Column Densities and Masses from Dust Emission

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Abstract

In this paper I derive formulas to convert the observed flux of dust emission into column densities and masses. These are evaluated for the standard assumptions for dust properties made by the c2d collaboration. This script is intended to serve as a future reference for the c2d project.

1 Molecular Weight

As one usually wishes to express column densities in terms of particles per area, the molecular weight needs to be introduced into the equations. The ***molecular weight per hydrogen molecule***, μ_{H_2}, is defined via $\mu_{\mathrm{H}_2} m_{\mathrm{H}} \mathcal{N}(\mathrm{H}_2) = \mathcal{M}$; here $\mathcal{M}$ is the total mass contained in a volume with $\mathcal{N}(\mathrm{H}_2)$ hydrogen molecules, and m_{H} is the H-atom mass. It can be calculated from cosmic abundance ratios. For hydrogen, helium, and metals the mass ratios are $\mathcal{M}(\mathrm{H})/\mathcal{M} \approx 0.71$, $\mathcal{M}(\mathrm{He})/\mathcal{M} \approx 0.27$, and $\mathcal{M}(\mathrm{Z})/\mathcal{M} \approx 0.02$, respectively, where $\mathcal{M} = \mathcal{M}(\mathrm{H}) + \mathcal{M}(\mathrm{He}) + \mathcal{M}(\mathrm{Z})$ (Cox 2000). As $\mathcal{N}(\mathrm{H}) = 2\mathcal{N}(\mathrm{H}_2)$, $\mathcal{M}(\mathrm{H}) = m_{\mathrm{H}}\mathcal{N}(\mathrm{H})$,

$$\mu_{\mathrm{H}_2} = \frac{\mathcal{M}}{m_{\mathrm{H}}\mathcal{N}(\mathrm{H}_2)} = \frac{2\mathcal{M}}{m_{\mathrm{H}}\mathcal{N}(\mathrm{H})} = \frac{2\mathcal{M}}{\mathcal{M}(\mathrm{H})} \approx 2.8\,. \tag{1}$$

Note the difference to the ***mean molecular weight per free particle***, μ_{p}, defined via $\mu_{\mathrm{p}} m_{\mathrm{H}} \mathcal{N} = \mathcal{M}$, where $\mathcal{N} \approx \mathcal{N}(\mathrm{H}_2) + \mathcal{N}(\mathrm{He})$ for gas with all H in molecules. As the helium contribution is dominated by ^{4}He, and each H_2 molecule has a weight of two H-atoms, $\mathcal{N}(\mathrm{H}_2) = \mathcal{M}(\mathrm{H})/(2m_{\mathrm{H}})$ and $\mathcal{N}(\mathrm{He}) = \mathcal{M}(\mathrm{He})/(4m_{\mathrm{H}})$. Thus

$$\begin{aligned} \mu_{\mathrm{p}} &= \frac{\mathcal{M}}{m_{\mathrm{H}}[\mathcal{N}(\mathrm{H}_2) + \mathcal{N}(\mathrm{He})]} = \frac{\mathcal{M}}{m_{\mathrm{H}}[\mathcal{M}(\mathrm{H})/(2m_{\mathrm{H}}) + \mathcal{M}(\mathrm{He})/(4m_{\mathrm{H}})]} && (2) \\ &= \frac{\mathcal{M}/\mathcal{M}(\mathrm{H})}{[1/2 + \mathcal{M}(\mathrm{He})/(4\mathcal{M}(\mathrm{H}))]} \approx 2.37\,; && (3) \end{aligned}$$

the classical value of $\mu_{\mathrm{p}} = 2.33$ holds for an abundance ratio $\mathcal{N}(\mathrm{H})/\mathcal{N}(\mathrm{He}) = 10$ and a negligible admixture of metals.

