac{1}{\nu}(\alpha x)^{\lambda}}$ with $\gamma > 0$. Furthermore, following Arnold and Arvanitis [1], we introduce a dependence parameter $\tau \in [0, 1]$ as before. Hence we choose our acceleration function as:

$$\beta(x) = \gamma e^{\frac{1}{\nu}(\alpha\tau x)^{\lambda}}.$$

4.1 Moments

Noting $X \sim Weibull(\alpha, \lambda)$ and $Y \sim Weibull(\gamma, \nu)$, we have:

$$E(X) = \frac{1}{\alpha\Gamma\left(1 + \frac{1}{\lambda}\right)}, \tag{21}$$

$$E(Y) = \frac{1}{\gamma\Gamma\left(1 + \frac{1}{\nu}\right)}, \tag{22}$$

$$Var(X) = \frac{1}{\alpha^2}\left(\Gamma\left(1 + \frac{2}{\lambda}\right) - \left(\Gamma\left(1 + \frac{1}{\lambda}\right)\right)^2\right) \tag{23}$$

$$Var(Y) = \frac{1}{\gamma^2}\left(\Gamma\left(1 + \frac{2}{\nu}\right) - \left(\Gamma\left(1 + \frac{1}{\nu}\right)\right)^2\right) \tag{24}$$

Accordingly:

$$\begin{aligned}
\operatorname{Cov}(X, Y) &= \int_0^{\infty}\int_0^{\infty} \big(P(Y > y, X > x) - P(X > x)P(Y > y)\big)\, dy\, dx \\
&= \int_0^{\infty}\int_0^{\infty} \bar{F}_0(x)\left(\bar{F}_1\big(\beta(x)y\big) - \bar{F}_1\big(\beta(0)y\big)\right)\, dy\, dx \\
&= \int_0^{\infty} e^{-(\alpha x)^{\lambda}}\left(\int_0^{\infty}\left(e^{-(\beta(x)y)^{\nu}} - e^{-(\beta(0)y)^{\nu}}\right) dy\right) dx \\
&= \int_0^{\infty} e^{-(\alpha x)^{\lambda}}\left(\frac{\Gamma\big(1 + \frac{1}{\nu}\big)}{\beta(x)} - \frac{\Gamma\big(1 + \frac{1}{\nu}\big)}{\beta(0)}\right) dx \\
&= \frac{1}{\gamma}\Gamma\left(1 + \frac{1}{\nu}\right)\left[\int_0^{\infty} e^{-(\alpha x)^{\lambda} - \frac{1}{\nu}(\alpha\tau x)^{\lambda}}\, dx - \int_0^{\infty} e^{-(\alpha x)^{\lambda}}\, dx\right] \\
&= \frac{1}{\alpha\gamma}\Gamma\left(1 + \frac{1}{\nu}\right)\Gamma\left(1 + \frac{1}{\lambda}\right)\left[\left(\frac{\nu}{\nu + \tau^{\lambda}}\right)^{\frac{1}{\lambda}} - 1\right].
\end{aligned} \tag{25}$$

Now, using (23), (24) and (25), the correlation is:

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$$
$$= \frac{\Gamma\left(1+\frac{1}{\nu}\right)\Gamma\left(1+\frac{1}{\lambda}\right)\left[\left(\frac{\nu}{\nu+\tau^{\lambda}}\right)^{\frac{1}{\lambda}}-1\right]}{\sqrt{\left(\Gamma\left(1+\frac{2}{\lambda}\right)-\left(\Gamma\left(1+\frac{1}{\lambda}\right)\right)^2\right)\left(\Gamma\left(1+\frac{2}{\nu}\right)-\left(\Gamma\left(1+\frac{1}{\nu}\right)\right)^2\right)}}, \tag{26}$$

Clearly, (26) is strictly decreasing in $\tau \in [0,1]$. Hence: 1 Now, $\inf_{\lambda,\nu>0}\rho_{\min}(\lambda,\nu) = \lim_{\nu\to\infty}\rho_{\min}(\lambda=1,\nu) = -\frac{\sqrt{6}}{\pi}$. Hence, this Weibull model with acceleration function $\beta(x) = \gamma e^{\frac{1}{\nu}(\alpha\tau x)^{\lambda}}$ will be able to accommodate correlations in the range $\rho(X,Y) \in \left[-\frac{\sqrt{6}}{\pi}, 0\right]$.

5 Log-Logistic

We restrict the survival function forms to the log-logistic family as such:

$$\bar{F}_0(x) = P(X > x) = 1 - \frac{1}{1+(\alpha x)^{-\lambda}} = \frac{1}{1+(\alpha x)^{\lambda}}, \quad x > 0,$$

where $\alpha, \lambda > 0$ and, for each $x > 0$:

$$\bar{F}_1(\beta(x)y) = P(Y > y \mid X > x) = 1 - \frac{1}{1+(\beta(x)y)^{-\nu}} = \frac{1}{1+(\beta(x)y)^{\nu}}, \quad y > 0,$$

where $\beta(x), \nu > 0$ for all $x > 0$ to ensure a valid survival function. The corresponding joint survival function will be:

$$\bar{F}_0(x)\bar{F}_1(\beta(x)y) = P(X > x, Y > y) = \frac{1}{(1+(\alpha x)^{\lambda})(1+(\beta(x)y)^{\nu})}, \quad x, y > 0. \tag{27}$$

Accordingly, $f_X(x) = \frac{\alpha\lambda(\alpha x)^{\lambda-1}}{((\alpha x)^{\lambda}+1)^2}$ for $x > 0$ and $f_Y(y) = \frac{\beta(0)\nu(\beta(0)y)^{\nu-1}}{((\beta(0)y)^{\nu}+1)^2}$ for $y > 0$. Hence, $X \sim \mathit{LogLogistic}(\alpha,\lambda)$ and $Y \sim \mathit{LogLogistic}(\beta(0),\nu)$.

