

Models with Accelerated Failure Conditionals

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1 Introduction

As per Arnold and Arvanitis [1], the general accelerated failure conditionals model is of the form:

$$\bar{F}_X(x) = P(X > x) = \bar{F}_0(x), \quad x > 0, \quad (1)$$

for some survival function $\bar{F}_0(x) \in (0, 1) \forall x > 0$, and for each $x > 0$:

$$P(Y > y \mid X > x) = \bar{F}_1(\beta(x)y), \quad x, y > 0, \quad (2)$$

for some survival function $\bar{F}_1(\beta(x)y) \in (0, 1) \forall x, y > 0$ and a suitable acceleration function $\beta(x)$. The joint survival function is then:

$$P(X > x, Y > y) = \bar{F}_0(x)\bar{F}_1(\beta(x)y), \quad x, y > 0 \quad (3)$$

Assuming differentiability and X, Y are continuous, we obtain the marginal densities:

$$f_X(x) = -\frac{d}{dx}P(X > x) = -\frac{d}{dx}\bar{F}_0(x) = f_0(x),$$

and, since $\lim_{x \rightarrow 0^+} \bar{F}_0(x) = 1$, where $\beta(0) := \lim_{x \rightarrow 0^+} \beta(x)$:

$$P(Y > y) = \lim_{x \rightarrow 0^+} P(Y > y, X > x) = \bar{F}_1(\beta(0)y).$$

Hence the marginal distribution of Y belongs to the same family as \bar{F}_1 , but with its argument scaled by $\beta(0)$. Now, the density of Y is:

$$f_Y(y) = -\frac{d}{dy}P(Y > y) = -\frac{d}{dy}\bar{F}_1(\beta(0)y) = \beta(0)f_1(\beta(0)y),$$

where $f_1(t) = -\bar{F}_1'(t)$ denotes the density associated with \bar{F}_1 . For (3) to be a valid survival function, it must have a non-negative mixed partial derivative. Denoting f_0 and f_1 as the densities corresponding to the survival function \bar{F}_0 and \bar{F}_1 , and differentiating, we obtain:

$$\begin{aligned} \frac{\partial}{\partial x} \frac{\partial}{\partial y} P(X > x, Y > y) &= f_0(x)f_1(\beta(x)y)\beta(x) - \bar{F}_0(x)f_1'(\beta(x)y)\beta(x)\beta'(x)y - \bar{F}_0(x)f_1(\beta(x)y)\beta'(x) \\ &= f_0(x)f_1(\beta(x)y)\beta(x) - \beta'(x)\bar{F}_0(x)[f_1(\beta(x)y) + f_1'(\beta(x)y)\beta(x)y]. \end{aligned}$$

Denoting $t = \beta(x)y$, we have:

$$\begin{aligned} \frac{\partial}{\partial x} \frac{\partial}{\partial y} P(X > x, Y > y) &= f_0(x)f_1(t)\beta(x) - \beta'(x)\bar{F}_0(x)[f_1(t) + f_1'(t)t] \\ &= f_0(x)f_1(t)\beta(x) - \beta'(x)\bar{F}_0(x)[tf_1(t)]'. \end{aligned}$$

Thus a necessary and sufficient condition for $\frac{\partial}{\partial x} \frac{\partial}{\partial y} P(X > x, Y > y) \geq 0$ is:

$$\beta'(x) \begin{cases} \leq \frac{f_0(x)}{\bar{F}_0(x)} \frac{f_1(t)}{[tf_1(t)]'} \beta(x), & \text{if } [tf_1(t)]' > 0, \\ \geq \frac{f_0(x)}{\bar{F}_0(x)} \frac{f_1(t)}{[tf_1(t)]'} \beta(x), & \text{if } [tf_1(t)]' < 0, \\ \text{unconstrained,} & \text{if } [tf_1(t)]' = 0. \end{cases}$$

Denoting the hazard function $h_0(x) = \frac{f_0(x)}{\bar{F}_0(x)}$ and $S(t) = \frac{f_1(t)}{[tf_1(t)]'}$, we may write the bounds as such:

$$\sup_{t:[tf_1(t)]' < 0} h_0(x)S(t)\beta(x) \leq \beta'(x) \leq \inf_{t:[tf_1(t)]' > 0} h_0(x)S(t)\beta(x). \quad (4)$$

A special case of the upper bound for $B'(x)$ arises for distribution families satisfying $f_1'(t) \leq 0$. In this case, we note that $S(t) = \frac{f_1(t)}{[tf_1(t)]'} = \frac{f_1(t)}{f_1(t) + tf_1'(t)} \geq 1$ for $t \geq 0$, hence $\beta'(x) \leq h_0(x)\beta(x)$.

It should be noted that multiple functional forms for $\beta(x)$ are possible. The question of selecting a form of $\beta(x)$ that provides the best model fit has already been investigated by Arnold and Arvanitis [1]. Accordingly, the present study restricts attention to those choices of $\beta(x)$ that arise from the extreme case in which $\beta'(x)$ attains its theoretical upper bound, that is, given $\inf_{t:[tf_1(t)]' > 0} S(t) = c^*$:

$$\begin{aligned} \beta'(x) &= c^* h_0(x) \beta(x) \\ \therefore \int \frac{1}{\beta(x)} d\beta(x) &= \int c^* h_0(x) dx \\ \therefore \log(\beta(x)) &= -c^* \log(\bar{F}_0(x)) + K \\ \therefore \beta(x) &= \gamma \frac{1}{[\bar{F}_0(x)]^{c^*}}, \end{aligned} \quad (5)$$

for $\gamma > 0$. This suggests $\beta(x) > 0$ for $x > 0$. Furthermore, $\beta(0) := \lim_{x \rightarrow 0^+} \beta(x) = \gamma$ since $\lim_{x \rightarrow 0^+} \bar{F}_0(x) = 1$.

The role of the acceleration function $\beta(x)$ in determining the sign of the correlation between X and Y is established in Theorem 1. In particular, since the present study considers only positive specifications of $\beta(x)$, the resulting framework is restricted to modelling data in which X and Y exhibit negative correlation.

Theorem 1. *Consider an accelerated conditional model of the form:*

$$P(X > x) = \bar{F}_0(x) \quad \text{and} \quad P(Y > y | X > x) = \bar{F}_1(\beta(x)y),$$

where \bar{F}_0 and \bar{F}_1 are survival functions. Then:

$$\beta'(x) \geq 0 \implies \text{Cov}(X, Y) \leq 0, \quad \beta'(x) \leq 0 \implies \text{Cov}(X, Y) \geq 0.$$

Proof. Since $P(Y > y, X > x) = \bar{F}_0(x)\bar{F}_1(\beta(x)y)$ then $P(Y > y) = \lim_{x \rightarrow 0^+} P(Y > y, X > x) = \bar{F}_0(0)\bar{F}_1(\beta(0)y) = \bar{F}_1(\beta(0)y)$ since $\lim_{x \rightarrow 0^+} \bar{F}_0(x) = 1$, we note:

$$\begin{aligned} \text{Cov}(X, Y) &= \int_0^\infty \int_0^\infty (P(Y > y, X > x) - P(X > x)P(Y > y)) \, dx dy \\ &= \int_0^\infty \int_0^\infty \bar{F}_0(x) (\bar{F}_1(\beta(x)y) - \bar{F}_1(\beta(0)y)) \, dx dy. \end{aligned}$$

Now since $\bar{F}_1'(\cdot) \leq 0$ (as \bar{F}_1 is a survival function), if $\beta'(x) \geq 0$ then $\bar{F}_1(\beta(x)y) \leq \bar{F}_1(\beta(0)y)$ and $\text{Cov}(X, Y) \leq 0$ for $x, y \geq 0$. Conversely, if $\beta'(x) \leq 0$ then $\bar{F}_1(\beta(x)y) \geq \bar{F}_1(\beta(0)y)$ and $\text{Cov}(X, Y) \geq 0$ for $x, y \geq 0$. Finally, if $\beta(x)$ is constant, say $\beta(x) \equiv c$, then $P(X > x, Y > y) = \bar{F}_0(x)\bar{F}_1(cy) = P(X > x)P(Y > y)$, so X and Y are independent and $\text{Cov}(X, Y) = 0$. \square

2 Exponential

We restrict the survival function forms to the exponential family as such:

$$\bar{F}_0(x) = P(X > x) = e^{-\alpha x}, \quad x > 0,$$

where $\alpha > 0$ and, for each $x > 0$:

$$\bar{F}_1(\beta(x)y) = P(Y > y | X > x) = e^{-\beta(x)y}, \quad y > 0,$$

where $\beta(x) > 0$ for all $x > 0$ to ensure a valid survival function. The corresponding joint survival function will be:

$$\bar{F}_0(x)\bar{F}_1(\beta(x)y) = P(X > x, Y > y) = e^{-\alpha x - \beta(x)y}, \quad x, y > 0. \quad (6)$$

Accordingly, $f_X(x) = \alpha e^{-(\alpha x)}$ for $x > 0$ and $f_Y(y) = \beta(0)e^{-(\beta(0)y)}$ for $y > 0$. Hence, $X \sim \text{Exp}(\alpha)$ and $Y \sim \text{Exp}(\beta(0))$.

