

# SOME REMARKS ON RESIDUAL-BASED STABILISATION OF INF-SUP STABLE DISCRETISATIONS OF THE GENERALISED OSEEN PROBLEM

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**Abstract** — We consider residual-based stabilised finite element methods for the generalised Oseen problem. The unique solvability based on a modified stability condition and an error estimate are proved for inf-sup stable discretisations of velocity and pressure. The analysis highlights the role of an additional stabilisation of the incompressibility constraint. It turns out that the stabilisation terms of the streamline-diffusion (SUPG) type play a less important role. In particular, there exists a conditional stability problem of the SUPG stabilisation if two relevant problem parameters tend to zero. The analysis extends a recent result to general shape-regular meshes and to discontinuous pressure interpolation. Some numerical observations support the theoretical results.

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## 1. Introduction

Let us consider the time-dependent, incompressible Navier — Stokes problem with homogeneous Dirichlet boundary conditions

$$\begin{aligned}\partial_t u - \nu \Delta u + (u \cdot \nabla)u + \nabla p &= \tilde{f} \quad \text{in } \Omega \times (0, T), \\ \operatorname{div} u &= 0 \quad \text{in } \Omega \times (0, T), \\ u &= 0 \quad \text{on } \partial\Omega \times (0, T], \\ u|_{t=0} &= u_0 \quad \text{in } \Omega,\end{aligned}\tag{1.1}$$

for the velocity  $u$  and the pressure  $p$  in the space-time cylinder  $\Omega \times (0, T)$  with a polyhedral domain  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$ , and a time  $T > 0$ . The given source term is denoted by  $\tilde{f}$ . A typical algorithmic approach for solving (1.1) is to first semidiscretise in time and then apply a fixed-point iteration within each time step. This leads in each step of this iteration to an auxiliary Oseen-type problem

$$L_O(b; u, p) := -\nu \Delta u + (b \cdot \nabla)u + \sigma u + \nabla p = f \quad \text{in } \Omega,$$

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$$\begin{aligned}\operatorname{div} u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega.\end{aligned}\tag{1.2}$$

Also, the iterative solution of the steady-state Navier — Stokes equations may lead to problems of type (1.2) with  $\sigma = 0$  if a fixed-point iteration is applied.

The basic Galerkin finite element method (FEM) for (1.2) may suffer from two problems: the dominating advection (and reaction) in the case of  $0 < \nu \ll \|b\|_{L^\infty(\Omega)}$ , and/or the violation of the discrete inf-sup (or Babuška — Brezzi) stability condition for the velocity and pressure approximations. The streamline-upwind/Petrov — Galerkin method (SUPG) introduced in [4] and the pressure-stabilisation/Petrov — Galerkin method (PSPG) introduced in [10, 11], opened the possibility of treating both problems within a unique framework using rather arbitrary FE approximations of velocity-pressure, including equal-order pairs. Additionally to the Galerkin part, the elementwise residual  $L_O(b; u, p) - f$  is tested against the (weighted) nonsymmetric SUPG/PSPG parts  $(b \cdot \nabla)v + \nabla q$  of  $L_O(b; v, q)$ . An additional elementwise stabilisation of the divergence constraint  $\operatorname{div} u = 0$  in (1.2), henceforth denoted as grad-div stabilisation, is important for the robustness if  $0 < \nu \ll \|b\|_{L^\infty(\Omega)}$  (see [6] for the analysis in the case of equal-order interpolation).

For a unified a priori analysis of classical residual-based stabilisation (RBS) techniques, we refer to [13]. We emphasize that the design of the stabilisation parameters for equal-order interpolation significantly differs from that for inf-sup stable pairs. In particular, the grad-div stabilisation is much more important in the advection-dominated case if an inf-sup stable interpolation is applied (see also [7, 17]).

One of the critical aspects of these RBS techniques for incompressible flows is the strong coupling between velocity and pressure in the stabilising terms. Several attempts have been made to relax this problem. In particular, we mention the promising idea of weakly-consistent, symmetric stabilisation techniques (e.g., via edge stabilisation or local projection) (see [3] for an overview).

Within the framework of strongly consistent RBS techniques, one natural idea is to skip the PSPG term in the case of inf-sup stable discretisations of velocity and pressure. We considered this possibility in [7]. The analysis of the so-called *reduced stabilised* scheme is so far restricted to the quasi-uniform case and to continuous pressure approximations. Moreover, in [7] there remained the question whether the analysis is optimal for the case  $\nu^2 + \sigma^2 \rightarrow +0$ . In particular, within our numerical simulations on equidistant meshes we did not observe numerical instabilities of the scheme.

The goal of this paper is to refine the analysis in [7] for the reduced stabilised scheme and to relax the assumptions of quasi-uniform meshes and continuous pressure discretisations. A technical ingredient is the application of quasi-local interpolation operators preserving the discrete divergence [9]. For brevity, we consider only conforming FEM. The main results are as follows:

- We prove a conditional inf-sup stability estimate of the scheme which requires a similar upper bound of the SUPG-stabilisation parameter as in [7]. Numerical tests on slightly distorted quasi-uniform meshes show that such upper bound may really exist. Moreover, we derive an a priori error estimate in terms of the stabilisation parameters.
- A discussion of the grad-div- and SUPG-stabilisation parameters, together with some numerical results, highlights the role of the additional stabilisation of the incompressibility constraint. Finally, it turns out that the SUPG-stabilisation is often less essential.

The paper is organised as follows. In Section 2, we introduce notation and the stabilised Galerkin discretisation of the Oseen problem. Then, we analyse the method in Section 3 and discuss the results in Section 4. Finally, in Section 5 we give a summary and consider some open problems.

## 2. Notation. The discrete problem

Let  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$  be a bounded polygonal or polyhedral domain. For a subdomain  $G \subset \Omega$ , the usual Sobolev spaces  $W^{m,p}(G)$  with norm  $\|\cdot\|_{m,p,G}$  and seminorm  $|\cdot|_{m,p,G}$  are used. In the case  $p = 2$ , we have  $H^m(G) = W^{m,2}(G)$  and the index  $p$  will be omitted. The  $L^2$  inner product on  $G$  is denoted by  $(\cdot, \cdot)_G$ . Note that the index  $G$  will be omitted for  $G = \Omega$ . This notation of norms, seminorms, and inner products is also used for the vector-valued and tensor-valued case. We set  $X := (H_0^1(\Omega))^d$ ,  $M := L_0^2(\Omega) := \{q \in L^2(\Omega) : (q, 1) = 0\}$  and  $H(\operatorname{div}, \Omega) := \{v \in (L^2(\Omega))^d : \operatorname{div} v \in L^2(\Omega)\}$ .

The generalised Oseen equations with homogeneous Dirichlet boundary conditions are given by problem (1.2) with constants  $\nu > 0$ ,  $\sigma \geq 0$  and a known convection field  $b \in H(\operatorname{div}, \Omega) \cap (L^\infty(\Omega))^d$  with  $\operatorname{div} b = 0$ . For  $u, v \in X$ ,  $p, q \in M$ , the bilinear forms  $A$ ,  $b$  and linear form  $L$  are defined by

$$A((u, p), (v, q)) := \nu(\nabla u, \nabla v) + ((b \cdot \nabla)u, v) + \sigma(u, v) - b(v, p) + b(u, q),$$

$$b(v, q) := (q, \operatorname{div} v),$$

$$L((v, q)) := (f, v).$$

Note that the following integration by parts

$$((b \cdot \nabla)v, w) = -((b \cdot \nabla)w, v) \quad (2.1)$$

holds true for all  $v, w \in X$  due to  $\operatorname{div} b = 0$ .

A weak formulation of the generalised Oseen equations (1.2) reads:

Find  $(u, p) \in X \times M$  such that

$$A((u, p), (v, q)) = L((v, q)) \quad \forall (v, q) \in X \times M. \quad (2.2)$$

Let  $\{\mathcal{T}_h\}$  be a family of shape-regular and exact triangulations of the domain  $\Omega$  such that

$$\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} \overline{K}$$

holds true for all triangulations  $\mathcal{T}_h$ .

Let  $X_h$  be a conforming finite element space based on  $\mathcal{T}_h$  for approximating the velocity. The space  $M_h$  for approximating the pressure may consist of continuous or generally discontinuous functions. We are interested in inf-sup stable discretisations, i.e., the condition

$$\inf_{q_h \in M_h} \sup_{v_h \in X_h} \frac{(\operatorname{div} v_h, q_h)}{|v_h|_1 \|q_h\|_0} \geq \beta_0 > 0 \quad (2.3)$$

is valid for all  $\mathcal{T}_h$  with a positive constant  $\beta_0$  which is independent of the mesh parameter  $h$ . Examples for such pairs are the Taylor — Hood family  $P_k/P_{k-1}$ ,  $k \geq 2$ , on simplices

and  $Q_k/Q_{k-1}$ ,  $k \geq 2$ , on quadrilaterals and hexahedra (see [8] and the references therein). Furthermore,  $Q_k/P_{k-1}^{\text{disc}}$ ,  $k \geq 2$  fulfils the inf-sup condition on quadrilaterals and hexahedra (see [8, 16]).

We assume that for all cells  $K \in \mathcal{T}_h$  the following inverse inequalities

$$\begin{aligned} \|\Delta v_h\|_{0,K} &\leq \mu h_K^{-1} \|\nabla v_h\|_{0,K} \quad \forall v_h \in X_h, \\ \frac{1}{\sqrt{d}} \|\operatorname{div} v_h\|_{0,K} &\leq \|\nabla v_h\|_{0,K} \leq \mu h_K^{-1} \|v_h\|_{0,K} \quad \forall v_h \in X_h, \\ \|\nabla q_h\|_{0,K} &\leq \mu h_K^{-1} \|q_h\|_{0,K} \quad \forall q_h \in M_h \end{aligned} \quad (2.4)$$

are valid with a constant  $\mu$  which depends only on the shape-regularity parameter of the family of triangulations.

We assume that the discrete velocity space  $X_h$  is based on finite elements of order  $k$ . One can think of the case where  $X_h$  consists of all continuous functions whose restrictions to a single cell  $K$  of the triangulation  $\mathcal{T}_h$  belong to  $P_k$  (for simplicial cells) or to  $Q_k$  (for quadrilateral and hexahedral cells). The discrete pressure space  $M_h$  is assumed to be based on finite elements of order  $\ell \geq 1$ . This means that the restriction of a function from  $M_h$  to a cell  $K \in \mathcal{T}_h$  belongs to  $P_\ell$  or  $Q_\ell$ . Note that  $P_\ell$  can be used also on quadrilaterals and hexahedra if no continuity is required in  $M_h$ .

The standard finite element interpolation operator  $J_h : M \rightarrow M_h$  fulfils for all  $K \in \mathcal{T}_h$  the estimate

$$|q - J_h q|_{m,K} \leq C h_K^{\ell+1-m} \|q\|_{\ell+1,K} \quad \forall q \in H^{\ell+1}(\Omega) \cap M, \quad m = 0, \dots, \ell + 1,$$

where the constant  $C$  is independent of  $h$  (see [5]). We choose from [9] for the velocity the quasi-local interpolation operator which preserves the discrete divergence. Hence, we have for the interpolation operator  $I_h : X \rightarrow X_h$  the estimate

$$|v - I_h v|_{m,K} \leq C h_K^{k+1-m} \|v\|_{k+1,\omega(K)} \quad \forall v \in (H^{k+1}(\Omega))^d \cap X, \quad m = 0, \dots, k + 1,$$

where  $\omega(K)$  is a suitable neighbourhood of  $K$  and  $C$  is independent of  $h$  (see [9]). Moreover,

$$(\operatorname{div} I_h v, q_h) = (\operatorname{div} v, q_h) \quad \forall q_h \in M_h, \quad \forall v \in X \quad (2.5)$$

holds true.

