Models with Accelerated Failure Conditionals

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1 Introduction

As per Arnold and Arvanitis [1], the general accelerated failure conditionals model is of the form:

$$\bar{F}_X(x) = P(X > x) = \bar{F}_0(x), \quad x > 0,$$

for some survival function $\bar{F}_0(x) \in (0,1)\ \forall x > 0$, and for each $x > 0$:

$$P(Y > y \mid X > x) = \bar{F}_1(\beta(x)y), \quad x, y > 0,$$

for some survival function $\bar{F}_1(\beta(x)y) \in (0,1)\ \forall x, y > 0$ and a suitable acceleration function $\beta(x)$. The joint survival function is then:

$$P(X > x, Y > y) = \bar{F}_0(x)\bar{F}_1(\beta(x)y), \quad x, y > 0 \tag{1}$$

Assuming differentiability and X, Y are continuous, we obtain the marginal densities:

$$f_X(x) = -\frac{d}{dx}P(X > x) = -\frac{d}{dx}\bar{F}_0(x) = f_0(x),$$

and, since $\lim_{x\to 0^+}\bar{F}_0(x) = 1$, where $\beta(0) := \lim_{x\to 0^+}\beta(x)$:

$$P(Y > y) = \lim_{x\to 0^+} P(Y > y, X > x) = \bar{F}_1\big(\beta(0)y\big).$$

Hence the marginal distribution of Y belongs to the same family as $\bar{F}_1$, but with its argument scaled by $\beta(0)$. Now, the density of Y is:

$$f_Y(y) = -\frac{d}{dy}P(Y > y) = -\frac{d}{dy}\bar{F}_1(\beta(0)y) = \beta(0)\, f_1(\beta(0)y),$$

where $f_1(t) = -\bar{F}_1'(t)$ denotes the density associated with $\bar{F}_1$. For (1) to be a valid survival function, it must have a non-negative mixed partial derivative. Denoting f_0 and f_1 as the densities corresponding to the survival function $\bar{F}_0$ and $\bar{F}_1$, and differentiating, we obtain:

$$\begin{aligned}\frac{\partial}{\partial x}\frac{\partial}{\partial y}P(X > x, Y > y) &= f_0(x)f_1(\beta(x)y)\beta(x) - \bar{F}_0(x)f_1'(\beta(x)y)\beta(x)\beta'(x)y - \bar{F}_0(x)f_1(\beta(x)y)\beta'(x)\\ &= f_0(x)f_1(\beta(x)y)\beta(x) - \beta'(x)\bar{F}_0(x)\left[f_1(\beta(x)y) + f_1'(\beta(x)y)\beta(x)y\right].\end{aligned}$$

Denoting $t = \beta(x)y$, we have:

$$\begin{aligned}\frac{\partial}{\partial x}\frac{\partial}{\partial y}P(X > x, Y > y) &= f_0(x)f_1(t)\beta(x) - \beta'(x)\bar{F}_0(x)\left[f_1(t) + f_1'(t)t\right]\\ &= f_0(x)f_1(t)\beta(x) - \beta'(x)\bar{F}_0(x)\left[tf_1(t)\right]'.\end{aligned}$$

Thus a necessary and sufficient condition for $\frac{\partial}{\partial x}\frac{\partial}{\partial y}P(X > x, Y > y) \geq 0$ is:

$$\beta'(x)\begin{cases}\leq \dfrac{f_0(x)}{\bar{F}_0(x)}\,\dfrac{f_1(t)}{[tf_1(t)]'}\,\beta(x), & \text{if } [tf_1(t)]' > 0,\\[2ex] \geq \dfrac{f_0(x)}{\bar{F}_0(x)}\,\dfrac{f_1(t)}{[tf_1(t)]'}\,\beta(x), & \text{if } [tf_1(t)]' < 0,\\ \text{unconstrained}, & \text{if } [tf_1(t)]' = 0.\end{cases}$$

Denoting the hazard function $h_0(x) = \frac{f_0(x)}{\bar{F}_0(x)}$ and $S(t) = \frac{f_1(t)}{[tf_1(t)]'}$, we may write the bounds as such:

$$\sup_{t:[tf_1(t)]'<0} h_0(x)S(t)\beta(x) \;\leq\; \beta'(x) \;\leq\; \inf_{t:[tf_1(t)]'>0} h_0(x)S(t)\beta(x). \tag{2}$$

A special case of the upper bound for $B'(x)$ arises for distribution families satisfying $f_1'(t) \leq 0$. In this case, we note that $S(t) = \frac{f_1(t)}{[tf_1(t)]'} = \frac{f_1(t)}{f_1(t)+tf_1'(t)} \geq 1$ for $t \geq 0$, hence $\beta'(x) \;\leq\; h_0(x)\beta(x)$.

It should be noted that multiple functional forms for $\beta(x)$ are possible. The question of selecting a form of $\beta(x)$ that provides the best model fit has already been investigated by Arnold and Arvanitis [1]. Accordingly, the present study restricts attention to those choices of $\beta(x)$ that arise from the extreme case in which $\beta'(x)$ attains its theoretical upper bound, that is, given $\inf_{t:[tf_1(t)]'>0} S(t) = c^*$:

$$\begin{aligned}
\beta'(x) &= c^* h_0(x)\beta(x) \\
\therefore \int \frac{1}{\beta(x)} d\beta(x) &= \int c^* h_0(x) dx \\
\therefore \log\left(\beta(x)\right) &= -c^* \log\left(\bar{F}_0(x)\right) + K \\
\therefore \beta(x) &= \gamma \frac{1}{\left[\bar{F}_0(x)\right]^{c^*}},
\end{aligned} \tag{3}$$

for $\gamma > 0$. This suggests $\beta(x) > 0$ for $x > 0$. Furthermore, $\beta(0) := \lim_{x\to 0^+} \beta(x) = \gamma$ since $\lim_{x\to 0^+} \bar{F}_0(x) = 1$.

The role of the acceleration function $\beta(x)$ in determining the sign of the correlation between X and Y is established in Theorem 1. In particular, since the present study considers only positive specifications of $\beta(x)$, the resulting framework is restricted to modelling data in which X and Y exhibit negative correlation.

Theorem 1. *Consider an accelerated conditional model of the form:*

$$P(X > x) = \bar{F}_0(x) \quad \textit{and} \quad P(Y > y \mid X > x) = \bar{F}_1\big(\beta(x)y\big),$$

where $\bar{F}_0$ and $\bar{F}_1$ are survival functions. Then:

$$\beta'(x) \geq 0 \quad \Longrightarrow \quad \mathrm{Cov}(X,Y) \leq 0, \qquad \beta'(x) \leq 0 \quad \Longrightarrow \quad \mathrm{Cov}(X,Y) \geq 0.$$

Proof. Since $P(Y > y, X > x) = \bar{F}_0(x)\bar{F}_1\big(\beta(x)y\big)$ then $P(Y > y) = \lim_{x\to 0^+} P(Y > y, X > x) = \bar{F}_0(0)\bar{F}_1\big(\beta(0)y\big) = \bar{F}_1\big(\beta(0)y\big)$ since $\lim_{x\to 0^+} \bar{F}_0(x) = 1$, we note:

$$\begin{aligned}
Cov(X,Y) &= \int_0^\infty \int_0^\infty \left(P(Y > y, X > x) - P(X > x)P(Y > y)\right)\, dxdy \\
&= \int_0^\infty \int_0^\infty \bar{F}_0(x)\left(\bar{F}_1\big(\beta(x)y\big) - \bar{F}_1\big(\beta(0)y\big)\right)\, dxdy.
\end{aligned}$$