Now since $[tf_1(t)]' = (1-t)e^{-t} > 0$ for $t < 1$, and $S(t) = -\frac{1}{t-1}$ is increasing on this interval, $\inf_{t < 1} S(t) = \lim_{t \rightarrow 0} S(t) = 1$. Additionally, since $[tf_1(t)]' = (1-t)e^{-t} < 0$ for $t > 1$, and $S(t) = -\frac{1}{t-1}$ is increasing on this interval, $\sup_{t > 1} S(t) = \lim_{t \rightarrow \infty} S(t) = 0$. Consequently, applying (4) yields:

$$0 \leq B'(x) \leq h_0(x)\beta(x) = \alpha\beta(x),$$

corresponding to Proposition 3.1 in Arnold and Arvanitis [1]. Now, we select the form of the acceleration function $\beta(x)$ such that $\beta'(x) = \alpha\beta(x)$ (that is, equal to $\beta'(x)$'s upper bound). Hence from (5) with $c^* = 1$, $\beta(x) = \gamma e^{\alpha x}$ with $\gamma > 0$. Furthermore, following Arnold and Arvanitis [1], we introduce a dependence parameter $\tau \in [0, 1]$ which governs the strength of the dependence between X and Y . The parameter τ is restricted to $[0, 1]$ since $\tau = 0$ implies $\beta(x) = \gamma$ is constant (so X and Y are independent with covariance 0), $\tau = 1$ corresponds to the case where $\beta'(x)$ attains its theoretical upper bound, where allowing $\tau > 1$ would imply $\beta'(x) > \alpha\beta(x)$, violating the theoretical upper bound. Hence we choose our acceleration function as:

$$\beta(x) = \gamma e^{\alpha \tau x}.$$

2.1 Moments

Noting $X \sim \text{Exp}(\alpha)$ and $Y \sim \text{Exp}(\gamma)$, we have:

$$E(X) = \frac{1}{\alpha}, \quad (7)$$

$$E(Y) = \frac{1}{\gamma}, \quad (8)$$

$$\text{Var}(X) = \frac{1}{\alpha^2}, \quad (9)$$

$$\text{Var}(Y) = \frac{1}{\gamma^2}. \quad (10)$$

Accordingly:

$$\begin{aligned} \text{Cov}(X, Y) &= \int_0^\infty \int_0^\infty (P(Y > y, X > x) - P(X > x)P(Y > y)) \, dx dy \\ &= \int_0^\infty \int_0^\infty \bar{F}_0(x) (\bar{F}_1(\beta(x)y) - \bar{F}_1(\beta(0)y)) \, dx dy \\ &= \int_0^\infty \int_0^\infty e^{-(\alpha x)} (e^{-\beta(x)y} - e^{-\beta(0)y}) \, dy dx \\ &= \int_0^\infty e^{-(\alpha x)} \left(\frac{1}{\beta(x)} - \frac{1}{\beta(0)} \right) \, dx \\ &= \frac{1}{\gamma} \int_0^\infty (e^{-\alpha(1+\tau)x} - e^{-\alpha x}) \, dx \\ &= -\frac{\tau}{\alpha\gamma(1+\tau)}. \end{aligned} \quad (11)$$

Using (9), (10) and (11), we have the correlation between X and Y as:

$$\begin{aligned} \rho(X, Y) &= \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} \\ &= -\frac{\tau}{1+\tau}. \end{aligned}$$

Seeing as $\tau \in [0, 1]$, this exponential model with particular acceleration function $\beta(x) = \gamma e^{\alpha \tau x}$, may only be used to model data for which $\rho(X, Y) \in [0, -\frac{1}{2}]$.

Additionally, to facilitate simulation of a Y conditional on X , it is convenient to formally note the conditional cumulative distribution function $F_{Y|X}$ as:

$$\begin{aligned} F_{Y|X=x}(y) &= \int_0^y \frac{f_{X,Y}(x, t)}{f_X(x)} dy \\ &= (-\gamma \tau y e^{\alpha \tau x} + e^{\gamma y e^{\alpha \tau x}} - 1) e^{-\gamma y e^{\alpha \tau x}}, \end{aligned} \quad (12)$$

where $f_{X,Y}(x, y)$ is provided in Appendix A.

3 Lomax

We restrict the survival function forms to the Lomax family as such:

$$\bar{F}_0(x) = P(X > x) = (1 + \alpha x)^{-\lambda}, \quad x > 0,$$

where $\alpha, \lambda > 0$ and, for each $x > 0$:

$$\bar{F}_1(\beta(x)y) = P(Y > y | X > x) = (1 + \beta(x)y)^{-\nu}, \quad y > 0,$$

where $\beta(x), \nu > 0$ for all $x > 0$ to ensure a valid survival function. The corresponding joint survival function will be:

$$\bar{F}_0(x) \bar{F}_1(\beta(x)y) = P(X > x, Y > y) = (1 + \alpha x)^{-\lambda} (1 + \beta(x)y)^{-\nu}, \quad x, y > 0. \quad (13)$$

Accordingly, $f_X(x) = \alpha \lambda (\alpha x + 1)^{-(\alpha+1)}$ for $x > 0$ and $f_Y(y) = \beta(0) \nu (\beta(0)y + 1)^{-(\nu+1)}$ for $y > 0$. Hence, $X \sim \text{Lomax}(\alpha, \lambda)$ and $Y \sim \text{Lomax}(\beta(0), \nu)$.

Now since $[tf_1(t)]' = \frac{\nu(t+1)^{-\nu}(1-\nu t)}{t^2+2t+1} > 0$ for $t < \frac{1}{\nu}$, and $S(t) = -\frac{t+1}{\nu t-1}$ is increasing on this interval, $\inf_{t < \frac{1}{\nu}} S(t) = \lim_{t \rightarrow 0} S(t) = 1$. Additionally, since $[tf_1(t)]' = \frac{\nu(t+1)^{-\nu}(1-\nu t)}{t^2+2t+1} < 0$ for $t > \frac{1}{\nu}$, and $S(t) = -\frac{t+1}{\nu t-1}$ is increasing on this interval, $\sup_{t > \frac{1}{\nu}} S(t) = \lim_{t \rightarrow \infty} S(t) = -\frac{1}{\nu}$. Consequently, applying (4) yields:

$$-\frac{1}{\nu} \frac{\alpha \lambda}{\alpha x + 1} \beta(x) \leq B'(x) \leq \frac{\alpha \lambda}{\alpha x + 1} \beta(x).$$

Now, we select the form of the acceleration function $\beta(x)$ such that $\beta'(x) = \frac{\alpha \lambda}{\alpha x + 1} \beta(x)$ (that is, equal to $\beta'(x)$'s upper bound). Hence from (5) with $c^* = 1$, $\beta(x) = \gamma (1 + \alpha x)^\lambda$ with $\gamma > 0$. Furthermore, following Arnold and Arvanitis [1], we introduce a dependence parameter $\tau \in [0, 1]$ which governs the strength of the dependence between X and Y . Hence we choose our acceleration function as:

$$\beta(x) = \gamma (1 + \alpha \tau x)^\lambda.$$

3.1 Moments

Noting $X \sim \text{Lomax}(\alpha, \lambda)$ and $Y \sim \text{Lomax}(\gamma, \nu)$, we have:

$$E(X) = \frac{1}{\alpha(\lambda - 1)}, \quad \lambda > 1, \quad (14)$$

$$E(Y) = \frac{1}{\gamma(\nu - 1)}, \quad \nu > 1, \quad (15)$$

$$\text{Var}(X) = \frac{\lambda}{\alpha^2(\lambda - 1)^2(\lambda - 2)}, \quad \lambda > 2, \quad (16)$$

$$\text{Var}(Y) = \frac{\nu}{\gamma^2(\nu - 1)^2(\nu - 2)}, \quad \nu > 2. \quad (17)$$

Accordingly:

$$\begin{aligned}
\text{Cov}(X, Y) &= \int_0^\infty \int_0^\infty (P(Y > y, X > x) - P(X > x)P(Y > y)) dy dx \\
&= \int_0^\infty \int_0^\infty \bar{F}_0(x) (\bar{F}_1(\beta(x)y) - \bar{F}_1(\beta(0)y)) dy dx \\
&= \int_0^\infty (1 + \alpha x)^{-\lambda} \left(\int_0^\infty ((1 + \beta(x)y)^{-\nu} - (1 + \beta(0)y)^{-\nu}) dy \right) dx \\
&= \frac{1}{\nu - 1} \int_0^\infty (1 + \alpha x)^{-\lambda} \left(\frac{1}{\beta(x)} - \frac{1}{\beta(0)} \right) dx \\
&= \frac{1}{\gamma(\nu - 1)} \left[\int_0^\infty \frac{dx}{(1 + \alpha x)^\lambda (1 + \alpha \tau x)^\lambda} - \int_0^\infty \frac{dx}{(1 + \alpha x)^\lambda} \right] \\
&= \frac{1}{\alpha \gamma(\nu - 1)} \left[\frac{1}{2\lambda - 1} {}_2F_1(\lambda, 1; 2\lambda; 1 - \tau) - \frac{1}{\lambda - 1} \right], \tag{18}
\end{aligned}$$

where the conditions $\nu > 1$ and $\lambda > 1$ ensure convergence of the inner and outer integrals, respectively. Now, using (16), (17) and (18), the correlation is:

$$\begin{aligned}
\rho(X, Y) &= \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}} \\
&= (\lambda - 1) \left(\frac{{}_2F_1(\lambda, 1; 2\lambda; 1 - \tau)}{2\lambda - 1} - \frac{1}{\lambda - 1} \right) \sqrt{\frac{(\lambda - 2)(\nu - 2)}{\lambda \nu}}, \tag{19}
\end{aligned}$$

valid for $\lambda, \nu > 2$. Since ${}_2F_1(\lambda, 1; 2\lambda; z)$ has a power series with strictly positive coefficients, it is strictly increasing in $z \in [0, 1]$; hence ${}_2F_1(\lambda, 1; 2\lambda; 1 - \tau)$ is strictly decreasing in $\tau \in [0, 1]$. Consequently, (19) attains its minimum at $\tau = 1$ and its maximum (zero) at $\tau = 0$. Therefore,

$$-\frac{\lambda}{2\lambda - 1} \sqrt{\frac{\lambda - 2}{\lambda}} \sqrt{\frac{\nu - 2}{\nu}} \leq \rho(X, Y) \leq 0.$$

Moreover, $\rho_{\min}(\lambda, \nu) := -\frac{\lambda}{2\lambda - 1} \sqrt{\frac{\lambda - 2}{\lambda}} \sqrt{\frac{\nu - 2}{\nu}}$ is strictly decreasing in each of λ and ν , so $\inf_{\lambda > 2, \nu > 2} \rho_{\min}(\lambda, \nu) = \lim_{\lambda, \nu \rightarrow \infty} \rho_{\min}(\lambda, \nu) = -\frac{1}{2}$. Thus, as in the exponential case, this Lomax model with acceleration $\beta(x) = \gamma(1 + \alpha \tau x)^\lambda$ can only accommodate correlations in the range $\rho(X, Y) \in [-\frac{1}{2}, 0]$.