Using the finite element spaces  $X_h$  and  $M_h$ , we can formulate the standard Galerkin discretisation of (2.2) which reads

Find  $(u_h, p_h) \in X_h \times M_h$  such that

$$A((u_h, p_h), (v_h, q_h)) = L((v_h, q_h)) \quad \forall (v_h, q_h) \in X_h \times M_h. \quad (2.6)$$

In the case of locally dominating convection, one may get solutions of (2.6) with spurious oscillations which are in general not localised to regions with dominating convection. In order to stabilise the discrete problem, we introduce a modified bilinear form and a modified linear form by

$$\begin{aligned} A_S((u, p), (v, q)) &:= A((u, p), (v, q)) + \\ &\sum_{K \in \mathcal{T}_h} \gamma_K (\operatorname{div} u, \operatorname{div} v)_K + \sum_{K \in \mathcal{T}_h} (-\nu \Delta u + (b \cdot \nabla) u + \sigma u + \nabla p, \delta_K (b \cdot \nabla) v)_K, \end{aligned}$$

$$L_S((v, q)) := L((v, q)) + \sum_{K \in \mathcal{T}_h} (f, \delta_K(b \cdot \nabla)v)_K$$

where  $\delta_K$  and  $\gamma_K$  are cell-dependent parameters. A detailed study of the choice of these parameters will be given later.

The stabilised discrete problem reads:

Find  $(u_h, p_h) \in X_h \times M_h$  such that

$$A_S((u_h, p_h), (v_h, q_h)) = L_S((v_h, q_h)) \quad \forall (v_h, q_h) \in X_h \times M_h. \quad (2.7)$$

Since the additional terms in  $A_S$  and  $L_S$  vanish in sum for a smooth solution, the stabilised problem is of the residual type. Hence, we have the Galerkin orthogonality

$$A_S((u - u_h, p - p_h), (v_h, q_h)) = 0 \quad \forall (v_h, q_h) \in X_h \times M_h \quad (2.8)$$

where  $(u_h, p_h) \in X_h \times M_h$  is the solution of (2.7) and the solution  $(u, p) \in X \times M$  of (2.2) satisfies additionally the regularity requirement  $u \in (H^2(\Omega))^d$  and  $p \in H^1(\Omega)$ .

**Remark 2.1.** It is possible to consider the fully stabilised discrete problem which includes a PSPG term. In this case, the bilinear form  $A_F$  and the linear form  $L_F$  are defined by

$$A_F((u_h, p_h), (v_h, q_h)) := A_S((u_h, p_h), (v_h, q_h)) + \sum_{K \in \mathcal{T}_h} (L_O(b; u_h, p_h), \alpha_K \nabla q)_K,$$

$$L_F((v_h, q_h)) := L_S((v_h, q_h)) + \sum_{K \in \mathcal{T}_h} (f, \alpha_K \nabla q)_K,$$

where  $\alpha_K$  is a user-chosen parameter. Using similar techniques as below, corresponding error estimates and parameter designs can be derived for the fully stabilised scheme (see [13]). Although the PSPG stabilisation is not needed for inf-sup stable discretisation from the point of stability, the additional term might improve the accuracy of the pressure approximation.

We introduce the norms

$$||[v]||^2 := \nu |v|_1^2 + \sigma \|v\|_0^2 + \sum_{K \in \mathcal{T}_h} \gamma_K \|\operatorname{div} v\|_{0,K}^2 + \sum_{K \in \mathcal{T}_h} \delta_K \|(b \cdot \nabla)v\|_{0,K}^2, \quad |||(v, q)|||^2 := |[v]|^2 + \alpha \|q\|_0^2$$

on  $X$  and  $X \times M$ , respectively. The positive constant  $\alpha$  will be chosen later on in the proof of Lemma 3.1. The lower and upper bounds are given in (3.9). Furthermore, we set

$$b_K := \|b\|_{0,\infty,K}, \quad b_\infty := \|b\|_{0,\infty}.$$

In this paper, the generic constant  $C$  may have different values at different places but it will be always independent of the mesh size  $h$  and the parameter  $\nu$ .

### 3. Analysis of the method

**3.1. Stability and solvability of the discrete problem.** To show that our stabilised discrete problem (2.7) is uniquely solvable and stable, we will prove for the bilinear form  $A_S$  an inf-sup condition on  $X_h \times M_h$  where the constant is independent of the mesh size  $h$  and the parameter  $\nu$ .

It turns out that our stability analysis requires the upper bound of the SUPG-parameter  $\delta_K$  which is basically dictated by an upper bound of the advective Galerkin term. We define

$$\varphi := \sqrt{\nu + \sigma C_F^2} + 2b_\infty \min\left(\frac{1}{\sqrt{\sigma}}, \frac{C_F}{\sqrt{\nu}}\right) + \sqrt{\gamma d} \quad (3.1)$$

where  $C_F$  is the Friedrichs constant for  $\Omega$  and  $\gamma := \max_K \gamma_K$ . We assume that the stabilisation parameters fulfil

$$0 \leq \gamma_K, \quad 0 \leq \delta_K \leq \min\left(\frac{1}{15} \min\left(\frac{1}{\sigma}, \frac{C_F^2}{\nu}\right), \frac{1}{30} \frac{h_K^2 \beta_0^2}{\mu^2 \varphi^2}\right) \quad (3.2)$$

where  $\mu$  is the constant from the inverse inequalities (2.4) and  $\beta_0$  the inf-sup constant for the pair  $(X_h, M_h)$  in (2.3).

**Lemma 3.1.** *Let the stabilisation parameters fulfil (3.2). Then, there exists a positive constant  $\beta_S$  independent of the mesh size  $h$  and the parameter  $\nu$  such that*

$$\inf_{(v_h, q_h)} \sup_{(w_h, r_h)} \frac{A_S((v_h, q_h), (w_h, r_h))}{\|(v_h, q_h)\| \|(w_h, r_h)\|} \geq \beta_S > 0 \quad (3.3)$$

holds true where the infimum and supremum are taken over  $X_h \times M_h$ .

*Proof.* Let  $(v_h, q_h)$  be an arbitrary element of  $X_h \times M_h$ . During the proof, we will use the following abbreviations:

$$X^2 := \sum_{K \in \mathcal{T}_h} \delta_K \|(b \cdot \nabla) v_h\|_{0,K}^2, \quad Z^2 := \sum_{K \in \mathcal{T}_h} \gamma_K \|\operatorname{div} v_h\|_{0,K}^2,$$

$$Y^2 := \sum_{K \in \mathcal{T}_h} \delta_K \|\nu \Delta v_h + \sigma v_h + \nabla q_h\|_{0,K}^2, \quad B^2 := \|q_h\|_0^2, \quad A^2 := \nu \|v_h\|_1^2 + \sigma \|v_h\|_0^2,$$

which give immediately that  $|[v_h]|^2 = A^2 + X^2 + Z^2$ .

The outline of the proof is as follows.

1. We show  $A_S((v_h, q_h), (v_h, q_h)) \geq C_1 |[v_h]|^2 - \bar{\delta} B^2$  with constants  $C_1$  and  $\bar{\delta}$ . The critical constant  $\bar{\delta}$  scales like  $\delta_K/h_K^2$  (see (3.5)).

2. We get from the inf-sup condition (2.3) the existence of a function  $z_h \in X_h$  such that  $A_S((v_h, q_h), (-z_h, 0)) \geq \frac{2}{3} \beta_0 B^2 - C_2 |[v_h]|^2$  with  $C_2$  scaling like  $\varphi^2$ , see (3.7).

3. The function  $(w_h, r_h) := (v_h, q_h) + \lambda(-z_h, 0) \in X_h \times M_h$  with a suitably chosen  $\lambda > 0$  satisfies  $A_S((v_h, q_h), (w_h, r_h)) \geq C_3 \|(v_h, q_h)\|^2$  and  $\|(w_h, r_h)\| \leq C_4 \|(v_h, q_h)\|$  which together result in the assertion of this lemma.

**Step 1.** Using the definition of the bilinear form  $A_S$ , we obtain via the Young inequality and integration by parts (see (2.1))

$$A_S((v_h, q_h), (v_h, q_h)) \geq |[v_h]|^2 - XY \geq |[v_h]|^2 - \frac{3}{4} X^2 - \frac{1}{3} Y^2.$$

The terms will be estimated separately. Exploiting (2.4), (3.2) and  $\nu^2/\varphi^2 \leq 1$ , we get

$$Y^2 \leq 2 \sum_{K \in \mathcal{T}_h} (\delta_K \|\nabla q_h\|_{0,K}^2 + \delta_K \|\nu \Delta v_h + \sigma v_h\|_{0,K}^2) \leq$$

$$\begin{aligned} & \sum_{K \in \mathcal{T}_h} \frac{2\delta_K \mu^2}{h_K^2} \|q_h\|_{0,K}^2 + 4 \left( \sum_{K \in \mathcal{T}_h} \delta_K \frac{\mu^2}{h_K^2} \nu^2 |v_h|_{1,K}^2 + \sum_{K \in \mathcal{T}_h} \delta_K \sigma^2 \|v_h\|_{0,K}^2 \right) \leq \\ & \sum_{K \in \mathcal{T}_h} \frac{2\delta_K \mu^2}{h_K^2} \|q_h\|_{0,K}^2 + 4 \left( \sum_{K \in \mathcal{T}_h} \frac{1}{30} \frac{\beta_0^2}{\varphi^2} \nu^2 |v_h|_{1,K}^2 + \sum_{K \in \mathcal{T}_h} \frac{1}{15} \sigma \|v_h\|_{0,K}^2 \right) \leq 2 \max_{K \in \mathcal{T}_h} \left( \frac{\delta_K \mu^2}{h_K^2} \right) B^2 + \frac{4}{15} A^2 \end{aligned} \quad (3.4)$$

where  $\beta_0 \leq 1$  was applied which is always possible to choose, see (2.3). Hence, we obtain

$$A_S((v_h, q_h), (v_h, q_h)) \geq \frac{1}{4} |[v_h]|^2 - \frac{2}{3} \max_{K \in \mathcal{T}_h} \left( \frac{\delta_K \mu^2}{h_K^2} \right) B^2. \quad (3.5)$$

**Step 2.** Due to the inf-sup condition (2.3) for  $(X_h, M_h)$ , there exists  $z_h \in X_h$  such that

$$|z_h|_1 = \|q_h\|_0 = B, \quad (\operatorname{div} z_h, q_h) \geq \beta_0 |z_h|_1 \|q_h\|_0 = \beta_0 B^2.$$

We have

$$A_S((v_h, q_h), (-z_h, 0)) \geq \beta_0 B^2 - \sum_{i=1}^4 T_i$$

where

$$\begin{aligned} T_1 &:= \nu(\nabla v_h, \nabla z_h) + \sigma(v_h, z_h) - ((b \cdot \nabla) z_h, v_h), \quad T_2 := \sum_{K \in \mathcal{T}_h} \gamma_K (\operatorname{div} v_h, \operatorname{div} z_h)_K, \\ T_3 &:= \sum_{K \in \mathcal{T}_h} \delta_K (-\nu \Delta v_h + \sigma v_h + \nabla q_h, (b \cdot \nabla) z_h)_K, \quad T_4 := \sum_{K \in \mathcal{T}_h} \delta_K ((b \cdot \nabla) v_h, (b \cdot \nabla) z_h)_K. \end{aligned}$$

These four terms will be estimated individually. Applying the Cauchy — Schwarz inequality, we obtain

$$|T_1| \leq (\nu |v_h|_1^2 + \sigma \|v_h\|_0^2)^{1/2} (\nu |z_h|_1^2 + \sigma C_F^2 |z_h|_1^2)^{1/2} + \sum_{K \in \mathcal{T}_h} b_K \|v_h\|_{0,K} |z_h|_{1,K} \leq$$

$$\sqrt{\nu + \sigma C_F^2} AB + \sum_{K \in \mathcal{T}_h} b_K \|v_h\|_{0,K} |z_h|_{1,K}.$$

The last term can be estimated in two ways

$$\sum_{K \in \mathcal{T}_h} b_K \|v_h\|_{0,K} |z_h|_{1,K} \leq \sum_{K \in \mathcal{T}_h} \frac{b_K}{\sqrt{\sigma}} (\sqrt{\sigma} \|v_h\|_{0,K}) |z_h|_{1,K} \leq b_\infty \frac{1}{\sqrt{\sigma}} AB$$

or

$$\sum_{K \in \mathcal{T}_h} b_K \|v_h\|_{0,K} |z_h|_{1,K} \leq \sum_{K \in \mathcal{T}_h} \frac{b_K}{\sqrt{\nu}} (\sqrt{\nu} \|v_h\|_{0,K}) |z_h|_{1,K} \leq b_\infty \frac{C_F}{\sqrt{\nu}} AB.$$