Now since $\bar{F}_1'(\cdot) \leq 0$ (as $\bar{F}_1$ is a survival function), if $\beta'(x) \geq 0$ then $\bar{F}_1\big(\beta(x)y\big) \leq \bar{F}_1\big(\beta(0)y\big)$ and $Cov(X,Y) \leq 0$ for $x, y \geq 0$. Conversely, if $\beta'(x) \leq 0$ then $\bar{F}_1\big(\beta(x)y\big) \geq \bar{F}_1\big(\beta(0)y\big)$ and $Cov(X,Y) \geq 0$ for $x, y \geq 0$. Finally, if $\beta(x)$ is constant, say $\beta(x) \equiv c$, then $P(X > x, Y > y) = \bar{F}_0(x)\,\bar{F}_1(cy) = P(X > x)\,P(Y > y)$, so X and Y are independent and $Cov(X,Y) = 0$. ☐

2 Exponential

We restrict the survival function forms to the exponential family as such:

$$\bar{F}_0(x) = P(X > x) = e^{-\alpha x}, \quad x > 0,$$

where $\alpha > 0$ and, for each $x > 0$:

$$\bar{F}_1(\beta(x)y) = P(Y > y \mid X > x) = e^{-\beta(x)y}, \quad y > 0,$$

where $\beta(x) > 0$ for all $x > 0$ to ensure a valid survival function. The corresponding joint survival function will be:

$$\bar{F}_0(x)\bar{F}_1(\beta(x)y) = P(X > x, Y > y) = e^{-\alpha x + \beta(x)y}, \quad x, y > 0. \tag{4}$$

Accordingly, $f_X(x) = \alpha e^{-(\alpha x)}$ for $x > 0$ and $f_Y(y) = \beta(0)e^{-(\beta(0)y)}$ for $y > 0$. Hence, $X \sim Exp(\alpha)$ and $Y \sim Exp(\beta(0))$.

Now since $[tf_1(t)]' = (1-t)e^{-t} > 0$ for $t < 1$, and $S(t) = -\frac{1}{t-1}$ is increasing on this interval, $\inf_{t<1} S(t) = \lim_{t\to 0} S(t) = 1$. Additionally, since $[tf_1(t)]' = (1-t)e^{-t} < 0$ for $t > 1$, and $S(t) = -\frac{1}{t-1}$ is increasing on this interval, $\sup_{t>1} S(t) = \lim_{t\to\infty} S(t) = 0$. Consequently, applying (2) yields:

$$0 \le B'(x) \le h_0(x)\beta(x) = \alpha\beta(x),$$

corresponding to Proposition 3.1 in Arnold and Arvanitis [1]. Now, we select the form of the acceleration function $\beta(x)$ such that $\beta'(x) = \alpha\beta(x)$ (that is, equal to $\beta'(x)'s$ upper bound). Hence from (3) with $c^* = 1$, $\beta(x) = \gamma e^{\alpha x}$ with $\gamma > 0$. Furthermore, following Arnold and Arvanitis [1], we introduce a dependence parameter $\tau \in [0,1]$ which governs the strength of the dependence between X and Y. The parameter τ is restricted to $[0,1]$ since $\tau = 0$ implies $\beta(x) = \gamma$ is constant (so X and Y are independent with covariance 0), $\tau = 1$ corresponds to the case where $\beta'(x)$ attains its theoretical upper bound, where allowing $\tau > 1$ would imply $\beta'(x) > \alpha\beta(x)$, violating the theoretical upper bound. Hence we choose our acceleration function as:

$$\beta(x) = \gamma e^{\alpha\tau x}.$$

2.1 Moments

Noting $X \sim Exp(\alpha)$ and $Y \sim Exp(\gamma)$, we have:

$$\begin{aligned} E(X) &= \frac{1}{\alpha}, \\ E(Y) &= \frac{1}{\gamma}, \\ Var(X) &= \frac{1}{\alpha^2}, && (5) \\ Var(Y) &= \frac{1}{\gamma^2}. && (6) \end{aligned}$$

Accordingly:

$$\begin{aligned} Cov(X,Y) &= \int_0^\infty \int_0^\infty \left(P(Y > y, X > x) - P(X > x)P(Y > y)\right) \, dxdy \\ &= \int_0^\infty \int_0^\infty \bar{F}_0(x) \left(\bar{F}_1(\beta(x)y) - \bar{F}_1(\beta(0)y)\right) \, dxdy \\ &= \int_0^\infty \int_0^\infty e^{-(\alpha x)} \left(e^{-\beta(x)y} - e^{-\beta(0)y}\right) \, dydx \\ &= \int_0^\infty e^{-(\alpha x)} \left(\frac{1}{\beta(x)} - \frac{1}{\beta(0)}\right) \, dx \\ &= \frac{1}{\gamma} \int_0^\infty \left(e^{-\alpha(1+\tau)x} - e^{-(\alpha x)}\right) \, dx \\ &= -\frac{\tau}{\alpha\tau(1+\tau)}. && (7) \end{aligned}$$

Using (5), (6) and (7), we have the correlation between X and Y as:

$$\begin{aligned} \rho(X,Y) &= \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} \\ &= -\frac{\tau}{1+\tau}. \end{aligned}$$

Seeing as $\tau \in [0,1]$, this exponential model with particular acceleration acceleration funciton $\beta(x) = \gamma e^{\alpha\tau x}$, may only be used to model data for which $\rho(X,Y) \in [0, -\frac{1}{2}]$.

3 Lomax

We restrict the survival function forms to the Lomax family as such:

$$\bar{F}_0(x) = P(X > x) = (1 + \alpha x)^{-\lambda}, \quad x > 0,$$

where $\alpha, \lambda > 0$ and, for each $x > 0$:

$$\bar{F}_1(\beta(x)y) = P(Y > y \mid X > x) = (1 + \beta(x)y)^{-\nu}, \quad y > 0,$$

where $\beta(x), \nu > 0$ for all $x > 0$ to ensure a valid survival function. The corresponding joint survival function will be:

$$\bar{F}_0(x)\bar{F}_1(\beta(x)y) = P(X > x, Y > y) = (1+\alpha x)^{-\lambda}(1+\beta(x)y)^{-\nu}, \quad x, y > 0. \tag{8}$$

Accordingly, $f_X(x) = \alpha\lambda(\alpha x+1)^{-(\alpha+1)}$ for $x > 0$ and $f_Y(y) = \beta(0)\nu(\beta(0)y+1)^{-(\nu+1)}$ for $y > 0$. Hence, $X \sim Lomax(\alpha, \lambda)$ and $Y \sim Lomax(\beta(0), \nu)$.