4 Weibull

We restrict the survival function forms to the Weibull family as such:

$$\bar{F}_0(x) = P(X > x) = e^{-(\alpha x)^\lambda}, \quad x > 0,$$

where $\alpha, \lambda > 0$ and, for each $x > 0$:

$$\bar{F}_1(\beta(x)y) = P(Y > y | X > x) = e^{-(\beta(x)y)^\nu}, \quad y > 0,$$

where $\beta(x), \nu > 0$ for all $x > 0$ to ensure a valid survival function. The corresponding joint survival function will be:

$$\bar{F}_0(x) \bar{F}_1(\beta(x)y) = P(X > x, Y > y) = e^{-(\alpha x)^\lambda - (\beta(x)y)^\nu}, \quad x, y > 0. \tag{20}$$

Accordingly, $f_X(x) = \alpha \lambda (\alpha x)^{\lambda-1} e^{-(\alpha x)^\lambda}$ for $x > 0$ and $f_Y(y) = \beta(0) \nu (\beta(0)y)^{\nu-1} e^{-(\beta(0)y)^\nu}$ for $y > 0$. Hence, $X \sim \text{Weibull}(\alpha, \lambda)$ and $Y \sim \text{Weibull}(\beta(0), \nu)$.

Now since $[tf_1(t)]' = \nu^2 t^{\nu-1} (1 - t^\nu) e^{-t^\nu} > 0$ for $t < 1$, and $S(t) = \frac{1}{\nu(1-t^\nu)}$ is increasing on this interval, $\inf_{t < 1} S(t) = \lim_{t \rightarrow 0} S(t) = \frac{1}{\nu}$. Additionally, since $[tf_1(t)]' = \nu^2 t^{\nu-1} (1 - t^\nu) e^{-t^\nu} < 0$ for $t > 1$, and $S(t) = \frac{1}{\nu(1-t^\nu)}$ is increasing on this interval, $\sup_{t > 1} S(t) = \lim_{t \rightarrow \infty} S(t) = 0$. Consequently, applying (4) yields:

$$0 \leq B'(x) \leq \frac{1}{\nu} \lambda \alpha (\alpha x)^{\lambda-1} \beta(x).$$

Now, we select the form of the acceleration function $\beta(x)$ such that $\beta'(x) = \frac{1}{\nu} \lambda \alpha (\alpha x)^{\lambda-1} \beta(x)$ (that is, equal to $\beta'(x)$'s upper bound). Hence from (5) with $c^* = \frac{1}{\nu}$, $\beta(x) = \gamma e^{\frac{1}{\nu}(\alpha x)^\lambda}$ with $\gamma > 0$. Furthermore, following Arnold and Arvanitis [1], we introduce a dependence parameter $\tau \in [0, 1]$ as before. Hence we choose our acceleration function as:

$$\beta(x) = \gamma e^{\frac{1}{\nu}(\alpha \tau x)^\lambda}.$$

4.1 Moments

Noting $X \sim Weibull(\alpha, \lambda)$ and $Y \sim Weibull(\gamma, \nu)$, we have:

$$E(X) = \frac{1}{\alpha \Gamma\left(1 + \frac{1}{\lambda}\right)}, \quad (21)$$

$$E(Y) = \frac{1}{\gamma \Gamma\left(1 + \frac{1}{\nu}\right)}, \quad (22)$$

$$Var(X) = \frac{1}{\alpha^2} \left(\Gamma\left(1 + \frac{2}{\lambda}\right) - \left(\Gamma\left(1 + \frac{1}{\lambda}\right) \right)^2 \right) \quad (23)$$

$$Var(Y) = \frac{1}{\gamma^2} \left(\Gamma\left(1 + \frac{2}{\nu}\right) - \left(\Gamma\left(1 + \frac{1}{\nu}\right) \right)^2 \right) \quad (24)$$

Accordingly:

$$\begin{aligned} Cov(X, Y) &= \int_0^\infty \int_0^\infty (P(Y > y, X > x) - P(X > x)P(Y > y)) dy dx \\ &= \int_0^\infty \int_0^\infty \bar{F}_0(x) (\bar{F}_1(\beta(x)y) - \bar{F}_1(\beta(0)y)) dy dx \\ &= \int_0^\infty e^{-(\alpha x)^\lambda} \left(\int_0^\infty (e^{-(\beta(x)y)^\nu} - e^{-(\beta(0)y)^\nu}) dy \right) dx \\ &= \int_0^\infty e^{-(\alpha x)^\lambda} \left(\frac{\Gamma(1 + \frac{1}{\nu})}{\beta(x)} - \frac{\Gamma(1 + \frac{1}{\nu})}{\beta(0)} \right) dx \\ &= \frac{1}{\gamma} \Gamma\left(1 + \frac{1}{\nu}\right) \left[\int_0^\infty e^{-(\alpha x)^\lambda - \frac{1}{\nu}(\alpha \tau x)^\lambda} dx - \int_0^\infty e^{-(\alpha x)^\lambda} dx \right] \\ &= \frac{1}{\alpha \gamma} \Gamma\left(1 + \frac{1}{\nu}\right) \Gamma\left(1 + \frac{1}{\lambda}\right) \left[\left(\frac{\nu}{\nu + \tau^\lambda} \right)^{\frac{1}{\lambda}} - 1 \right]. \end{aligned} \quad (25)$$

Now, using (23), (24) and (25), the correlation is:

$$\begin{aligned} \rho(X, Y) &= \frac{Cov(X, Y)}{\sqrt{Var(X) Var(Y)}} \\ &= \frac{\Gamma\left(1 + \frac{1}{\nu}\right) \Gamma\left(1 + \frac{1}{\lambda}\right) \left[\left(\frac{\nu}{\nu + \tau^\lambda} \right)^{\frac{1}{\lambda}} - 1 \right]}{\sqrt{\left(\Gamma\left(1 + \frac{2}{\lambda}\right) - \left(\Gamma\left(1 + \frac{1}{\lambda}\right) \right)^2 \right) \left(\Gamma\left(1 + \frac{2}{\nu}\right) - \left(\Gamma\left(1 + \frac{1}{\nu}\right) \right)^2 \right)}}, \end{aligned} \quad (26)$$

Clearly, (26) is strictly decreasing in $\tau \in [0, 1]$. Hence: 1 Now, $\inf_{\lambda, \nu > 0} \rho_{\min}(\lambda, \nu) = \lim_{\nu \rightarrow \infty} \rho_{\min}(\lambda = 1, \nu) = -\frac{\sqrt{6}}{\pi}$. Hence, this Weibull model with acceleration function $\beta(x) = \gamma e^{\frac{1}{\nu}(\alpha \tau x)^\lambda}$ will be able to accommodate correlations in the range $\rho(X, Y) \in \left[-\frac{\sqrt{6}}{\pi}, 0 \right]$.

5 Log-Logistic

We restrict the survival function forms to the log-logistic family as such:

$$\bar{F}_0(x) = P(X > x) = 1 - \frac{1}{1 + (\alpha x)^{-\lambda}} = \frac{1}{1 + (\alpha x)^\lambda}, \quad x > 0,$$

where $\alpha, \lambda > 0$ and, for each $x > 0$:

$$\bar{F}_1(\beta(x)y) = P(Y > y \mid X > x) = 1 - \frac{1}{1 + (\beta(x)y)^{-\nu}} = \frac{1}{1 + (\beta(x)y)^\nu}, \quad y > 0,$$

where $\beta(x), \nu > 0$ for all $x > 0$ to ensure a valid survival function. The corresponding joint survival function will be:

$$\bar{F}_0(x)\bar{F}_1(\beta(x)y) = P(X > x, Y > y) = \frac{1}{(1 + (\alpha x)^\lambda)(1 + (\beta(x)y)^\nu)}, \quad x, y > 0. \quad (27)$$

Accordingly, $f_X(x) = \frac{\alpha\lambda(\alpha x)^{\lambda-1}}{((\alpha x)^\lambda + 1)^2}$ for $x > 0$ and $f_Y(y) = \frac{\beta(0)\nu(\beta(0)y)^{\nu-1}}{((\beta(0)y)^\nu + 1)^2}$ for $y > 0$. Hence, $X \sim \text{LogLogistic}(\alpha, \lambda)$ and $Y \sim \text{LogLogistic}(\beta(0), \nu)$.

Now since $[tf_1(t)]' = \frac{\nu^2 t^{\nu-1}(1-t^\nu)}{t^{3\nu} + 3t^{2\nu} + 3t^\nu + 1} > 0$ for $t < 1$, and $S(t) = -\frac{t^\nu + 1}{\nu(t^\nu - 1)}$ is increasing on this interval, $\inf_{t < 1} S(t) = \lim_{t \rightarrow 0} S(t) = \frac{1}{\nu}$. Additionally, since $[tf_1(t)]' = \frac{\nu^2 t^{\nu-1}(1-t^\nu)}{t^{3\nu} + 3t^{2\nu} + 3t^\nu + 1} < 0$ for $t > 1$, and $S(t) = -\frac{t^\nu + 1}{\nu(t^\nu - 1)}$ is increasing on this interval, $\sup_{t > 1} S(t) = \lim_{t \rightarrow \infty} S(t) = -\frac{1}{\nu}$. Consequently, applying (4) yields:

$$-\frac{1}{\nu} \frac{\alpha\lambda(\alpha x)^{\lambda-1}}{(\alpha x)^\lambda + 1} \beta(x) \leq B'(x) \leq \frac{1}{\nu} \frac{\alpha\lambda(\alpha x)^{\lambda-1}}{(\alpha x)^\lambda + 1} \beta(x).$$

Now, we select the form of the acceleration function $\beta(x)$ such that $\beta'(x) = \frac{1}{\nu} \frac{\alpha\lambda(\alpha x)^{\lambda-1}}{(\alpha x)^\lambda + 1} \beta(x)$ (that is, equal to $\beta'(x)$'s upper bound). Hence from (5) with $c^* = \frac{1}{\nu}$, $\beta(x) = \gamma(1 + (\alpha x)^\lambda)^{\frac{1}{\nu}}$ with $\gamma > 0$. Furthermore, following Arnold and Arvanitis [1], we introduce a dependence parameter $\tau \in [0, 1]$ as before. Hence we choose our acceleration function as:

$$\beta(x) = \gamma(1 + (\alpha\tau x)^\lambda)^{\frac{1}{\nu}}.$$

5.1 Moments

Noting $X \sim \text{LogLogistic}(\alpha, \lambda)$ and $Y \sim \text{LogLogistic}(\gamma, \nu)$, we have:

$$E(X) = \frac{\pi}{\alpha\lambda \sin(\frac{\pi}{\lambda})}, \quad \lambda > 1, \quad (28)$$

$$E(Y) = \frac{\pi}{\gamma\nu \sin(\frac{\pi}{\nu})}, \quad \nu > 1, \quad (29)$$

$$\text{Var}(X) = \frac{1}{\alpha^2} \left[\frac{2\pi}{\lambda \sin(\frac{2\pi}{\lambda})} - \left(\frac{\pi}{\lambda \sin(\frac{\pi}{\lambda})} \right)^2 \right], \quad \lambda > 2, \quad (30)$$

$$\text{Var}(Y) = \frac{1}{\gamma^2} \left[\frac{2\pi}{\nu \sin(\frac{2\pi}{\nu})} - \left(\frac{\pi}{\nu \sin(\frac{\pi}{\nu})} \right)^2 \right], \quad \nu > 2. \quad (31)$$