Now since $[tf_1(t)]' = \frac{\nu^2 t^{\nu-1}(1-t^{\nu})}{t^{3\nu}+3t^{2\nu}+3t^{\nu}+1} > 0$ for $t < 1$, and $S(t) = -\frac{t^{\nu}+1}{\nu(t^{\nu}-1)}$ is increasing on this interval, $\inf_{t<1} S(t) = \lim_{t\to 0} S(t) = \frac{1}{\nu}$. Additionally, since $[tf_1(t)]' = \frac{\nu^2 t^{\nu-1}(1-t^{\nu})}{t^{3\nu}+3t^{2\nu}+3t^{\nu}+1} < 0$ for $t > 1$, and $S(t) = -\frac{t^{\nu}+1}{\nu(t^{\nu}-1)}$ is increasing on this interval, $\sup_{t>1} S(t) = \lim_{t\to\infty} S(t) = -\frac{1}{\nu}$. Consequently, applying (4) yields:

$$-\frac{1}{\nu}\frac{\alpha\lambda(\alpha x)^{\lambda-1}}{(\alpha x)^{\lambda}+1}\beta(x) \le B'(x) \le \frac{1}{\nu}\frac{\alpha\lambda(\alpha x)^{\lambda-1}}{(\alpha x)^{\lambda}+1}\beta(x).$$

Now, we select the form of the acceleration function $\beta(x)$ such that $\beta'(x) = \frac{1}{\nu}\frac{\alpha\lambda(\alpha x)^{\lambda-1}}{(\alpha x)^{\lambda}+1}\beta(x)$ (that is, equal to $\beta'(x)'s$ upper bound). Hence from (5) with $c^* = \frac{1}{\nu}$, $\beta(x) = \gamma(1+(\alpha x)^{\lambda})^{\frac{1}{\nu}}$ with $\gamma > 0$. Furthermore, following Arnold and Arvanitis [1], we introduce a dependence parameter $\tau \in [0,1]$ as before. Hence we choose our acceleration function as:

$$\beta(x) = \gamma(1+(\alpha\tau x)^{\lambda})^{\frac{1}{\nu}}.$$

5.1 Moments

Noting $X \sim LogLogistic(\alpha, \lambda)$ and $Y \sim LogLogistic(\gamma, \nu)$, we have:

$$E(X) = \frac{\pi}{\alpha\lambda\sin\left(\frac{\pi}{\lambda}\right)}, \quad \lambda > 1, \tag{28}$$

$$E(Y) = \frac{\pi}{\gamma\nu\sin\left(\frac{\pi}{\nu}\right)}, \quad \nu > 1, \tag{29}$$

$$Var(X) = \frac{1}{\alpha^2}\left[\frac{2\pi}{\lambda\sin\left(\frac{2\pi}{\lambda}\right)} - \left(\frac{\pi}{\lambda\sin\left(\frac{\pi}{\lambda}\right)}\right)^2\right], \quad \lambda > 2, \tag{30}$$

$$Var(Y) = \frac{1}{\gamma^2}\left[\frac{2\pi}{\nu\sin\left(\frac{2\pi}{\nu}\right)} - \left(\frac{\pi}{\nu\sin\left(\frac{\pi}{\nu}\right)}\right)^2\right], \quad \nu > 2. \tag{31}$$

Accordingly:

$$\begin{aligned}
\operatorname{Cov}(X,Y) &= \int_0^\infty\int_0^\infty \big(P(Y>y, X>x) - P(X>x)P(Y>y)\big)\,dy\,dx \\
&= \int_0^\infty\int_0^\infty \bar{F}_0(x)\left(\bar{F}_1\big(\beta(x)y\big) - \bar{F}_1\big(\beta(0)y\big)\right)\,dy\,dx \\
&= \int_0^\infty \frac{1}{1+(\alpha x)^\lambda}\left(\int_0^\infty\left(\frac{1}{1+(\beta(x)y)^\nu} - \frac{1}{1+(\beta(0)y)^\nu}\right)dy\right)dx \\
&= \frac{\pi}{\nu\sin\left(\frac{\pi}{\nu}\right)}\int_0^\infty \frac{1}{1+(\alpha x)^\lambda}\left(\frac{1}{\beta(x)} - \frac{1}{\beta(0)}\right)dx \\
&= \frac{\pi}{\gamma\nu\sin\left(\frac{\pi}{\nu}\right)}\left[\int_0^\infty \frac{dx}{((1+(\alpha x)^\lambda)\,((1+(\alpha\tau x)^\lambda)^{\frac{1}{\nu}}} - \int_0^\infty\frac{dx}{1+(\alpha x)^\lambda}\right] \\
&= \frac{\pi}{\alpha\lambda\nu\gamma\sin\left(\frac{\pi}{\nu}\right)}\left[\frac{\Gamma\left(\frac{1}{\lambda}\right)\Gamma\left(1+\frac{1}{\nu}-\frac{1}{\lambda}\right)}{\Gamma\left(1+\frac{1}{\nu}\right)}\,{}_2F_1\left(\frac{1}{\nu},\frac{1}{\lambda};1+\frac{1}{\nu};1-\tau^\lambda\right) - \frac{\pi}{\sin\left(\frac{\pi}{\lambda}\right)}\right].
\end{aligned} \tag{32}$$

