Single-Laser Lambda Three-Level Atom Resonance Fluorescence

H. M. Castro-Beltrán
Centro de Investigación en Ingeniería y Ciencias Aplicadas, Universidad Autónoma del Estado de Morelos, Avenida Universidad 1001, 62209 Cuernavaca, Morelos, México

(Dated: June 9, 2022)

Here we obtain analytical results for the density operator, photon correlations, and full spectrum of resonance fluorescence of a three-level atom with only one driven transition and two decay channels from the excited state to the ground state. We use both direct and fluctuation operator approaches. These notes were the starting point for the paper H. M. Castro-Beltran, R. Roman-Ancheyta, and L. Gutierrez, Phys. Rev. A, **93**, 033801 (2016). Some calculations are found in only one of these documents. NOTES IN PROGRESS

Keywords: resonance fluorescence, antibunching, electron shelving, non-Gaussian fluctuations.

I. ATOM-LASER INTERACTION

We consider a single three-level atom where only one transition is driven by a laser, but the excited state $|e\rangle$ has two decay channels: one directly to the ground state $|g\rangle$, and one via a long-lived intermediate (ancilla) state $|a\rangle$, see Fig. **??**. In the latter, the excited state decays at the rate γ_d to $|a\rangle$ and from this to the ground state at the rate γ_a.

With changes in notation we reproduce results from Ref. [**?**] and generate many new ones reported in Ref.[**?**].

The Hamiltonian and master equation are, in the frame rotating at the laser frequency,

$$\mathcal{H} = \Delta\,\sigma_{eg}\sigma_{ge} + \frac{\Omega}{2}(\sigma_{eg} + \sigma_{ge})\,, \tag{1.1}$$

$$\begin{aligned}\dot{\rho}(t) &= -i\Delta[\sigma_{eg}\sigma_{ge},\rho] - i\frac{\Omega}{2}[\sigma_{eg}+\sigma_{ge},\rho] \\ &\quad + \frac{\gamma}{2}(2\sigma_{ge}\rho\sigma_{eg} - \sigma_{eg}\sigma_{ge}\rho - \rho\sigma_{eg}\sigma_{ge}) \\ &\quad + \frac{\gamma_d}{2}(2\sigma_{ae}\rho\sigma_{ea} - \sigma_{ea}\sigma_{ae}\rho - \rho\sigma_{ea}\sigma_{ae}) \\ &\quad + \frac{\gamma}{2}(2\sigma_{ga}\rho\sigma_{ag} - \sigma_{ag}\sigma_{ga}\rho - \rho\sigma_{ag}\sigma_{ga})\,,\end{aligned} \tag{1.2}$$

where Ω is the Rabi frequency, Δ is the detuning of the laser from the atomic transition, and $\sigma_{\pm}^{j}$ are Pauli spin operators for the undriven transitions. In the notation used in [**?**] the *rho*'s and *sigma*'s wear a tilde to indicate they are in the frame rotating at the laser frequency. For simplicity, *we do not use such tildes.*

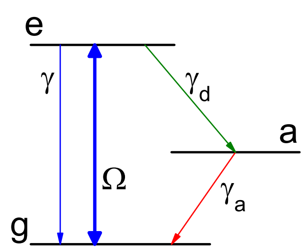

FIG. 1: Scheme of the three-level atom-laser interaction and spontaneous decay rates.

We need a full set of spin operators in projector form:

$$\sigma_{jk} = |j\rangle\langle k|\,, \qquad \sigma_{kj} = \sigma_{jk}^{+}\,, \qquad j,k = g,e,a$$

from which we derive some important products:

$$\sigma_{jk}\sigma_{lm} = \sigma_{jm}\delta_{kl}\,, \qquad j,k,l,m = g,e,a\,,$$

allowing to simplify the master equation as

$$\begin{aligned}\dot{\rho}(t) &= -i\Delta[\sigma_{ee},\rho] - i\frac{\Omega}{2}[\sigma_{eg}+\sigma_{ge},\rho] \\ &\quad - \frac{(\gamma+\gamma_d)}{2}(\sigma_{ee}\rho + \rho\sigma_{ee}) - \frac{\gamma_a}{2}(\sigma_{aa}\rho + \rho\sigma_{aa}) \\ &\quad + \gamma\rho_{ee}\sigma_{gg} + \gamma_d\rho_{ee}\sigma_{aa} + \gamma_a\rho_{aa}\sigma_{gg}\,.\end{aligned} \tag{1.3}$$

We obtain two sets of equations. The first one is:

$$\begin{aligned}\dot{\rho}_{eg} &= i\frac{\Omega}{2}(\rho_{ee}-\rho_{gg}) - \left(\frac{\gamma_+}{2} + i\Delta\right)\rho_{eg} \\ \dot{\rho}_{ge} &= -i\frac{\Omega}{2}(\rho_{ee}-\rho_{gg}) - \left(\frac{\gamma_+}{2} - i\Delta\right)\rho_{ge} \\ \dot{\rho}_{ee} &= -i\frac{\Omega}{2}(\rho_{ge}-\rho_{eg}) - \gamma_+\rho_{ee} \\ \dot{\rho}_{gg} &= i\frac{\Omega}{2}(\rho_{ge}-\rho_{eg}) + \gamma_-\rho_{ee} - \gamma_a\rho_{gg} + \gamma_a\,.\end{aligned} \tag{1.4}$$

where

$$\gamma_+ = \gamma + \gamma_d\,, \qquad \gamma_- = \gamma - \gamma_a\,. \tag{1.5}$$

This system of equations is a reduced one since the equation $\dot{\rho}_{aa} = \gamma_d\rho_{ee} - \gamma_a\rho_{aa}$ was eliminated due conservation of population, $\rho_{gg} + \rho_{ee} + \rho_{aa} = 1$, replaced in the equation $\dot{\rho}_{gg} = i(\Omega/2)(\rho_{ge} - \rho_{eg}) + \gamma\rho_{ee} + \gamma_a\rho_{aa}$.

The other set of equations involves the coherences linking the states $|e\rangle$ and $|g\rangle$ to the state $|a\rangle$,

$$\begin{aligned}\dot{\rho}_{ga} &= -i\frac{\Omega}{2}\rho_{ea} - \frac{\gamma_a}{2}\rho_{ga} \\ \dot{\rho}_{ea} &= -i\frac{\Omega}{2}\rho_{ga} - \left(i\Delta + \frac{\gamma+\gamma_d+\gamma_a}{2}\right)\rho_{ea} \\ \dot{\rho}_{\nu\mu} &= \dot{\rho}_{\mu\nu}^{*}\,.\end{aligned}$$

These coherences have zero mean and vanish for times $t \sim \gamma^{-1}$. The two sets are decoupled, and only the first one is relevant for the observables of interest, which involve only the driven transition.

Noticing that $\langle\sigma_{jk}\rangle = \rho_{kj}$ we rewrite the Bloch equations (**??**) as

$$\begin{aligned}
\langle\dot{\sigma}_-\rangle &= -\left(\frac{\gamma_+}{2} + i\Delta\right)\langle\sigma_-\rangle + i\frac{\Omega}{2}(\langle\sigma_{ee}\rangle - \langle\sigma_{gg}\rangle) \\
\langle\dot{\sigma}_+\rangle &= -\left(\frac{\gamma_+}{2} - i\Delta\right)\langle\sigma_+\rangle - i\frac{\Omega}{2}(\langle\sigma_{ee}\rangle - \langle\sigma_{gg}\rangle) \\
\langle\dot{\sigma}_{ee}\rangle &= i\frac{\Omega}{2}(\langle\sigma_-\rangle - \langle\sigma_+\rangle) - \gamma_+\langle\sigma_{ee}\rangle \\
\langle\dot{\sigma}_{gg}\rangle &= -i\frac{\Omega}{2}(\langle\sigma_-\rangle - \langle\sigma_+\rangle) + \gamma_-\langle\sigma_{ee}\rangle - \gamma_a\langle\sigma_{gg}\rangle + \gamma_a\,.
\end{aligned} \tag{1.6}$$