Accordingly:

$$\begin{aligned} \text{Cov}(X, Y) &= \int_0^\infty \int_0^\infty (P(Y > y, X > x) - P(X > x)P(Y > y)) dy dx \\ &= \int_0^\infty \int_0^\infty \bar{F}_0(x) (\bar{F}_1(\beta(x)y) - \bar{F}_1(\beta(0)y)) dy dx \\ &= \int_0^\infty \frac{1}{1 + (\alpha x)^\lambda} \left(\int_0^\infty \left(\frac{1}{1 + (\beta(x)y)^\nu} - \frac{1}{1 + (\beta(0)y)^\nu} \right) dy \right) dx \\ &= \frac{\pi}{\nu \sin(\frac{\pi}{\nu})} \int_0^\infty \frac{1}{1 + (\alpha x)^\lambda} \left(\frac{1}{\beta(x)} - \frac{1}{\beta(0)} \right) dx \\ &= \frac{\pi}{\gamma\nu \sin(\frac{\pi}{\nu})} \left[\int_0^\infty \frac{dx}{((1 + (\alpha x)^\lambda)((1 + (\alpha\tau x)^\lambda)^{\frac{1}{\nu}})} - \int_0^\infty \frac{dx}{1 + (\alpha x)^\lambda} \right] \\ &= \frac{\pi}{\alpha\lambda\nu\gamma \sin(\frac{\pi}{\nu})} \left[\frac{\Gamma(\frac{1}{\lambda}) \Gamma(1 + \frac{1}{\nu} - \frac{1}{\lambda})}{\Gamma(1 + \frac{1}{\nu})} {}_2F_1\left(\frac{1}{\nu}, \frac{1}{\lambda}; 1 + \frac{1}{\nu}; 1 - \tau^\lambda\right) - \frac{\pi}{\sin(\frac{\pi}{\lambda})} \right]. \end{aligned} \quad (32)$$

where the conditions $\nu > 1$ and $\lambda > 1$ ensure convergence of the inner and outer integrals, respectively. Now, using (30), (31) and (32), the correlation is:

$$\begin{aligned}\rho(X, Y) &= \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}} \\ &= \frac{\pi \left[\frac{\Gamma(\frac{1}{\lambda})\Gamma(1+\frac{1}{\nu}-\frac{1}{\lambda})}{\Gamma(1+\frac{1}{\nu})} {}_2F_1\left(\frac{1}{\nu}, \frac{1}{\lambda}; 1+\frac{1}{\nu}; 1-\tau^\lambda\right) - \frac{\pi}{\sin(\frac{\pi}{\lambda})} \right]}{\lambda\nu \sin\left(\frac{\pi}{\nu}\right) \sqrt{\left[\frac{2\pi}{\lambda \sin(\frac{2\pi}{\lambda})} - \left(\frac{\pi}{\lambda \sin(\frac{\pi}{\lambda})}\right)^2 \right] \left[\frac{2\pi}{\nu \sin(\frac{2\pi}{\nu})} - \left(\frac{\pi}{\nu \sin(\frac{\pi}{\nu})}\right)^2 \right]}},\end{aligned}\quad (33)$$

valid for $\lambda, \nu > 2$. Since ${}_2F_1\left(\frac{1}{\nu}, \frac{1}{\lambda}; 1+\frac{1}{\nu}; z\right)$ has a power series with strictly positive coefficients, it is strictly increasing in $z \in [0, 1]$; hence ${}_2F_1\left(\frac{1}{\nu}, \frac{1}{\lambda}; 1+\frac{1}{\nu}; 1-\tau^\lambda\right)$ is strictly decreasing in $\tau \in [0, 1]$. Consequently, (33) attains its minimum at $\tau = 1$ and its maximum (zero) at $\tau = 0$. Therefore,

$$\rho_{\min(\lambda, \nu)} := \frac{\pi \left[\frac{\Gamma(\frac{1}{\lambda})\Gamma(1+\frac{1}{\nu}-\frac{1}{\lambda})}{\Gamma(1+\frac{1}{\nu})} - \frac{\pi}{\sin(\frac{\pi}{\lambda})} \right]}{\lambda\nu \sin\left(\frac{\pi}{\nu}\right) \sqrt{\left[\frac{2\pi}{\lambda \sin(\frac{2\pi}{\lambda})} - \left(\frac{\pi}{\lambda \sin(\frac{\pi}{\lambda})}\right)^2 \right] \left[\frac{2\pi}{\nu \sin(\frac{2\pi}{\nu})} - \left(\frac{\pi}{\nu \sin(\frac{\pi}{\nu})}\right)^2 \right]}} \leq \rho(X, Y) \leq 0.$$

Now denoting $[\lambda^*, \nu^*]' = \text{argmin}_{\lambda, \nu} \rho_{\min(\lambda, \nu)}$, we obtain $\inf_{\lambda, \nu > 2} \rho_{\min(\lambda, \nu)} = \lim_{\nu \rightarrow \infty} \rho_{\min}(\lambda = \lambda^* \approx 5.06, \nu) \approx -0.54076$. Hence, this log-logistic model with acceleration function $\beta(x) = \gamma(1 + (\alpha\tau x)^\lambda)^{\frac{1}{\nu}}$ will be able to accommodate correlations in the range $\rho(X, Y) \in [-0.54076, 0]$.

6 Half-Cauchy

We restrict the survival function forms to the Half-Cauchy family as such:

$$\bar{F}_0(x) = P(X > x) = 1 - \frac{2}{\pi} \arctan(\alpha x), \quad x > 0,$$

where $\alpha > 0$ and, for each $x > 0$:

$$\bar{F}_1(\beta(x)y) = P(Y > y \mid X > x) = 1 - \frac{2}{\pi} \arctan(\beta(x)y), \quad y > 0,$$

where $\beta(x) > 0$ for all $x > 0$ to ensure a valid survival function. The corresponding joint survival function will be:

$$\bar{F}_0(x)\bar{F}_1(\beta(x)y) = P(X > x, Y > y) = \frac{1}{\pi^2} (2 \arctan(\alpha x) - \pi) (2 \arctan(\beta(x)y) - \pi), \quad x, y > 0. \quad (34)$$

Accordingly, $f_X(x) = \frac{2\alpha}{\pi(1+(\alpha x)^2)}$ for $x > 0$ and $f_Y(y) = \frac{2\beta(0)}{\pi(1+(\beta(0)y)^2)}$ for $y > 0$. Hence, $X \sim \text{Half-Cauchy}(\alpha)$ and $Y \sim \text{Half-Cauchy}(\beta(0))$.

Now since $[tf_1(t)]' = \frac{2(1-t^2)}{\pi(t^2+1)^2} > 0$ for $t < 1$, and $S(t) = \frac{t^2+1}{1-t^2}$ is increasing on this interval, $\inf_{t < 1} S(t) = \lim_{t \rightarrow 0} S(t) = 1$. Additionally, since $[tf_1(t)]' = \frac{2(1-t^2)}{\pi(t^2+1)^2} < 0$ for $t > 1$, and $S(t) = \frac{t^2+1}{1-t^2}$ is increasing on this interval, $\sup_{t > 1} S(t) = \lim_{t \rightarrow \infty} S(t) = -1$. Consequently, applying (4) yields:

$$-\frac{\alpha}{(1+(\alpha x)^2)(\frac{\pi}{2} - \arctan(\alpha x))} \beta(x) \leq B'(x) \leq \frac{\alpha}{(1+(\alpha x)^2)(\frac{\pi}{2} - \arctan(\alpha x))} \beta(x).$$

Now, we select the form of the acceleration function $\beta(x)$ such that $\beta'(x) = \frac{\alpha}{(1+(\alpha x)^2)(\frac{\pi}{2} - \arctan(\alpha x))} \beta(x)$ (that is, equal to $\beta'(x)$'s upper bound). Hence from (5) with $c^* = 1$, $\beta(x) = \gamma \frac{1}{1 - \frac{2}{\pi} \arctan(\alpha x)}$ with $\gamma > 0$. Furthermore, following Arnold and Arvanitis [1], we introduce a dependence parameter $\tau \in [0, 1]$ as before. Hence we choose our acceleration function as:

$$\beta(x) = \gamma \frac{1}{1 - \frac{2}{\pi} \arctan(\alpha \tau x)}.$$

7 Gamma

We restrict the survival function forms to the Gamma family as such:

$$\bar{F}_0(x) = P(X > x) = 1 - \frac{1}{\Gamma(\lambda)} \gamma^*(\lambda, \alpha x) = \frac{\Gamma(\lambda, \alpha x)}{\Gamma(\lambda)}, \quad x > 0,$$

where $\alpha, \lambda > 0$ and denoting γ^* as the lower incomplete gamma function. Now for each $x > 0$:

$$\bar{F}_1(\beta(x)y) = P(Y > y \mid X > x) = 1 - \frac{1}{\Gamma(\nu)} \gamma^*(\nu, \beta(x)y) = \frac{\Gamma(\nu, \beta(x)y)}{\Gamma(\nu)}, \quad y > 0,$$

where $\beta(x), \nu > 0$ for all $x > 0$ to ensure a valid survival function. The corresponding joint survival function will be:

$$\bar{F}_0(x) \bar{F}_1(\beta(x)y) = P(X > x, Y > y) = \frac{\Gamma(\lambda, \alpha x) \Gamma(\nu, \beta(x)y)}{\Gamma(\lambda) \Gamma(\nu)}, \quad x, y > 0. \quad (35)$$

Accordingly, $f_X(x) = \frac{\alpha^\lambda}{\Gamma(\lambda)} x^{\lambda-1} e^{-\alpha x}$ for $x > 0$ and $f_Y(y) = \frac{\beta(0)^\nu}{\Gamma(\nu)} y^{\nu-1} e^{-\beta(0)y}$ for $y > 0$. Hence, $X \sim \text{Gamma}(\alpha, \lambda)$ and $Y \sim \text{Gamma}(\beta(0), \nu)$.