where the conditions $\nu > 1$ and $\lambda > 1$ ensure convergence of the inner and outer integrals, respectively. Now, using (30), (31) and (32), the correlation is:

$$\begin{aligned}
\rho(X,Y) &= \frac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}} \\
&= \frac{\pi\left[\frac{\Gamma\left(\frac{1}{\lambda}\right)\Gamma\left(1+\frac{1}{\nu}-\frac{1}{\lambda}\right)}{\Gamma\left(1+\frac{1}{\nu}\right)}\,{}_2F_1\left(\frac{1}{\nu},\frac{1}{\lambda};1+\frac{1}{\nu};1-\tau^\lambda\right) - \frac{\pi}{\sin\left(\frac{\pi}{\lambda}\right)}\right]}{\lambda\nu\sin\left(\frac{\pi}{\nu}\right)\sqrt{\left[\frac{2\pi}{\lambda\sin\left(\frac{2\pi}{\lambda}\right)} - \left(\frac{\pi}{\lambda\sin\left(\frac{\pi}{\lambda}\right)}\right)^2\right]\left[\frac{2\pi}{\nu\sin\left(\frac{2\pi}{\nu}\right)} - \left(\frac{\pi}{\nu\sin\left(\frac{\pi}{\nu}\right)}\right)^2\right]}},
\end{aligned} \tag{33}$$

valid for $\lambda, \nu > 2$. Since ${}_2F_1\left(\frac{1}{\nu},\frac{1}{\lambda};1+\frac{1}{\nu};z\right)$ has a power series with strictly positive coefficients, it is strictly increasing in $z \in [0,1]$; hence ${}_2F_1\left(\frac{1}{\nu},\frac{1}{\lambda};1+\frac{1}{\nu};1-\tau^\lambda\right)$ is strictly decreasing in $\tau \in [0,1]$. Consequently, (33) attains its minimum at $\tau = 1$ and its maximum (zero) at $\tau = 0$. Therefore,

$$\rho_{\min(\lambda,\nu)} := \frac{\pi\left[\frac{\Gamma\left(\frac{1}{\lambda}\right)\Gamma\left(1+\frac{1}{\nu}-\frac{1}{\lambda}\right)}{\Gamma\left(1+\frac{1}{\nu}\right)} - \frac{\pi}{\sin\left(\frac{\pi}{\lambda}\right)}\right]}{\lambda\nu\sin\left(\frac{\pi}{\nu}\right)\sqrt{\left[\frac{2\pi}{\lambda\sin\left(\frac{2\pi}{\lambda}\right)} - \left(\frac{\pi}{\lambda\sin\left(\frac{\pi}{\lambda}\right)}\right)^2\right]\left[\frac{2\pi}{\nu\sin\left(\frac{2\pi}{\nu}\right)} - \left(\frac{\pi}{\nu\sin\left(\frac{\pi}{\nu}\right)}\right)^2\right]}} \le \rho(X,Y) \le 0.$$

Now denoting $[\lambda^*, \nu^*]' = \operatorname{argmin}_{\lambda,\nu}\rho_{\min(\lambda,\nu)}$, we obtain $\inf_{\lambda,\nu>2}\rho_{\min}(\lambda,\nu) = \lim_{\nu\to\infty}\rho_{\min}(\lambda = \lambda^* \approx 5.06, \nu) \approx -0.54076$. Hence, this log-logistic model with acceleration function $\beta(x) = \gamma(1+(\alpha\tau x)^\lambda)^{\frac{1}{\nu}}$ will be able to accommodate correlations in the range $\rho(X,Y) \in \big[-0.54076,\, 0\big]$.

6 Half-Cauchy

We restrict the survival function forms to the Half-Cauchy family as such:

$$\bar{F}_0(x) = P(X > x) = 1 - \frac{2}{\pi}\arctan(\alpha x), \quad x > 0,$$

where $\alpha > 0$ and, for each $x > 0$:

$$\bar{F}_1(\beta(x)y) = P(Y > y \mid X > x) = 1 - \frac{2}{\pi}\arctan(\beta(x)y), \quad y > 0,$$

where $\beta(x) > 0$ for all $x > 0$ to ensure a valid survival function. The corresponding joint survival function will be:

$$\bar{F}_0(x)\bar{F}_1(\beta(x)y) = P(X > x, Y > y) = \frac{1}{\pi^2}\left(2\arctan(\alpha x) - \pi\right)\left(2\arctan(\beta(x)y) - \pi\right), \quad x, y > 0. \tag{34}$$

Accordingly, $f_X(x) = \frac{2\alpha}{\pi(1+(\alpha x)^2)}$ for $x > 0$ and $f_Y(y) = \frac{2\beta(0)}{\pi(1+(\beta(0)y)^2)}$ for $y > 0$. Hence, $X \sim Half - Cauchy(\alpha)$ and $Y \sim Half - Cauchy(\beta(0))$.