We write this set in compact form as

$$\begin{aligned}
\langle\dot{\mathbf{s}}\rangle &= \mathbf{M}\langle\mathbf{s}\rangle + \mathbf{b}\,, && (1.7)\\
\mathbf{s} &\equiv (\sigma_-, \sigma_+, \sigma_{ee}, \sigma_{gg})^T\,, && (1.8)\\
\mathbf{b} &= (0,0,0,\gamma_a)^T\,,
\end{aligned}$$

$$\mathbf{M} = \begin{pmatrix} -\frac{1}{2}(\gamma_+ + i2\Delta) & 0 & i\Omega/2 & -i\Omega/2 \\ 0 & -\frac{1}{2}(\gamma_+ - i2\Delta) & -i\Omega/2 & i\Omega/2 \\ i\Omega/2 & -i\Omega/2 & -\gamma_+ & 0 \\ -i\Omega/2 & i\Omega/2 & \gamma_- & -\gamma_a \end{pmatrix} \tag{1.9}$$

Formal solutions for the Bloch equations

$$\frac{d}{dt}\left(\langle\mathbf{s}\rangle + \mathbf{M}^{-1}\mathbf{b}\right) = \mathbf{M}\left(\langle\mathbf{s}\rangle + \mathbf{M}^{-1}\mathbf{b}\right) \tag{1.10}$$

are given by

$$\begin{aligned}
\langle\mathbf{s}(t)\rangle &= e^{\mathbf{M}t}\left[\langle\mathbf{s}(0)\rangle + \mathbf{M}^{-1}\mathbf{b}\right] - \mathbf{M}^{-1}\mathbf{b} \\
&= S^{-1}e^{\lambda t}S\left[\langle\mathbf{s}(0)\rangle + \mathbf{M}^{-1}\mathbf{b}\right] - \mathbf{M}^{-1}\mathbf{b}\,,
\end{aligned}$$

where S is a matrix that diagonalizes $\mathbf{M}$, whose eigenvalues are given in the matrix $\lambda = diag(\lambda_1, \lambda_2, \lambda_+, \lambda_-)$. An exact analytical solution is not possible.

It looks hopeless to perform a full matrix treatment (inverse, eigenvectors, eigenvalues, diagonalization matrix) analytically. Even the on-resonance case, $\Delta = 0$, is very difficult. In this case the determinant is $|\mathbf{M}| = (\gamma_+/4)[(2+q)\Omega^2 + \gamma_+^2]$, the inverse matrix is $\mathbf{M}^{-1}$ ... and only one eigenvalue and its corresponding eigenvector are obtained easily: $\lambda_1 = -\gamma_+/2$, $\mathbf{r}_1 = (1,1,0,0)^T$. The characteristic equation for the remaining eigenvalues is given in a section below. Mathematica gives very complicated exact expressions for such eigenvalues.

However, approximate solutions, with simple expressions of the eigenvalues for $\Delta = 0$, are obtained later.

II. STEADY STATE OF DENSITY OPERATOR

We start with the equations for ρ including ρ_{aa},

$$\begin{aligned}
\dot{\rho}_{ee} &= 0 = -i\frac{\Omega}{2}(\rho_{ge} - \rho_{eg}) - \gamma_+\rho_{ee} && (2.1)\\
\dot{\rho}_{aa} &= 0 = \gamma_d\rho_{ee} - \gamma_a\rho_{aa} && (2.2)\\
\dot{\rho}_{gg} &= 0 = i\frac{\Omega}{2}(\rho_{ge} - \rho_{eg}) + \gamma_-\rho_{ee} - \gamma_a\rho_{gg} + \gamma_a && (2.3)\\
\dot{\rho}_{ge} &= 0 = -i\frac{\Omega}{2}(\rho_{ee} - \rho_{gg}) - \left(\frac{\gamma_+}{2} - i\Delta\right)\rho_{ge} && (2.4)\\
\dot{\rho}_{eg} &= 0 = i\frac{\Omega}{2}(\rho_{ee} - \rho_{gg}) - \left(\frac{\gamma_+}{2} + i\Delta\right)\rho_{eg}\,. && (2.5)
\end{aligned}$$

For the rest of this Section we mean ρ to be in the steady state (ρ^{st}). Adding B1 and B3

$$\begin{aligned}
0 &= -\gamma_+\rho_{ee} - \gamma_-\rho_{ee} - \gamma_a\rho_{gg} + \gamma_a \\
\rightarrow \rho_{gg} &= 1 - \frac{\gamma_d + \gamma_a}{\gamma_a}\rho_{ee}\,. && (2.6)
\end{aligned}$$

Inserting B6 in B4

$$\begin{aligned}
0 &= -\left(\frac{\gamma_+}{2} - i\Delta\right)\rho_{ge} \\
&\quad -i\frac{\Omega}{2}\left[\rho_{ee} - \left(1 - \frac{\gamma_d + \gamma_a}{\gamma_a}\rho_{ee}\right)\right] \\
&= -\left(\frac{\gamma_+}{2} - i\Delta\right)\rho_{ge} - i\frac{\Omega}{2}\left[(2+q)\rho_{ee} - 1\right] \\
\rho_{ge} &= \frac{-i\Omega\left[(2+q)\rho_{ee} - 1\right]}{\gamma_+ - i2\Delta}\,, \qquad \rho_{eg} = \rho_{ge}^*\,,
\end{aligned}$$

donde $q = \gamma_d/\gamma_a$. Then

$$\begin{aligned}
\rho_{ge} - \rho_{eg} &= -\frac{i2\gamma_+\Omega\left[(2+q)\rho_{ee} - 1\right]}{\gamma_+^2 + 4\Delta^2} \\
-i\frac{\Omega}{2}\left(\rho_{ge} - \rho_{eg}\right) &= -\gamma_+\Omega^2\left[\frac{(2+q)\rho_{ee} - 1}{\gamma_+^2 + 4\Delta^2}\right] && (2.7)
\end{aligned}$$

We used $q = \gamma_d/\gamma_a$. Substituting B7 into B1

$$\begin{aligned}
0 &= \frac{\Omega^2\gamma_+}{\gamma_+^2 + 4\Delta^2} - \gamma_+\rho_{ee} - \frac{\Omega^2\gamma_+(2+q)\rho_{ee}}{\gamma_+^2 + 4\Delta^2} \\
&= -\left[1 + \frac{\Omega^2(2+q)}{\gamma_+^2 + 4\Delta^2}\right]\rho_{ee} + \frac{\Omega^2}{\gamma_+^2 + 4\Delta^2} \\
&= -\left[\gamma_+^2 + 4\Delta^2 + \Omega^2(2+q)\right]\rho_{ee} + \Omega^2 \\
\rho_{ee}^{st} &= \frac{\Omega^2}{(2+q)\Omega^2 + \gamma_+^2 + 4\Delta^2}\,. && (2.8)
\end{aligned}$$

Substituting B8 into B2

$$\begin{aligned}
0 &= \gamma_d\rho_{ee} - \gamma_a\rho_{aa} \qquad \rightarrow \rho_{aa} = q\rho_{ee} \\
\rho_{aa}^{st} &= \frac{q\Omega^2}{(2+q)\Omega^2 + \gamma_+^2 + 4\Delta^2}\,. && (2.9)
\end{aligned}$$

Substituting B8 and B9 into $1 = \rho_{ee} + \rho_{aa} + \rho_{gg}$

$$\begin{aligned} \rho_{gg}^{st} &= 1 - \rho_{ee} - \rho_{aa} = 1 - (1+q)\rho_{ee} \\ &= 1 - \frac{(1+q)\Omega^2}{(2+q)\Omega^2 + \gamma_+^2 + 4\Delta^2} \\ &= \frac{\Omega^2 + \gamma_+^2 + 4\Delta^2}{(2+q)\Omega^2 + \gamma_+^2 + 4\Delta^2} \,. \end{aligned} \tag{2.10}$$

Substituting B8 and B10 into B4,

$$\begin{aligned} \left(\frac{\gamma_+}{2} - i\Delta\right)\rho_{ge} &= -i\frac{\Omega}{2}(\rho_{ee} - \rho_{gg}) \\ (\gamma_+ - i2\Delta)\rho_{ge} &= \frac{i\Omega[\gamma_+^2 + 4\Delta^2]}{(2+q)\Omega^2 + \gamma_+^2 + 4\Delta^2} \\ &= \frac{i\Omega(\gamma_+ - i2\Delta)(\gamma_+ + i2\Delta)}{(2+q)\Omega^2 + \gamma_+^2 + 4\Delta^2} \end{aligned}$$

$$\rho_{ge}^{st} = \frac{-\Omega[2\Delta - i\gamma_+]}{(2+q)\Omega^2 + \gamma_+^2 + 4\Delta^2} \tag{2.11}$$

$$\rho_{eg}^{st} = \frac{-\Omega[2\Delta + i\gamma_+]}{(2+q)\Omega^2 + \gamma_+^2 + 4\Delta^2} \,. \tag{2.12}$$