Hence, we get the estimate

$$|T_1| \leq \sqrt{\nu + \sigma C_F^2} AB + b_\infty \min \left( \frac{1}{\sqrt{\sigma}}, \frac{C_F}{\sqrt{\nu}} \right) AB$$

which is governed by the bound of the advective term  $((b \cdot \nabla)z_h, v_h)$ . Furthermore, we have

$$|T_2| \leq \sum_{K \in \mathcal{T}_h} \sqrt{\gamma_K} \|\operatorname{div} v_h\|_{0,K} \sqrt{\gamma_K} \|\operatorname{div} z_h\|_{0,K} \leq Z \sqrt{\gamma d} \left( \sum_{K \in \mathcal{T}_h} |z_h|_{1,K}^2 \right)^{1/2} = \sqrt{\gamma d} Z B$$

and

$$|T_3| \leq \left( \sum_{K \in \mathcal{T}_h} \delta_K \| -\nu \Delta v_h + \sigma v_h + \nabla q_h \|_{0,K}^2 \right)^{1/2} \left( \sum_{K \in \mathcal{T}_h} \delta_K \|(b \cdot \nabla)z_h\|_{0,K}^2 \right)^{1/2} \leq \\ Y \left( \sum_{K \in \mathcal{T}_h} \delta_K b_K^2 |z_h|_{1,K}^2 \right)^{1/2} \leq \left( \max_{K \in \mathcal{T}_h} (b_K \sqrt{\delta_K}) \right) Y B.$$

Using (3.2) and (3.4), we obtain  $Y^2 \leq \frac{2}{30} \frac{\beta_0^2}{\varphi^2} B^2 + \frac{4}{15} A^2$  which gives  $Y \leq \sqrt{\frac{2}{30}} \frac{\beta_0}{\varphi} B + \frac{2}{\sqrt{15}} A$ . Furthermore, we have

$$|T_3| \leq \frac{2}{\sqrt{15}} \left( \max_{K \in \mathcal{T}_h} (b_K \sqrt{\delta_K}) \right) AB + \sqrt{\frac{2}{30}} \frac{\beta_0}{\varphi} \frac{1}{2} \frac{1}{\sqrt{15}} 2b_\infty \min \left( \frac{1}{\sqrt{\sigma}}, \frac{C_F}{\sqrt{\nu}} \right) B^2 \leq \\ \frac{2}{\sqrt{15}} \left( \max_{K \in \mathcal{T}_h} (b_K \sqrt{\delta_K}) \right) AB + \frac{1}{30} \beta_0 B^2.$$

It remains to bound  $T_4$ . We obtain

$$|T_4| \leq \left( \sum_{K \in \mathcal{T}_h} \delta_K \|(b \cdot \nabla)v_h\|_{0,K}^2 \right)^{1/2} \left( \sum_{K \in \mathcal{T}_h} \delta_K \|(b \cdot \nabla)z_h\|_{0,K}^2 \right)^{1/2} \leq \max_{K \in \mathcal{T}_h} (b_K \sqrt{\delta_K}) X B.$$

We proceed with estimating the max-term via the first argument of the min-term in (3.2)

$$\max_{K \in \mathcal{T}_h} (b_K \sqrt{\delta_K}) \leq \frac{1}{\sqrt{15}} b_\infty \min \left( \frac{1}{\sqrt{\sigma}}, \frac{C_F}{\sqrt{\nu}} \right). \quad (3.6)$$

Note that, due to the upper bound of  $|T_1|$ , no gain is obtained if the second argument of the min-term in (3.2) is used. Using (3.6) and the estimates for  $T_1, \dots, T_4$ , we end up with

$$\sum_{i=1}^4 T_i \leq \sum_{i=1}^4 |T_i| \leq \left( \sqrt{\nu + \sigma C_F^2} + \frac{17}{15} b_\infty \min \left( \frac{1}{\sqrt{\sigma}}, \frac{C_F}{\sqrt{\nu}} \right) \right) AB + \\ \frac{1}{\sqrt{15}} b_\infty \min \left( \frac{1}{\sqrt{\sigma}}, \frac{C_F}{\sqrt{\nu}} \right) X B + \sqrt{\gamma d} Z B + \frac{1}{30} \beta_0 B^2 \leq (A + X + Z) \varphi B + \frac{1}{30} \beta_0 B^2.$$

To summarise, we have

$$A_S((v_h, q_h), (-z_h, 0)) \geq \beta_0 B^2 - (A + X + Z) \varphi B - \frac{1}{30} \beta_0 B^2 \geq \\ \frac{29}{30} \beta_0 B^2 - \frac{3 \cdot 1}{10} \beta_0 B^2 - \frac{5 \varphi^2}{2 \beta_0} (A^2 + X^2 + Z^2) = \frac{2}{3} \beta_0 B^2 - \frac{5 \varphi^2}{2 \beta_0} |[v_h]|^2. \quad (3.7)$$



**Step 3.** We define  $(w_h, r_h) := (v_h, q_h) + \lambda(-z_h, 0)$  with  $\lambda > 0$ . Using estimates (3.5) and (3.7), we obtain

$$A_S((v_h, q_h), (w_h, r_h)) \geq \left(\frac{1}{4} - \frac{5\lambda\varphi^2}{2\beta_0}\right) \|v_h\|^2 + \left(\frac{2\lambda\beta_0}{3\alpha} - \frac{2}{3} \max_{K \in \mathcal{T}_h} \left(\frac{\delta_K \mu^2}{\alpha h_K^2}\right)\right) \alpha B^2.$$

By choosing  $\lambda$  and  $\alpha$  such that

$$\frac{1}{4} - \frac{5\lambda\varphi^2}{2\beta_0} = \frac{1}{30} \quad \text{and} \quad \frac{2\lambda\beta_0}{3\alpha} - \frac{2}{3} \max_{K \in \mathcal{T}_h} \left(\frac{\delta_K \mu^2}{\alpha h_K^2}\right) = \frac{1}{30}, \quad (3.8)$$

we derive

$$\lambda\beta_0 = \frac{13}{150} \frac{\beta_0^2}{\varphi^2} \quad \text{and} \quad \alpha = \frac{26}{15} \frac{\beta_0^2}{\varphi^2} - 20 \max_{K \in \mathcal{T}_h} \left(\frac{\delta_K \mu^2}{h_K^2}\right).$$

We can bound  $\alpha$  from below and above via (3.2) as follows:

$$\frac{16}{15} \frac{\beta_0^2}{\varphi^2} \leq \alpha \leq \frac{26}{15} \frac{\beta_0^2}{\varphi^2}. \quad (3.9)$$

Our choice of  $\lambda$  and  $\alpha$  results in

$$A_S((v_h, q_h), (w_h, r_h)) \geq \frac{1}{30} \|(v_h, q_h)\|^2.$$

Finally, we will show that  $\|(w_h, r_h)\| \leq C \|(v_h, q_h)\|$ . To this end, we start with

$$\|(w_h, r_h)\| \leq \|(v_h, q_h)\| + \lambda \|(-z_h, 0)\|,$$

and see that it suffices to estimate  $\|(-z_h, 0)\|$ . We have

$$\begin{aligned} \|(-z_h, 0)\|^2 &= \nu |z_h|_1^2 + \sigma \|z_h\|_0^2 + \sum_{K \in \mathcal{T}_h} \gamma_K \|\operatorname{div} z_h\|_{0,K}^2 + \sum_{K \in \mathcal{T}_h} \delta_K \|(b \cdot \nabla) z_h\|_{0,K}^2 \leq \\ &\sum_{K \in \mathcal{T}_h} (\nu + \sigma C_F^2 + \gamma d + b_K^2 \delta_K) |z_h|_{1,K}^2 \leq \varphi^2 B^2 \leq \frac{\varphi^2}{\alpha} \|(v_h, q_h)\|^2 \end{aligned}$$

where we have used (3.6) to bound  $b_K^2 \delta_K$ . Using the above estimate, we obtain

$$\|(w_h, r_h)\| \leq \left(1 + \frac{\lambda\varphi}{\sqrt{\alpha}}\right) \|(v_h, q_h)\|.$$

Exploiting the choice of  $\lambda$  in (3.8) and the lower bound of  $\alpha$  in (3.9), we have

$$Q := 1 + \frac{\lambda\varphi}{\sqrt{\alpha}} \leq 1 + \frac{13}{150} \frac{\beta_0}{\varphi^2} \varphi \sqrt{\frac{15\varphi^2}{16\beta_0^2}} = 1 + \frac{13}{150} \sqrt{\frac{15}{16}}$$

which results in

$$A_S((v_h, q_h), (w_h, r_h)) \geq \frac{1}{30Q} \|(v_h, q_h)\| \|(w_h, r_h)\|.$$

Hence, the inf-sup constant  $\beta_S := 1/(30Q)$  is independent of  $\nu$  and  $h$ .  $\square$

**3.2. Is there a conditional stability problem for the SUPG-parameter?** The condition on the SUPG-parameter according to (3.1), (3.2) in the stability analysis above gives a strong upper bound. In particular, one obtains  $\delta_K \rightarrow 0$  in the critical case  $\nu^2 + \sigma^2 \rightarrow 0$ . This leads to the question whether such conditional stability problem of the method with respect to the SUPG-stabilisation really exists. The following example of a vortex flow exemplarily shows that this is indeed the case, although the results do not show the very strict upper bound for  $\nu^2 + \sigma^2 \rightarrow 0$ . (For further examples, see Section 4.)

**Example 3.1.** The flow and pressure fields

$$u(x) = (\sin(2\pi x_1) \cos(2\pi x_2), -\cos(2\pi x_1) \sin(2\pi x_2))^\top, \quad p(x) := \frac{1}{4}(\cos(4\pi x_1) + \cos(4\pi x_2))$$

solve the Oseen problem (1.2) in  $\Omega = (0, 1)^2$  with  $b(x) := u(x)$ ,  $\sigma = 0$ ,  $f(x) := 8\pi^2 \nu b(x)$ , and with inhomogeneous Dirichlet data  $u(x) = b(x)$  on  $\partial\Omega$ . The field  $u$  (extended to  $\mathbb{R}^2$ ) has stagnation points of the saddle point type, e.g., around  $(\frac{1}{2}, \frac{1}{2})$ . We emphasize that this flow was considered in [14] (Example 2.3) as a solution of the stationary Euler problem. It is mentioned there that "this flow is dynamically instable so that small perturbations result in very chaotic motion".

For the numerical simulations, the inf-sup stable  $Q_2/Q_1$  elements pair is applied on a sequence of unstructured quasi-uniform, quadrilateral meshes with  $h \in \{\frac{1}{12}, \frac{1}{32}, \frac{1}{64}, \frac{1}{122}\}$  for the small viscosity  $\nu = 10^{-6}$  and  $\sigma = 0$ .

First, we consider the stabilised problem (2.7) with the SUPG parameter  $\delta_K = \delta_0 h_K^2$  and without grad-div stabilisation, i.e., with  $\gamma_K = 0$ . In Fig. 3.1 we plot the  $H^1$ -seminorm and  $L^2$ -norm of the discrete solution vs. the scaling parameter  $\delta_0$  on a relatively small range of  $\delta_0$  (sampled at 257 points, equidistantly distributed on the logarithmic scale). One clearly observes the arising instability of the discrete solution for certain values of  $\delta_0 \geq 20$ .