Now since $[tf_1(t)]' = \frac{t^{\nu-1}(\nu-t)e^{-t}}{\Gamma(\nu)} > 0$ for $t < \nu$, and $S(t) = \frac{1}{\nu-t}$ is increasing on this interval, $\inf_{t < \nu} S(t) = \lim_{t \rightarrow 0} S(t) = \frac{1}{\nu}$. Additionally, since $[tf_1(t)]' = \frac{t^{\nu-1}(\nu-t)e^{-t}}{\Gamma(\nu)} < 0$ for $t > \nu$, and $S(t) = \frac{1}{\nu-t}$ is increasing on this interval, $\sup_{t > \nu} S(t) = \lim_{t \rightarrow \infty} S(t) = 0$. Consequently, applying (4) yields:

$$0 \leq B'(x) \leq \frac{1}{\nu} \frac{\alpha^\lambda}{\Gamma(\lambda, \alpha x)} x^{\lambda-1} e^{-\alpha x}.$$

Now, we select the form of the acceleration function $\beta(x)$ such that $\beta'(x) = \frac{1}{\nu} \frac{\alpha^\lambda}{\Gamma(\lambda, \alpha x)} x^{\lambda-1} e^{-\alpha x}$ (that is, equal to $\beta'(x)$'s upper bound). Hence from (5) with $c^* = \frac{1}{\nu}$, $\beta(x) = \gamma \left[\frac{\Gamma(\lambda)}{\Gamma(\lambda, \alpha x)} \right]^\frac{1}{\nu}$ with $\gamma > 0$. Furthermore, following Arnold and Arvanitis [1], we introduce a dependence parameter $\tau \in [0, 1]$ as before. Hence we choose our acceleration function as:

$$\beta(x) = \gamma \left[\frac{\Gamma(\lambda)}{\Gamma(\lambda, \alpha \tau x)} \right]^\frac{1}{\nu}.$$

8 Simulation

8.1 Method

To generate dependent pairs (X, Y) for simulation, we employ a copula-based simulation approach for the method of moments estimation and a Metropolis-Hastings (MH) algorithm for maximum likelihood estimation. m.m.e's Rely exclusively on a limited set of moments, such as means, variances, and covariances - begin such, MH samples may fail to reproduce these accurately (unless the Markov chains are extremely long and well-mixed) leading to poor moment-based estimates. Copula-based simulation provides a more effective solution in this context by generating synthetic (X, Y) pairs that preserve the specified marginal distributions and dependence structure, ensuring that simulated moments are consistent with their theoretical counterparts. In contrast, copula-based simulation is unsuitable for likelihood-based estimation because it alters the joint density of (X, Y) , whereas m.l.e's are defined relative to the true joint distribution; using copula-generated samples for maximum likelihood estimation would therefore imply optimizing a likelihood corresponding to a different model.

Additionally, we note that since we were able to derive $F_{Y|X=x}(y)$ for the exponential model, we may merely utilise the inverse transform method to simulate a Y conditioned on X , given $X \sim \text{Exp}(\alpha)$ and $Y \sim \text{Exp}(\beta(0))$. Specifically, making use of (12), and conditional on a draw of $X = x$, we may obtain:

$$Y = - \frac{\tau W \left(\frac{(U-1)e^{-\frac{1}{\tau}}}{\tau} \right) + 1}{\gamma \tau e^{\alpha \tau x}}$$

where W is the lower branch of the Lambert W function and $U \sim \text{Uniform}(0, 1)$.

Furthermore, we conduct simulations based on 10000¹ data sets of size $n = 1000, 5000$ and 10000 for fixed parameter values: $\alpha = 1, \gamma = 2, \lambda = 3, \nu = 4$ and $\tau = 0.5$.

8.1.1 Method of moments

The copula preserves the marginal distributions of X and Y while introducing dependence through ρ_{latent} . Specifically, we use a Student- t copula to capture both tail dependence and negative correlations. Here, $F_X(x) = 1 - \bar{F}_0(x)$ and $F_Y(y) = 1 - \bar{F}_1(\beta(0)y)$.

Step 1: Construct a t -copula with correlation ρ_{latent} and degrees of freedom v_c . Formally, for $U = F_X(X)$ and $V = F_Y(Y)$, we have $(U, V) \sim C_{v_c, \rho_{\text{latent}}}$, where the copula distribution function is:

$$C_{v_c, \rho_{\text{latent}}}(u, v) = t_{2, v_c}(t_{v_c}^{-1}(u), t_{v_c}^{-1}(v); \rho_{\text{latent}}), \quad (u, v) \in [0, 1]^2,$$

with $t_{v_c}^{-1}$ the quantile function of a univariate Student- $t(v_c)$ and t_{2, v_c} the bivariate Student- t distribution function with correlation ρ_{latent} . From Sklar's Theorem, the joint density of (X, Y) with marginal distributions F_X and F_Y is:

$$f_{X, Y}(x, y) = c_{v_c, \rho_{\text{latent}}}(F_X(x), F_Y(y)) f_X(x) f_Y(y),$$

where the copula density is $c_{v_c, \rho_{\text{latent}}}(u, v) = \frac{\partial^2}{\partial u \partial v} C_{v_c, \rho_{\text{latent}}}(u, v)$.

Step 2: Map the model Pearson correlation ρ_{target} into the copula correlation parameter ρ_{latent} that drives dependence in the t -copula. This requires re-expressing the correlation in terms of $c_{v_c, \rho_{\text{latent}}}(u, v)$ and solving numerically to obtain ρ_{latent} :

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}} = \frac{\int_0^1 \int_0^1 F_X^{-1}(u) F_Y^{-1}(v) c_{v_c, \rho_{\text{latent}}}(u, v) du dv - E(X)E(Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}},$$

with $E(X) = \int_0^1 F_X^{-1}(u) du$, $E(Y) = \int_0^1 F_Y^{-1}(v) dv$, and $\text{Var}(X) = E(X^2) - E(X)^2$.

Step 3: Simulate n pairs (U, V) from $(U, V) \sim C_{v_c, \rho_{\text{latent}}}$, then transform back via the inverse distributions to obtain $X = F_X^{-1}(U)$ and $Y = F_Y^{-1}(V)$.

Furthermore, Appendix B provides a guide on how to obtain m.m.e's for the models (noting that moments were not provided for the Half-Cauchy and gamma models, we exclude m.m.e's for said models).

8.1.2 Maximum Likelihood Estimation

We utilise the MH algorithm to sample from the joint density of the models considered in this study, where the target distribution is given by $\pi(x, y) = f_{X, Y}(x, y)$, noting that $X \sim \text{Family}(\alpha, \lambda)$ and $Y \sim \text{Family}(\beta(0), \nu)$, with $f_{X, Y}(x, y)$ given in Appendix A

Step 1 (Initialization): Choose any starting point $(x^{(0)}, y^{(0)})$ in the support of $f_{X, Y}$. Set $j = 0$.

Step 2 (Proposal): Given the current state $(x^{(j)}, y^{(j)})$, draw a candidate

$$(x^*, y^*) \sim q(\cdot \mid x^{(j)}, y^{(j)}).$$

The study utilises an independence proposal:

$$q(x^*, y^* \mid x^{(j)}, y^{(j)}) = q(x^*, y^*) = f_X(x^*) f_Y(y^*).$$

¹For the Half-Cauchy and gamma models, only 1000 datasets were used, as the computation of the m.l.e's required substantially more time when employing R's `optim` function with the `L-BFGS-B` algorithm.

Step 3 (Acceptance ratio α_a): Compute:

$$\begin{aligned}\alpha &= \min \left(\frac{f_{X,Y}(x^*, y^*) q(x^{(j)}, y^{(j)} | x^*, y^*)}{f_{X,Y}(x^{(j)}, y^{(j)}) q(x^*, y^* | x^{(j)}, y^{(j)})}, 1 \right) \\ &= \min \left(\frac{f_{X,Y}(x^*, y^*)}{f_X(x^*) f_Y(y^*)} \bigg/ \frac{f_{X,Y}(x^{(j)}, y^{(j)})}{f_X(x^{(j)}) f_Y(y^{(j)})}, 1 \right).\end{aligned}$$

Equivalently, using logs for numerical stability,

$$\begin{aligned}\log(\alpha_a) &= \min \left\{ \left[\log(f_{X,Y}(x^*, y^*)) - \log(f_X(x^*)) - \log(f_Y(y^*)) \right] \right. \\ &\quad \left. - \left[\log(f_{X,Y}(x^{(j)}, y^{(j)})) - \log(f_X(x^{(j)})) - \log(f_Y(y^{(j)})) \right], 0 \right\}.\end{aligned}$$

Subsequently, (x^*, y^*) is accepted as the new pair in the Markov chain under the following acceptance criterion:

$$(x^{(j+1)}, y^{(j+1)}) = \begin{cases} (x^*, y^*), & \text{if } U < \alpha_a \\ (x^{(j)}, y^{(j)}), & \text{otherwise.} \end{cases}$$

where $U \sim \text{Uniform}(0, 1)$.

Furthermore, Appendix A presents the log-likelihoods for the study's models - m.l.e's are obtained numerically as $\text{argmax}_{\theta} \ell(\theta)$ where $\theta = [\alpha, \lambda, \gamma, \nu, \tau]'$.

8.2 Simulation Data

The corresponding m.m.e and m.l.e results are reported in Tables 1-6, including 95% confidence intervals computed as $\hat{\theta} \pm Z_{\alpha/2} \text{S.E.}(\hat{\theta})$, where $\hat{\theta}$ is the point estimator ² We also display the pearson correlations, where PC^{C} denotes the pearson correlation from datasets derived from the copula and PC^{MH} denotes the pearson correlation from datasets derived from the MH algorithm.