Defining $\alpha_- = \langle\sigma_-\rangle_{st} = \rho_{eg}^{st}$, $\alpha_+ = \alpha_-^*$, $\alpha_{ee} = \langle\sigma_+\sigma_-\rangle_{st} = \rho_{ee}^{st}$, $\alpha_{gg} = \langle\sigma_-\sigma_+\rangle_{st} = \rho_{gg}^{st}$, $\alpha_{aa} = \langle\sigma_+^a\sigma_-^a\rangle_{st} = \rho_{aa}^{st}$, the steady state of the relevant elements of the density matrix are

$$\alpha_\mp = \frac{-\Omega[2\Delta \pm i\gamma_+]}{D} \,, \tag{2.13a}$$

$$\alpha_{gg} = \frac{\Omega^2 + \gamma_+^2 + 4\Delta^2}{D} \,, \tag{2.13b}$$

$$\alpha_{ee} = \frac{\Omega^2}{D} \,, \tag{2.13c}$$

$$\alpha_{aa} = q\frac{\Omega^2}{D} \,, \tag{2.13d}$$

where

$$\begin{aligned} D &= (2+q)\Omega^2 + \gamma_+^2 + 4\Delta^2 \,, \\ D_0 &= (2+q)\Omega^2 + \gamma_+^2 \,, \qquad \Delta = 0 \,, \end{aligned}$$

and

$$q = \gamma_d/\gamma_a \,. \tag{2.14}$$

For $\gamma_d = 0$ we recover the 2LA results. Evers and Keitel [?] find that $\gamma_d \sim 3.3\gamma_a$ is very close to the optimum value for the appearance of the extra narrow peak.

Side comment: the system is ergodic, with a non-zero average time between two succesive photons from the driven transition

$$\bar{\tau} = \frac{1}{\gamma\alpha_{ee}} = \frac{(2+q)\Omega^2 + \gamma_+^2 + 4\Delta^2}{\gamma\Omega^2} \,, \tag{2.15}$$

increased by the additional coupling of the excited state to the long-lived state. Of course, as an average time, it includes the effect of dark periods. This enhances the fluctuations of the emission, as seen in Mandel's Q parameter [?].

III. INITIAL CONDITIONS OF CORRELATIONS

A. Direct Approach

The initial conditions of the correlations are:

$$\langle\sigma_+\mathbf{s}\rangle = \{\alpha_{ee}, 0, 0, \alpha_+\} \,, \tag{3.1a}$$

$$\langle\mathbf{s}\sigma_-\rangle = \{0, \alpha_{ee}, 0, \alpha_-\} \,, \tag{3.1b}$$

$$\langle\sigma_+\mathbf{s}\sigma_-\rangle = \{0, 0, 0, \alpha_{ee}\} \,. \tag{3.1c}$$

B. Fluctuation Operator Approach

We define the vector $\Delta\mathbf{s} \equiv (\Delta\sigma_-, \Delta\sigma_+, \Delta\sigma_{ee}, \Delta\sigma_{gg})^T$. The initial conditions of the correlations are:

$$\begin{aligned} \langle\Delta\sigma_+\Delta\mathbf{s}\rangle &= \begin{pmatrix} \alpha_{ee} - \alpha_+\alpha_- \\ -\alpha_+^2 \\ -\alpha_+\alpha_{ee} \\ \alpha_+(1-\alpha_{gg}) \end{pmatrix} \\ &= \frac{\Omega^2}{D^2}\begin{pmatrix} (2+q)\Omega^2 \\ \gamma_+^2 + i4\Delta\gamma_+ - 4\Delta^2 \\ \Omega[2\Delta - i\gamma_+] \\ -(1+q)\Omega[2\Delta - i\gamma_+] \end{pmatrix} . \end{aligned} \tag{3.2}$$

The first element, for example, is calculated as

$$\begin{aligned} \langle\Delta\sigma_+\Delta\sigma_-\rangle_{st} &= \langle[\sigma_+ - \langle\sigma_+\rangle_{st}][\sigma_- - \langle\sigma_-\rangle_{st}]\rangle_{st} \\ &= \alpha_{ee} - |\alpha_+|^2 \\ &= \frac{\Omega^2}{(2+q)\Omega^2 + \gamma_+^2 + 4\Delta^2} \\ &\quad - \frac{\Omega^2[\gamma_+^2 + 4\Delta^2]}{[(2+q)\Omega^2 + \gamma_+^2 + 4\Delta^2]^2} \\ &= \frac{(2+q)\Omega^2}{[(2+q)\Omega^2 + \gamma_+^2 + 4\Delta^2]^2} \,. \end{aligned}$$

Also we can calculate

$$\begin{aligned} \langle\Delta\mathbf{s}\Delta\sigma_-\rangle &= \begin{pmatrix} -\alpha_-^2 \\ \alpha_{ee} - \alpha_+\alpha_- \\ -\alpha_-\alpha_{ee} \\ \alpha_-(1-\alpha_{gg}) \end{pmatrix} \\ &= \frac{\Omega^2}{D^2}\begin{pmatrix} \gamma_+^2 - i4\Delta\gamma_+ - 4\Delta^2 \\ (2+q)\Omega^2 \\ \Omega[2\Delta + i\gamma_+] \\ -(1+q)\Omega[2\Delta + i\gamma_+] \end{pmatrix} , \end{aligned} \tag{3.3}$$

For the third-order correlations,

$$\begin{aligned} \langle\Delta\sigma_+\Delta\mathbf{s}\Delta\sigma_-\rangle &= \begin{pmatrix} 2\alpha_-(\alpha_+\alpha_- - \alpha_{ee}) \\ 2\alpha_+(\alpha_+\alpha_- - \alpha_{ee}) \\ \alpha_e(2\alpha_+\alpha_- - \alpha_{ee}) \\ (\alpha_g - 1)(2\alpha_+\alpha_- - \alpha_{ee}) \end{pmatrix} \\ &= \frac{\Omega^4}{D^3}\begin{pmatrix} 2(2+q)\Omega[2\Delta + i\gamma_+] \\ 2(2+q)\Omega[2\Delta - i\gamma_+] \\ -[(2+q)\Omega^2 - \gamma_+^2 - 4\Delta^2] \\ -(1+q)[(2+q)\Omega^2 - \gamma_+^2 - 4\Delta^2] \end{pmatrix} . \end{aligned} \tag{3.4}$$

IV. TWO-TIME CORRELATIONS

The quantum regression formula allows to write equations for the correlation functions:

$$\begin{aligned}\frac{d}{d\tau}\langle\sigma_+(0)\mathbf{s}(\tau)\rangle_{st} &= \mathbf{M}\langle\sigma_+(0)\mathbf{s}(\tau)\rangle_{st}+\mathbf{b}\langle\sigma_+\rangle_{st}\\ \frac{d}{d\tau}\langle\mathbf{s}(\tau)\sigma_-(0)\rangle_{st} &= \mathbf{M}\langle\mathbf{s}(\tau)\sigma_-(0)\rangle_{st}+\mathbf{b}\langle\sigma_-\rangle_{st}\\ \frac{d}{d\tau}\langle\sigma_+(0)\mathbf{s}(\tau)\sigma_-(0)\rangle_{st} &= \mathbf{M}\langle\sigma_+(0)\mathbf{s}(\tau)\sigma_-(0)\rangle_{st}\\ &\quad +\mathbf{b}\langle\sigma_+\sigma_-\rangle_{st}\,,\end{aligned}$$

A. Fluctuation Operator Approach

From the Bloch equations $\langle\dot{\mathbf{s}}\rangle=\mathbf{M}\langle\mathbf{s}\rangle+\mathbf{b}$, the steady state ($t\to\infty$) is

$$0=\mathbf{M}\langle\mathbf{s}\rangle_{st}+\mathbf{b}\,,\qquad \to\langle\mathbf{s}\rangle_{st}=\mathbf{M}^{-1}\mathbf{b}\,. \tag{4.1}$$