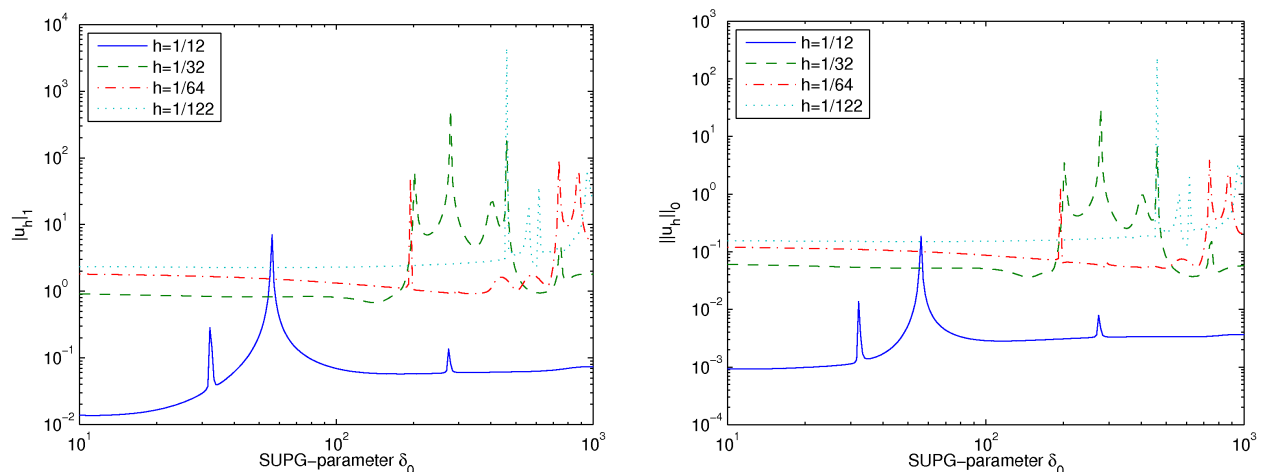


Fig. 3.1.  $H^1$ -seminorm and  $L^2$ -norm vs. scaling SUPG-parameter  $\delta_0$  (without grad-div stabilisation) for Example 3.1 with  $\nu = 10^{-6}$ ,  $\sigma = 0$  and different values of  $h$

For comparison, we plot in Fig. 3.2 the corresponding results for the SUPG/PSPG scheme with a common parameter  $\alpha_K = \delta_K = \delta_0 h_K^2$  (see Remark 2.1) on a much larger range of the scaling parameter  $\delta_0$ . The results are sampled only at 49 points (equidistantly distributed on the logarithmic scale) as the discrete problem is coercive on a much larger range of the SUPG/PSPG-parameter with  $\delta_K \lesssim \min\{h_K^2/\nu, 1/\sigma\}$  (see [13]). Indeed, we observe a robust behaviour of the  $H^1$ -seminorm and  $L^2$ -norm for a large range of  $\delta_0$ . The strongly decreasing values of the norms for sufficiently large  $\delta_0$  are due to the excessive numerical diffusion.

Moreover, we present in Fig. 3.3 the corresponding plots for the solution of the stabilised scheme (2.7) with grad-div stabilisation, i.e., with  $\gamma_K \equiv \gamma_0$ , and without SUPG, i.e., with  $\delta_0 = 0$ . We observe that the  $H^1$ -seminorm and  $L^2$ -norm of the solution are uniformly bounded for a wide range of the parameter  $\gamma_0$  (sampled only at 49 points, again equidistantly distributed on the logarithmic scale, as the discrete problem is coercive for arbitrary  $\gamma_K \geq 0$  and again excessive numerical diffusion for large values of  $\gamma_0$ ).

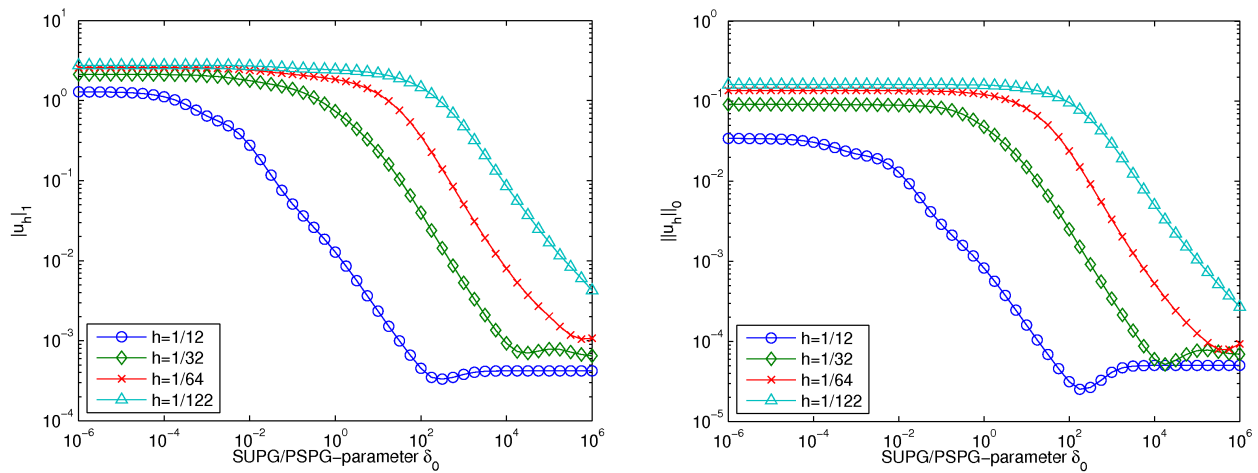


Fig. 3.2.  $H^1$ -seminorm and  $L^2$ -norm vs. scaling SUPG/PSPG-parameter  $\delta_0$  (without grad-div stabilisation) for Example 3.1 with  $\nu = 10^{-6}$ ,  $\sigma = 0$  and different values of  $h$

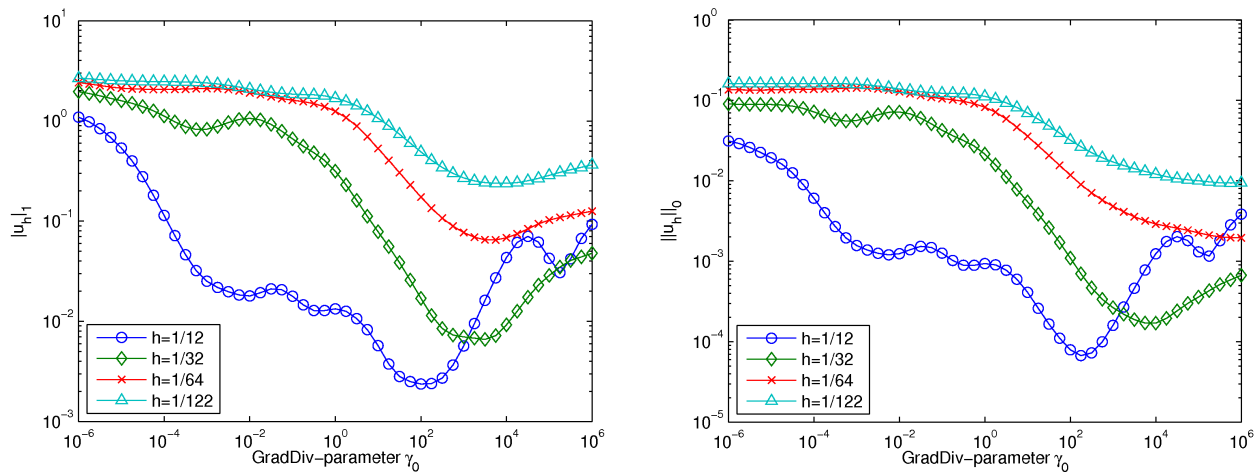


Fig. 3.3.  $H^1$ -seminorm and  $L^2$ -norm vs. scaling parameter  $\gamma_0$  of grad-div stabilisation (without SUPG) for Example 3.1 with  $\nu = 10^{-6}$ ,  $\sigma = 0$  and different values of  $h$

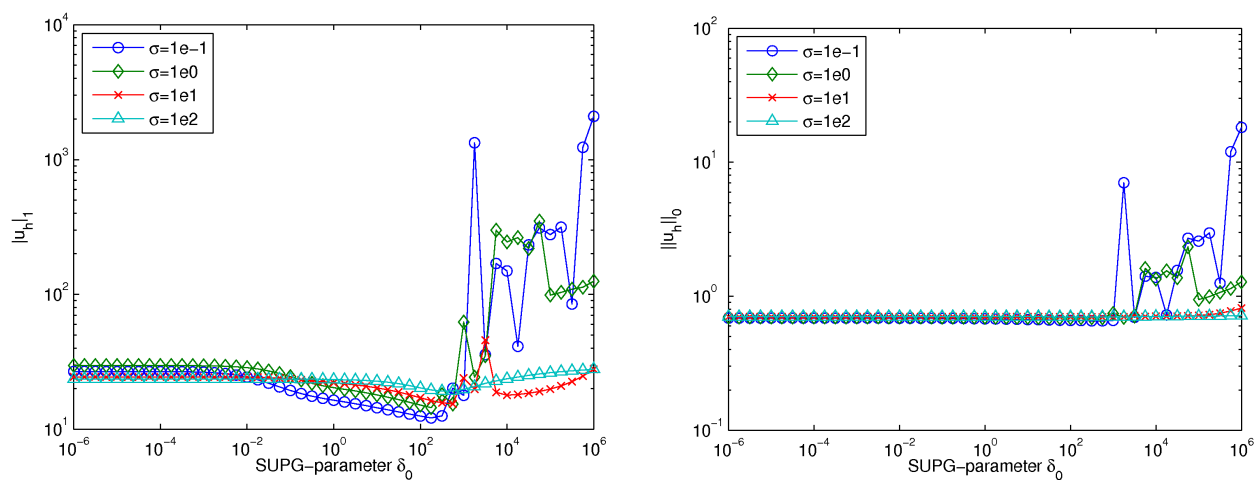


Fig. 3.4.  $H^1$ -seminorm and  $L^2$ -norm vs. scaling SUPG-parameter  $\delta_0$  (without grad-div stabilisation) for Example 3.1 with  $\nu = 10^{-6}$ ,  $h = 1/64$  and different values of  $\sigma$

Finally, we consider the influence of the parameter  $\sigma$  on the stability of the discrete problem with SUPG-stabilisation (without grad-div stabilisation). Therefore, we study Example 3.1 for  $\nu = 10^{-6}$ ,  $h = \frac{1}{64}$  and different values of  $\sigma = 10^i, i \in \{-1, 0, 1, 2\}$ . This, together with the source term  $\tilde{f}(x) := f(x) + \sigma b(x)$ , mimics the behaviour of a simple implicit time discretisation. In Fig. 3.4, we observe that the  $H^1$ -seminorm and  $L^2$ -norm of the velocity are robust in an increasing range of the scaling parameter  $\delta_0$  for increasing values of  $\sigma$ . This is in agreement with the increasing upper bound of  $\delta_K$  in (3.2) for increasing values of  $\sigma$ . Nevertheless, again numerical instabilities are obtained for larger values of  $\delta_0$ . In fact, from (3.1), (3.2) we observe decreasing upper bound of  $\delta_K$  if  $\sigma C_F^2$  is too large.  $\square$

**3.3. Preliminary a priori error estimate.** First, we will state and prove a continuity estimate for the bilinear form  $A_S$ .

**Lemma 3.2.** *Let  $u \in (H^{k+1}(\Omega))^d \cap X$  and  $p \in H^{\ell+1}(\Omega) \cap M$ . Moreover,  $I_h u$  is the interpolant of  $u$  which preserves the discrete divergence (see (2.5)) while  $J_h p$  is the standard finite element interpolant of  $p$ . Then, for all  $(w_h, r_h) \in X_h \times M_h$ , the following estimate holds true:*

$$\frac{A_S((u - I_h u, p - J_h p), (w_h, r_h))}{\|(w_h, r_h)\|} \leq C \left( \sum_{K \in \mathcal{T}_h} \left[ \nu + \sigma h_K^2 + \gamma_K d + \delta_K b_K^2 + \frac{3 b_K^2 h_K^2}{\delta_K b_K^2 + \nu + \sigma h_K^2} \right] \times \right. \\ \left. h_K^{2k} \|u\|_{k+1, \omega(K)}^2 + \sum_{K \in \mathcal{T}_h} \left[ \delta_K + \frac{2 d h_K^2}{\nu + \gamma_K d} \right] h_K^{2\ell} \|p\|_{\ell+1, K}^2 \right)^{1/2}.$$

*Proof.* Let  $w := u - I_h u$  and  $r := p - J_h p$ . As the following estimate of  $A_S((w, r), (w_h, r_h))$  is straightforward, we only emphasize some important aspects. By separation of symmetric and nonsymmetric terms and using the definitions of  $\|[w]\|$  and  $\|(w_h, r_h)\|$ , we obtain

$$A_S((w, r), (w_h, r_h)) \leq \|[w]\| \|(w_h, r_h)\| + \left| \sum_{K \in \mathcal{T}_h} \delta_K (-\nu \Delta w + \sigma w + \nabla r, (b \cdot \nabla) w_h)_K \right| + \\ |(r_h, \operatorname{div} w)| + |(r, \operatorname{div} w_h)| + |((b \cdot \nabla) w, w_h)|.$$

The estimate for the interpolation is

$$\|[w]\| \leq C \left( \sum_{K \in \mathcal{T}_h} \left[ (\nu + \sigma h_K^2 + \delta_K b_K^2 + \gamma_K d) h_K^{2k} \|u\|_{k+1, \omega(K)}^2 \right] \right)^{1/2}.$$

Now, the remaining terms are estimated separately. We obtain

$$\left| \sum_{K \in \mathcal{T}_h} \delta_K (-\nu \Delta w + \sigma w + \nabla r, (b \cdot \nabla) w_h)_K \right| \leq \\ C \left( \sum_{K \in \mathcal{T}_h} [(\nu + \sigma h_K^2) h_K^{2k} \|u\|_{k+1, \omega(K)}^2 + \delta_K h_K^{2\ell} \|p\|_{\ell+1, K}^2] \right)^{1/2} \|(w_h, r_h)\|$$

where we have used the fact that  $\nu \delta_K \leq C h_K^2$  and  $\delta_K \sigma \leq C$  by (3.1), (3.2). Since the interpolation operator  $I_h$  preserves the discrete divergence (see (2.5)) we have  $(r_h, \operatorname{div} w) = 0$ . Note that this term is in general nonzero for standard interpolation operators. An estimate

would involve a negative power of  $\alpha$  causing additional difficulties. Please note that also the Ritz projection of the Stokes problem would not be sufficient.