Now since $[tf_1(t)]' = \frac{\nu(t+1)^{-\nu}(1-\nu t)}{t^2+2t+1} > 0$ for $t < \frac{1}{v}$, and $S(t) = -\frac{t+1}{vt-1}$ is increasing on this interval, $\inf_{t<\frac{1}{v}} S(t) = \lim_{t\to 0} S(t) = 1$. Additionally, since $[tf_1(t)]' = \frac{\nu(t+1)^{-\nu}(1-\nu t)}{t^2+2t+1} < 0$ for $t > \frac{1}{\nu}$, and $S(t) = -\frac{t+1}{vt-1}$ is increasing on this interval, $\sup_{t>\frac{1}{\nu}} S(t) = \lim_{t\to\infty} S(t) = -\frac{1}{\nu}$. Consequently, applying (2) yields:

$$-\frac{1}{\nu}\frac{\alpha\lambda}{\alpha x+1}\beta(x) \le B'(x) \le \frac{\alpha\lambda}{\alpha x+1}\beta(x).$$

Now, we select the form of the acceleration function $\beta(x)$ such that $\beta'(x) = \frac{\alpha\lambda}{\alpha x+1}\beta(x)$ (that is, equal to $\beta'(x)'s$ upper bound). Hence from (3) with $c^* = 1$, $\beta(x) = \gamma\left(1+\alpha x\right)^{\lambda}$ with $\gamma > 0$. Furthermore, following Arnold and Arvanitis [1], we introduce a dependence parameter $\tau \in [0, 1]$ which governs the strength of the dependence between X and Y. Hence we choose our acceleration function as:

$$\beta(x) = \gamma\left(1+\alpha\tau x\right)^{\lambda}.$$

3.1 Moments

Noting $X \sim Lomax(\alpha, \lambda)$ and $Y \sim Lomax(\gamma, \nu)$, we have:

$$E(X) = \frac{1}{\alpha(\lambda-1)}, \quad \lambda > 1,$$

$$E(Y) = \frac{1}{\gamma(\nu-1)}, \quad \nu > 1,$$

$$Var(X) = \frac{\lambda}{\alpha^2(\lambda-1)^2(\lambda-2)}, \quad \lambda > 2, \tag{9}$$

$$Var(Y) = \frac{\nu}{\gamma^2(\nu-1)^2(\nu-2)}, \quad \nu > 2. \tag{10}$$

Accordingly:

$$\begin{aligned}
\mathrm{Cov}(X,Y) &= \int_0^\infty \int_0^\infty \left(P(Y > y, X > x) - P(X > x)P(Y > y)\right) dy\, dx \\
&= \int_0^\infty \int_0^\infty \bar{F}_0(x)\left(\bar{F}_1\big(\beta(x)y\big) - \bar{F}_1\big(\beta(0)y\big)\right) dy\, dx \\
&= \int_0^\infty (1+\alpha x)^{-\lambda}\left(\int_0^\infty \left((1+\beta(x)y)^{-\nu} - (1+\beta(0)y)^{-\nu}\right) dy\right) dx \\
&= \frac{1}{\nu-1}\int_0^\infty (1+\alpha x)^{-\lambda}\left(\frac{1}{\beta(x)} - \frac{1}{\beta(0)}\right) dx \\
&= \frac{1}{\gamma(\nu-1)}\left[\int_0^\infty \frac{dx}{(1+\alpha x)^{\lambda}(1+\alpha\tau x)^{\lambda}} - \int_0^\infty \frac{dx}{(1+\alpha x)^{\lambda}}\right] \\
&= \frac{1}{\alpha\gamma(\nu-1)}\left[\frac{1}{2\lambda-1}\, {}_2F_1(\lambda, 1; 2\lambda; 1-\tau) - \frac{1}{\lambda-1}\right],
\end{aligned} \tag{11}$$

where the conditions $\nu > 1$ and $\lambda > 1$ ensure convergence of the inner and outer integrals, respectively. Now, using (9), (10) and (11), the correlation is:

$$\begin{aligned}
\rho(X,Y) &= \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} \\
&= (\lambda-1)\left(\frac{{}_2F_1(\lambda, 1; 2\lambda; 1-\tau)}{2\lambda-1} - \frac{1}{\lambda-1}\right)\sqrt{\frac{(\lambda-2)(\nu-2)}{\lambda\nu}},
\end{aligned} \tag{12}$$

valid for $\lambda, \nu > 2$. Since ${}_2F_1(\lambda, 1; 2\lambda; z)$ has a power series with strictly positive coefficients, it is strictly increasing in $z \in [0, 1]$; hence ${}_2F_1(\lambda, 1; 2\lambda; 1-\tau)$ is strictly decreasing in $\tau \in [0, 1]$. Consequently, (12) attains its minimum at $\tau = 1$ and its maximum (zero) at $\tau = 0$. Therefore,

$$-\frac{\lambda}{2\lambda-1}\sqrt{\frac{\lambda-2}{\lambda}}\sqrt{\frac{\nu-2}{\nu}} \le \rho(X,Y) \le 0.$$

Moreover, $\rho_{\min}(\lambda,\nu) := -\frac{\lambda}{2\lambda-1}\sqrt{\frac{\lambda-2}{\lambda}}\sqrt{\frac{\nu-2}{\nu}}$ is strictly decreasing in each of λ and ν, so $\inf_{\lambda>2,\,\nu>2}\rho_{\min}(\lambda,\nu) = \lim_{\lambda,\nu\to\infty}\rho_{\min}(\lambda,\nu) = -\frac{1}{2}$. Thus, as in the exponential case, this Lomax model with acceleration $\beta(x) = \gamma\,(1+\alpha\tau x)^{\lambda}$ can only accommodate correlations in the range $\rho(X,Y) \in \left[-\frac{1}{2},\, 0\right]$.

4 Weibull

We restrict the survival function forms to the Weibull family as such:

$$\bar{F}_0(x) = P(X > x) = e^{-(\alpha x)^{\lambda}}, \quad x > 0,$$

where $\alpha, \lambda > 0$ and, for each $x > 0$:

$$\bar{F}_1(\beta(x)y) = P(Y > y \mid X > x) = e^{-(\beta(x)y)^{\nu}}, \quad y > 0,$$

where $\beta(x), \nu > 0$ for all $x > 0$ to ensure a valid survival function. The corresponding joint survival function will be:

$$\bar{F}_0(x)\bar{F}_1(\beta(x)y) = P(X > x, Y > y) = e^{-(\alpha x)^{\lambda}-(y\beta(x))^{\nu}}, \quad x, y > 0. \tag{13}$$

Accordingly, $f_X(x) = \alpha\lambda(\alpha\lambda)^{\lambda-1}e^{-(\alpha x)^{\lambda}}$ for $x > 0$ and $f_Y(y) = \beta(0)\nu(\beta(0)\nu)^{\nu-1}e^{-(\beta(0)y)^{\nu}}$ for $y > 0$. Hence, $X \sim Weibull(\alpha,\lambda)$ and $Y \sim Weibull(\beta(0),\nu)$.

Now since $[tf_1(t)]' = \nu^2 t^{\nu-1}(1-t^{\nu})e^{-t^{\nu}} > 0$ for $t < 1$, and $S(t) = \frac{1}{\nu(1-t^{\nu})}$ is increasing on this interval, $\inf_{t<1} S(t) = \lim_{t\to 0} S(t) = \frac{1}{\nu}$. Additionally, since $[tf_1(t)]' = \nu^2 t^{\nu-1}(1-t^{\nu})e^{-t^{\nu}} < 0$ for $t > 1$, and $S(t) = \frac{1}{\nu(1-t^{\nu})}$ is increasing on this interval, $\sup_{t>1} S(t) = \lim_{t\to\infty} S(t) = 0$. Consequently, applying (2) yields:

$$0 \le B'(x) \le \frac{1}{\nu}\lambda\alpha(\alpha x)^{\lambda-1}\beta(x).$$

Now, we select the form of the acceleration function $\beta(x)$ such that $\beta'(x) = \frac{1}{\nu}\lambda\alpha(\alpha x)^{\lambda-1}\beta(x)$ (that is, equal to $\beta'(x)'s$ upper bound). Hence from (3) with $c^* = \frac{1}{\nu}$, $\beta(x) = \gamma e^{\frac{1}{\nu}(\alpha x)^{\lambda}}$ with $\gamma > 0$. Furthermore, following Arnold and Arvanitis [1], we introduce a dependence parameter $\tau \in [0,1]$ as before. Hence we choose our acceleration function as:

$$\beta(x) = \gamma e^{\frac{1}{\nu}(\alpha\tau x)^{\lambda}}.$$

4.1 Moments

Noting $X \sim Weibull(\alpha,\lambda)$ and $Y \sim Weibull(\gamma,\nu)$, we have:

$$E(X) = \frac{1}{\alpha\Gamma\left(1+\frac{1}{\lambda}\right)},$$

$$E(Y) = \frac{1}{\gamma\Gamma\left(1+\frac{1}{\nu}\right)},$$

$$Var(X) = \frac{1}{\alpha^2}\left(\Gamma\left(1+\frac{2}{\lambda}\right) - \left(\Gamma\left(1+\frac{1}{\lambda}\right)\right)^2\right) \tag{14}$$

$$Var(Y) = \frac{1}{\gamma^2}\left(\Gamma\left(1+\frac{2}{\nu}\right) - \left(\Gamma\left(1+\frac{1}{\nu}\right)\right)^2\right) \tag{15}$$