We split the field (dipole) into a mean plus fluctuations, $\mathbf{s}(t)=\langle\mathbf{s}\rangle_{st}+\Delta\mathbf{s}(t)$ leading to

$$\begin{aligned}\frac{d}{dt}\langle\Delta\mathbf{s}\rangle+0 &= \mathbf{M}\left[\langle\mathbf{s}\rangle_{st}+\langle\Delta\mathbf{s}\rangle\right]+\mathbf{b}\\ &= \mathbf{M}\langle\Delta\mathbf{s}\rangle+\mathbf{M}\langle\mathbf{s}\rangle_{st}+\mathbf{b}\\ \frac{d}{dt}\langle\Delta\mathbf{s}\rangle &= \mathbf{M}\langle\Delta\mathbf{s}\rangle\,.\end{aligned} \tag{4.2}$$

Of course, this is meaningless because, by definition, $\langle\Delta\mathbf{s}\rangle=0$. We have to resort to the QRF to calculate two-time correlations: (i) remove the brackets, multiply $\Delta\mathbf{s}$ from the left and from the right by suitable operators ΔA_+ and ΔA_-, respectively, and (iii) put again the brackets, obtaining

$$\frac{d}{d\tau}\langle\Delta A_+\Delta\mathbf{s}(\tau)\Delta A_-\rangle = \mathbf{M}\langle\Delta A_+\Delta\mathbf{s}(\tau)\Delta A_-\rangle\,, \tag{4.3}$$

with solutions

$$\begin{aligned}\langle\Delta A_+\Delta\mathbf{s}(\tau)\Delta A_-\rangle &= e^{\mathbf{M}\tau}\langle\Delta A_+\Delta\mathbf{s}\Delta A_-\rangle\\ &= S^{-1}e^{\lambda t}S\langle\Delta A_+\Delta\mathbf{s}\Delta A_-\rangle\,.\end{aligned}$$

Again, no exact analytical solutions for the full time-dependent problem are possible.

V. STATIONARY POWER SPECTRUM

The stationary power spectrum is given by the Wiener-Khintchine spectrum, which consists of the Fourier transform of the field autocorrelation function,

$$S(\omega) = \frac{1}{\pi\alpha_{ee}}\mathrm{Re}\int_0^\infty d\tau e^{-i\omega\tau}\langle\sigma_+(0)\sigma_-(\tau)\rangle_{st}\,. \tag{5.1}$$

The factor $(\pi\alpha_{ee})^{-1}$ allows to normalize the spectrum as $\int_{-\infty}^{\infty}S(\omega)d\omega=1$.

To reveal more clearly the role of fluctuations we split the atomic operators into a mean, α_{jk}, plus fluctuations, $\Delta\sigma_{jk}(t)$, that is,

$$\sigma_{jk}(t)=\langle\sigma_{jk}\rangle_{st}+\Delta\sigma_{jk}(t)\,, \tag{5.2}$$

where, by definition, $\langle\Delta\sigma_\pm(t)\rangle=0$, such that

$$\langle\sigma_+(0)\sigma_-(\tau)\rangle_{st} = |\alpha_+|^2+\langle\Delta\sigma_+(0)\Delta\sigma_-(\tau)\rangle_{st}\,. \tag{5.3}$$

At zero interval, $\tau=0$,

$$\langle\sigma_+\sigma_-\rangle_{st} = |\alpha_+|^2+\langle\Delta\sigma_+\Delta\sigma_-\rangle_{st}\,,$$

where the first term on the rhs is the coherent part of the dipole emission, and the second term is the incoherent intensity,

$$\begin{aligned}\langle\Delta\sigma_+\Delta\sigma_-\rangle_{st} &= \alpha_{ee}-|\alpha_+|^2\\ &= \frac{(2+q)\Omega^4}{[(2+q)\Omega^2+\gamma_+^2+4\Delta^2]^2}\,.\end{aligned} \tag{5.4}$$

An important relation is the ratio of coherent (elastic) to incoherent (inelastic) intensity:

$$\begin{aligned}\frac{I_{coh}}{I_{inc}} &= \frac{|\langle\sigma_+\rangle_{st}|^2}{\langle\Delta\sigma_+\Delta\sigma_-\rangle_{st}}\\ &= \frac{\Omega^2[\gamma_+^2+4\Delta^2]/D^2}{(2+q)\Omega^4/D^2}=\frac{\gamma_+^2+4\Delta^2}{(2+q)\Omega^2}\,.\end{aligned} \tag{5.5}$$

This is the inverse of the saturation parameter s, which gives the intensity of the driving field necessary to saturate the atom, where the incoherent intensity is larger than the coherent one. Later, we will explain some results in terms of this parameter.

The splitting (**??**) separates the spectrum in two parts:

$$S(\omega) = S_{coh}(\omega)+S_{inc}(\omega)\,, \tag{5.6}$$

where

$$S_{coh}(\omega) = \frac{|\alpha_+|^2}{\pi\alpha_{ee}}\mathrm{Re}\int_0^\infty e^{-i\omega\tau}d\tau=\frac{|\alpha_+|^2}{\pi\alpha_{ee}}\delta(\omega)\,, \tag{5.7}$$

is the coherent spectrum, due to elastic scattering, and

$$S_{inc}(\omega) = \frac{1}{\pi\alpha_{ee}}\mathrm{Re}\int_0^\infty d\tau e^{-i\omega\tau}\langle\Delta\sigma_+(0)\Delta\sigma_-(\tau)\rangle_{st}\,, \tag{5.8}$$

is the incoherent (inelastic) spectrum, due to atomic fluctuations.

The coherent spectrum is

$$\begin{aligned}S_{coh}(\omega) &= \frac{\Omega^2[\gamma_+^2+4\Delta^2]/D^2}{\pi\Omega^2/D}\delta(\omega)\\ &= \frac{\gamma_+^2+4\Delta^2}{\pi[(2+q)\Omega^2+\gamma_+^2+4\Delta^2]}\delta(\omega)\,.\end{aligned} \tag{5.9}$$

This is smaller than the elastic peak of the 2LA spectrum (where $\gamma_d=q=0$). Evers and Keitel [**?**] shown that the difference is transferred to the incoherent spectrum as a very narrow peak due to the shelving of electronic population in the long-lived state. This was first predicted by Hegerfeldt and Plenio [**?**] and Garraway et al [**?**] for the V and Λ 3LA configurations.

Writing $\pi S_{coh}^{NLA} = I^{NLA}\delta(\omega)$, $N = 2, 3$, the relative intensity of the narrow inelastic peak is

$$\begin{aligned} I_{ep} &= \frac{|\alpha_+^{2LA}|^2}{\alpha_{ee}^{2LA}} - \frac{|\alpha_+^{3LA}|^2}{\alpha_{ee}^{3LA}} \\ &= \frac{\Omega^2[q(\gamma^2+4\Delta^2) - 2\gamma_d\gamma_+]}{[2\Omega^2+\gamma^2+4\Delta^2][(2+q)\Omega^2+\gamma_+^2+4\Delta^2]} . \end{aligned} \tag{5.10}$$

The Rabi frequency for the maximum intensity of the extra peak, from $dI_{ep}/d\Omega = 0$, is given by

$$\Omega_{opt} = \left[\frac{(\gamma^2+4\Delta^2)\left[\gamma_+^2+4\Delta^2\right]}{2(2+q)}\right]^{1/4} . \tag{5.11}$$

Note: Had we chosen not to normalize the spectrum I_{np} would be

$$\begin{aligned} I_{ep} &= \left(|\langle\sigma_{eg}\rangle_{st}|^2\right)_{2LA} - \left(|\langle\sigma_{eg}\rangle_{st}|^2\right)_{3LA} \\ &= \frac{\Omega^4[(2+q)\gamma^2 - 2\gamma_+^2 + 4\Delta^2 q]}{[2\Omega^2+\gamma^2+4\Delta^2]^2[(2+q)\Omega^2+\gamma_+^2+4\Delta^2]^2} . \end{aligned}$$

A. Effects of Shelving

This system features long bright and dark periods if $\gamma \gg \gamma_d, \gamma_a$. A random telegraph model gives the average time of a dark and a bright period [**? ?**]. Here we only write the last few steps. We only need the equation for the trapping state, $\dot{\rho}_{aa} = \gamma_d\rho_{ee} - \gamma_a\rho_{aa}$, evaluated in the quasi-steady state (steady state of the two-level system of the bright transition), here denoted as $t = 0$. For the bright period $\rho_{aa}(0) = 0$ and $\rho_{ee}(0) \to (\rho_{ee}^{st})^{2LA}$,

$$T_B^{-1} = (\dot{\rho}_{aa})_{t=0} , \quad T_B = \frac{2\Omega^2+\gamma^2+4\Delta^2}{\gamma_d\Omega^2} . \tag{5.12}$$