The term  $|(r, \operatorname{div} w_h)|$  can be handled in two ways

$$|(r, \operatorname{div} w_h)| \leq \sum_{K \in \mathcal{T}_h} \gamma_K^{-1/2} \|r\|_{0,K} \sqrt{\gamma_K} \|\operatorname{div} w_h\|_{0,K} \leq C \left( \sum_{K \in \mathcal{T}_h} \gamma_K^{-1} h_K^{2\ell+2} \|p\|_{\ell+1,K}^2 \right)^{1/2} \|(w_h, r_h)\|$$

or

$$\begin{aligned} |(r, \operatorname{div} w_h)| &\leq \left( \sum_{K \in \mathcal{T}_h} d\nu^{-1} \|r\|_{0,K}^2 \right)^{1/2} \left( \sum_{K \in \mathcal{T}_h} \nu |w_h|_{1,K}^2 \right)^{1/2} \leq \\ &C \left( \sum_{K \in \mathcal{T}_h} d\nu^{-1} h_K^{2\ell+2} \|p\|_{\ell+1,K}^2 \right)^{1/2} \|(w_h, r_h)\|. \end{aligned}$$

This gives

$$|(r, \operatorname{div} w_h)| \leq C \left( \sum_{K \in \mathcal{T}_h} \frac{2d}{\nu + \gamma_K d} h_K^{2\ell+2} \|p\|_{\ell+1,K}^2 \right)^{1/2} \|(w_h, r_h)\|.$$

There are several ways for estimating the remaining term

$$\begin{aligned} |((b \cdot \nabla)w, w_h)| &\leq \sum_{K \in \mathcal{T}_h} b_K |w|_{1,K} \|w_h\|_{0,K} \leq \\ &\left( \sum_K \frac{b_K^2}{\sigma} |w|_{1,K}^2 \right)^{1/2} \left( \sum_{K \in \mathcal{T}_h} \sigma \|w_h\|_{0,K}^2 \right)^{1/2} \leq C \left( \sum_K \frac{b_K^2}{\sigma} h_K^{2k} \|u\|_{k+1,\omega(K)}^2 \right)^{1/2} \|(w_h, r_h)\| \end{aligned}$$

or using integration by parts

$$\begin{aligned} |((b \cdot \nabla)w, w_h)| &= |((b \cdot \nabla)w_h, w)| \leq \sum_{K \in \mathcal{T}_h} b_K |w_h|_{1,K} \|w\|_{0,K} \leq \\ &\left( \sum_K \frac{b_K^2}{\nu} \|w\|_{0,K}^2 \right)^{1/2} \left( \sum_{K \in \mathcal{T}_h} \nu |w_h|_{1,K}^2 \right)^{1/2} \leq C \left( \sum_K \frac{b_K^2}{\nu} h_K^{2k+2} \|u\|_{k+1,\omega(K)}^2 \right)^{1/2} \|(w_h, r_h)\| \end{aligned}$$

or

$$\begin{aligned} |((b \cdot \nabla)w, w_h)| &= |((b \cdot \nabla)w_h, w)| \leq \left( \sum_{K \in \mathcal{T}_h} \delta_K^{-1} \|w\|_{0,K}^2 \right)^{1/2} \left( \sum_{K \in \mathcal{T}_h} \delta_K \|(b \cdot \nabla)w_h\|_{0,K}^2 \right)^{1/2} \leq \\ &C \left( \sum_{K \in \mathcal{T}_h} \delta_K^{-1} h_K^{2k+2} \|u\|_{k+1,\omega(K)}^2 \right)^{1/2} \|(w_h, r_h)\|. \end{aligned}$$

These three estimates give together

$$|((b \cdot \nabla)w, w_h)| \leq C \left( \sum_{K \in \mathcal{T}_h} \frac{3b_K^2 h_K^2}{\delta_K b_K^2 + \nu + \sigma h_K^2} h_K^{2k} \|u\|_{k+1,\omega(K)}^2 \right)^{1/2} \|(w_h, r_h)\|.$$

The combination of all the above estimates gives the assertion of the Lemma.  $\square$

We are now in a position to derive a preliminary a priori error estimate using the previous stability and continuity estimates.

**Theorem 3.1.** Let  $(u, p) \in (X \cap H^{k+1}(\Omega)^d) \times ((M \cap H^{\ell+1}(\Omega)))$  and  $(u_h, p_h) \in X_h \times M_h$  be the solutions of (2.2) and (2.7), respectively. Moreover, we assume that conditions (3.1), (3.2) are valid. Then, the following estimate holds true:

$$\begin{aligned} \|(u - u_h, p - p_h)\|^2 \leq & C \sum_{K \in \mathcal{T}_h} \left( \frac{dh_K^{2(\ell+1)}}{\nu + \gamma_K d} \|p\|_{\ell+1, K}^2 + \right. \\ & \left. \left[ \nu + \sigma h_K^2 + \delta_K b_K^2 + \gamma_K d + \frac{b_K^2 h_K^2}{\delta_K b_K^2 + \nu + \sigma h_K^2} \right] h_K^{2k} \|u\|_{k+1, \omega(K)}^2 \right). \end{aligned}$$

*Proof.* Using the triangle inequality, we obtain

$$\|(u - u_h, p - p_h)\| \leq \|(u - I_h u, p - J_h p)\| + \|(I_h u - u_h, J_h p - p_h)\|$$

where  $J_h p$  is the standard finite element interpolant of  $p$  and  $I_h u$  the interpolant of  $u$  which additionally preserves the discrete divergence (see (2.5)). The inf-sup condition for  $A_S$  given by Lemma 3.1 ensures the existence of  $(w_h, r_h) \in X_h \times M_h$  such that

$$\beta_S \|(I_h u - u_h, J_h p - p_h)\| \leq \frac{A_S((I_h u - u_h, J_h p - p_h), (w_h, r_h))}{\|(w_h, r_h)\|} = \frac{A_S((I_h u - u, J_h p - p), (w_h, r_h))}{\|(w_h, r_h)\|}$$

where we also used the Galerkin orthogonality (2.8). The application of Lemma 3.2 yields

$$\begin{aligned} \beta_S \|(I_h u - u_h, J_h p - p_h)\| \leq & C \left( \sum_{K \in \mathcal{T}_h} \left[ \nu + \sigma h_K^2 + \delta_K b_K^2 + \gamma_K d + \frac{3 b_K^2 h_K^2}{\delta_K b_K^2 + \nu + \sigma h_K^2} \right] \times \right. \\ & \left. h_K^{2k} \|u\|_{k+1, \omega(K)}^2 + \left[ \delta_K + \frac{2 d h_K^2}{\nu + \gamma_K d} \right] h_K^{2\ell} \|p\|_{\ell+1, K}^2 \right)^{1/2}. \end{aligned}$$

We use (3.1), (3.2) for the estimates

$$\frac{h_K^2}{\delta_K} \geq C \varphi^2 \geq C(\nu + \gamma_K d), \quad \delta_K \leq C \frac{d h_K^2}{\varphi^2} \leq C \frac{d h_K^2}{\nu + \gamma_K d}.$$

This allows a simplification of the  $[\cdot]$ -factors of the previous estimate.

The interpolation error estimates for  $I_h$  and  $J_h$  give

$$\|(u - I_h u, p - J_h p)\|^2 \leq C \sum_{K \in \mathcal{T}_h} \left[ (\nu + \sigma h_K^2 + \gamma_K d + \delta_K b_K^2) h_K^{2k} \|u\|_{k+1, \omega(K)}^2 + \alpha h_K^2 h_K^{2\ell} \|p\|_{\ell+1, K}^2 \right].$$

We can simplify the right-hand side by using (3.1)–(3.2), (3.9) and

$$\delta_K \leq C \frac{d h_K^2}{\nu + \gamma_K d}, \quad \alpha h_K^2 \leq C \frac{d h_K^2}{\varphi^2} \leq C \frac{d h_K^2}{\nu + \gamma_K d}.$$

Putting together all estimates and applying the triangle inequality from the beginning of this proof gives the assertion.  $\square$

**Remark 3.1.** Theorem 3.1 clarifies and generalises several aspects of the result of Theorem 4.1 in [7]. The new result relaxes the assumption of quasi-uniformity of the mesh to shape-regularity and the assumption of continuous pressure approximation to a (potentially) discontinuous ansatz. Finally, the  $H^2$ -regularity result for the Stokes problem which is used in [7] can be avoided (as a technical tool).

#### 4. Discussion of the parameter action. Numerical experiments

Here we will apply Theorem 3.1 in order to discuss the action of the stabilisation parameters  $\delta_K$  and  $\gamma_K$ . For simplicity, we discuss cases with  $k = \ell + 1$ , including conforming  $P_k$ - or  $Q_k$ -interpolation of velocity and continuous or discontinuous  $P_\ell$ - or  $Q_\ell$ -interpolation of pressure.

First of all, the error estimate is optimal with respect to the mesh size  $h_K$  for fixed data. Nevertheless, the error estimate may deteriorate in the critical case  $\nu^2 + \sigma^2 \rightarrow 0$ . We observe from (3.1) that positive  $\gamma_K$  and  $\delta_K$ , respectively, would prevent a degeneration of the  $[\cdot]$ -factor of the  $p$ -dependent term if  $\nu \rightarrow +0$  and of the  $[\cdot]$ -factor of the  $u$ -dependent term if  $\nu^2 + \sigma^2 \rightarrow 0$ , respectively. The standard approach to design stabilisation parameters is the equilibration of corresponding terms on the right-hand side of the error estimate (3.1).

(i) The critical terms in the  $[\cdot]$ -factor of the  $u$ -dependent term are

$$\delta_K b_K^2 + \nu + \sigma h_K^2 \quad \text{and} \quad \frac{b_K^2 h_K^2}{\delta_K b_K^2 + \nu + \sigma h_K^2}. \quad (4.1)$$

The first observation is that eventually no SUPG-stabilisation is required if the latter term remains of order  $\mathcal{O}(1)$  for  $\nu^2 + \sigma^2 \rightarrow 0$ . This occurs if

$$\nu \geq C b_K^2 h_K^2, \quad \text{i.e.,} \quad Re_K := \frac{b_K h_K}{\nu} \leq \frac{1}{C \sqrt{\nu}} \quad (4.2)$$

or

$$\sigma h_K^2 \geq C b_K^2 h_K^2, \quad \text{i.e.,} \quad \sigma \geq C b_K^2. \quad (4.3)$$

On the other hand, if  $\nu + \sigma h_K^2 \leq C b_K^2 h_K^2$ , a choice with  $\delta_K \geq C h_K^2$  would be desirable. Unfortunately, this is not possible under the stability condition (3.2). Example 3.1 shows that an upper bound on the parameter  $\delta_K$  may exist in the general case.