As anticipated, the standard errors of both the m.m.e. and m.l.e. decrease with increasing sample size n , resulting in correspondingly narrower 95% confidence intervals. We refrain from comparing the accuracy of the m.m.e's with m.l.e's since they were calculated on different datasets. A salient factor to note is the somewhat inaccuracy of correlations of datasets derived from the copula for the log-logistic model; noting PC^{C} in Table 4, as well as the somewhat inaccuracy of the correlations of datasets derived from the MH algorithm for the Lomax model; noting PC^{MH} in Table 2. The only other bad remark from the simulations would be the slight inaccuracy of estimates of the Lomax model in Table 2: reliable estimation of estimates appears to require substantially larger sample sizes.

²The reported standard errors (S.E.) are computed as the empirical standard deviation of the parameter estimates across n simulated datasets, representing the sampling variability of the estimator.

n	Parameter	m.m.e	m.l.e	SE(m.m.e)	SE(m.l.e)	95%CI (m.m.e)	95%CI (m.l.e)	PC
1000	α	1.001	1.000	0.032	0.031	(0.938, 1.063)	(0.939, 1.061)	-0.334
	γ	2.001	2.000	0.063	0.063	(1.877, 2.125)	(1.878, 2.123)	
	τ	0.499	0.501	0.045	0.028	(0.410, 0.588)	(0.445, 0.557)	
	ρ	-0.333	-0.334	0.020	0.013	(-0.373, -0.294)	(-0.359, -0.309)	
5000	α	1.001	1.000	0.014	0.0138	(0.973, 1.028)	(0.973, 1.027)	-0.334
	γ	2.001	2.000	0.028	0.0279	(1.945, 2.056)	(1.945, 2.055)	
	τ	0.500	0.500	0.020	0.0126	(0.460, 0.539)	(0.475, 0.525)	
	ρ	-0.333	-0.333	0.009	0.0056	(-0.351, -0.316)	(-0.344, -0.322)	
10000	α	1.000	1.000	0.0100	0.0097	(0.980, 1.020)	(0.981, 1.019)	-0.334
	γ	2.000	2.000	0.0203	0.0199	(1.960, 2.040)	(1.961, 2.039)	
	τ	0.500	0.500	0.0143	0.0090	(0.472, 0.528)	(0.482, 0.518)	
	ρ	-0.333	-0.333	0.0063	0.0040	(-0.346, -0.321)	(-0.341, -0.326)	

Table 1: Simulations for exponential model ($\alpha = 1$, $\gamma = 2$, $\tau = 0.5$ with $\rho = -0.333$).

n	Parameter	m.m.e	m.l.e	SE(m.m.e)	SE(m.l.e)	95%CI (m.m.e)	95%CI (m.l.e)	PC ^C	PC ^{MH}
1000	α	0.843	0.848	0.199	0.268	(0.452, 1.234)	(0.323, 1.374)	-0.212	-0.236
	γ	1.795	1.889	0.495	0.622	(0.824, 2.766)	(0.670, 3.109)		
	λ	3.383	3.406	0.591	0.888	(2.224, 4.542)	(1.664, 5.146)		
	ν	4.349	4.178	0.945	1.120	(2.496, 6.201)	(1.984, 6.373)		
	τ	0.559	0.481	0.145	0.090	(0.276, 0.843)	(0.304, 0.657)		
	ρ	-0.212	-0.195	0.044	0.041	(-0.298, -0.125)	(-0.276, -0.113)		
5000	α	0.911	0.926	0.148	0.157	(0.622, 1.201)	(0.618, 1.235)	-0.1987	-0.224
	γ	1.906	1.963	0.315	0.332	(1.289, 2.524)	(1.313, 2.613)		
	λ	3.198	3.202	0.323	0.465	(2.565, 3.832)	(2.291, 4.113)		
	ν	4.153	4.062	0.456	0.440	(3.259, 5.046)	(3.200, 4.924)		
	τ	0.522	0.491	0.089	0.051	(0.347, 0.696)	(0.391, 0.592)		
	ρ	-0.199	-0.192	0.030	0.023	(-0.257, -0.141)	(-0.237, -0.148)		
10000	α	0.931	0.946	0.131	0.125	(0.674, 1.189)	(0.700, 1.192)	-0.1948	-0.220
	γ	1.935	1.974	0.245	0.227	(1.454, 2.416)	(1.529, 2.419)		
	λ	3.149	3.143	0.266	0.349	(2.628, 3.670)	(2.458, 3.828)		
	ν	4.102	4.038	0.349	0.317	(3.419, 4.785)	(3.417, 4.658)		
	τ	0.512	0.494	0.073	0.040	(0.368, 0.656)	(0.416, 0.572)		
	ρ	-0.195	-0.191	0.025	0.017	(-0.243, -0.146)	(-0.225, -0.157)		

Table 2: Simulations for Lomax model ($\alpha = 1$, $\gamma = 2$, $\lambda = 3$, $\nu = 4$, $\tau = 0.5$ with $\rho = -0.187$).

n	Parameter	m.m.e	m.l.e	SE(m.m.e)	SE(m.l.e)	95%CI (m.m.e)	95%CI (m.l.e)	PC ^C	PC ^{MH}
1000	α	1.000	1.000	0.011	0.012	(0.978, 1.022)	(0.976, 1.025)	-0.101	-0.100
	γ	2.000	2.000	0.017	0.018	(1.968, 2.033)	(1.964, 2.036)		
	λ	3.003	3.005	0.076	0.083	(2.854, 3.151)	(2.842, 3.167)		
	ν	4.004	4.004	0.103	0.109	(3.802, 4.205)	(3.790, 4.218)		
	τ	0.501	0.501	0.075	0.050	(0.354, 0.648)	(0.402, 0.600)		
	ρ	-0.101	-0.100	0.037	0.029	(-0.173, -0.029)	(-0.157, -0.044)		
5000	α	1.000	1.000	0.0050	0.0056	(0.990, 1.010)	(0.989, 1.011)	-0.100	-0.100
	γ	2.000	2.000	0.0075	0.0082	(1.985, 2.015)	(1.984, 2.016)		
	λ	3.001	3.001	0.034	0.038	(2.934, 3.067)	(2.927, 3.074)		
	ν	4.001	4.000	0.047	0.049	(3.909, 4.092)	(3.904, 4.096)		
	τ	0.500	0.500	0.028	0.021	(0.444, 0.556)	(0.459, 0.541)		
	ρ	-0.100	-0.100	0.016	0.013	(-0.132, -0.068)	(-0.125, -0.075)		
10000	α	1.000	1.000	0.0050	0.0039	(0.990, 1.010)	(0.992, 1.008)	-0.100	-0.100
	γ	2.000	2.000	0.0075	0.0058	(1.985, 2.015)	(1.989, 2.011)		
	λ	3.001	3.000	0.0337	0.0265	(2.934, 3.067)	(2.948, 3.052)		
	ν	4.001	4.000	0.0467	0.0344	(3.909, 4.092)	(3.933, 4.067)		
	τ	0.500	0.500	0.0284	0.0147	(0.444, 0.556)	(0.471, 0.529)		
	ρ	-0.100	-0.100	0.0163	0.0089	(-0.132, -0.068)	(-0.118, -0.083)		

Table 3: Simulations for Weibull model ($\alpha = 1$, $\gamma = 2$, $\lambda = 3$, $\nu = 4$, $\tau = 0.5$ with $\rho = -0.100$).

n	Parameter	m.m.e	m.l.e	SE(m.m.e)	SE(m.l.e)	95%CI (m.m.e)	95%CI (m.l.e)	PC ^C	PC ^{MH}
1000	α	0.987	1.001	0.139	0.024	(0.715, 1.259)	(0.954, 1.048)	-0.150	-0.117
	γ	1.996	1.998	0.034	0.033	(1.929, 2.063)	(1.934, 2.063)		
	λ	3.146	3.022	0.228	0.125	(2.700, 3.593)	(2.778, 3.266)		
	ν	4.084	4.013	0.238	0.125	(3.618, 4.549)	(3.767, 4.259)		
	τ	0.573	0.488	0.121	0.079	(0.336, 0.810)	(0.332, 0.644)		
	ρ	-0.150	-0.119	0.045	0.030	(-0.239, -0.061)	(-0.178, -0.061)		
5000	α	0.992	1.001	0.102	0.012	(0.791, 1.192)	(0.978, 1.024)	-0.137	-0.122
	γ	1.998	1.999	0.019	0.016	(1.961, 2.034)	(1.968, 2.030)		
	λ	3.080	3.011	0.156	0.068	(2.774, 3.386)	(2.878, 3.144)		
	ν	4.036	4.006	0.141	0.066	(3.760, 4.312)	(3.876, 4.136)		
	τ	0.537	0.494	0.081	0.037	(0.378, 0.696)	(0.421, 0.567)		
	ρ	-0.137	-0.122	0.029	0.014	(-0.193, -0.080)	(-0.150, -0.094)		
10000	α	0.993	1.000	0.098	0.008	(0.802, 1.185)	(0.984, 1.016)	-0.134	-0.124
	γ	1.998	2.000	0.013	0.011	(1.972, 2.024)	(1.978, 2.021)		
	λ	3.060	3.007	0.137	0.052	(2.792, 3.327)	(2.906, 3.108)		
	ν	4.023	4.004	0.106	0.049	(3.816, 4.231)	(3.907, 4.100)		
	τ	0.528	0.496	0.068	0.028	(0.395, 0.662)	(0.442, 0.550)		
	ρ	-0.134	-0.123	0.024	0.011	(-0.181, -0.088)	(-0.144, -0.102)		

Table 4: Simulations for log-logistic model ($\alpha = 1$, $\gamma = 2$, $\lambda = 3$, $\nu = 4$, $\tau = 0.5$ with $\rho = -0.125$).

n	Parameter	m.l.e	SE(m.l.e)	95%CI (m.l.e)	PC ^{MH}
1000	α	1.009	0.070	(0.872, 1.146)	-0.032
	γ	1.987	0.116	(1.759, 2.215)	
	τ	0.488	0.069	(0.352, 0.624)	
5000	α	1.003	0.041	(0.922, 1.085)	-0.014
	γ	1.996	0.073	(1.853, 2.140)	
	τ	0.496	0.040	(0.418, 0.573)	
10000	α	1.002	0.031	(0.941, 1.062)	-0.009
	γ	1.998	0.051	(1.898, 2.098)	
	τ	0.497	0.029	(0.440, 0.555)	

Table 5: Simulations for Half-Cauchy model ($\alpha = 1$, $\gamma = 2$, $\tau = 0.5$).

n	Parameter	m.l.e	SE(m.l.e)	95%CI (m.l.e)	PC ^{MH}
1000	α	1.007	0.058	(0.894, 1.120)	-0.122
	γ	2.007	0.105	(1.800, 2.213)	
	λ	3.016	0.146	(2.729, 3.303)	
	ν	4.012	0.197	(3.626, 4.398)	
	τ	0.497	0.069	(0.362, 0.633)	
5000	α	1.003	0.026	(0.951, 1.054)	-0.122
	γ	2.002	0.045	(1.913, 2.090)	
	λ	3.005	0.066	(2.876, 3.134)	
	ν	4.004	0.085	(3.838, 4.169)	
	τ	0.499	0.032	(0.437, 0.561)	
10000	α	1.003	0.019	(0.966, 1.039)	-0.123
	γ	2.000	0.033	(1.936, 2.064)	
	λ	3.006	0.048	(2.913, 3.099)	
	ν	3.998	0.062	(3.877, 4.120)	
	τ	0.499	0.022	(0.455, 0.543)	

Table 6: Simulations for gamma model ($\alpha = 1$, $\gamma = 2$, $\lambda = 3$, $\nu = 4$ and $\tau = 0.5$).