For the dark period $\rho_{aa}(0) = 1$ and $\rho_{ee}(0) \to 0$,

$$T_D^{-1} = (\dot{\rho}_{aa})_{t=0} , \qquad T_D = \gamma_a^{-1} . \tag{5.13}$$

The width of the extra peak is accurately given by

$$\begin{aligned} \Gamma_{ep} &= T_D^{-1} + T_B^{-1} = \frac{(2+q)\Omega^2+\gamma^2+4\Delta^2}{2\Omega^2+\gamma^2+4\Delta^2}\gamma_a \\ &= \gamma_a\left[1 + \frac{q\Omega^2}{2\Omega^2+\gamma^2+4\Delta^2}\right] . \end{aligned} \tag{5.14}$$

Evers and Keitel then assume $I_{ep} = \pi\Gamma_{ep}A_{ep}$, where the width obtained above is rewritten as

$$\Gamma_{ep} = \frac{\Omega^2 + q^{-1}(2\Omega^2+\gamma^2+4\Delta^2)}{2\Omega^2+\gamma^2+4\Delta^2}\gamma_d , \tag{5.15}$$

and the amplitude is

$$A_{ep} = \frac{\Omega^2(\gamma^2+4\Delta^2)/\pi\gamma_d}{[\Omega^2+q^{-1}(2\Omega^2+\gamma^2+4\Delta^2)]^2} . \tag{5.16}$$

Assuming a Lorentzian shape of the extra peak,

$$S_{ep}(\omega) = \frac{I_{ep}}{\pi}\frac{\Gamma_{ep}}{(\omega-\omega_L)^2+\Gamma_{ep}^2} . \tag{5.17}$$

Later we calculate the full spectrum analytically in the resonance case in the limit $\gamma \gg \gamma_d, \gamma_a$. In this case the spectral components have such Lorentzian shape, where $\Gamma_{ep}, A_{ep}, I_{ep}$ are readily identified.

In [**?**] it is also found that $\gamma_d \sim 3.3\gamma_a$ is very close to the optimum value for the appearance of the extra peak.

B. Handling the Integrals - Integrated Spectra

The incoherent spectrum requires solving for the time dependent correlation $\langle\Delta\sigma_+(0)\Delta\sigma_-(\tau)\rangle_{st}$, which can be written as a sum of exponentials. Hence, we have integrals such as

$$\begin{aligned} \int_0^\infty d\tau e^{(-i\omega+\lambda_m)\tau} &= \left.\frac{e^{(-i\omega+\lambda_m)\tau}}{-i\omega+\lambda_m}\right|_0^\infty = \frac{-1}{-i\omega+\lambda_m} \\ \mathrm{Re}\int_0^\infty d\tau e^{(-i\omega+\lambda_m)\tau} &= -\mathrm{Re}\left[\frac{i\omega+\lambda_m}{\omega^2+\lambda_m^2}\right] = \frac{-\lambda_m}{\omega^2+\lambda_m^2} . \end{aligned} \tag{5.18}$$

However, for numerical work, a matrix analysis saves the work of solving the correlation explicitly followed by time integration, that is

$$\begin{aligned} S_{inc}(\omega) &= \frac{1}{\pi\alpha_{ee}}\mathrm{Re}\left[\int_0^\infty d\tau e^{-(i\omega\mathbf{1}-\mathbf{M})\tau}\langle\Delta\sigma_+\Delta\sigma_-\rangle_{st}\right]_1 \\ &= \frac{1}{\pi\alpha_{ee}}\mathrm{Re}\left[-(i\omega\mathbf{1}-\mathbf{M})^{-1}e^{-(i\omega\mathbf{1}-\mathbf{M})\tau}\,|_0^\infty\right. \\ &\quad \left.\times\langle\Delta\sigma_+\Delta\sigma_-\rangle_{st}\right]_1 \\ &= \frac{1}{\pi\alpha_{ee}}\mathrm{Re}\left[(i\omega\mathbf{1}-\mathbf{M})^{-1}\langle\Delta\sigma_+\Delta\sigma_-\rangle_{st}\right]_1 , \end{aligned} \tag{5.19}$$

where $\mathbf{1}$ is the 4×4 identity matrix. The subindex 1 outside the square brackets indicates that only the first element of the resulting vector is to be taken.

Now we sketch the calculation of the intensity of a peak. First, the components of the incoherent spectra can be well approximated by Lorentzians,

$$S_m(\omega) = \frac{\lambda_m}{\omega^2+\lambda_m^2} = \frac{-a_m}{\omega^2+a_m^2} \tag{5.20}$$

since the eigenvalues (**??**) have negative real parts, $\lambda_m = -a_m$. Then, we integrate over all frequencies,

$$\begin{aligned} \int_{-\infty}^\infty \frac{d\omega}{\omega^2+a^2} &= \left.\frac{1}{a}\tan^{-1}\left(\frac{\omega}{a}\right)\right|_{-\infty}^\infty = \frac{1}{a}\left[\frac{\pi}{2} - \frac{-\pi}{2}\right] \\ &= \pi/a . \end{aligned} \tag{5.21}$$

This result is also used to calculate (alternatively) the variance (noise) of a quadrature,

$$\int_{-\infty}^{\infty} S_\phi(\omega)d\omega \propto V_\phi = \langle : (\Delta\sigma_\phi)^2 : \rangle_{st} \,. \qquad (5.22)$$

VI. PHOTON CORRELATIONS

$$g^{(2)}(\tau) = \frac{\langle\sigma_+(0)\sigma_{ee}(\tau)\sigma_-(0)\rangle_{st}}{\alpha_{ee}^2} = \frac{\langle\sigma_{ee}(\tau)\rangle}{\alpha_{ee}} \qquad (6.1)$$

$$= 1 - e^{\lambda_2\tau} + \frac{1+Y^2+(q/2)Y^2}{1+Y^2} f(\tau) \,. \qquad (6.2)$$

The last equation is valid for $\Delta = 0$.

Photon correlation by fluctuation operator approach; using $\sigma_{jk} = \alpha_{jk} + \Delta\sigma_{jk}$ we have

$$\begin{aligned} G^{(2)}(\tau) &= \alpha_{ee}\left(\alpha_+\alpha_- + \langle\Delta\sigma_+\Delta\sigma_-\rangle_{st}\right) \\ &\quad +\alpha_+\langle\Delta\sigma_{ee}(\tau)\Delta\sigma_-(0)\rangle_{st} \\ &\quad +\alpha_-\langle\Delta\sigma_+(0)\Delta\sigma_{ee}(\tau)\rangle_{st} \\ &\quad +\langle\Delta\sigma_+(0)\Delta\sigma_{ee}(\tau)\Delta\sigma_-(0)\rangle_{st} \\ &= \alpha_{ee}^2 + 2\mathrm{Re}\left[\alpha_-\langle\Delta\sigma_+(0)\Delta\sigma_{ee}(\tau)\rangle_{st}\right] \\ &\quad +\langle\Delta\sigma_+(0)\Delta\sigma_{ee}(\tau)\Delta\sigma_-(0)\rangle_{st} \end{aligned}$$

VII. APPROXIMATE ANALYTIC SOLUTIONS

Solutions obtained by Ricardo Román Ancheyta for the on-resonance case, $\Delta = 0$, in the limit $\gamma \gg \gamma_d, \gamma_a$, and reorganized by HMCB.

We define $x_{jk}(\tau) = \langle A_+\sigma_{jk}(\tau)A_-\rangle$, $A_\pm = A_\pm(0)$ for the expectation values of atomic operators and two-time correlations (see Table 1).

$$\begin{aligned} \frac{d}{d\tau}x_- &= -\frac{\gamma_+}{2}x_- + iw(x_{ee} - x_{gg}) \,, \\ \frac{d}{d\tau}x_+ &= -\frac{\gamma_+}{2}x_+ - iw(x_{ee} - x_{gg}) \,, \\ \frac{d}{d\tau}x_{ee} &= iw(x_- - x_+) - \gamma_+ x_{ee} \,, \\ \frac{d}{d\tau}x_{gg} &= -iw(x_- - x_+) + \gamma_- x_{ee} - \gamma_a x_{gg} + \gamma_a\beta \,, \end{aligned}$$

where $w = \Omega/2$. See Table 1 for definitions of β and μ.