Accordingly:

$$\begin{aligned}
\operatorname{Cov}(X,Y) &= \int_0^{\infty}\int_0^{\infty}\big(P(Y > y, X > x) - P(X > x)P(Y > y)\big)\,dy\,dx \\
&= \int_0^{\infty}\int_0^{\infty}\bar{F}_0(x)\left(\bar{F}_1\big(\beta(x)y\big) - \bar{F}_1\big(\beta(0)y\big)\right)\,dy\,dx \\
&= \int_0^{\infty} e^{-(\alpha x)^{\lambda}}\left(\int_0^{\infty}\left(e^{-(\beta(x)y)^{\nu}} - e^{-(\beta(0)y)^{\nu}}\right)dy\right)dx \\
&= \int_0^{\infty} e^{-(\alpha x)^{\lambda}}\left(\frac{\Gamma\big(1+\frac{1}{\nu}\big)}{\beta(x)} - \frac{\Gamma\big(1+\frac{1}{\nu}\big)}{\beta(0)}\right)dx \\
&= \frac{1}{\gamma}\Gamma\left(1+\frac{1}{\nu}\right)\left[\int_0^{\infty} e^{-(\alpha x)^{\lambda}-\frac{1}{\nu}(\alpha\tau x)^{\lambda}}\,dx - \int_0^{\infty} e^{-(\alpha x)^{\lambda}}\,dx\right] \\
&= \frac{1}{\alpha\gamma}\Gamma\left(1+\frac{1}{\nu}\right)\Gamma\left(1+\frac{1}{\lambda}\right)\left[\left(\frac{\nu}{\nu+\tau^{\lambda}}\right)^{\frac{1}{\lambda}} - 1\right].
\end{aligned} \tag{16}$$

Now, using (14), (15) and (16), the correlation is:

$$\rho(X,Y) = \frac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}}$$
$$= \frac{\Gamma\left(1+\frac{1}{\nu}\right)\Gamma\left(1+\frac{1}{\lambda}\right)\left[\left(\frac{\nu}{\nu+\tau^{\lambda}}\right)^{\frac{1}{\lambda}}-1\right]}{\sqrt{\left(\Gamma\left(1+\frac{2}{\lambda}\right)-\left(\Gamma\left(1+\frac{1}{\lambda}\right)\right)^2\right)\left(\Gamma\left(1+\frac{2}{\nu}\right)-\left(\Gamma\left(1+\frac{1}{\nu}\right)\right)^2\right)}}, \tag{17}$$

Clearly, (17) is strictly decreasing in $\tau \in [0,1]$. Hence:

$$\rho_{\min}(\lambda,\nu) := \frac{\Gamma\left(1+\frac{1}{\nu}\right)\Gamma\left(1+\frac{1}{\lambda}\right)\left[\left(\frac{\nu}{\nu+1}\right)^{\frac{1}{\lambda}}-1\right]}{\sqrt{\left(\Gamma\left(1+\frac{2}{\lambda}\right)-\left(\Gamma\left(1+\frac{1}{\lambda}\right)\right)^2\right)\left(\Gamma\left(1+\frac{2}{\nu}\right)-\left(\Gamma\left(1+\frac{1}{\nu}\right)\right)^2\right)}} \leq \rho(X,Y) \leq 0. \tag{18}$$

Now, $\inf_{\lambda,\nu>0}\rho_{\min}(\lambda,\nu) = \lim_{\nu\to\infty}\rho_{\min}(\lambda=1,\nu) = -\frac{\sqrt{6}}{\pi}$. Hence, this Weibull model with acceleration function $\beta(x) = \gamma e^{\frac{1}{\nu}(\alpha\tau x)^{\lambda}}$ will be able to accommodate correlations in the range $\rho(X,Y) \in \left[-\frac{\sqrt{6}}{\pi}, 0\right]$.

5 Log-Logistic

We restrict the survival function forms to the log-logistic family as such:

$$\bar{F}_0(x) = P(X>x) = 1 - \frac{1}{1+(\alpha x)^{-\lambda}} = \frac{1}{1+(\alpha x)^{\lambda}}, \quad x>0,$$

where $\alpha,\lambda>0$ and, for each $x>0$:

$$\bar{F}_1(\beta(x)y) = P(Y>y \mid X>x) = 1 - \frac{1}{1+(\beta(x)y)^{-\nu}} = \frac{1}{1+(\beta(x)y)^{\nu}}, \quad y>0,$$

where $\beta(x),\nu>0$ for all $x>0$ to ensure a valid survival function. The corresponding joint survival function will be:

$$\bar{F}_0(x)\bar{F}_1(\beta(x)y) = P(X>x, Y>y) = \frac{1}{(1+(\alpha x)^{\lambda})(1+(\beta(x)y)^{\nu})}, \quad x,y>0. \tag{19}$$

Accordingly, $f_X(x) = \frac{\alpha\lambda(\alpha x)^{\lambda-1}}{((\alpha x)^{\lambda}+1)^2}$ for $x>0$ and $f_Y(y) = \frac{\beta(0)\nu(\beta(0)y)^{\nu-1}}{((\beta(0)y)^{\nu}+1)^2}$ for $y>0$. Hence, $X \sim \mathit{LogLogistic}(\alpha,\lambda)$ and $Y \sim \mathit{LogLogistic}(\beta(0),\nu)$.

Now since $[tf_1(t)]' = \frac{\nu^2 t^{\nu-1}(1-t^{\nu})}{t^{3\nu}+3t^{2\nu}+3t^{\nu}+1} > 0$ for $t<1$, and $S(t) = -\frac{t^{\nu}+1}{\nu(t^{\nu}-1)}$ is increasing on this interval, $\inf_{t<1} S(t) = \lim_{t\to 0} S(t) = \frac{1}{\nu}$. Additionally, since $[tf_1(t)]' = \frac{\nu^2 t^{\nu-1}(1-t^{\nu})}{t^{3\nu}+3t^{2\nu}+3t^{\nu}+1} < 0$ for $t>1$, and $S(t) = -\frac{t^{\nu}+1}{\nu(t^{\nu}-1)}$ is increasing on this interval, $\sup_{t>1} S(t) = \lim_{t\to\infty} S(t) = -\frac{1}{\nu}$. Consequently, applying (2) yields:

$$-\frac{1}{\nu}\frac{\alpha\lambda(\alpha x)^{\lambda-1}}{(\alpha x)^{\lambda}+1}\beta(x) \leq B'(x) \leq \frac{1}{\nu}\frac{\alpha\lambda(\alpha x)^{\lambda-1}}{(\alpha x)^{\lambda}+1}\beta(x).$$