Now since $[tf_1(t)]' = \frac{2(1-t^2)}{\pi(t^2+1)^2} > 0$ for $t < 1$, and $S(t) = \frac{t^2+1}{1-t^2}$ is increasing on this interval, $\inf_{t<1} S(t) = \lim_{t\to 0} S(t) = 1$. Additionally, since $[tf_1(t)]' = \frac{2(1-t^2)}{\pi(t^2+1)^2} < 0$ for $t > 1$, and $S(t) = \frac{t^2+1}{1-t^2}$ is increasing on this interval, $\sup_{t>1} S(t) = \lim_{t\to\infty} S(t) = -1$. Consequently, applying (4) yields:

$$-\frac{\alpha}{(1+(\alpha x)^2)(\frac{\pi}{2} - \arctan(\alpha x))}\beta(x) \le B'(x) \le \frac{\alpha}{(1+(\alpha x)^2)(\frac{\pi}{2} - \arctan(\alpha x))}\beta(x).$$

Now, we select the form of the acceleration function $\beta(x)$ such that $\beta'(x) = \frac{\alpha}{(1+(\alpha x)^2)(\frac{\pi}{2} - \arctan(\alpha x))}\beta(x)$ (that is, equal to $\beta'(x)'s$ upper bound). Hence from (5) with $c^* = 1$, $\beta(x) = \gamma\frac{1}{1-\frac{2}{\pi}\arctan(\alpha x)}$ with $\gamma > 0$. Furthermore, following Arnold and Arvanitis [1], we introduce a dependence parameter $\tau \in [0, 1]$ as before. Hence we choose our acceleration function as:

$$\beta(x) = \gamma\frac{1}{1 - \frac{2}{\pi}\arctan(\alpha\tau x)}.$$

7 Gamma

We restrict the survival function forms to the Gamma family as such:

$$\bar{F}_0(x) = P(X > x) = 1 - \frac{1}{\Gamma(\lambda)}\gamma^*(\lambda, \alpha x) = \frac{\Gamma(\lambda, \alpha x)}{\Gamma(\lambda)}, \quad x > 0,$$

where $\alpha, \lambda > 0$ and denoting γ^* as the lower incomplete gamma function. Now for each $x > 0$:

$$\bar{F}_1(\beta(x)y) = P(Y > y \mid X > x) = 1 - \frac{1}{\Gamma(\nu)}\gamma^*(\nu, \beta(x)y) = \frac{\Gamma(\nu, \beta(x)y)}{\Gamma(\nu)}, \quad y > 0,$$

where $\beta(x), \nu > 0$ for all $x > 0$ to ensure a valid survival function. The corresponding joint survival function will be:

$$\bar{F}_0(x)\bar{F}_1(\beta(x)y) = P(X > x, Y > y) = \frac{\Gamma(\lambda, \alpha x)\Gamma(\nu, \beta(x)y)}{\Gamma(\lambda)\Gamma(\nu)}, \quad x, y > 0. \tag{35}$$

Accordingly, $f_X(x) = \frac{\alpha^\lambda}{\Gamma(\lambda)}x^{\lambda-1}e^{-\alpha x}$ for $x > 0$ and $f_Y(y) = \frac{\beta(0)^\nu}{\Gamma(\nu)}y^{\nu-1}e^{-\beta(0)y}$ for $y > 0$. Hence, $X \sim Gamma(\alpha, \lambda)$ and $Y \sim Gamma(\beta(0), \nu)$.

Now since $[tf_1(t)]' = \frac{t^{\nu-1}(\nu-t)e^{-t}}{\Gamma(\nu)} > 0$ for $t < \nu$, and $S(t) = \frac{1}{\nu - t}$ is increasing on this interval, $\inf_{t<\nu} S(t) = \lim_{t\to 0} S(t) = \frac{1}{\nu}$. Additionally, since $[tf_1(t)]' = \frac{t^{\nu-1}(\nu-t)e^{-t}}{\Gamma(\nu)} < 0$ for $t > \nu$, and $S(t) = \frac{1}{\nu-t}$ is increasing on this interval, $\sup_{t>\nu} S(t) = \lim_{t\to\infty} S(t) = 0$. Consequently, applying (4) yields:

$$0 \le B'(x) \le \frac{1}{\nu}\frac{\alpha^\lambda}{\Gamma(\lambda, \alpha x)}x^{\lambda-1}e^{-\alpha x}.$$

Now, we select the form of the acceleration function $\beta(x)$ such that $\beta'(x) = \frac{1}{\nu}\frac{\alpha^\lambda}{\Gamma(\lambda,\alpha x)}x^{\lambda-1}e^{-\alpha x}$ (that is, equal to $\beta'(x)'s$ upper bound). Hence from (5) with $c^* = \frac{1}{\nu}$, $\beta(x) = \gamma\left[\frac{\Gamma(\lambda)}{\Gamma(\lambda,\alpha x)}\right]^{\frac{1}{\nu}}$ with $\gamma > 0$. Furthermore, following Arnold and Arvanitis [1], we introduce a dependence parameter $\tau \in [0,1]$ as before. Hence we choose our acceleration function as:

$$\beta(x) = \gamma\left[\frac{\Gamma(\lambda)}{\Gamma(\lambda,\alpha\tau x)}\right]^{\frac{1}{\nu}}.$$

8 Simulation Data

To generate dependent pairs (X,Y) for simulation, we employ a copula-based simulation approach for the method of moments estimation and a Metropolis–Hastings (MH) algorithm for maximum likelihood estimation. m.m.e's rely exclusively on a limited set of moments, such as means, variances, and covariances, and MH samples may fail to reproduce these accurately unless chains are extremely long and well-mixed, leading to poor moment-based estimates. Copula-based simulation provides a more effective solution in this context by generating synthetic (X,Y) pairs that preserve the specified marginal distributions and dependence structure, ensuring that simulated moments are consistent with their theoretical counterparts. In contrast, copula-based simulation is unsuitable for likelihood-based estimation because it alters the joint density of (X,Y), whereas m.l.e's are defined relative to the true joint distribution; using copula-generated samples for maximum likelihood estimation would therefore imply optimizing a likelihood corresponding to a different model.