Both molecular weights are applied in different contexts. The mean molecular weight per free particle, μ_{p}, is e.g. used to evaluate the thermal gas pressure, $P = \varrho k_{\mathrm{B}} T/\mu_{\mathrm{p}}$, where T, ϱ, and k_{B} are the gas temperature, density, and Boltzmanns constant, respectively. The molecular mass per hydrogen molecule, μ_{H_2}, is needed below to derive particle column densities.

2 Radiative Transfer

2.1 Equation of Radiative Transfer

The intensity emitted by a medium of temperature T and of optical depth τ_ν at the frequency ν is given by the ***equation of radiative transfer***, which reads

$$I_\nu = B_\nu(T)\left(1 - \mathrm{e}^{-\tau_\nu}\right) \tag{4}$$

in the case of local thermal equilibrium. Here B_ν is the Planck function. For the optical depth,

$$\tau_\nu = \int \kappa_\nu \varrho \, \mathrm{d}s\,, \tag{5}$$

where κ_ν is the ***(specific) absorption coefficient*** (i.e., per mass) or ***dust opacity***. If most hydrogen is in molecules, the optical depth can be related to the column density of molecular hydrogen,

$$N_{\mathrm{H}_2} = \int n_{\mathrm{H}_2}\,\mathrm{d}s = \int \frac{\varrho}{\mu_{\mathrm{H}_2} m_\mathrm{H}}\,\mathrm{d}s = \frac{1}{\mu_{\mathrm{H}_2} m_\mathrm{H} \kappa_\nu} \int \kappa_\nu \varrho \,\mathrm{d}s$$

$$\Rightarrow \quad \boxed{N_{\mathrm{H}_2} = \frac{\tau_\nu}{\mu_{\mathrm{H}_2} m_\mathrm{H} \kappa_\nu}}\,, \tag{6}$$

where n_{H_2} is the particle density of hydrogen molecules. This reads in a usable form

$$\boxed{N_{\mathrm{H}_2} = 2.14 \cdot 10^{25}\ \mathrm{cm}^{-2}\,\tau_\nu \left(\frac{\kappa_\nu}{0.01\,\mathrm{cm}^2\,\mathrm{g}^{-1}}\right)^{-1}}\,, \tag{7}$$

where the chosen numerical value of κ_ν is characteristic for wavelength $\lambda \approx 1\,\mathrm{mm}$. As typical H_2 column densities are below 10^{23} cm^2, τ_ν is expected to be by far smaller that 1. Thermal dust emission in the (sub-)millimeter regime is therefore optically thin.

For optically thin conditions the equation of radiative transfer can be simplified:

$$\boxed{I_\nu \approx B_\nu(T)\,\tau_\nu}\,. \tag{8}$$

As the optical depth is related to the column density (Eq. 7), Eq. (8) relates the observed intensity to the column density.

2.2 The Planck Function

The Planck function reads

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{\mathrm{e}^{h\nu/(k_\mathrm{B}T)} - 1}\,, \tag{9}$$

in which c is the speed of light and h is Planck's constant. In the Rayleigh-Jeans limit, $h\nu \ll k_\mathrm{B}T$, this simplifies to

$$B_\nu(T) = \frac{2\nu^2}{c^2} k_\mathrm{B} T\,. \tag{10}$$

However, the limiting condition, which reads

$$\lambda \gg 1.44 \text{ mm} \left(\frac{T}{10 \text{ K}}\right)^{-1} \tag{11}$$

in useful units, is under typical dust temperatures of about 10 K not fulfilled for observations at about 1 mm wavelength. The exact value of the Planck function is

$$B_\nu(T) = 1.475 \cdot 10^{-23} \text{ W m}^{-2} \text{ Hz}^{-1}\text{sr}^{-1} \left(\frac{\nu}{\text{GHz}}\right)^3 \frac{1}{\mathrm{e}^{0.0048(\nu/\text{GHz})(T/10\text{ K})^{-1}} - 1} \tag{12}$$

in useful units.