A. Solutions in the Laplace Domain

We define the vector in the Laplace space as $(a, b, c, d)^T$. Upon Laplace transform

$$\begin{aligned} sa - x_-(0) &= iw(c-d) - (\gamma_+/2)a \\ sb - x_+(0) &= -iw(c-d) - (\gamma_+/2)b \\ sc - x_{ee}(0) &= iw(a-b) - \gamma_+ c \\ sd - x_{gg}(0) &= -iw(a-b) + \gamma_- c - \gamma_a d + \frac{\gamma_a\beta}{s} \,. \end{aligned}$$

TABLE I: What's wrong with this table? Latex error: Bad math environment. $\mathbf{s} \equiv (\sigma_{ge}, \sigma_{eg}, \sigma_{ee}, \sigma_{gg)^T}$, $\mathbf{x}(0) = (\mu, 0, 0, \beta)^T$, β is a factor in the inhomogeneous term. Add "st" subindex to two-time functions

function $x_{jk}(\tau)$	$x_{jk}(0)$	β	μ	A_-
$\langle\sigma_{jk}(t)\rangle$	$(0,0,0,1)^T$	1	0	**1**
$\langle\sigma_{eg}(0)\sigma_{jk}(\tau)\rangle$	$(\alpha_{ee},0,0,\alpha_{eg})^T$	α_{eg}	α_{ee}	**1**
$\langle\sigma_{eg}(0)\sigma_{jk}(\tau)\sigma_{ge}(0)\rangle$	$(0,0,0,\alpha_{ee})^T$	α_{ee}	0	σ_{ge}
$\langle\Delta\sigma_{eg}(0)\Delta\sigma_{jk}(\tau)\rangle$	see Eq. (**??**)	0	−−	**1**
$\langle\Delta\sigma_{eg}(0)\Delta\sigma_{jk}(\tau)\Delta\sigma_{ge}(0)\rangle$	see Eq. (**??**)	0	−−	$\Delta\sigma_{ge}$

The initial conditions are $\mathbf{x}(0) = (\mu, 0, 0, \beta)^T$,

$$\begin{aligned} a(s+\gamma_+/2) &= iw(c-d) + \mu \\ b(s+\gamma_+/2) &= -iw(c-d) \\ c(s+\gamma_+) &= iw(a-b) \\ d(s+\gamma_a) &= -iw(a-b) + \gamma_- c + \frac{\gamma_a\beta}{s} + \beta\frac{s}{s} \,. \end{aligned}$$

$$a = iw\frac{c-d}{s+\gamma_+/2} + \frac{\mu}{s+\gamma_+/2} \qquad (7.1a)$$

$$b = -iw\frac{c-d}{s+\gamma_+/2} \qquad (7.1b)$$

$$c = iw\frac{a-b}{s+\gamma_+} \qquad (7.1c)$$

$$d = -iw\frac{a-b}{s+\gamma_a} + \gamma_-\frac{c}{s+\gamma_a} + \frac{\beta}{s} \,. \qquad (7.1d)$$

From (8.1a,b)

$$a - b = 2iw\frac{c-d}{s+\gamma_+/2} + \frac{\mu}{s+\gamma_+/2} \,,$$

and substituting it in (8.1c,d):

$$c = -\frac{2w^2(c-d)}{(s+\gamma_+)(s+\gamma_+/2)} + \frac{iw\mu}{(s+\gamma_+)(s+\gamma_+/2)} \,, \qquad (7.2)$$

$$d = \frac{2w^2(c-d)}{(s+\gamma_a)(s+\gamma_+/2)} - \frac{iw\mu}{(s+\gamma_a)(s+\gamma_+/2)} + \gamma_-\frac{c}{s+\gamma_a} + \frac{\beta}{s} \,. \qquad (7.3)$$

From (8.2)

$$\frac{c-d}{s+\gamma_+/2} = -\frac{c(s+\gamma_+)}{2w^2} + \frac{i\mu/2w}{s+\gamma_+/2} \,, \qquad (7.4)$$

and substituting it in (8.3):

$$\begin{aligned} d &= \frac{2w^2}{s+\gamma_a}\left[-\frac{c(s+\gamma_+)}{2w^2}+\frac{i\mu/2w}{s+\gamma_+/2}\right]-\frac{iw\mu}{(s+\gamma_a)(s+\gamma_+/2)}+\gamma_-\frac{c}{s+\gamma_a}+\frac{\beta}{s} \\ &= -\frac{c(s+\gamma_+-\gamma_-)}{s+\gamma_a}+\frac{\beta}{s}+c-c \\ c-d &= c\left[\frac{s+\gamma_d+\gamma_a}{s+\gamma_a}+1\right]-\frac{\beta}{s} \end{aligned} \tag{7.5}$$

Substituting $c-d$ from (8.4) in (8.5)

$$\begin{aligned} c\left[\frac{s+\gamma_d+\gamma_a}{s+\gamma_a}+1+\frac{(s+\gamma_+)(s+\gamma_+/2)}{2w^2}\right] &= \frac{i\mu}{2w}+\frac{\beta}{s} \\ c\left[(s+\gamma_d+\gamma_a)+(s+\gamma_a)+\frac{1}{2w^2}(s+\gamma_a)(s+\gamma_+)(s+\gamma_+/2)\right] &= \frac{i\mu}{2w}(s+\gamma_a)+\frac{\beta(s+\gamma_a)}{s} \\ c\left[2(s+\gamma_a+\gamma_d/2)+\frac{1}{2w^2}(s+\gamma_a)(s+\gamma_+)(s+\gamma_+/2)\right] &= \frac{i\mu}{2w}(s+\gamma_a)+\frac{\beta(s+\gamma_a)}{s}\,. \end{aligned} \tag{7.6a}$$

We've managed to obtain a decoupled expression for $c(s)$. We substitute (8.4) into (8.1a,b):

$$a(s) = iw\left[-\frac{c(s)(s+\gamma_+)}{2w^2}+\frac{i\mu/2w}{s+\gamma_+/2}\right]+\frac{\mu}{s+\gamma_+/2} = -\frac{i}{2w}(s+\gamma_+)c(s)+\frac{\mu/2}{s+\gamma_+/2}\,, \tag{7.6b}$$

$$b(s) = -iw\left[-\frac{c(s)(s+\gamma_+)}{2w^2}+\frac{i\mu/2w}{s+\gamma_+/2}\right] = \frac{i}{2w}(s+\gamma_+)c(s)+\frac{\mu/2}{s+\gamma_+/2}\,, \tag{7.6c}$$

and substitute (8.6b,c) into (8.1d)

$$d(s) = -iw\frac{-ic(s)}{w}\frac{s+\gamma_+}{s+\gamma_a}+\gamma_-\frac{c(s)}{s+\gamma_a}+\frac{\beta}{s} = -\frac{s+\gamma_+-\gamma_-}{s+\gamma_a}c(s)+\frac{\beta}{s}\,. \tag{7.6d}$$

We can simply substitute (8.6a) into (8.6b-d). However, it would be very difficult to obtain the inverse Laplace transform. This does not look amenable for further analytical work. Here comes a brilliant insight from Ricardo (which comes with a price, but he found the way to deal with it –see further below). We first rewrite (8.6a) as

$$\begin{aligned} c(s)\left[\frac{2}{\Omega^2}(s+\gamma_a)(s+\gamma_+)(s+\gamma_+/2)+2(s+\gamma_a+\gamma_d/2)\right] &= \frac{i\mu}{\Omega}(s+\gamma_a)+\frac{\beta(s+\gamma_a)}{s} \\ c(s)\left[(s+\gamma_a)(s+\gamma_+)(s+\gamma_+/2)+\Omega^2(s+\gamma_a+\gamma_d/2)\right] &= \frac{i\mu\Omega}{2}(s+\gamma_a)+\frac{\Omega^2\beta}{2}\frac{(s+\gamma_a)}{s}\,. \end{aligned}$$