Figure 1 ³

Figure 1: Bivariate densities of models given $\alpha = 1, \gamma = 2, \lambda = 3, \nu = 4$ and $\tau = 0.5$.

9 Two Small Applications

The first dataset employs the song metric speechiness as the independent variable X and song popularity as the dependent variable Y conditioned on X , based on the most popular TikTok songs in 2020 available on ??⁴. The pearson correlation is -0.322 ; which ensures all models in this study may be utilised. Additionally, it is such that $S_1 < M_1^2$ and $S_2 < M_2^2$ for this dataset, hence m.m.e's for the Lomax model were excluded.

The second dataset employs the logarithm of the volatility index (VIX) as the independent variable X and the logarithm of the price of gold futures (GC) as the dependent variable Y conditioned on X , covering the period from 1 January 2020 to 17 October 2025⁵. The data was obtained from Yahoo Finance ???. The pearson correlation is -0.311 ; which ensures all models in this study may be utilised. Again, it is such that $S_1 < M_1^2$ and $S_2 < M_2^2$ for this dataset, hence m.m.e's for the Lomax model were excluded.

³Interactive content: this figure contains an embedded GIF. To view the animation, please open the PDF with **Adobe Acrobat Reader**.

⁴Song popularity (Y) was originally reported on a $0 - 100$ scale. For analysis, these values were normalized to the unit interval by dividing by 100, with songs assigned a score of 0 removed from the dataset.

⁵Rows containing missing values were removed - arising primarily from non-trading days (weekends and public holidays) and differences in trading calendars across the VIX and gold futures contracts.

Model	Parameter	TikTok			VIX/GC		
		m.m.e	m.l.e	AIC	m.m.e	m.l.e	AIC
Exponential	α	7.003	6.174	-232.041	0.334	0.327	13847.219
	γ	1.567	1.585		0.131	0.098	
	τ	0.090	0.384		0.001	1.000	
	ρ	-0.083	-0.278		-0.001	-0.500	
Lomax	α	-	1.477	-231.381	-	0.000	13851.245
	λ	-	4.565		-	∞	
	γ	-	0.000		-	0	
	ν	-	∞		-	∞	
	τ	-	0.484		-	1.000	
	ρ	-	-0.301		-	-0.500	
Weibull	α	6.936	6.391	-715.220	0.319	0.318	1417.524
	λ	1.024	1.459		11.565	0.129	
	γ	1.426	4.449		0.129	30.050	
	ν	4.297	1.123		44.197	8.713	
	τ	0.406	0.425		0.938	0.079	
	ρ	-0.322	-0.232		-0.311	-0.035	
Log-logistic	α	1.618	1.531	-592.039	0.336	0.337	281.396
	λ	2.725	2.000		17.416	17.103	
	γ	1.618	1.531		0.131	0.132	
	ν	7.179	5.189		63.541	66.386	
	τ	0.855	0.456		0.954	0.405	
	ρ	-0.322	0.000		-0.311	0.000	
Half-Cauchy	α	-	9.968	-129.483	-	0.334	15032.367
	γ	-	1.576		-	0.114	
	τ	-	0.570		-	1.000	
Gamma	α	-	9.432	-517.098	-	6.670	6183.966
	λ	-	1.370		-	20.958	
	γ	-	8.440		-	2.656	
	ν	-	5.383		-	19.505	
	τ	-	0.641		-	0.232	

Table 7: m.m.e's, m.l.e's and AIC's for models using TikTok and VIX/GC datasets.

A Joint Densities and Log-likelihoods

A.1 Exponential

We obtain the joint density function by differentiating the joint survival function (6):

$$f_{X,Y}(x, y) = \alpha\gamma (\gamma\tau y e^{\alpha\tau x} - \tau + 1) e^{(\alpha x(\tau-1) - \gamma y e^{\alpha\tau x})}, \quad x, y > 0.$$

where $\alpha, \gamma > 0$ and $\tau \in [0, 1]$. The log-likelihood is thus:

$$\begin{aligned} \ell(\alpha, \gamma, \tau) = & n \log \alpha + n \log \gamma + \sum_{i=1}^n \log (\gamma\tau y_i e^{\alpha\tau x_i} - \tau + 1) \\ & + \alpha(\tau - 1) \sum_{i=1}^n x_i - \gamma \sum_{i=1}^n y_i e^{\alpha\tau x_i}. \end{aligned}$$

A.2 Lomax

We obtain the joint density function by differentiating the joint survival function (13):

$$\begin{aligned}
f_{X,Y}(x,y) &= \nu (1 + \alpha x)^{-3\lambda-1} (1 + \gamma y(1 + \alpha \tau x)^\lambda)^{-3\nu-4} \\
&\times \left[\frac{\alpha \gamma^2 \lambda \tau y (\nu + 1) (1 + \alpha x)^{2\lambda+1} (1 + \alpha \tau x)^{2\lambda} (1 + \gamma y(1 + \alpha \tau x)^\lambda)^{2\nu+2}}{1 + \alpha \tau x} \right. \\
&\quad - \frac{\alpha \gamma \lambda \tau (1 + \alpha x)^{2\lambda+1} (1 + \alpha \tau x)^\lambda (1 + \gamma y(1 + \alpha \tau x)^\lambda)^{2\nu+3}}{1 + \alpha \tau x} \\
&\quad \left. + \alpha \gamma \lambda ((1 + \alpha x)^2 (1 + \alpha \tau x))^\lambda (1 + \gamma y(1 + \alpha \tau x)^\lambda)^{2\nu+3} \right], \quad x, y > 0,
\end{aligned}$$

where $\alpha, \lambda, \nu, \gamma > 0$ and $\tau \in [0, 1]$. The log-likelihood is thus:

$$\begin{aligned}
\ell(\alpha, \lambda, \nu, \gamma, \tau) &= n \log \nu - (3\lambda + 1) \sum_{i=1}^n \log(1 + \alpha x_i) \\
&\quad - (3\nu + 4) \sum_{i=1}^n \log(1 + \gamma y_i (1 + \alpha \tau x_i)^\lambda) \\
&\quad + \sum_{i=1}^n \log \left\{ \frac{\alpha \gamma^2 \lambda \tau y_i (\nu + 1) (1 + \alpha x_i)^{2\lambda+1} (1 + \alpha \tau x_i)^{2\lambda}}{1 + \alpha \tau x_i} (1 + \gamma y_i (1 + \alpha \tau x_i)^\lambda)^{2\nu+2} \right. \\
&\quad \quad - \frac{\alpha \gamma \lambda \tau (1 + \alpha x_i)^{2\lambda+1} (1 + \alpha \tau x_i)^\lambda}{1 + \alpha \tau x_i} (1 + \gamma y_i (1 + \alpha \tau x_i)^\lambda)^{2\nu+3} \\
&\quad \quad \left. + \alpha \gamma \lambda ((1 + \alpha x_i)^2 (1 + \alpha \tau x_i))^\lambda (1 + \gamma y_i (1 + \alpha \tau x_i)^\lambda)^{2\nu+3} \right\}.
\end{aligned}$$

A.3 Weibull

We obtain the joint density function by differentiating the joint survival function (20):

$$f_{X,Y}(x,y) = \alpha^\lambda \gamma^\nu \lambda \nu x^{\lambda-1} y^{\nu-1} \left(\tau^\lambda (\gamma y)^\nu e^{(\alpha \tau x)^\lambda} - \tau^\lambda + 1 \right) e^{(\alpha x)^\lambda (\tau^\lambda - 1) - (\gamma y)^\nu e^{(\alpha \tau x)^\lambda}}, \quad x, y > 0,$$

where $\alpha, \lambda, \nu, \gamma > 0$ and $\tau \in [0, 1]$. The log-likelihood is thus:

$$\begin{aligned}
\ell(\alpha, \lambda, \nu, \gamma, \tau) &= n \lambda \log \alpha + n \nu \log \gamma + n \log \lambda + n \log \nu \\
&\quad + (\lambda - 1) \sum_{i=1}^n \log x_i + (\nu - 1) \sum_{i=1}^n \log y_i \\
&\quad + \sum_{i=1}^n \log \left(\tau^\lambda (\gamma y_i)^\nu e^{(\alpha \tau x_i)^\lambda} - \tau^\lambda + 1 \right) \\
&\quad + (\tau^\lambda - 1) \sum_{i=1}^n (\alpha x_i)^\lambda - \sum_{i=1}^n (\gamma y_i)^\nu e^{(\alpha \tau x_i)^\lambda}.
\end{aligned}$$