3 Observed Quantities

The received flux per beam is related to the intensity by

$$S_\nu^{\text{beam}} = \int I_\nu \, P \, \mathrm{d}\Omega \,, \tag{13}$$

where P is the normalised power pattern of the telescope (i.e. $\max(P) = 1$). Defining the ***beam solid angle*** as the integral

$$\Omega_\text{A} = \int P \, \mathrm{d}\Omega \,, \tag{14}$$

one can derive a beam-averaged intensity

$$\boxed{\langle I_\nu \rangle = S_\nu^{\text{beam}} / \Omega_\text{A}} \,. \tag{15}$$

The beam solid angle can be conveniently approximated for telescopes with a beam profile similar to a Gaussian function,

$$P(\theta) = \mathrm{e}^{-\theta^2/(2\theta_0^2)} \tag{16}$$

(where the angle θ gives the distance from the beam center). The parameter θ_0 is related to the half power beam width of the telescope, θ_HPBW, via

$$\theta_0 = \frac{\theta_\text{HPBW}}{\sqrt{8 \ln(2)}} \,. \tag{17}$$

For these idealisations the beam solid angle is

$$\Omega_\text{A} = 2\pi \int_0^\infty P(\theta) \, \theta \, \mathrm{d}\theta = 2\pi \int_0^\infty \mathrm{e}^{-\theta^2/(2\theta_0^2)} \theta \, \mathrm{d}\theta \tag{18}$$

$$= 2\pi\theta_0^2 \tag{19}$$

$$= \frac{\pi}{4 \ln(2)} \theta_\text{HPBW}^2 \,. \tag{20}$$

Using the conversion

$$1 \text{ arcsec} = 4.85 \cdot 10^{-6} \text{ rad} \,,$$

one obtains

$$\Omega_{\mathrm{A}} = 2.665 \cdot 10^{-11}\ \mathrm{sr} \left(\frac{\theta_{\mathrm{HPBW}}}{\mathrm{arcsec}} \right)^2 . \tag{21}$$

The parameter θ_{HPBW} does not need to be the real telescope beam, but depends on the calibration of the data. To give an example, some software packages (e.g., MOPSIC) apply scaling factors to the data (i.e., S_ν^{beam}) when spatially smoothing a map, so that the beam width to which the calibration refers is equal to the spatial resolution of the map after smoothing.

For the idealisations made here the average intensity derived from Eq. (15) is a good approximation to the actual intensity only if the source has an extension of the order of the main lobe of the telescope. Otherwise radiation received via the side lobes may have a significant contribution to S_ν^{beam}, and the idealisation of the telescope beam by Gaussian functions is too simple. Inclusion of the side lobes increases Ω_{A}, and thus reduces $\langle I_\nu \rangle$. The nature of the average intensity derived in Eq. (15) is therefore comparable to the one of main beam brightness temperatures used in spectroscopic radio observations.

4 Derivation of Conversion Laws

4.1 Flux per Beam and Column Density

Equations (6, 8, 15) relate optical depth, column density, intensity, and the observed flux per beam with each other. Rearrangement yields

$$N_{\mathrm{H}_2} = \frac{S_\nu^{\mathrm{beam}}}{\Omega_{\mathrm{A}} \mu_{\mathrm{H}_2} m_{\mathrm{H}} \kappa_\nu B_\nu(T)} , \tag{22}$$

which reads

$$\boxed{\begin{aligned} N_{\mathrm{H}_2} \quad = \quad & 2.02 \cdot 10^{20}\ \mathrm{cm}^{-2} \left(\mathrm{e}^{1.439 (\lambda/\mathrm{mm})^{-1} (T/10\ \mathrm{K})^{-1}} - 1 \right) \\ & \left(\frac{\kappa_\nu}{0.01\ \mathrm{cm}^2\ \mathrm{g}^{-1}} \right)^{-1} \left(\frac{S_\nu^{\mathrm{beam}}}{\mathrm{mJy\ beam}^{-1}} \right) \left(\frac{\theta_{\mathrm{HPBW}}}{10\ \mathrm{arcsec}} \right)^{-2} \left(\frac{\lambda}{\mathrm{mm}} \right)^3 \end{aligned}} \tag{23}$$

in useful units.