Method of Moments The copula preserves the marginal distributions of X and Y while introducing dependence through ρ_{latent}. Specifically, we use a Student-t copula to capture both tail dependence and negative correlations. Here, $F_X(x) = 1-\bar{F}_0(x)$ and $F_Y(y) = 1-\bar{F}_1(\beta(0)y)$.

Step 1: Construct a t-copula with correlation ρ_{latent} and degrees of freedom v_c. Formally, for $U = F_X(X)$ and $V = F_Y(Y)$, we have $(U,V) \sim C_{v_c,\rho_{\text{latent}}}$, where the copula distribution function is:

$$C_{v_c,\rho_{\text{latent}}}(u,v) = t_{2,v_c}\big(t_{v_c}^{-1}(u),\, t_{v_c}^{-1}(v);\, \rho_{\text{latent}}\big), \quad (u,v) \in [0,1]^2,$$

with $t_{v_c}^{-1}$ the quantile function of a univariate Student-$t(v_c)$ and t_{2,v_c} the bivariate Student-t distribution function with correlation ρ_{latent}. From Sklar's Theorem, the joint density of (X,Y) with marginal distributions F_X and F_Y is:

$$f_{X,Y}(x,y) = c_{v_c,\rho_{\text{latent}}}\big(F_X(x), F_Y(y)\big)\, f_X(x)\, f_Y(y),$$

where the copula density is $c_{v_c,\rho_{\text{latent}}}(u,v) = \frac{\partial^2}{\partial u\,\partial v}C_{v_c,\rho_{\text{latent}}}(u,v)$.

Step 2: Map the model Pearson correlation ρ_{target} into the copula correlation parameter ρ_{latent} that drives dependence in the t-copula. This requires re-expressing the correlation in terms of $c_{v_c,\rho_{\text{latent}}}(u,v)$ and solving numerically to obtain ρ_{latent}:

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \frac{\int_0^1\int_0^1 F_X^{-1}(u)\,F_Y^{-1}(v)\,c_{v_c,\rho_{\text{latent}}}(u,v)\,du\,dv - E(X)E(Y)}{\sqrt{Var(X)\,Var(Y)}},$$

with $E(X) = \int_0^1 F_X^{-1}(u)\,du$, $E(Y) = \int_0^1 F_Y^{-1}(v)\,dv$, and $Var(X) = E(X^2) - E(X)^2$.

Step 3: Simulate n pairs (U,V) from $(U,V) \sim C_{v_c,\rho_{\text{latent}}}$, then transform back via the inverse distributions to obtain $X = F_X^{-1}(U)$ and $Y = F_Y^{-1}(V)$.

Furthermore, Appendix B provides a guide on how to obtain m.m.e's for the models (noting that moments were not provided for the Half-Cauchy and Gamma models).

Maximum Likelihood Estimation We utilise the MH algorithm to sample from the joint density of the models considered in this study, where the target distribution is given by $\pi(x,y) = f_{X,Y}(x,y)$, noting that $X \sim Family(\alpha,\lambda)$ and $Y \sim Family(\beta(0),\nu)$, with $f_{X,Y}(x,y)$ given in Appendix A

Step 1 (Initialization): Choose any starting point $(x^{(0)}, y^{(0)})$ in the support of $f_{X,Y}$. Set $j = 0$.

Step 2 (Proposal): Given the current state $(x^{(j)}, y^{(j)})$, draw a candidate

$$(x^*, y^*) \sim q(\cdot \mid x^{(j)}, y^{(j)}).$$

The study utilises an independence proposal:

$$q(x^*, y^* \mid x^{(j)}, y^{(j)}) = q(x^*, y^*) = f_X(x^*)\, f_Y(y^*).$$

Step 3 (Acceptance ratio α_a): Compute:

$$\begin{aligned}\alpha &= \min\left(\frac{f_{X,Y}(x^*, y^*)\, q(x^{(j)}, y^{(j)} \mid x^*, y^*)}{f_{X,Y}(x^{(j)}, y^{(j)})\, q(x^*, y^* \mid x^{(j)}, y^{(j)})}, 1\right)\\ &= \min\left(\frac{f_{X,Y}(x^*, y^*)}{f_X(x^*)\, f_Y(y^*)}\Big/\frac{f_{X,Y}(x^{(j)}, y^{(j)})}{f_X(x^{(j)})\, f_Y(y^{(j)})}, 1\right).\end{aligned}$$

Equivalently, using logs for numerical stability,

$$\begin{aligned}\log(\alpha_a) = \min\Bigg\{&\Big[\log\big(f_{X,Y}(x^*, y^*)\big) - \log\big(f_X(x^*)\big) - \log\big(f_Y(y^*)\big)\Big]\\ &- \Big[\log\big(f_{X,Y}(x^{(j)}, y^{(j)})\big) - \log\big(f_X(x^{(j)})\big) - \log\big(f_Y(y^{(j)})\big)\Big], 0\Bigg\}.\end{aligned}$$

Subsequently, (x^*, y^*) is accepted as the new pair in the Markov chain under the following acceptance criterion:

$$(x^{(j+1)}, y^{(j+1)}) = \begin{cases}(x^*, y^*), & \text{if } U < \alpha_a\\ (x^{(j)}, y^{(j)}), & \text{otherwise.}\end{cases}$$

where $U \sim Uniform(0, 1)$.