(ii) The critical terms in the  $[\cdot]$ -factors of the  $u$ - and  $p$ -dependent terms in (3.1) with  $k = \ell + 1$  are  $(\nu + \gamma_K d) \|u\|_{k+1, \omega(K)}^2$  and  $(\nu + \gamma_K d)^{-1} \|p\|_{\ell+1, K}^2$ . A formal equilibration yields

$$\|p\|_{\ell+1, K} \sim \gamma_K d \|u\|_{k+1, \omega(K)} \quad (4.4)$$

and thus the behavior of the parameter  $\gamma_K$  depending on the local norms of the solution. Unfortunately, these quantities are not available. As the Oseen problem is only an auxiliary problem within a simulation of time-dependent flows, one could think of a "dynamic" recovery of  $\|p\|_{\ell+1, K}$  and  $\|u\|_{k+1, \omega(K)}$  from a previous time step or iteration. We will report on such approach elsewhere.

Let us emphasize that the dominating approach in the literature is to assume that the Sobolev norms of  $p$  and  $u$  in (4.4) have the same size with respect to  $\nu \rightarrow 0$ . This is indeed not valid in general. For simplicity, let us consider the stationary Navier — Stokes equation

$$-\nu \Delta u + (u \cdot \nabla) u + \nabla p = 0$$

where we assume that the source term  $f$  is a gradient field and can be hidden in  $\nabla p$ . Let us look at two extremal situations:

- Consider a situation with  $\|(u \cdot \nabla) u\|_{L^2(D)} \gg \nu \|\Delta u\|_{L^2(D)}$ . This occurs typically in vortices of the flow away from boundary or interior layers. Then the choice of a positive value of  $\gamma_K$  for  $K \subset D$  seems to be appropriate. This will be discussed in Examples 3.1 and 4.2 below.

- In a simple shear flow with  $(u \cdot \nabla)u = 0$  (like the Poiseuille flow in a straight channel), we obtain  $\nabla p = \nu \Delta u$ , i.e.  $\|\nabla p\|_{L^2(D)} \ll \|\Delta u\|_{L^2(D)}$  for arbitrary  $D \subset \Omega$ . As a conclusion from (4.4), the choice  $\gamma_K = 0$  is reasonable. This will be discussed in Example 4.1 below.

We continue the discussion with some numerical experiments to check some aspects of the a priori analysis. To be as close to the Navier — Stokes model as possible, the solution  $u$  of the Oseen problem (1.2) is chosen as the convective field  $b$ . Unfortunately, it is not possible to discuss the dependence of the scheme with respect to all parameters and data in this paper. In particular, we restrict ourselves to the simplest Taylor — Hood pair  $Q_2/Q_1$ . The stabilisation parameters are chosen, for simplicity, as  $\delta_K = \delta_0 h_K^2$  and  $\gamma_K = \gamma_0$  with scaling parameters  $\delta_0, \gamma_0 \geq 0$ . We will always consider a sequence of unstructured, quasi-uniform, quadrilateral meshes with  $h \in \{\frac{1}{12}, \frac{1}{32}, \frac{1}{64}, \frac{1}{122}\}$  and sampling points for  $\delta_0$  and  $\gamma_0$  as in Subsection 3.2.

First, we look again at Example 3.1 with a smooth and  $\nu$ -independent solution.

We solve the Oseen problem (1.2) on  $\Omega = (0, 1)^2$  for the small viscosity  $\nu = 10^{-6}$  and  $\sigma = 0$ , with the flow field  $b(x) = (\sin(2\pi x_1) \cos(2\pi x_2), -\cos(2\pi x_1) \sin(2\pi x_2))^T$ , source term  $f(x) := 8\pi^2 \nu b(x)$ , and with inhomogeneous Dirichlet data  $u(x) = b(x)$  on  $\partial\Omega$ . The exact solution is  $u(x) := b(x)$  and  $p(x) := \frac{1}{4}(\cos(4\pi x_1) + \cos(4\pi x_2))$ .

First, we consider the stabilised problem (2.7) with the SUPG-parameter  $\delta_K = \delta_0 h_K^2$  and without grad-div stabilisation, i.e., with  $\gamma_K = 0$ . In Fig. 4.1, we plot the errors in the  $H^1$ -seminorm and  $L^2$ -norm vs. the scaling parameter  $\delta_0$  in a relatively small range of  $\delta_0$ . One clearly observes again the arising instability of the discrete solution for certain values of  $\delta_0 \geq 20$ .

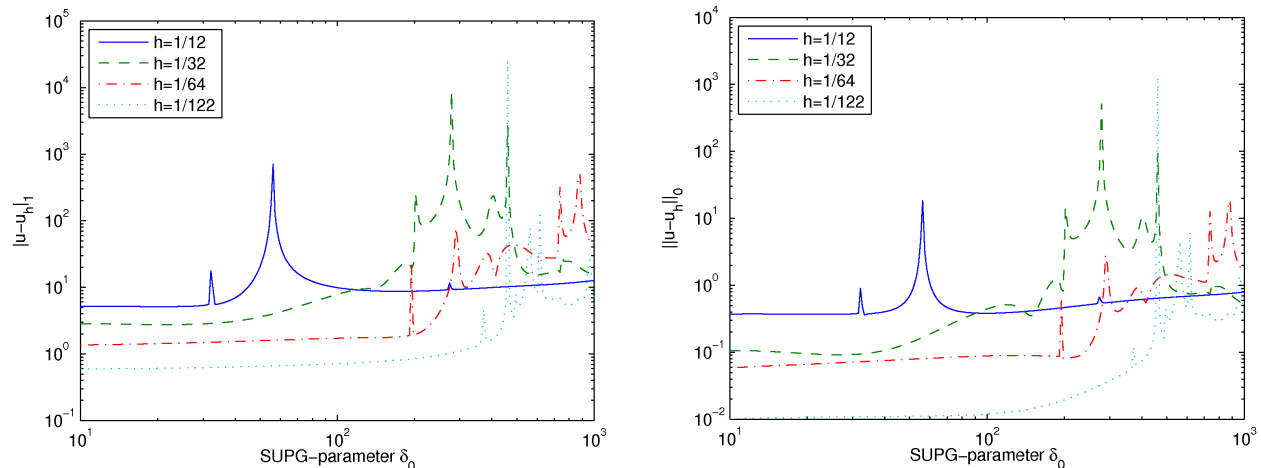


Fig. 4.1. Errors in the  $H^1$ -seminorm and  $L^2$ -norm vs. the SUPG-scaling parameter  $\delta_0$  (without grad-div stabilisation) for Example 3.1 with  $\nu = 10^{-6}$ ,  $\sigma = 0$  and different values of  $h$

For comparison, we plot in Fig. 4.2 the corresponding results for the SUPG/PSPG scheme with  $\alpha_K = \delta_K = \delta_0 h_K^2$  in a much larger range of  $\delta_0$ . We observe a minimum of the error in the  $H^1$ -seminorm for  $\delta_0 \approx 5$ . Moreover, we present in Fig. 4.3 the corresponding plots for the solution of the stabilised scheme (2.7) with grad-div stabilisation, i.e., with  $\gamma_K \equiv \gamma_0$  and without SUPG ( $\delta_0 = 0$ ). We observe a distinguished minimum of the errors in the  $H^1$ -seminorm and  $L^2$ -norm for parameter  $\gamma_0 \approx 10^{-1}$  which leads (as compared to the unstabilised case) to improved values of the norms by a factor of nearly  $10^{-2}$  on the finest grid. Notably, compared to the SUPG/PSPG case, the errors are better by a factor  $10^{-2} \dots 10^{-1}$  on the finest grids.



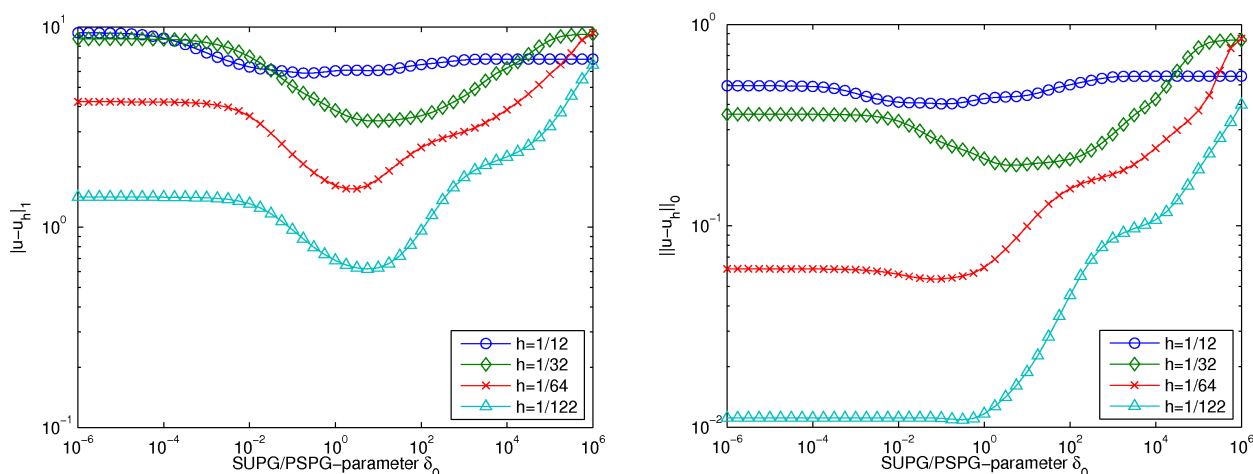


Fig. 4.2. Errors in the  $H^1$ -seminorm and  $L^2$ -norm vs. the SUPG/PSPG-scaling parameter  $\delta_0$  (without grad-div stabilisation) for Example 3.1 with  $\nu = 10^{-6}$ ,  $\sigma = 0$  and different values of  $h$

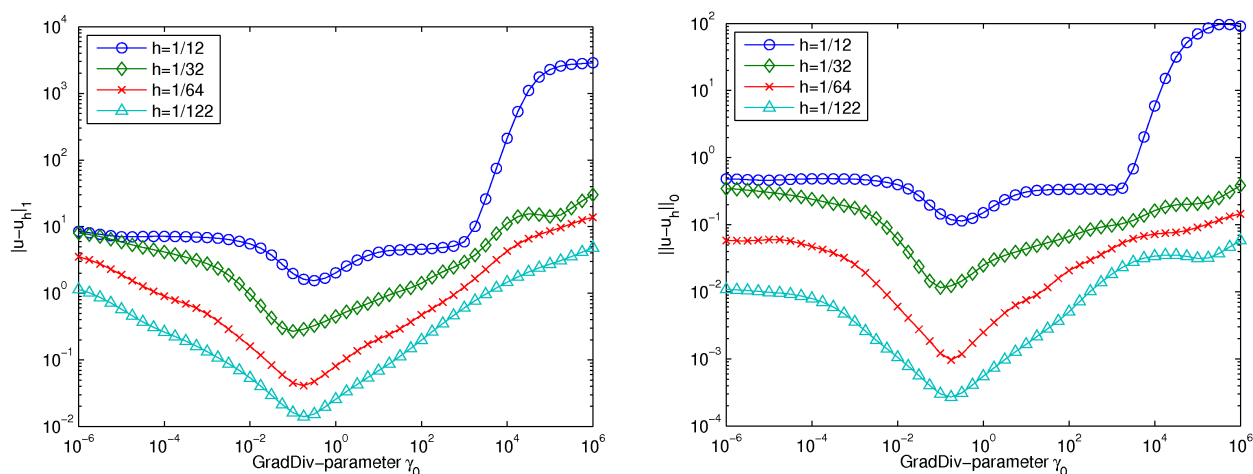


Fig. 4.3. Plots of  $H^1$ - and  $L^2$ -errors vs. the scaling parameter  $\gamma_0$  of grad-div stabilisation (without SUPG) for Example 3.1 with  $\nu = 10^{-6}$ ,  $\sigma = 0$  and different values of  $h$

For this example there holds  $(u \cdot \nabla)u = \pi(\sin(4\pi x_1), -\sin(4\pi x_2))^T$ . According to the discussion in (i) and (ii) above, a SUPG-stabilisation is questionable whereas an "optimised" value of the grad-div stabilisation leads to even (much) better results as for the standard SUPG/PSPG stabilisation. Nevertheless, we have to admit that the simplified parameter design  $\delta_K = \delta_0 h_K^2$ ,  $\gamma_K = \gamma_0$  may be not optimal.

Now, we consider the influence of the parameter  $\sigma$  on the error of the discrete problem with SUPG-stabilisation (without grad-div stabilisation). From Fig. 4.4 we draw mainly the same conclusions for the errors in the  $H^1$ -seminorm and  $L^2$ -norm of the velocity as for the corresponding norms in Subsection 3.2: both errors are robust in an increasing range of the scaling parameter  $\delta_0$  for increasing values of  $\sigma$  and again numerical instabilities for larger values of  $\delta_0$ .