By dropping the small term $\Omega^2\gamma_d/2$, the factor $s+\gamma_a$ is eliminated, greatly simplifying the expression for inverse Laplace transform:

$$c(s) = \frac{\Omega^2\beta}{2}\frac{1}{s\left[\left(s+\frac{3\gamma_+}{4}\right)^2-\delta^2\right]}+\frac{i\mu\Omega}{2}\frac{1}{\left[\left(s+\frac{3\gamma_+}{4}\right)^2-\delta^2\right]}\,, \tag{7.7a}$$

where $\delta^2=(\gamma_+/4)^2-\Omega^2$. Substituting (8.7a) into (8.6b-d):

$$a(s) = -i\frac{\Omega\beta}{2}\frac{(s+\gamma_+)}{s\left[\left(s+\frac{3\gamma_+}{4}\right)^2-\delta^2\right]}+\frac{\mu}{2}\frac{(s+\gamma_+)}{\left[\left(s+\frac{3\gamma_+}{4}\right)^2-\delta^2\right]}+\frac{\mu/2}{s+\gamma_+/2}\,, \tag{7.7b}$$

$$b(s) = \frac{i\Omega\beta}{2}\frac{(s+\gamma_+)}{s\left[\left(s+\frac{3\gamma_+}{4}\right)^2-\delta^2\right]}-\frac{\mu}{2}\frac{(s+\gamma_+)}{\left[\left(s+\frac{3\gamma_+}{4}\right)^2-\delta^2\right]}+\frac{\mu/2}{s+\gamma_+/2}\,, \tag{7.7c}$$

$$d(s) = -\frac{\Omega^2\beta}{2}\frac{(s+\gamma_d+\gamma_a)}{s(s+\gamma_a)\left[\left(s+\frac{3\gamma_+}{4}\right)^2-\delta^2\right]}-\frac{i\mu\Omega}{2}\frac{(s+\gamma_d+\gamma_a)}{(s+\gamma_a)\left[\left(s+\frac{3\gamma_+}{4}\right)^2-\delta^2\right]}+\frac{\beta}{s}\,. \tag{7.7d}$$

B. Time-Domain Solutions for Bloch Equations and Third Order Correlations

Equations (8.7) are approximate solutions in the Laplace space. We now proceed to obtain the corresponding solutions in the time domain for $\mu = 0$. We begin with the inverse transform of $c(s)$, the simplest of these solutions:

$$c(s) = \frac{\Omega^2\beta}{2}\frac{1}{s\left[\left(s+\frac{3\gamma_+}{4}\right)^2-\delta^2\right]} = \frac{\Omega^2\beta}{2}\left[\frac{A}{s}+\frac{Bs+C}{\left(s+\frac{3\gamma_+}{4}\right)^2-\delta^2}\right] .$$

We find $A = -B = (\Omega^2+\gamma_+^2/2)^{-1}$, $C = -3\gamma_+ A/2$, so

$$c(s) = \frac{\Omega^2\beta A}{2}\left[\frac{1}{s}-\frac{s+3\gamma_+/2}{\left(s+\frac{3\gamma_+}{4}\right)^2-\delta^2}\right] = \frac{\Omega^2\beta/2}{\Omega^2+\gamma_+^2/2}\left[\frac{1}{s}-\frac{s+3\gamma_+/4}{\left(s+\frac{3\gamma_+}{4}\right)^2-\delta^2}-\frac{3\gamma_+}{4\delta}\frac{\delta}{\left(s+\frac{3\gamma_+}{4}\right)^2-\delta^2}\right] , \quad (7.8)$$

The inverse Laplace transform, $\mathcal{L}^{-1}[c(s)]$, is readily obtained

$$\begin{aligned}\langle\sigma_{ee}(t)\rangle &= \frac{\Omega^2/2}{\Omega^2+\gamma_+^2/2}\left[1-e^{-3\gamma_+t/4}\cosh\delta t-\frac{3\gamma_+}{4\delta}e^{-3\gamma_+t/4}\sinh\delta t\right] \\ &= \frac{\Omega^2/2}{\Omega^2+\gamma_+^2/2}\left\{1-\frac{1}{2}\left[\left(1+\frac{3\gamma_+}{4\delta}\right)e^{\lambda_+t}+\left(1-\frac{3\gamma_+}{4\delta}\right)e^{\lambda_-t}\right]\right\} . \quad (7.9)\end{aligned}$$

$$\begin{aligned}\langle\sigma_+(0)\sigma_{ee}(\tau)\sigma_-(0)\rangle_{st} &= \frac{\Omega^2\alpha_{ee}/2}{\Omega^2+\gamma_+^2/2}\left[1-e^{-3\gamma_+\tau/4}\cosh\delta\tau-\frac{3\gamma_+}{4\delta}e^{-3\gamma_+\tau/4}\sinh\delta\tau\right] \\ &= \frac{\Omega^2\alpha_{ee}/2}{\Omega^2+\gamma_+^2/2}\left\{1-\frac{1}{2}\left[\left(1+\frac{3\gamma_+}{4\delta}\right)e^{\lambda_+\tau}+\left(1-\frac{3\gamma_+}{4\delta}\right)e^{\lambda_-\tau}\right]\right\} . \quad (7.10)\end{aligned}$$

where $\lambda_\pm = -(3\gamma_+/4)\pm\delta$. Now we solve for $a(s)$:

$$a(s) = -\frac{i\Omega\beta}{2}\frac{s+\gamma_+}{s\left[\left(s+\frac{3\gamma_+}{4}\right)^2-\delta^2\right]} = -\frac{i\Omega\beta}{2}\left[\frac{A}{s}+\frac{Bs+C}{\left(s+\frac{3\gamma_+}{4}\right)^2-\delta^2}\right] .$$

We find $A = -B = \gamma_+/(\Omega^2+\gamma_+^2/2)$, $C = 1-\frac{3\gamma_+}{2}A = 1-\frac{3\gamma_+}{4}A-\frac{3\gamma_+}{4}A$, so

$$a(s) = -\frac{i\gamma_+\Omega\beta/2}{\Omega^2+\gamma_+^2/2}\left[\frac{1}{s}-\frac{s+3\gamma_+/4}{\left(s+\frac{3\gamma_+}{4}\right)^2-\delta^2}-\frac{3\gamma_+/4}{\left(s+\frac{3\gamma_+}{4}\right)^2-\delta^2}\right]-\frac{i\Omega\beta}{2}\frac{1}{\left(s+\frac{3\gamma_+}{4}\right)^2-\delta^2} , \quad (7.11)$$

The inverse Laplace transform, $\mathcal{L}^{-1}[a(s)]$, is then obtained

$$\begin{aligned}\langle\sigma_-(t)\rangle &= -\frac{i\gamma_+\Omega/2}{\Omega^2+\gamma_+^2/2}\left[1-e^{-3\gamma_+t/4}\cosh\delta t-\frac{3\gamma_+}{4\delta}e^{-3\gamma_+t/4}\sinh\delta t\right]-\frac{i\Omega}{2\delta}\sinh\delta t \\ &= -\frac{i\gamma_+\Omega/2}{\Omega^2+\gamma_+^2/2}\left\{1-\frac{1}{2}\left[\left(1+\frac{3\gamma_+}{4\delta}\right)e^{\lambda_+t}+\left(1-\frac{3\gamma_+}{4\delta}\right)e^{\lambda_-t}\right]\right\}-\frac{i\Omega}{4\delta}\left(e^{\lambda_+t}-e^{\lambda_-t}\right) , \quad (7.12)\end{aligned}$$

$$\begin{aligned}\langle\sigma_+(0)\sigma_-(\tau)\sigma_-(0)\rangle_{st} &= -\frac{i\gamma_+\Omega\alpha_{ee}/2}{\Omega^2+\gamma_+^2/2}\left[1-e^{-3\gamma_+\tau/4}\cosh\delta\tau-\frac{3\gamma_+}{4\delta}e^{-3\gamma_+\tau/4}\sinh\delta\tau\right]-\frac{i\Omega\alpha_{ee}}{2\delta}\sinh\delta\tau \\ &= -\frac{i\gamma_+\Omega\alpha_{ee}/2}{\Omega^2+\gamma_+^2/2}\left\{1-\frac{1}{2}\left[\left(1+\frac{3\gamma_+}{4\delta}\right)e^{\lambda_+\tau}+\left(1-\frac{3\gamma_+}{4\delta}\right)e^{\lambda_-\tau}\right]\right\}-\frac{i\Omega\alpha_{ee}}{4\delta}\left(e^{\lambda_+\tau}-e^{\lambda_-\tau}\right) . \quad (7.13)\end{aligned}$$

We recall that $\langle\sigma_+(t)\rangle = \langle\sigma_-(t)\rangle^*$ and $\langle\sigma_+(0)\sigma_+(\tau)\sigma_-(0)\rangle_{st} = \langle\sigma_+(0)\sigma_-(\tau)\sigma_-(0)\rangle^*_{st}$. Also

$$\langle\sigma_{gg}(t)\rangle = ?? = 1 - \frac{\Omega^2/2}{\Omega^2+\gamma_+^2/2}\left[1 - e^{-3\gamma_+ t/4}\left(\cosh\delta t + \frac{3\gamma_+}{4\delta}\sinh\delta t\right)\right] \tag{7.14}$$

$$\langle\sigma_+(0)\sigma_{gg}(\tau)\sigma_-(0)\rangle_{st} = ?? \tag{7.15}$$

We mentioned above that dropping a small term came with a price: the system solved is just that of the two-level atom (with γ replaced by $\gamma_+ = \gamma + \gamma_d$), thus missing the small eigenvalue λ_2 corresponding to the slow time scale of our three-level system. To compensate for said approximation, Ricardo proposed to:

- replace the *one* of the first term of the expression between braces, $\{1 + \frac{1}{2}(\cdots)\}$, by $e^{\lambda_2 t}$, i.e., $\{e^{\lambda_2 t} + \frac{1}{2}(\cdots)\}$, where λ_2 is a small eigenvalue defined below;

- add a term $\alpha_{jk}[1 - e^{\lambda_2 t}]$ to the corresponding $\langle\sigma_{jk}(t)\rangle$ and the associated correlation.