A Joint Densities and Log-likelihoods

A.1 Exponential

We obtain the joint density function by differentiating the joint survival function (6):

$$f_{X,Y}(x, y) = \alpha\gamma\left(\gamma\tau y e^{\alpha\tau x} - \tau + 1\right)e^{(\alpha x(\tau-1) - \gamma y e^{\alpha\tau x})}, \qquad x, y > 0.$$

where $\alpha, \gamma > 0$ and $\tau \in [0, 1]$. The log-likelihood is thus:

$$\begin{aligned}\ell(\alpha, \gamma, \tau) = n\log\alpha + n\log\gamma + \sum_{i=1}^{n}\log\Big(\gamma\tau y_i e^{\alpha\tau x_i} - \tau + 1\Big)\\ + \alpha(\tau - 1)\sum_{i=1}^{n} x_i - \gamma\sum_{i=1}^{n} y_i e^{\alpha\tau x_i}.\end{aligned}$$

A.2 Lomax

We obtain the joint density function by differentiating the joint survival function (13):

$$\begin{aligned}
f_{X,Y}(x,y) = {}& \nu\,(1+\alpha x)^{-3\lambda-1}\left(1+\gamma y(1+\alpha\tau x)^{\lambda}\right)^{-3\nu-4} \\
& \times \left[\frac{\alpha\gamma^2\lambda\tau y(\nu+1)\,(1+\alpha x)^{2\lambda+1}(1+\alpha\tau x)^{2\lambda}\left(1+\gamma y(1+\alpha\tau x)^{\lambda}\right)^{2\nu+2}}{1+\alpha\tau x}\right. \\
& -\frac{\alpha\gamma\lambda\tau\,(1+\alpha x)^{2\lambda+1}(1+\alpha\tau x)^{\lambda}\left(1+\gamma y(1+\alpha\tau x)^{\lambda}\right)^{2\nu+3}}{1+\alpha\tau x} \\
& \left. +\,\alpha\gamma\lambda\left((1+\alpha x)^2(1+\alpha\tau x)\right)^{\lambda}\left(1+\gamma y(1+\alpha\tau x)^{\lambda}\right)^{2\nu+3}\right], \quad x,y>0,
\end{aligned}$$

where $\alpha, \lambda, \nu, \gamma > 0$ and $\tau \in [0,1]$. The log-likelihood is thus:

$$\begin{aligned}
\ell(\alpha,\lambda,\nu,\gamma,\tau) = {}& n\log\nu - (3\lambda+1)\sum_{i=1}^{n}\log(1+\alpha x_i) \\
& -(3\nu+4)\sum_{i=1}^{n}\log\!\left(1+\gamma y_i(1+\alpha\tau x_i)^{\lambda}\right) \\
& +\sum_{i=1}^{n}\log\!\left\{\frac{\alpha\gamma^2\lambda\tau y_i(\nu+1)\,(1+\alpha x_i)^{2\lambda+1}(1+\alpha\tau x_i)^{2\lambda}}{1+\alpha\tau x_i}\left(1+\gamma y_i(1+\alpha\tau x_i)^{\lambda}\right)^{2\nu+2}\right. \\
& -\frac{\alpha\gamma\lambda\tau(1+\alpha x_i)^{2\lambda+1}(1+\alpha\tau x_i)^{\lambda}}{1+\alpha\tau x_i}\left(1+\gamma y_i(1+\alpha\tau x_i)^{\lambda}\right)^{2\nu+3} \\
& \left. +\,\alpha\gamma\lambda\left((1+\alpha x_i)^2(1+\alpha\tau x_i)\right)^{\lambda}\left(1+\gamma y_i(1+\alpha\tau x_i)^{\lambda}\right)^{2\nu+3}\right\}.
\end{aligned}$$

A.3 Weibull

We obtain the joint density function by differentiating the joint survival function (20):

$$f_{X,Y}(x,y) = \alpha^{\lambda}\gamma^{\nu}\lambda\nu x^{\lambda-1}y^{\nu-1}\left(\tau^{\lambda}(\gamma y)^{\nu}e^{(\alpha\tau x)^{\lambda}} - \tau^{\lambda}+1\right)e^{(\alpha x)^{\lambda}(\tau^{\lambda}-1)-(\gamma y)^{\nu}e^{(\alpha\tau x)^{\lambda}}}, \quad x,y>0,$$

where $\alpha, \lambda, \nu, \gamma > 0$ and $\tau \in [0,1]$. The log-likelihood is thus:

$$\begin{aligned}
\ell(\alpha,\lambda,\nu,\gamma,\tau) = {}& n\lambda\log\alpha + n\nu\log\gamma + n\log\lambda + n\log\nu \\
& +(\lambda-1)\sum_{i=1}^{n}\log x_i + (\nu-1)\sum_{i=1}^{n}\log y_i \\
& +\sum_{i=1}^{n}\log\!\left(\tau^{\lambda}(\gamma y_i)^{\nu}e^{(\alpha\tau x_i)^{\lambda}} - \tau^{\lambda}+1\right) \\
& +(\tau^{\lambda}-1)\sum_{i=1}^{n}(\alpha x_i)^{\lambda} - \sum_{i=1}^{n}(\gamma y_i)^{\nu}e^{(\alpha\tau x_i)^{\lambda}}.
\end{aligned}$$