A.4 Log-logistic

We obtain the joint density function by differentiating the joint survival function (27):

$$f_{X,Y}(x,y) = \frac{\lambda\nu(\gamma y)^\nu}{x y ((\alpha x)^\lambda + 1)^2 ((\gamma y)^\nu ((\alpha \tau x)^\lambda + 1) + 1)^3} \\ \times \left\{ (\alpha x)^\lambda ((\alpha \tau x)^\lambda + 1) ((\gamma y)^\nu ((\alpha \tau x)^\lambda + 1) + 1) \right. \\ \left. + (\alpha \tau x)^\lambda ((\alpha x)^\lambda + 1) ((\gamma y)^\nu ((\alpha \tau x)^\lambda + 1) + 1) \right. \\ \left. - 2 (\alpha \tau x)^\lambda ((\alpha x)^\lambda + 1) \right\}, \quad x, y > 0,$$

where $\alpha, \lambda, \nu, \gamma > 0$ and $\tau \in [0, 1]$. Thus the log-likelihood is:

$$\ell(\alpha, \lambda, \nu, \gamma, \tau) = n \log \lambda + n \log \nu + n \nu \log \gamma + (\nu - 1) \sum_{i=1}^n \log y_i - \sum_{i=1}^n \log x_i \\ - 2 \sum_{i=1}^n \log ((\alpha x_i)^\lambda + 1) - 3 \sum_{i=1}^n \log ((\gamma y_i)^\nu ((\alpha \tau x_i)^\lambda + 1) + 1) \\ + \sum_{i=1}^n \log \left\{ (\alpha x_i)^\lambda ((\alpha \tau x_i)^\lambda + 1) ((\gamma y_i)^\nu ((\alpha \tau x_i)^\lambda + 1) + 1) \right. \\ \left. + (\alpha \tau x_i)^\lambda ((\alpha x_i)^\lambda + 1) ((\gamma y_i)^\nu ((\alpha \tau x_i)^\lambda + 1) + 1) \right. \\ \left. - 2 (\alpha \tau x_i)^\lambda ((\alpha x_i)^\lambda + 1) \right\}.$$

A.5 Half-Cauchy

We obtain the joint density function by differentiating the joint survival function (34):

$$f_{X,Y}(x,y) = \frac{4\alpha\gamma}{\pi ((\alpha x)^2 + 1) \left((\pi\gamma y)^2 + (2 \arctan(\alpha \tau x) - \pi)^2 \right)^2 ((\alpha \tau x)^2 + 1)} \times \\ \left[- 2(\pi\gamma)^2 \tau y^2 ((\alpha x)^2 + 1) (2 \arctan(\alpha x) - \pi) \right. \\ \left. + \tau ((\alpha x)^2 + 1) \left((\pi\gamma y)^2 + (2 \arctan(\alpha \tau x) - \pi)^2 \right) (2 \arctan(\alpha x) - \pi) \right. \\ \left. - \left((\pi\gamma y)^2 + (2 \arctan(\alpha \tau x) - \pi)^2 \right) ((\alpha \tau x)^2 + 1) (2 \arctan(\alpha \tau x) - \pi) \right]$$

where $\alpha, \gamma > 0$ and $\tau \in [0, 1]$. Thus the log-likelihood is:

$$\begin{aligned} \ell(\alpha, \gamma, \tau) = \sum_{i=1}^n & \left\{ \log(4\alpha\gamma) - \log \pi \right. \\ & - \log((\alpha x_i)^2 + 1) \\ & - \log\left((\pi\gamma y_i)^2 + (2\arctan(\alpha\tau x_i) - \pi)^2\right) \\ & - \log((\alpha\tau x_i)^2 + 1) \\ & + \log \left[-2(\pi\gamma)^2 \tau y_i^2 ((\alpha x_i)^2 + 1) (2\arctan(\alpha x_i) - \pi) \right. \\ & \quad + \tau ((\alpha x_i)^2 + 1) \left((\pi\gamma y_i)^2 + (2\arctan(\alpha\tau x_i) - \pi)^2 \right) (2\arctan(\alpha x_i) - \pi) \\ & \quad \left. \left. - \left((\pi\gamma y_i)^2 + (2\arctan(\alpha\tau x_i) - \pi)^2 \right) ((\alpha\tau x_i)^2 + 1) (2\arctan(\alpha\tau x_i) - \pi) \right] \right\}. \end{aligned}$$

A.6 Gamma

We obtain the joint density function by differentiating the joint survival function (35):

$$\begin{aligned} f_{X,Y}(x, y) = & \frac{\alpha^\lambda \gamma^\nu x^{\lambda-1} y^\nu}{\lambda^{1+\frac{1}{\nu}} y \Gamma(\lambda) \Gamma(\nu+1) \Gamma(\lambda, \alpha\tau x)^2} e^{-\alpha x - \alpha\tau x - \gamma \lambda^{-\frac{1}{\nu}} y \Gamma(\lambda, \alpha\tau x)^{-\frac{1}{\nu}} \Gamma(\lambda+1)^{\frac{1}{\nu}}} \\ & \times \left[\gamma \tau^\lambda y \Gamma(\lambda, \alpha\tau x)^{-\frac{1}{\nu}} e^{\alpha x} \Gamma(\lambda+1)^{1+\frac{1}{\nu}} \Gamma(\lambda, \alpha x) \right. \\ & \quad \left. - \lambda^{\frac{1}{\nu}} \nu \tau^\lambda e^{\alpha x} \Gamma(\lambda) \Gamma(\lambda+1) + \lambda^{1+\frac{1}{\nu}} \nu \tau^\lambda e^{\alpha x} \Gamma(\lambda) \gamma^*(\lambda, \alpha x) + \lambda^{1+\frac{1}{\nu}} \nu e^{\alpha\tau x} \Gamma(\lambda) \Gamma(\lambda, \alpha\tau x) \right], \quad x, y > 0, \end{aligned}$$

where $\alpha, \gamma > 0$ and $\tau \in [0, 1]$. Thus the log-likelihood is:

$$\begin{aligned} \ell(\alpha, \gamma, \lambda, \nu, \tau) = \sum_{i=1}^n & \left\{ \lambda \log \alpha + \nu \log \gamma + (\lambda-1) \log x_i + (\nu-1) \log y_i \right. \\ & - \left(1 + \frac{1}{\nu} \right) \log \lambda - \log \Gamma(\lambda) - \log \Gamma(\nu+1) - 2 \log \Gamma(\lambda, \alpha\tau x_i) \\ & - \alpha x_i - \alpha\tau x_i - \gamma \lambda^{-\frac{1}{\nu}} y_i \Gamma(\lambda, \alpha\tau x_i)^{-\frac{1}{\nu}} \Gamma(\lambda+1)^{\frac{1}{\nu}} \\ & + \log \left[\gamma \tau^\lambda y_i \Gamma(\lambda, \alpha\tau x_i)^{-\frac{1}{\nu}} e^{\alpha x_i} \Gamma(\lambda+1)^{1+\frac{1}{\nu}} \Gamma(\lambda, \alpha x_i) \right. \\ & \quad - \lambda^{\frac{1}{\nu}} \nu \tau^\lambda e^{\alpha x_i} \Gamma(\lambda) \Gamma(\lambda+1) + \lambda^{1+\frac{1}{\nu}} \nu \tau^\lambda e^{\alpha x_i} \Gamma(\lambda) \gamma^*(\lambda, \alpha x_i) \\ & \quad \left. \left. + \lambda^{1+\frac{1}{\nu}} \nu e^{\alpha\tau x_i} \Gamma(\lambda) \Gamma(\lambda, \alpha\tau x_i) \right] \right\}. \end{aligned}$$

B Method of Moment Estimators

Suppose data in the form of $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \dots, \mathbf{X}^{(n)}$ (where for each $i = 1, \dots, n$, $\mathbf{X}^{(i)} = (X_{1i}, X_{2i})^T$) are obtained, which are independent and identically distributed according to 3 and 2. If $M_1, M_2 > 0$, consistent asymptotically normal method of moment estimates (m.m.e's) are acquired.

Define:

$$\begin{aligned}
M_1 &= \frac{1}{n} \sum_{i=1}^n x_{1i} \\
M_2 &= \frac{1}{n} \sum_{i=1}^n x_{2i} \\
S_{12} &= \frac{1}{n} \sum_{i=1}^n (x_{1i} - M_1)(x_{2i} - M_2) \\
S_1 &= \frac{1}{n} \sum_{i=1}^n (x_{2i} - M_2)^2 \\
S_2 &= \frac{1}{n} \sum_{i=1}^n (x_{2i} - M_2)^2.
\end{aligned}$$

We denote the method of moment estimators for α, λ, γ and ν as $\tilde{\alpha}, \tilde{\lambda}, \tilde{\gamma}$ and $\tilde{\nu}$ respectively.

B.1 Exponential

Using (7) and (8), we obtain:

$$\begin{aligned}
\tilde{\alpha} &= \frac{1}{M_1}, \\
\tilde{\lambda} &= \frac{1}{M_2}.
\end{aligned}$$

And using (11):

$$\tilde{\tau} = -\frac{1}{\frac{M_1 M_2}{S_{12}} + 1}.$$

B.2 Lomax

Using (14) and (16), we obtain:

$$\begin{aligned}
\tilde{\lambda} &= \frac{2S_1}{S_1 - M_1^2}, \\
\tilde{\alpha} &= \frac{1}{M_1(\tilde{\lambda} - 1)}.
\end{aligned}$$

Likewise using (15) and (17), we obtain:

$$\begin{aligned}
\tilde{\nu} &= \frac{2S_2}{S_2 - M_2^2}, \\
\tilde{\gamma} &= \frac{1}{M_2(\tilde{\nu} - 1)}.
\end{aligned}$$

And using (18) we may solve for $\tilde{\tau}$ numerically. Additionally, to ensure existence of positive m.m.e's $\tilde{\lambda}$ and $\tilde{\nu}$, it must be such that $S_1 > M_1^2$ and $S_2 > M_2^2$.

C Weibull

Solve for $\tilde{\lambda}$ and $\tilde{\alpha}$ numerically using (21) and (23). Likewise, solve for $\tilde{\nu}$ and $\tilde{\gamma}$ numerically using (22) and (24). Finally, solve for $\tilde{\tau}$ numerically using (25).

D Log-Logistic

Solve for $\tilde{\lambda}$ and $\tilde{\alpha}$ numerically using (28) and (30). Likewise, solve for $\tilde{\nu}$ and $\tilde{\gamma}$ numerically using (29) and (31). Finally, solve for $\tilde{\tau}$ numerically using (32).

References

- [1] Barry C Arnold and Matthew A Arvanitis. On bivariate pseudo-exponential distributions. *Journal of Applied Statistics*, 47 (13-15):2299–2311, 2020.