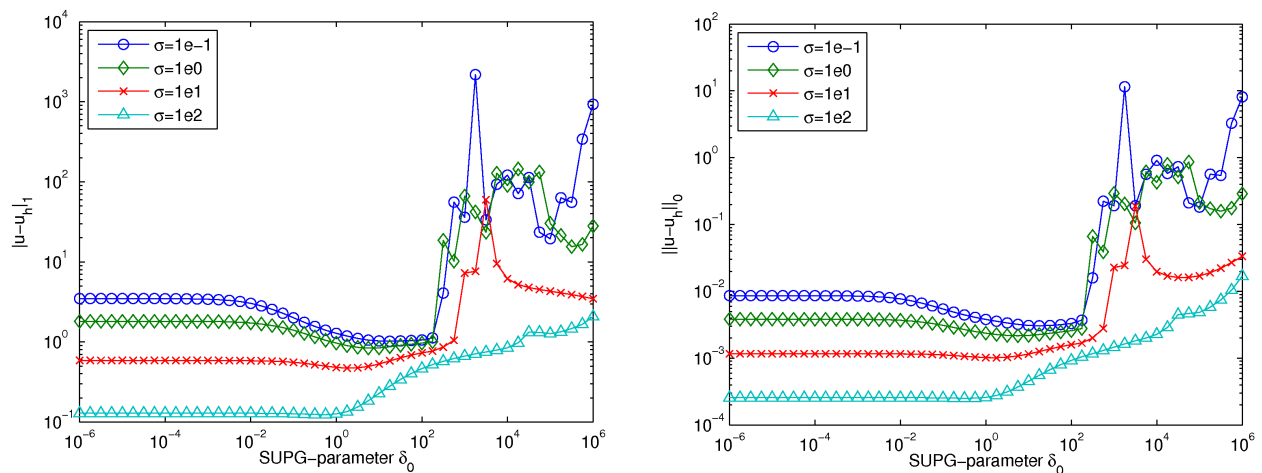


Fig. 4.4. Errors in  $H^1$ -seminorm and  $L^2$ -norm vs. the scaling SUPG-parameter  $\delta_0$  (without grad-div stabilisation) for Example 3.1 with  $\nu = 10^{-6}$ ,  $h = 1/64$  and different values of  $\sigma$

**Remark 4.1.** Additionally, we checked the behaviour of the grad-div stabilisation for an inf-sup pair with discontinuous pressure approximation. For the  $Q_2/P_1^{\text{disc}}$ -pair we observed similar results of the errors as in Fig. 4.3.

As a second example, let us consider a problem with  $(u \cdot \nabla)u = (b \cdot \nabla)u \equiv 0$ .

**Example 4.1.** We solve the Oseen problem (1.2) on  $\Omega = (0, 1)^2$  for viscosity  $\nu = 10^{-6}$  and  $\sigma = 0$ , with the flow field  $b(x) = (\sin(\pi x_2), 0)^\top$ , the source term  $f(x) \equiv \nu \pi^2 b(x) - (2\pi\nu, 0)^\top$ , and with inhomogeneous Dirichlet data  $u(x) = b(x)$  on  $\partial\Omega$ . The exact solution is  $u(x) = b(x)$  and  $p(x) = -2\pi\nu x_1 + \pi\nu$ . (We did not use the standard Poiseuille flow as the flow solution is contained in the discrete space.)

The errors in the  $H^1$ -seminorm and  $L^2$ -norm are again plotted against the scaling parameters  $\delta_0$  and  $\gamma_0$ . Surprisingly, we observe in Fig. 4.5 a similar error behaviour for increasing values of  $\delta_0$  if only SUPG-stabilisation (with  $\gamma_0 = 0$ ) is applied. This means, in particular, that the method is unable to reflect the behaviour of the continuous solution with  $\nabla p = \nu \Delta u$  and  $(b \cdot \nabla)u \equiv 0$ . There is seemingly some gain for an increasing value of the SUPG-parameter but this observation is corrupted by the arising instabilities of the discrete solution. (Let us remark that the maximal value of the errors on the coarsest grid with  $\delta_0 \approx 10$  was of the order of  $10^{12}$  (not shown in Fig. 4.5)).

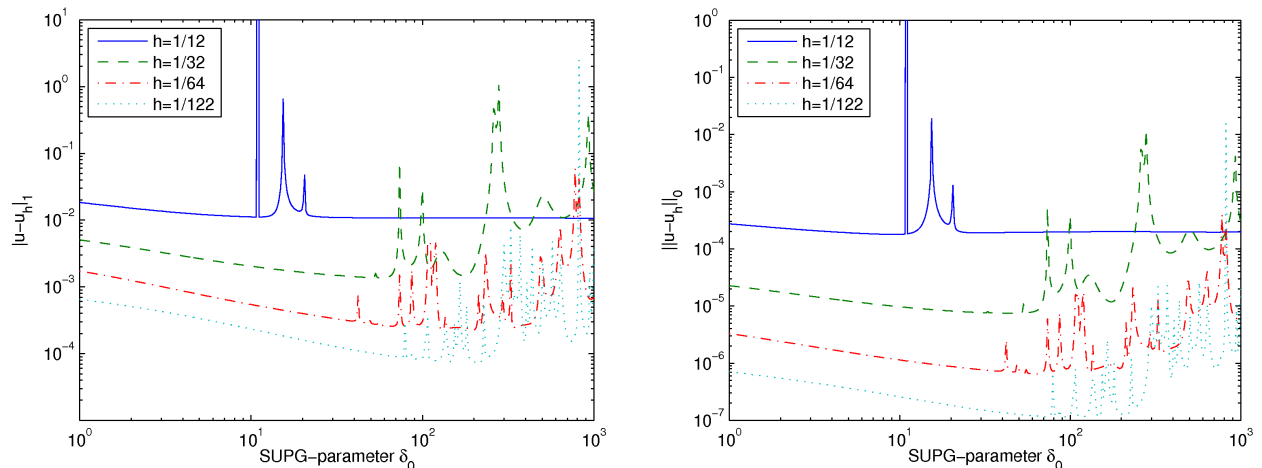


Fig. 4.5. Errors in the  $H^1$ -seminorm and  $L^2$ -norm vs. the SUPG-scaling parameter  $\delta_0$  (without grad-div stabilisation) for Example 4.1 with  $\nu = 10^{-6}$ ,  $\sigma = 0$  and different values of  $h$

The tests for the grad-div stabilisation, i.e., with  $\delta_0 = 0$ , reflect again the robustness of the discrete solution with respect to  $\gamma_0 \in (0, 100)$  (see Figure 4.6). Compared to the unstabilised case  $\gamma_0 = 0$ , we observe for an optimal value of  $\gamma_0$  a reduction of the  $H^1$ -seminorm error by a factor 5 – 8. This reduction is much less pronounced as in Example 3.1 according to a discussion in case (ii) and formula (4.4) above. This is also reflected by comparison of Figs. 4.6 and 4.5.

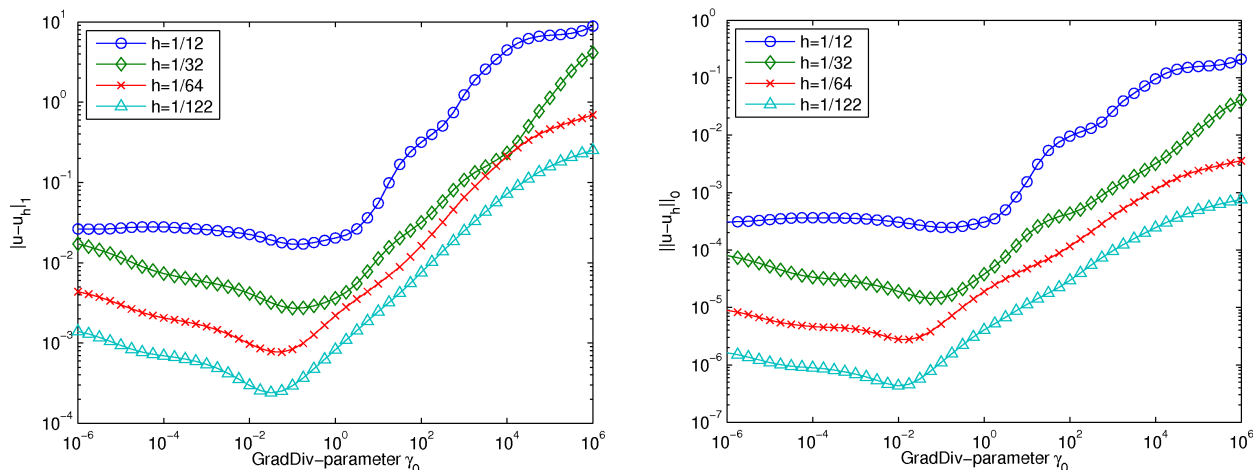


Fig. 4.6. Errors in the  $H^1$ -seminorm and  $L^2$ -norm vs. the scaling parameter  $\gamma_0$  of grad-div stabilisation (without SUPG) for Example 4.1 with  $\nu = 10^{-6}$ ,  $\sigma = 0$  and different values of  $h$

Finally, we consider the influence of the parameter  $\sigma$  on the error of the discrete problem with SUPG-stabilisation (without grad-div stabilisation). From Fig. 4.7 we mainly draw the same conclusions as for Example 3.1.  $\square$

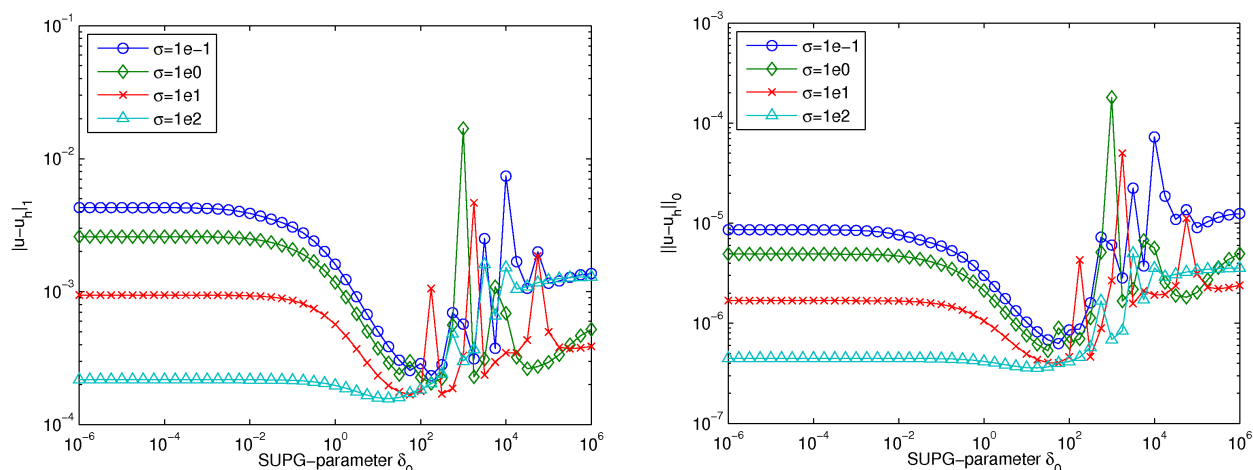


Fig. 4.7. Errors in the  $H^1$ -seminorm and  $L^2$ -norm vs. the scaling SUPG-parameter  $\delta_0$  (without grad-div stabilisation) for Example 4.1 with  $\nu = 10^{-6}$ ,  $h = 1/64$  and different values of  $\sigma$

As the last example, we consider the problem with a boundary layer proposed by S. Berrone [2].