Now, we select the form of the acceleration function $\beta(x)$ such that $\beta'(x) = \frac{1}{\nu}\frac{\alpha\lambda(\alpha x)^{\lambda-1}}{(\alpha x)^{\lambda}+1}\beta(x)$ (that is, equal to $\beta'(x)'s$ upper bound). Hence from (3) with $c^* = \frac{1}{\nu}$, $\beta(x) = \gamma(1+(\alpha x)^{\lambda})^{\frac{1}{\nu}}$ with $\gamma>0$. Furthermore, following Arnold and Arvanitis [1], we introduce a dependence parameter $\tau \in [0,1]$ as before. Hence we choose our acceleration function as:

$$\beta(x) = \gamma(1+(\alpha\tau x)^{\lambda})^{\frac{1}{\nu}}.$$

5.1 Moments

Noting $X \sim LogLogistic(\alpha, \lambda)$ and $Y \sim LogLogistic(\gamma, \nu)$, we have:

$$E(X) = \frac{\pi}{\alpha\lambda \sin\left(\frac{\pi}{\lambda}\right)}, \quad \lambda > 1,$$

$$E(Y) = \frac{\pi}{\gamma\nu \sin\left(\frac{\pi}{\nu}\right)}, \quad \nu > 1,$$

$$Var(X) = \frac{1}{\alpha^2}\left[\frac{2\pi}{\lambda \sin\left(\frac{2\pi}{\lambda}\right)} - \left(\frac{\pi}{\lambda \sin\left(\frac{\pi}{\lambda}\right)}\right)^2\right], \quad \lambda > 2, \tag{20}$$

$$Var(Y) = \frac{1}{\gamma^2}\left[\frac{2\pi}{\nu \sin\left(\frac{2\pi}{\nu}\right)} - \left(\frac{\pi}{\nu \sin\left(\frac{\pi}{\nu}\right)}\right)^2\right], \quad \nu > 2. \tag{21}$$

Accordingly:

$$\begin{aligned}
\operatorname{Cov}(X,Y) &= \int_0^\infty \int_0^\infty \big(P(Y>y, X>x) - P(X>x)P(Y>y)\big)\, dy\, dx \\
&= \int_0^\infty \int_0^\infty \bar{F}_0(x)\left(\bar{F}_1\big(\beta(x)y\big) - \bar{F}_1\big(\beta(0)y\big)\right)\, dy\, dx \\
&= \int_0^\infty \frac{1}{1+(\alpha x)^\lambda}\left(\int_0^\infty \left(\frac{1}{1+(\beta(x)y)^\nu} - \frac{1}{1+(\beta(0)y)^\nu}\right) dy\right) dx \\
&= \frac{\pi}{\nu \sin\left(\frac{\pi}{\nu}\right)} \int_0^\infty \frac{1}{1+(\alpha x)^\lambda}\left(\frac{1}{\beta(x)} - \frac{1}{\beta(0)}\right) dx \\
&= \frac{\pi}{\gamma\nu \sin\left(\frac{\pi}{\nu}\right)}\left[\int_0^\infty \frac{dx}{((1+(\alpha x)^\lambda)\,((1+(\alpha\tau x)^\lambda)^{\frac{1}{\nu}}} - \int_0^\infty \frac{dx}{1+(\alpha x)^\lambda}\right] \\
&= \frac{\pi}{\alpha\lambda\nu\gamma \sin\left(\frac{\pi}{\nu}\right)}\left[\frac{\Gamma\left(\frac{1}{\lambda}\right)\Gamma\left(1+\frac{1}{\nu}-\frac{1}{\lambda}\right)}{\Gamma\left(1+\frac{1}{\nu}\right)}\, {}_2F_1\left(\frac{1}{\nu}, \frac{1}{\lambda}; 1+\frac{1}{\nu}; 1-\tau^\lambda\right) - \frac{\pi}{\sin\left(\frac{\pi}{\lambda}\right)}\right].
\end{aligned} \tag{22}$$

where the conditions $\nu > 1$ and $\lambda > 1$ ensure convergence of the inner and outer integrals, respectively. Now, using (20), (21) and (22), the correlation is:

$$\begin{aligned}
\rho(X,Y) &= \frac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}} \\
&= \frac{\pi\left[\frac{\Gamma\left(\frac{1}{\lambda}\right)\Gamma\left(1+\frac{1}{\nu}-\frac{1}{\lambda}\right)}{\Gamma\left(1+\frac{1}{\nu}\right)}\, {}_2F_1\left(\frac{1}{\nu}, \frac{1}{\lambda}; 1+\frac{1}{\nu}; 1-\tau^\lambda\right) - \frac{\pi}{\sin\left(\frac{\pi}{\lambda}\right)}\right]}{\lambda\nu \sin\left(\frac{\pi}{\nu}\right)\sqrt{\left[\frac{2\pi}{\lambda \sin\left(\frac{2\pi}{\lambda}\right)} - \left(\frac{\pi}{\lambda \sin\left(\frac{\pi}{\lambda}\right)}\right)^2\right]\left[\frac{2\pi}{\nu \sin\left(\frac{2\pi}{\nu}\right)} - \left(\frac{\pi}{\nu \sin\left(\frac{\pi}{\nu}\right)}\right)^2\right]}},
\end{aligned} \tag{23}$$

valid for $\lambda, \nu > 2$. Since ${}_2F_1\left(\frac{1}{\nu}, \frac{1}{\lambda}; 1+\frac{1}{\nu}; z\right)$ has a power series with strictly positive coefficients, it is strictly increasing in $z \in [0,1]$; hence ${}_2F_1\left(\frac{1}{\nu}, \frac{1}{\lambda}; 1+\frac{1}{\nu}; 1-\tau^\lambda\right)$ is strictly decreasing in $\tau \in [0,1]$. Consequently, (23) attains its minimum at $\tau = 1$ and its maximum (zero) at $\tau = 0$. Therefore,

$$\rho_{\min(\lambda,\nu)} := \frac{\pi\left[\frac{\Gamma\left(\frac{1}{\lambda}\right)\Gamma\left(1+\frac{1}{\nu}-\frac{1}{\lambda}\right)}{\Gamma\left(1+\frac{1}{\nu}\right)} - \frac{\pi}{\sin\left(\frac{\pi}{\lambda}\right)}\right]}{\lambda\nu \sin\left(\frac{\pi}{\nu}\right)\sqrt{\left[\frac{2\pi}{\lambda \sin\left(\frac{2\pi}{\lambda}\right)} - \left(\frac{\pi}{\lambda \sin\left(\frac{\pi}{\lambda}\right)}\right)^2\right]\left[\frac{2\pi}{\nu \sin\left(\frac{2\pi}{\nu}\right)} - \left(\frac{\pi}{\nu \sin\left(\frac{\pi}{\nu}\right)}\right)^2\right]}} \le \rho(X,Y) \le 0.$$

Now denoting $[\lambda^*, \nu^*]' = \operatorname{argmin}_{\lambda,\nu} \rho_{\min(\lambda,\nu)}$, we obtain $\inf_{\lambda,\nu>2} \rho_{\min}(\lambda,\nu) = \lim_{\nu\to\infty} \rho_{\min}(\lambda = \lambda^* \approx 5.06, \nu) \approx -0.54076$. Hence, this log-logistic model with acceleration function $\beta(x) = \gamma(1+(\alpha\tau x)^\lambda)^{\frac{1}{\nu}}$ will be able to accommodate correlations in the range $\rho(X,Y) \in \big[-0.54076,\ 0\big]$.