A.4 Log-logistic

We obtain the joint density function by differentiating the joint survival function (27):

$$
\begin{aligned}
f_{X,Y}(x,y) = {} & \frac{\lambda\nu\,(\gamma y)^{\nu}}{x\,y\,\big((\alpha x)^{\lambda}+1\big)^{2}\Big((\gamma y)^{\nu}\big((\alpha\tau x)^{\lambda}+1\big)+1\Big)^{3}} \\
& \times \Bigg\{(\alpha x)^{\lambda}\big((\alpha\tau x)^{\lambda}+1\big)\Big((\gamma y)^{\nu}\big((\alpha\tau x)^{\lambda}+1\big)+1\Big) \\
& \qquad + (\alpha\tau x)^{\lambda}\big((\alpha x)^{\lambda}+1\big)\Big((\gamma y)^{\nu}\big((\alpha\tau x)^{\lambda}+1\big)+1\Big) \\
& \qquad - 2\,(\alpha\tau x)^{\lambda}\big((\alpha x)^{\lambda}+1\big)\Bigg\}, \qquad x,y>0,
\end{aligned}
$$

where $\alpha,\lambda,\nu,\gamma>0$ and $\tau\in[0,1]$. Thus the log-likelihood is:

$$
\begin{aligned}
\ell(\alpha,\lambda,\nu,\gamma,\tau) = {} & n\log\lambda + n\log\nu + n\,\nu\log\gamma + (\nu-1)\sum_{i=1}^{n}\log y_i - \sum_{i=1}^{n}\log x_i \\
& - 2\sum_{i=1}^{n}\log\big((\alpha x_i)^{\lambda}+1\big) - 3\sum_{i=1}^{n}\log\Big((\gamma y_i)^{\nu}\big((\alpha\tau x_i)^{\lambda}+1\big)+1\Big) \\
& + \sum_{i=1}^{n}\log\Big\{(\alpha x_i)^{\lambda}\big((\alpha\tau x_i)^{\lambda}+1\big)\big((\gamma y_i)^{\nu}\big((\alpha\tau x_i)^{\lambda}+1\big)+1\big) \\
& \qquad + (\alpha\tau x_i)^{\lambda}\big((\alpha x_i)^{\lambda}+1\big)\big((\gamma y_i)^{\nu}\big((\alpha\tau x_i)^{\lambda}+1\big)+1\big) \\
& \qquad - 2\,(\alpha\tau x_i)^{\lambda}\big((\alpha x_i)^{\lambda}+1\big)\Big\}.
\end{aligned}
$$

A.5 Half-Cauchy

We obtain the joint density function by differentiating the joint survival function (34):

$$
\begin{aligned}
f_{X,Y}(x,y) = {} & \frac{4\alpha\gamma}{\pi\left((\alpha x)^2+1\right)\Big((\pi\gamma y)^2+(2\arctan(\alpha\tau x)-\pi)^2\Big)^2\left((\alpha\tau x)^2+1\right)} \times \\
& \Bigg[-2(\pi\gamma)^2\tau y^2\left((\alpha x)^2+1\right)(2\arctan(\alpha x)-\pi) \\
& + \tau\left((\alpha x)^2+1\right)\Big((\pi\gamma y)^2+(2\arctan(\alpha\tau x)-\pi)^2\Big)(2\arctan(\alpha x)-\pi) \\
& - \Big((\pi\gamma y)^2+(2\arctan(\alpha\tau x)-\pi)^2\Big)\left((\alpha\tau x)^2+1\right)(2\arctan(\alpha\tau x)-\pi)\Bigg]
\end{aligned}
$$

where $\alpha, \gamma > 0$ and $\tau \in [0, 1]$. Thus the log-likelihood is:

$$\begin{aligned}\ell(\alpha,\gamma,\tau) = \sum_{i=1}^{n} \Bigg\{ & \log(4\alpha\gamma) - \log\pi \\ & - \log\big((\alpha x_i)^2 + 1\big) \\ & - \log\Big((\pi\gamma y_i)^2 + (2\arctan(\alpha\tau x_i) - \pi)^2\Big)^2 \\ & - \log\big((\alpha\tau x_i)^2 + 1\big) \\ & + \log\Bigg[-2(\pi\gamma)^2\tau y_i^2 \left((\alpha x_i)^2 + 1\right)(2\arctan(\alpha x_i) - \pi) \\ & \quad + \tau\left((\alpha x_i)^2 + 1\right)\Big((\pi\gamma y_i)^2 + (2\arctan(\alpha\tau x_i) - \pi)^2\Big)(2\arctan(\alpha x_i) - \pi) \\ & \quad - \Big((\pi\gamma y_i)^2 + (2\arctan(\alpha\tau x_i) - \pi)^2\Big)\left((\alpha\tau x_i)^2 + 1\right)(2\arctan(\alpha\tau x_i) - \pi)\Bigg]\Bigg\}.\end{aligned}$$

A.6 Gamma

We obtain the joint density function by differentiating the joint survival function (35):

$$\begin{aligned} f_{X,Y}(x,y) = & \frac{\alpha^\lambda\,\gamma^\nu\,x^{\lambda-1}\,y^\nu}{\lambda^{1+\frac{1}{\nu}}\,y\,\Gamma(\lambda)\,\Gamma(\nu+1)\,\Gamma(\lambda,\alpha\tau x)^2}\, e^{-\alpha x - \alpha\tau x - \gamma\,\lambda^{-\frac{1}{\nu}}\,y\,\Gamma(\lambda,\alpha\tau x)^{-\frac{1}{\nu}}\,\Gamma(\lambda+1)^{\frac{1}{\nu}}} \\ & \times\Bigg[\gamma\tau^\lambda y\,\Gamma(\lambda,\alpha\tau x)^{-\frac{1}{\nu}} e^{\alpha x}\,\Gamma(\lambda+1)^{1+\frac{1}{\nu}}\Gamma(\lambda,\alpha x) \\ & - \lambda^{\frac{1}{\nu}}\,\nu\,\tau^\lambda\,e^{\alpha x}\,\Gamma(\lambda)\,\Gamma(\lambda+1) + \lambda^{1+\frac{1}{\nu}}\,\nu\,\tau^\lambda\,e^{\alpha x}\,\Gamma(\lambda)\,\gamma^*(\lambda,\alpha x) + \lambda^{1+\frac{1}{\nu}}\,\nu\,e^{\alpha\tau x}\,\Gamma(\lambda)\,\Gamma(\lambda,\alpha\tau x)\Bigg], \quad x,y>0,\end{aligned}$$