4.2 Flux and Mass

The mass is given by the integral of the column densities across the source,

$$M = \mu_{\mathrm{H}_2} m_{\mathrm{H}} \int N_{\mathrm{H}_2}\, \mathrm{d}A . \tag{24}$$

Substitution of Eqs. (6, 8) yields

$$M = \frac{1}{\kappa_\nu B_\nu(T)} \int I_\nu\, \mathrm{d}A . \tag{25}$$

The surface element $\mathrm{d}A$ is related to the solid angle element $\mathrm{d}\Omega$ by $\mathrm{d}A = D^2\, \mathrm{d}\Omega$, where D is the distance of the source. Thus

$$M = \frac{D^2}{\kappa_\nu B_\nu(T)} \int I_\nu\, \mathrm{d}\Omega \tag{26}$$

$$= \frac{D^2 S_\nu}{\kappa_\nu B_\nu(T)} , \tag{27}$$

Table 1: Standard c2d conversion factors for masses and column densities from dust emission.

Instrument	λ	$\theta_{\rm HPBW}$	κ_ν	$N_{\rm H_2}/S_\nu^{\rm beam}$	$M/S_\nu(D=100~{\rm pc})$
	μm	arcsec	cm^2 g^{-1}	cm^{-2} (mJy beam^{-1})$^{-1}$	$M_\odot$ Jy^{-1}
SHARC	350	8.5	0.101	$7.13 \cdot 10^{19}$	0.031
SCUBA	450	7	0.0619	$1.42 \cdot 10^{19}$	0.041
	850	15	0.0182	$1.34 \cdot 10^{20}$	0.18
BOLOCAM	1120	31	0.0114	$6.77 \cdot 10^{19}$	0.39
MAMBO	1200	11	0.0102	$6.69 \cdot 10^{20}$	0.47
SIMBA	1200	24	0.0102	$1.41 \cdot 10^{20}$	0.47

where $S_\nu = \int I_\nu \, {\rm d}\Omega$ is the integrated flux. This reads

$$\boxed{\begin{aligned} M &= 0.12\,M_\odot \left({\rm e}^{1.439(\lambda/{\rm mm})^{-1}(T/10~{\rm K})^{-1}} - 1\right) \\ &\quad \left(\frac{\kappa_\nu}{0.01~{\rm cm^2~g^{-1}}}\right)^{-1} \left(\frac{S_\nu}{\rm Jy}\right) \left(\frac{D}{100~{\rm pc}}\right)^2 \left(\frac{\lambda}{\rm mm}\right)^3 \end{aligned}} \tag{28}$$

in useful units.

5 Conversion Factors

5.1 Conversion from Dust Emission

Table 1 summarises the c2d standard conversion factors for masses and column densities from dust emission. The wavelength-dependent dust opacities are from Ossenkopf & Henning (1994) and hold for dust with thin ice mantles coagulating for 10^5 yr at an H-density of 10^6 cm^{-3}. We adopt a dust temperature of 10 K. This choice for the dust temperature and opacity are the standard assumptions made by the c2d collaboration. The values listed in Tab. 1 are thus thought to serve as a standard reference within the collaboration.

5.2 Conversion to Extinction

The c2d standard conversion factor between column densities and visual extinction,

$$N_{\rm H_2} = 9.4 \cdot 10^{20}~{\rm cm^{-2}}\,(A_V/{\rm mag})\,, \tag{29}$$

is taken from Bohlin et al. (1978). They combined measurements of H_2 and HI from the Copernicus satellite for lightly reddened stars to get

$$\langle [N_{\rm HI} + 2N_{\rm H_2}]/E(B-V)\rangle = 5.8 \cdot 10^{21}~{\rm cm^{-2}~mag^{-1}}\,. \tag{30}$$

For a standard total-to-selective extinction ratio $R_V = A_V/E(B-V) = 3.1$ this yields the above conversion factor.

References

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