**Example 4.2.** We solve the Oseen problem (1.2) on  $\Omega = (0, 1)^2$  with  $b = u$  and the solution

$$u_1(x) = \left(1 - \cos\left(\frac{2\pi(e^{R_1 x_1} - 1)}{e^{R_1} - 1}\right)\right) \sin\left(\frac{2\pi(e^{R_2 x_2} - 1)}{e^{R_2} - 1}\right) \frac{R_2}{2\pi} \frac{e^{R_2 x_2}}{(e^{R_2} - 1)},$$

$$u_2(x) = -\sin\left(\frac{2\pi(e^{R_1x_1} - 1)}{e^{R_1} - 1}\right)\left(1 - \cos\left(\frac{2\pi(e^{R_2x_2} - 1)}{e^{R_2} - 1}\right)\right)\frac{R_1}{2\pi}\frac{e^{R_1x_1}}{(e^{R_1} - 1)},$$

$$p(x) = R_1R_2\sin\left(\frac{2\pi(e^{R_1x_1} - 1)}{e^{R_1} - 1}\right)\sin\left(\frac{2\pi(e^{R_2x_2} - 1)}{e^{R_2} - 1}\right)\frac{e^{R_2x_1}e^{R_2x_2}}{(e^{R_1} - 1)(e^{R_2} - 1)}.$$

The velocity field resembles a counter-clockwise vortex with the center at

$$(x_{01}, x_{02}) = \left(\frac{1}{R_1}\log\left(\frac{e^{R_1} + 1}{2}\right), \frac{1}{R_2}\log\left(\frac{e^{R_2} + 1}{2}\right)\right).$$

The parameters are chosen as  $R_2 = 0.1$  leading to  $x_{02} = 0.5125$  and  $R_1$  such that  $x_{01} = 1 - \nu^{1/4}$ , i.e., the centre moves with decreasing  $\nu$  to the right boundary. This leads to a  $\nu$ -dependent solution with  $\|\nabla u\|_0 \sim \nu^{-0.35}$  and  $\|p\|_0 \sim \nu^{-0.12}$ .

First we present the results for  $\sigma = 0$  and  $\nu = 10^{-4}$ . The value of the viscosity allows a resolution of the boundary layer on the finest meshes. The errors in the  $H^1$ -seminorm and  $L^2$ -norm are again plotted against the scaling parameters  $\delta_0$  and  $\gamma_0$ . We again observe in Fig. 4.8 (with sampling for 385 values of  $\delta_0$ ) a similar behavior of the errors in the  $H^1$ -seminorm and  $L^2$ -norm for increasing values of  $\delta_0$  if only SUPG-stabilisation (with  $\gamma_0 = 0$ ) is applied. A gain by a factor of 3 is obtained for an optimal value of  $\delta_0$  compared to the unstabilised case  $\delta_0 = 0$  (not shown). Let us remark that the maximal value of the errors on the coarsest grid with  $\delta_0 \approx 10$  was of the order of  $10^{12}$  (not shown in Fig. 4.8).

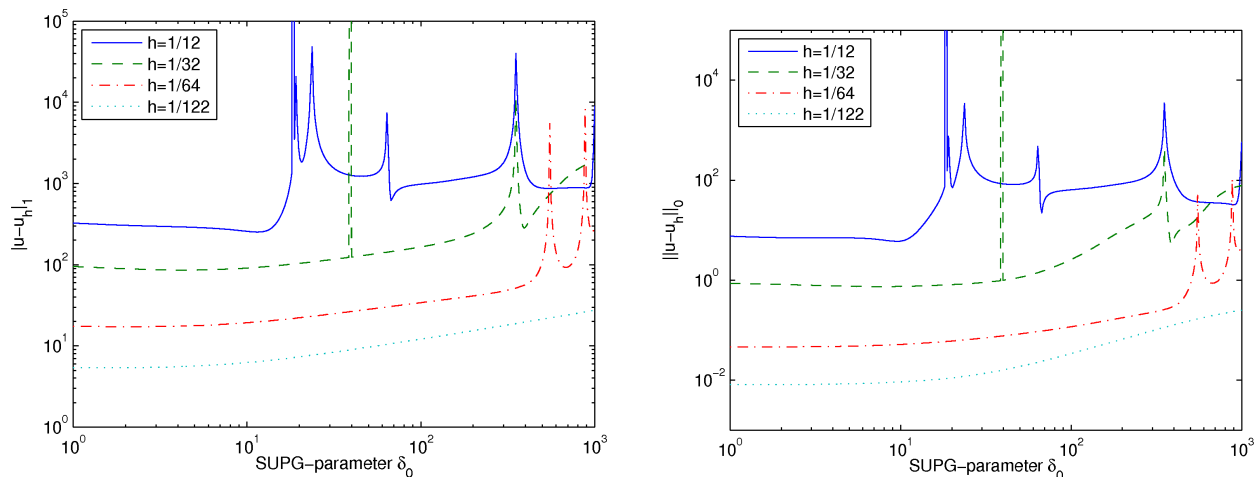


Fig. 4.8. Errors in the  $H^1$ -seminorm and  $L^2$ -norm vs. the SUPG-scaling parameter  $\delta_0$  (without grad-div stabilisation) for Example 4.2 with  $\nu = 10^{-4}$ ,  $\sigma = 0$  and different values of  $h$

The tests for the grad-div stabilisation, i.e., with  $\delta_0 = 0$ , and sampling for 49 values of  $\gamma_0$  reflect again the robustness of the discrete solution with respect to  $\gamma_0$  (see Fig. 4.9). In comparison to the unstabilised case  $\gamma_0 = 0$ , we observe for an optimal value of  $\gamma_0$  a reduction of the errors on the finer meshes by a factor of nearly  $10^{-2}$ . This reduction is clearly pronounced as in Example 3.1.

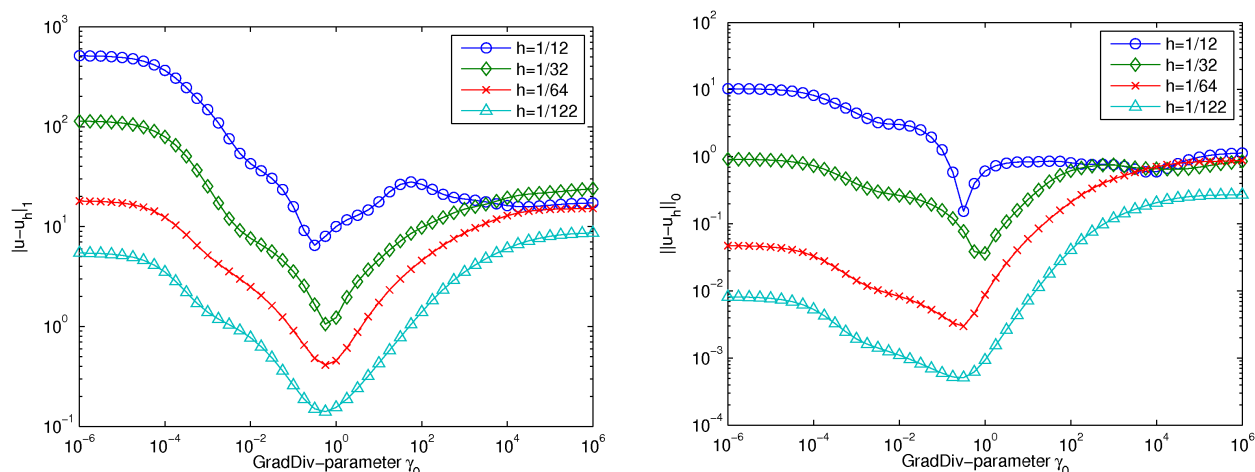


Fig. 4.9. Errors in the  $H^1$ -seminorm and  $L^2$ -norm vs. the scaling parameter  $\gamma_0$  of grad-div stabilisation (without SUPG) for Example 4.2 with  $\nu = 10^{-4}$ ,  $\sigma = 0$  and different values of  $h$

Finally, let us consider the influence of the viscosity parameter  $\nu$ . Figure 4.10 shows the  $h$ -convergence for  $\|u - u_h\|$  and  $\|p - p_h\|_0$  (scaled by appropriate Sobolev norms of the solution) for the optimised value of  $\gamma_0$  and  $\delta_0$  as above for different values of  $\nu = 10^{-i}$ ,  $i = 2, 3, 4, 5, 6$  and  $\sigma = 0$ . We observe that the second-order accuracy is reached for the larger values of  $\nu$  and for the smaller values at least on sufficiently fine grids as an absolute accuracy can only be obtained for a mesh which resolves the boundary layer at  $x_1 = 1$ .

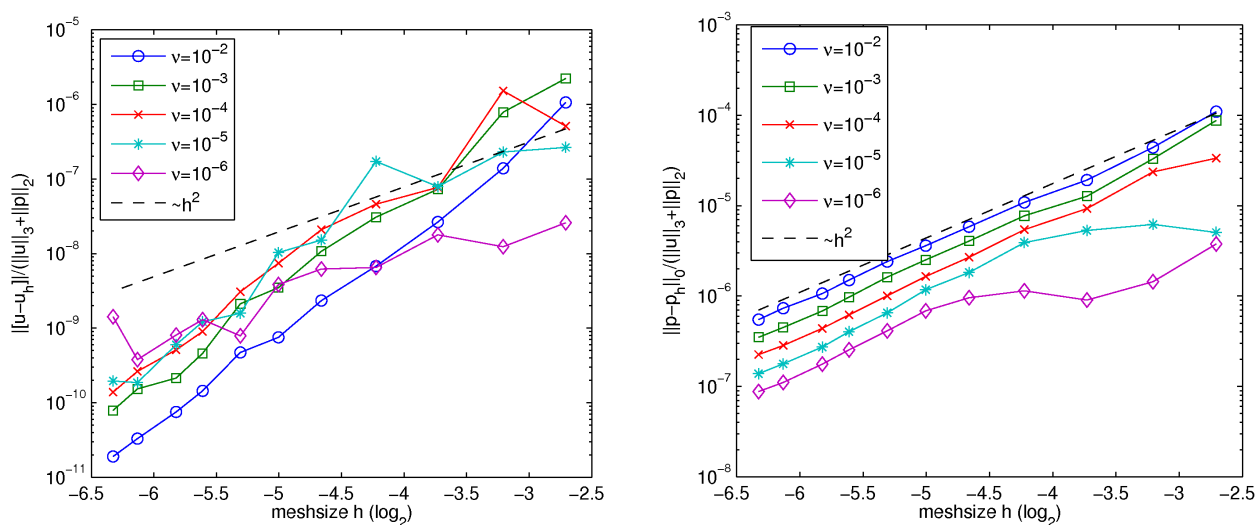


Fig. 4.10. Convergence plots for different values of  $\nu$ ,  $\sigma = 0$  and  $\delta_0 = 0.1$ ,  $\gamma_0 = 0.2$

## 5. Summary. Outlook

In the present paper, we have considered stabilised finite element methods for the generalised Oseen problem. For inf-sup stable discretisations of velocity and pressure, we have proved the unique solvability based on a modified stability condition and an error estimate. The main results are as follows:

- First of all, we emphasize the role of an additional stabilisation of the divergence constraint via grad-div stabilisation. It is robust in a wide range of the stabilisation

parameter  $\gamma_K$  and might be important, in particular, for flows with strong inertia if the parameters are optimised.

- Secondly, the analysis of the streamline-diffusion (SUPG) stabilisation requires an upper bound on the SUPG-stabilisation parameter  $\delta_K \leq Ch_K^2$  with a data-dependent constant  $C = C(\nu, \sigma, \|b\|_{L^\infty(\Omega)^d})$ . This scaling implies that SUPG-stabilisation is less important in the case of inf-sup velocity-pressure pairs. Numerical results on slightly distorted quasi-uniform meshes indeed show that numerical instabilities may occur.
- Thirdly, our analysis extends the result in [7] on quasi-uniform meshes and continuous pressure approximations to general shape-regular meshes and to discontinuous pressure interpolation.

Summarising, the application of the SUPG-stabilisation for inf-sup stable elements without PSPG-stabilisation and/or without grad-div-stabilisation cannot be recommended to practitioners.

Let us finally mention some open problems:

- We didn't discuss the dependence on the polynomial degree of the finite elements. This appears in the stability estimate of Lemma 3.1 and in the upper bound of  $\delta_K$ .
- The upper bound of the SUPG-parameter  $\delta_K$  in formula (3.2) that stems from the stability analysis is not convincing. Let us emphasize that such restriction does not exist for the symmetric stabilisation of local projection type (see e.g., [15]).
- Further research is necessary for the design of the grad-div stabilisation parameters  $\gamma_K$ . The results of [12] indicate that this might depend on the local behaviour of the flow.
- The grad-div stabilisation with  $\gamma \sim \mathcal{O}(1)$  may lead to problems for iterative solvers of the mixed algebraic problem as the kernel of the div-operator is large. To a certain extent, this is discussed for the Stokes model in [17] and for the Oseen problem in [1].

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