where $\alpha, \gamma > 0$ and $\tau \in [0, 1]$. Thus the log-likelihood is:

$$\begin{aligned}\ell(\alpha,\gamma,\lambda,\nu,\tau) = \sum_{i=1}^{n}\Bigg\{ & \lambda\log\alpha + \nu\log\gamma + (\lambda-1)\log x_i + (\nu-1)\log y_i \\ & - \left(1+\tfrac{1}{\nu}\right)\log\lambda - \log\Gamma(\lambda) - \log\Gamma(\nu+1) - 2\log\Gamma(\lambda,\alpha\tau x_i) \\ & - \alpha x_i - \alpha\tau x_i - \gamma\,\lambda^{-\frac{1}{\nu}}\,y_i\,\Gamma(\lambda,\alpha\tau x_i)^{-\frac{1}{\nu}}\,\Gamma(\lambda+1)^{\frac{1}{\nu}} \\ & + \log\Big[\gamma\tau^\lambda y_i\,\Gamma(\lambda,\alpha\tau x_i)^{-\frac{1}{\nu}} e^{\alpha x_i}\,\Gamma(\lambda+1)^{1+\frac{1}{\nu}}\,\Gamma(\lambda,\alpha x_i) \\ & \quad - \lambda^{\frac{1}{\nu}}\,\nu\,\tau^\lambda\,e^{\alpha x_i}\,\Gamma(\lambda)\,\Gamma(\lambda+1) + \lambda^{1+\frac{1}{\nu}}\,\nu\,\tau^\lambda\,e^{\alpha x_i}\,\Gamma(\lambda)\,\gamma^*(\lambda,\alpha x_i) \\ & \quad + \lambda^{1+\frac{1}{\nu}}\,\nu\,e^{\alpha\tau x_i}\,\Gamma(\lambda)\,\Gamma(\lambda,\alpha\tau x_i)\Big]\Bigg\}.\end{aligned}$$

B Method of Moments Estimators

Suppose data in the form of $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, ..., \mathbf{X}^{(n)}$ (where for each $i = 1, \ldots, n$, $\mathbf{X}^{(i)} = (X_{1i}, X_{2i})^T$) are obtained, which are independent and identically distributed according to 3 and 2. If $M_1, M_2 > 0$, consistent asymptotically normal method of moment estimates (m.m.e's) are acquired.

Define:

$$
\begin{aligned}
M_1 &= \frac{1}{n}\sum_{i=1}^{n} x_{1i} \\
M_2 &= \frac{1}{n}\sum_{i=1}^{n} x_{2i} \\
S_{12} &= \frac{1}{n}\sum_{i=1}^{n} (x_{1i} - M_1)(x_{2i} - M_2) \\
S_1 &= \frac{1}{n}\sum_{i=1}^{n} (x_{2i} - M_2)^2 \\
S_2 &= \frac{1}{n}\sum_{i=1}^{n} (x_{2i} - M_2)^2.
\end{aligned}
$$

We denote the method of moment estimators for α, λ, γ and ν as $\tilde{\alpha}, \tilde{\lambda}, \tilde{\gamma}$ and $\tilde{\nu}$ respectively.

B.1 Exponential

Using (7) and (8), we obtain:

$$
\tilde{\alpha} = \frac{1}{M_1},
$$
$$
\tilde{\lambda} = \frac{1}{M_2}.
$$

And using (11):

$$
\tilde{\tau} = -\frac{1}{\frac{M_1 M_2}{S_{12}} + 1}.
$$

B.2 Lomax

Using (14) and (16), we obtain:

$$
\tilde{\lambda} = \frac{2S_1}{S_1 - M_1^2},
$$
$$
\tilde{\alpha} = \frac{1}{M_1(\tilde{\lambda} - 1)}.
$$

Likewise using (15) and (17), we obtain:

$$
\tilde{\nu} = \frac{2S_2}{S_2 - M_2^2},
$$
$$
\tilde{\gamma} = \frac{1}{M_2(\tilde{\nu} - 1)}.
$$

And using (18) we may solve for $\tilde{\tau}$ numerically.

C Weibull

Solve for $\tilde{\lambda}$ and $\tilde{\alpha}$ numerically using (21) and (23). Likewise, solve for $\tilde{\nu}$ and $\tilde{\gamma}$ numerically using (22) and (24). Finally, solve for $\tilde{\tau}$ numerically using (25).

D Log-Logistic

Solve for $\tilde{\lambda}$ and $\tilde{\alpha}$ numerically using (28) and (30). Likewise, solve for $\tilde{\nu}$ and $\tilde{\gamma}$ numerically using (29) and (31). Finally, solve for $\tilde{\tau}$ numerically using (32).

References

[1] Barry C Arnold and Matthew A Arvanitis. On bivariate pseudo-exponential distributions. *Journal of Applied Statistics*, 47 (13-15):2299–2311, 2020.