6 Cauchy

We restrict the survival function forms to the Cauchy family as such:

$$\bar{F}_0(x) = P(X > x) = \frac{1}{2} - \frac{1}{\pi}\arctan(\alpha x), \quad x > 0,$$

where $\alpha > 0$ and, for each $x > 0$:

$$\bar{F}_1(\beta(x)y) = P(Y > y \mid X > x) = \frac{1}{2} - \frac{1}{\pi}\arctan(\beta(x)y), \quad y > 0,$$

where $\beta(x) > 0$ for all $x > 0$ to ensure a valid survival function. The corresponding joint survival function will be:

$$\bar{F}_0(x)\bar{F}_1(\beta(x)y) = P(X > x, Y > y) = \frac{1}{\pi^2}\left(\arctan(\alpha x) - \frac{1}{2}\pi\right)\left(\arctan(\beta(x)y) - \frac{1}{2}\pi\right), \quad x, y > 0. \tag{24}$$

Accordingly, $f_X(x) = \frac{\alpha}{\pi(1+(\alpha x)^2)}$ for $x > 0$ and $f_Y(y) = \frac{\beta(0)}{\pi(1+(\beta(0)x)^2)}$ for $y > 0$. Hence, $X \sim Cauchy(\alpha)$ and $Y \sim Cauchy(\beta(0))$.

Now since $[tf_1(t)]' = \frac{1-t^2}{\pi(t^2+1)^2} > 0$ for $t < 1$, and $S(t) = \frac{t^2+1}{1-t^2}$ is increasing on this interval, $\inf_{t<1} S(t) = \lim_{t\to 0} S(t) = 1$. Additionally, since $[tf_1(t)]' = \frac{1-t^2}{\pi(t^2+1)^2} < 0$ for $t > 1$, and $S(t) = \frac{t^2+1}{1-t^2}$ is increasing on this interval, $\sup_{t>1} S(t) = \lim_{t\to\infty} S(t) = -1$. Consequently, applying (2) yields:

$$-\frac{\alpha}{(1+(\alpha x)^2)(\frac{\pi}{2} - \arctan(\alpha x))}\beta(x) \leq B'(x) \leq \frac{\alpha}{(1+(\alpha x)^2)(\frac{\pi}{2} - \arctan(\alpha x))}\beta(x).$$

Now, we select the form of the acceleration function $\beta(x)$ such that $\beta'(x) = \frac{\alpha}{(1+(\alpha x)^2)(\frac{\pi}{2} - \arctan(\alpha x))}\beta(x)$ (that is, equal to $\beta'(x)'s$ upper bound). Hence from (3) with $c^* = 1$, $\beta(x) = \gamma\frac{1}{\frac{1}{2} - \frac{1}{\pi}\arctan(\alpha x)}$ with $\gamma > 0$. Furthermore, following Arnold and Arvanitis [1], we introduce a dependence parameter $\tau \in [0, 1]$ as before. Hence we choose our acceleration function as:

$$\beta(x) = \gamma\frac{1}{\frac{1}{2} - \frac{1}{\pi}\arctan(\alpha\tau x)}.$$

7 Gamma

We restrict the survival function forms to the Gamma family as such:

$$\bar{F}_0(x) = P(X > x) = 1 - \frac{1}{\Gamma(\lambda)}\gamma^*(\lambda, \alpha x) = \frac{\Gamma(\lambda, \alpha x)}{\Gamma(\lambda)}, \quad x > 0,$$

where $\alpha, \lambda > 0$ and denoting γ^* as the lower incomplete gamma function. Now for each $x > 0$:

$$\bar{F}_1(\beta(x)y) = P(Y > y \mid X > x) = 1 - \frac{1}{\Gamma(\nu)}\gamma^*(\nu, \beta(x)y) = \frac{\Gamma(\nu, \beta(x)y)}{\Gamma(\nu)}, \quad y > 0,$$

where $\beta(x), \nu > 0$ for all $x > 0$ to ensure a valid survival function. The corresponding joint survival function will be:

$$\bar{F}_0(x)\bar{F}_1(\beta(x)y) = P(X > x, Y > y) = \frac{\Gamma(\lambda, \alpha x)\Gamma(\nu, \beta(x)y)}{\Gamma(\lambda)\Gamma(\nu)}, \quad x, y > 0. \tag{25}$$

Accordingly, $f_X(x) = \frac{\alpha^\lambda}{\Gamma(\lambda)}x^{\lambda-1}e^{-\alpha x}$ for $x > 0$ and $f_Y(y) = \frac{\beta(0)^\nu}{\Gamma(\nu)}x^{\nu-1}e^{-\beta(0)y}$ for $y > 0$. Hence, $X \sim Gamma(\alpha, \lambda)$ and $Y \sim Gamma(\beta(0), \nu)$.

Now since $[tf_1(t)]' = \frac{t^{\nu-1}(\nu-t)e^{-t}}{\Gamma(\nu)} > 0$ for $t < \nu$, and $S(t) = \frac{1}{\nu-t}$ is increasing on this interval, $\inf_{t<\nu} S(t) = \lim_{t\to 0} S(t) = \frac{1}{\nu}$. Additionally, since $[tf_1(t)]' = \frac{t^{\nu-1}(\nu-t)e^{-t}}{\Gamma(\nu)} < 0$ for $t > \nu$, and $S(t) = \frac{1}{\nu-t}$ is increasing on this interval, $\sup_{t>\nu} S(t) = \lim_{t\to\infty} S(t) = 0$. Consequently, applying (2) yields:

$$0 \leq B'(x) \leq \frac{1}{\nu}\frac{\alpha^\lambda}{\Gamma(\lambda, \alpha x)}x^{\lambda-1}e^{-\alpha x}.$$

Now, we select the form of the acceleration function $\beta(x)$ such that $\beta'(x) = \frac{1}{\nu}\frac{\alpha^\lambda}{\Gamma(\lambda,\alpha x)}x^{\lambda-1}e^{-\alpha x}$ (that is, equal to $\beta'(x)'s$ upper bound). Hence from (3) with $c^* = \frac{1}{\nu}$, $\beta(x) = \gamma\left[\frac{\Gamma(\lambda)}{\Gamma(\lambda,\alpha x)}\right]^{\frac{1}{\nu}}$ with $\gamma > 0$. Furthermore, following Arnold and Arvanitis [1], we introduce a dependence parameter $\tau \in [0, 1]$ as before. Hence we choose our acceleration function as:

$$\beta(x) = \gamma\left[\frac{\Gamma(\lambda)}{\Gamma(\lambda, \alpha\tau x)}\right]^{\frac{1}{\nu}}.$$

A Joint Densities and Log-likelihoods

Exponential

We obtain the joint density function by differentiating the joint survival function (4):

$$f_{X,Y}(x,y) = \alpha\gamma\left(\gamma\tau y e^{\alpha\tau x} - \tau + 1\right)e^{(\alpha x(\tau-1) - \gamma y e^{\alpha\tau x})}, \qquad x, y > 0.$$

where $\alpha, \gamma > 0$ and $\tau \in [0,1]$. The log-likelihood is thus:

$$\begin{aligned}\ell(\alpha,\gamma,\tau) = n\log\alpha + n\log\gamma + \sum_{i=1}^{n}\log\Big(\gamma\tau y_i e^{\alpha\tau x_i} - \tau + 1\Big) \\ + \alpha(\tau-1)\sum_{i=1}^{n}x_i - \gamma\sum_{i=1}^{n}y_i e^{\alpha\tau x_i}.\end{aligned}$$

Lomax

We obtain the joint density function by differentiating the joint survival function (8):

$$\begin{aligned}f_{X,Y}(x,y) = \nu\,(1+\alpha x)^{-3\lambda-1}\left(1+\gamma y(1+\alpha\tau x)^{\lambda}\right)^{-3\nu-4} \\ \times\Bigg[\frac{\alpha\gamma^2\lambda\tau y(\nu+1)\,(1+\alpha x)^{2\lambda+1}(1+\alpha\tau x)^{2\lambda}\left(1+\gamma y(1+\alpha\tau x)^{\lambda}\right)^{2\nu+2}}{1+\alpha\tau x} \\ - \frac{\alpha\gamma\lambda\tau\,(1+\alpha x)^{2\lambda+1}(1+\alpha\tau x)^{\lambda}\left(1+\gamma y(1+\alpha\tau x)^{\lambda}\right)^{2\nu+3}}{1+\alpha\tau x} \\ + \alpha\gamma\lambda\left((1+\alpha x)^2(1+\alpha\tau x)\right)^{\lambda}\left(1+\gamma y(1+\alpha\tau x)^{\lambda}\right)^{2\nu+3}\Bigg], \quad x, y > 0,\end{aligned}$$

where $\alpha, \lambda, \nu, \gamma > 0$ and $\tau \in [0,1]$. The log-likelihood is thus:

$$\begin{aligned}\ell(\alpha,\lambda,\nu,\gamma,\tau) = n\log\nu - (3\lambda+1)\sum_{i=1}^{n}\log(1+\alpha x_i) \\ - (3\nu+4)\sum_{i=1}^{n}\log\Big(1+\gamma y_i(1+\alpha\tau x_i)^{\lambda}\Big) \\ + \sum_{i=1}^{n}\log\Bigg\{\frac{\alpha\gamma^2\lambda\tau y_i(\nu+1)\,(1+\alpha x_i)^{2\lambda+1}(1+\alpha\tau x_i)^{2\lambda}}{1+\alpha\tau x_i}\left(1+\gamma y_i(1+\alpha\tau x_i)^{\lambda}\right)^{2\nu+2} \\ - \frac{\alpha\gamma\lambda\tau(1+\alpha x_i)^{2\lambda+1}(1+\alpha\tau x_i)^{\lambda}}{1+\alpha\tau x_i}\left(1+\gamma y_i(1+\alpha\tau x_i)^{\lambda}\right)^{2\nu+3} \\ + \alpha\gamma\lambda\left((1+\alpha x_i)^2(1+\alpha\tau x_i)\right)^{\lambda}\left(1+\gamma y_i(1+\alpha\tau x_i)^{\lambda}\right)^{2\nu+3}\Bigg\}.\end{aligned}$$

Weibull

We obtain the joint density function by differentiating the joint survival function (13):

$$f_{X,Y}(x,y) = \alpha^{\lambda}\gamma^{\nu}\lambda\nu x^{\lambda-1}y^{\nu-1}\left(\tau^{\lambda}(\gamma y)^{\nu}e^{(\alpha\tau x)^{\lambda}} - \tau^{\lambda} + 1\right)e^{(\alpha x)^{\lambda}(\tau^{\lambda}-1) - (\gamma y)^{\nu}e^{(\alpha\tau x)^{\lambda}}}, \quad x, y > 0,$$

where $\alpha, \lambda, \nu, \gamma > 0$ and $\tau \in [0,1]$. The log-likelihood is thus:

$$\begin{aligned}\ell(\alpha,\lambda,\nu,\gamma,\tau) = n\lambda\log\alpha + n\nu\log\gamma + n\log\lambda + n\log\nu \\ + (\lambda-1)\sum_{i=1}^{n}\log x_i + (\nu-1)\sum_{i=1}^{n}\log y_i \\ + \sum_{i=1}^{n}\log\Big(\tau^{\lambda}(\gamma y_i)^{\nu}e^{(\alpha\tau x_i)^{\lambda}} - \tau^{\lambda} + 1\Big) \\ + (\tau^{\lambda}-1)\sum_{i=1}^{n}(\alpha x_i)^{\lambda} - \sum_{i=1}^{n}(\gamma y_i)^{\nu}e^{(\alpha\tau x_i)^{\lambda}}.\end{aligned}$$

Log-logistic

We obtain the joint density function by differentiating the joint survival function (19):

$$
\begin{aligned}
f_{X,Y}(x,y) = {} & \frac{\lambda\nu\,(\gamma y)^{\nu}}{x\,y\,\big((\alpha x)^{\lambda}+1\big)^{2}\Big((\gamma y)^{\nu}\big((\alpha\tau x)^{\lambda}+1\big)+1\Big)^{3}} \\
& \times \Bigg\{(\alpha x)^{\lambda}\big((\alpha\tau x)^{\lambda}+1\big)\Big((\gamma y)^{\nu}\big((\alpha\tau x)^{\lambda}+1\big)+1\Big) \\
& \qquad + (\alpha\tau x)^{\lambda}\big((\alpha x)^{\lambda}+1\big)\Big((\gamma y)^{\nu}\big((\alpha\tau x)^{\lambda}+1\big)+1\Big) \\
& \qquad - 2\,(\alpha\tau x)^{\lambda}\big((\alpha x)^{\lambda}+1\big)\Bigg\}, \qquad x,y>0,
\end{aligned}
$$

where $\alpha,\lambda,\nu,\gamma>0$ and $\tau\in[0,1]$. Thus the log-likelihood is:

$$
\begin{aligned}
\ell(\alpha,\lambda,\nu,\gamma,\tau) = {} & n\log\lambda + n\log\nu + n\,\nu\log\gamma + (\nu-1)\sum_{i=1}^{n}\log y_i - \sum_{i=1}^{n}\log x_i \\
& - 2\sum_{i=1}^{n}\log\big((\alpha x_i)^{\lambda}+1\big) - 3\sum_{i=1}^{n}\log\big((\gamma y_i)^{\nu}\big((\alpha\tau x_i)^{\lambda}+1\big)+1\big) \\
& + \sum_{i=1}^{n}\log\Big\{(\alpha x_i)^{\lambda}\big((\alpha\tau x_i)^{\lambda}+1\big)\big((\gamma y_i)^{\nu}\big((\alpha\tau x_i)^{\lambda}+1\big)+1\big) \\
& \qquad + (\alpha\tau x_i)^{\lambda}\big((\alpha x_i)^{\lambda}+1\big)\big((\gamma y_i)^{\nu}\big((\alpha\tau x_i)^{\lambda}+1\big)+1\big) \\
& \qquad - 2\,(\alpha\tau x_i)^{\lambda}\big((\alpha x_i)^{\lambda}+1\big)\Big\}.
\end{aligned}
$$

Cauchy

We obtain the joint density function by differentiating the joint survival function (24):

$$
\begin{aligned}
f_{X,Y}(x,y) = \Bigg(& -2\,\alpha\gamma\,(\pi\gamma y)^{2}\,\tau\,\big((\alpha x)^{2}+1\big)\,\big(\arctan(\alpha x)-\tfrac{\pi}{2}\big) \\
& + \alpha\gamma\,\tau\,\big((\alpha x)^{2}+1\big)\,\Big((\pi\gamma y)^{2}+\big(\arctan(\alpha\tau x)-\tfrac{\pi}{2}\big)^{2}\Big)\,\big(\arctan(\alpha x)-\tfrac{\pi}{2}\big) \\
& - \alpha\gamma\,\Big((\pi\gamma y)^{2}+\big(\arctan(\alpha\tau x)-\tfrac{\pi}{2}\big)^{2}\Big)\,\big((\alpha\tau x)^{2}+1\big)\,\big(\arctan(\alpha\tau x)-\tfrac{\pi}{2}\big)\Bigg) \\
& \times \frac{1}{\pi\,\big((\alpha x)^{2}+1\big)\,\Big((\pi\gamma y)^{2}+\big(\arctan(\alpha\tau x)-\tfrac{\pi}{2}\big)^{2}\Big)^{2}\,\big((\alpha\tau x)^{2}+1\big)}, \quad x,y>0,
\end{aligned}
$$

where $\alpha,\gamma>0$ and $\tau\in[0,1]$. Thus the log-likelihood is:

$$
\begin{aligned}
\ell(\alpha,\gamma,\tau) = \sum_{i=1}^{n}\log\Bigg[& -2\,\alpha\gamma\,(\pi\gamma y_i)^{2}\,\tau\,\big((\alpha x_i)^{2}+1\big)\,\big(\arctan(\alpha x_i)-\tfrac{\pi}{2}\big) \\
& + \alpha\gamma\,\tau\,\big((\alpha x_i)^{2}+1\big)\,\Big((\pi\gamma y_i)^{2}+\big(\arctan(\alpha\tau x_i)-\tfrac{\pi}{2}\big)^{2}\Big)\,\big(\arctan(\alpha x_i)-\tfrac{\pi}{2}\big) \\
& - \alpha\gamma\,\Big((\pi\gamma y_i)^{2}+\big(\arctan(\alpha\tau x_i)-\tfrac{\pi}{2}\big)^{2}\Big)\,\big((\alpha\tau x_i)^{2}+1\big)\,\big(\arctan(\alpha\tau x_i)-\tfrac{\pi}{2}\big)\Bigg] \\
& - n\log\pi - \sum_{i=1}^{n}\log\big((\alpha x_i)^{2}+1\big) - 2\sum_{i=1}^{n}\log\Big((\pi\gamma y_i)^{2}+\big(\arctan(\alpha\tau x_i)-\tfrac{\pi}{2}\big)^{2}\Big) \\
& - \sum_{i=1}^{n}\log\big((\alpha\tau x_i)^{2}+1\big).
\end{aligned}
$$

Gamma

We obtain the joint density function by differentiating the joint survival function (25):

$$
\begin{aligned}
f_{X,Y}(x,y) = \Bigg(& \alpha^{\lambda}\gamma^{\nu}\lambda^{-1-\frac{1}{\nu}} x^{\lambda-1} y^{\nu}\, \gamma\, \tau^{\lambda} y\, \Gamma(\lambda,\alpha\tau x)^{-\frac{1}{\nu}} e^{\alpha x}\, \Gamma(\lambda)\, \Gamma(\lambda+1)^{\,1+\frac{1}{\nu}} \\
& - \alpha^{\lambda}\gamma^{\nu}\lambda^{-1-\frac{1}{\nu}} x^{\lambda-1} y^{\nu}\, \gamma\, \tau^{\lambda} y\, \Gamma(\lambda,\alpha\tau x)^{-\frac{1}{\nu}} e^{\alpha x}\, \Gamma(\lambda+1)^{\,1+\frac{1}{\nu}}\, \gamma^{*}(\lambda,\alpha x) \\
& - \alpha^{\lambda}\gamma^{\nu}\lambda^{-1-\frac{1}{\nu}} x^{\lambda-1} y^{\nu}\, \lambda^{\frac{1}{\nu}} \nu \tau^{\lambda} e^{\alpha x} \Gamma(\lambda)\, \Gamma(\lambda+1) \\
& + \alpha^{\lambda}\gamma^{\nu}\lambda^{-1-\frac{1}{\nu}} x^{\lambda-1} y^{\nu}\, \lambda^{1+\frac{1}{\nu}} \nu\, \tau^{\lambda} e^{\alpha x} \Gamma(\lambda)\, \gamma^{*}(\lambda,\alpha x) \\
& + \alpha^{\lambda}\gamma^{\nu}\lambda^{-1-\frac{1}{\nu}} x^{\lambda-1} y^{\nu}\, \lambda^{1+\frac{1}{\nu}} \nu\, e^{\alpha\tau x} \Gamma(\lambda)\, \Gamma(\lambda,\alpha\tau x) \Bigg) \\
& \times \left[\frac{e^{-\alpha\tau x - \alpha x - \gamma\, \lambda^{-\frac{1}{\nu}} y \Gamma(\lambda,\alpha\tau x)^{-\frac{1}{\nu}} \Gamma(\lambda+1)^{\frac{1}{\nu}}}}{y\, \Gamma(\lambda)\, \Gamma(\nu+1)\, \Gamma(\lambda,\alpha\tau x)^{2}} \right], \quad x, y > 0,
\end{aligned}
$$

where $\alpha, \gamma > 0$ and $\tau \in [0,1]$. Thus the log-likelihood is:

$$
\begin{aligned}
\ell(\alpha,\lambda,\nu,\gamma,\tau) = \sum_{i=1}^{n} \Bigg\{ & \lambda \log\alpha + \nu \log\gamma - \left(1 + \tfrac{1}{\nu}\right) \log\lambda + (\lambda-1)\log x_i + \nu \log y_i \\
& + \log \Big[\alpha^{\lambda}\gamma^{\nu}\lambda^{-1-\frac{1}{\nu}} x_i^{\lambda-1} y_i^{\nu}\, \gamma\, \tau^{\lambda} y_i\, \Gamma(\lambda,\alpha\tau x_i)^{-\frac{1}{\nu}} e^{\alpha x_i}\, \Gamma(\lambda)\, \Gamma(\lambda+1)^{\,1+\frac{1}{\nu}} \\
& \quad - \alpha^{\lambda}\gamma^{\nu}\lambda^{-1-\frac{1}{\nu}} x_i^{\lambda-1} y_i^{\nu}\, \gamma\, \tau^{\lambda} y_i\, \Gamma(\lambda,\alpha\tau x_i)^{-\frac{1}{\nu}} e^{\alpha x_i}\, \Gamma(\lambda+1)^{\,1+\frac{1}{\nu}}\, \gamma^{*}(\lambda,\alpha x_i) \\
& \quad - \alpha^{\lambda}\gamma^{\nu}\lambda^{-1-\frac{1}{\nu}} x_i^{\lambda-1} y_i^{\nu}\, \lambda^{\frac{1}{\nu}} \nu \tau^{\lambda} e^{\alpha x_i} \Gamma(\lambda)\, \Gamma(\lambda+1) \\
& \quad + \alpha^{\lambda}\gamma^{\nu}\lambda^{-1-\frac{1}{\nu}} x_i^{\lambda-1} y_i^{\nu}\, \lambda^{1+\frac{1}{\nu}} \nu\, \tau^{\lambda} e^{\alpha x_i} \Gamma(\lambda)\, \gamma^{*}(\lambda,\alpha x_i) \\
& \quad + \alpha^{\lambda}\gamma^{\nu}\lambda^{-1-\frac{1}{\nu}} x_i^{\lambda-1} y_i^{\nu}\, \lambda^{1+\frac{1}{\nu}} \nu\, e^{\alpha\tau x_i} \Gamma(\lambda)\, \Gamma(\lambda,\alpha\tau x_i) \Big] \\
& - \alpha\tau x_i - \alpha x_i - \gamma\, \lambda^{-\frac{1}{\nu}} y_i\, \Gamma(\lambda,\alpha\tau x_i)^{-\frac{1}{\nu}}\, \Gamma(\lambda+1)^{\frac{1}{\nu}} - \log y_i \\
& - \log\Gamma(\lambda) - \log\Gamma(\nu+1) - 2\log\Gamma(\lambda,\alpha\tau x_i) \Bigg\}.
\end{aligned}
$$

References

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