These amendments may seem arbitrary but work extremely well provided $\gamma_d, \gamma_a \ll \gamma$ (at least to one order in magnitude). Later we will try to justify them more formally.

The expectation values of the atomic operators are (with $Y = \sqrt{2}\Omega/\gamma_+$):

$$\langle\sigma_\mp(t)\rangle = \mp i\frac{Y/\sqrt{2}}{1+Y^2}f(t) \mp i\frac{\sqrt{2}\gamma_+ Y}{8\delta}\left(e^{\lambda_+ t} - e^{\lambda_- t}\right) + \alpha_\mp\left(1 - e^{\lambda_2 t}\right), \tag{7.16a}$$

$$\langle\sigma_{ee}(t)\rangle = \frac{Y^2/2}{1+Y^2}f(t) + \alpha_{ee}\left(1 - e^{\lambda_2 t}\right), \tag{7.16b}$$

$$\langle\sigma_{gg}(t)\rangle = e^{\lambda_2 t} - \frac{Y^2/2}{1+Y^2}f(t) + \alpha_{gg}\left(1 - e^{\lambda_2 t}\right), \tag{7.16c}$$

where

$$f(t) = e^{\lambda_2 t} - \frac{1}{2}\left[\left(1 + \frac{3\gamma_+}{4\delta}\right)e^{\lambda_+ t} + \left(1 - \frac{3\gamma_+}{4\delta}\right)e^{\lambda_- t}\right], \tag{7.17}$$

$$\lambda_1 = -\gamma_+/2\,, \tag{7.18a}$$

$$\lambda_2 = -\gamma_a\left(1 + q\frac{\Omega^2}{2\Omega^2+\gamma^2}\right), \tag{7.18b}$$

$$\lambda_\pm = -\frac{3\gamma_+}{4} \pm \delta\,, \tag{7.18c}$$

and

$$\delta = (\gamma_+/4)\sqrt{1-8Y^2}\,. \tag{7.19}$$

λ_1 is exact (indeed absent from these results, but appear in the second order correlation); $\lambda_2 = -\Gamma_{ep}$ is approximated from the random telegraph approach, Eq.(**??**), Refs.([**? ?**]); and $\lambda_\pm$ are obtained as a result of the approximations in the Laplace space. These $\lambda_{2,\pm}$ are very close to the exact eigenvalues.

$$\langle\sigma_\mp(t)\rangle = \mp i\frac{Y/\sqrt{2}}{1+Y^2}\left[e^{\lambda_2 t} - e^{-3\gamma_+ t/4}\left(\cosh\delta t + \frac{3\gamma_+}{4\delta}\sinh\delta t\right)\right] \mp i\sqrt{2}Y\frac{\gamma_+}{4\delta}e^{-3\gamma_+ t/4}\sinh\delta t + \alpha_\mp\left(1 - e^{\lambda_2 t}\right), \tag{7.20a}$$

$$\langle\sigma_{ee}(t)\rangle = \frac{Y^2/2}{1+Y^2}\left[e^{\lambda_2 t} - e^{-3\gamma_+ t/4}\left(\cosh\delta t + \frac{3\gamma_+}{4\delta}\sinh\delta t\right)\right] + \frac{Y^2/2}{1+Y^2+(q/2)Y^2}\left(1 - e^{\lambda_2 t}\right), \tag{7.20b}$$

$$\langle\sigma_{gg}(t)\rangle = e^{\lambda_2 t} - \frac{Y^2/2}{1+Y^2}\left[e^{\lambda_2 t} - e^{-3\gamma_+ t/4}\left(\cosh\delta t + \frac{3\gamma_+}{4\delta}\sinh\delta t\right)\right] + \frac{1+(Y^2/2)}{1+Y^2+(q/2)Y^2}\left(1 - e^{\lambda_2 t}\right). \tag{7.20c}$$

C. Time-Domain Solutions for the Second Order Correlations

IN PROCESS. Similarly, we obtain

$$\begin{aligned}
\langle\sigma_+(0)\sigma_\mp(\tau)\rangle_{st} &= \pm\frac{Y^2/2}{(1+Y^2+(q/2)Y^2)^2} \pm \frac{qY^4/4}{(1+Y^2)\,(1+Y^2+(q/2)Y^2)^2}e^{\lambda_2\tau} + \frac{Y^2/4}{1+Y^2+(q/2)Y^2}e^{\lambda_1\tau} \\
&\mp\frac{Y^2[1-Y^2-(1-5Y^2)(\gamma_+/4\delta)]}{8(1+Y^2)\,(1+Y^2+(q/2)Y^2)}e^{\lambda_+\tau} \mp \frac{Y^2[1-Y^2+(1-5Y^2)(\gamma_+/4\delta)]}{8(1+Y^2)\,(1+Y^2+(q/2)Y^2)}e^{\lambda_-\tau} \\
&= \pm\frac{Y^2/2}{(1+Y^2+(q/2)Y^2)^2} + \frac{Y^2/4}{1+Y^2+(q/2)Y^2}e^{\lambda_1\tau} \pm \frac{qY^4/4}{(1+Y^2)\,(1+Y^2+(q/2)Y^2)^2}e^{\lambda_2\tau} \\
&\mp\frac{Y^2/4}{(1+Y^2)\,(1+Y^2+(q/2)Y^2)}e^{-3\gamma_+\tau/4}\left[(1-Y^2)\cosh\delta\tau \mp \frac{1-5Y^2}{4\delta/\gamma_+}\sinh\delta\tau\right]. \qquad (7.21)
\end{aligned}$$

which we use to obtain the solutions for the correlations of fluctuations,

$$\begin{aligned}
\langle\Delta\sigma_+(0)\Delta\sigma_\mp(\tau)\rangle_{st} &= \langle\sigma_+(0)\sigma_\mp(\tau)\rangle_{st} - \langle\sigma_+\rangle_{st}\langle\sigma_\mp\rangle_{st}\,, \\
&= C_1e^{\lambda_1\tau} \mp C_+e^{\lambda_+\tau} \mp C_-e^{\lambda_-\tau} \pm C_2e^{\lambda_2\tau}\,, \qquad (7.22)
\end{aligned}$$

where

$$C_1 = \frac{1}{4}\frac{Y^2}{1+Y^2+(q/2)Y^2}\,, \qquad (7.23a)$$

$$C_2 = \frac{q}{4}\frac{Y^4}{(1+Y^2)\,(1+Y^2+(q/2)Y^2)^2}\,, \qquad (7.23b)$$

$$C_\mp = \frac{1}{8}\frac{Y^2[1-Y^2\pm(1-5Y^2)(\gamma_+/4\delta)]}{(1+Y^2)\,(1+Y^2+(q/2)Y^2)}\,. \qquad (7.23c)$$

VIII. EIGENVALUES AND EIGENVECTORS ON-RESONANCE

$$|\mathbf{M}-\lambda\mathbf{1}| = \begin{vmatrix} -\frac{1}{2}\gamma_+-\lambda & 0 & i\Omega/2 & -i\Omega/2 \\ 0 & -\frac{1}{2}\gamma_+-\lambda & -i\Omega/2 & i\Omega/2 \\ i\Omega/2 & -i\Omega/2 & -\gamma_+-\lambda & 0 \\ -i\Omega/2 & i\Omega/2 & \gamma_- & -\gamma_a-\lambda \end{vmatrix} = 0\,.$$

We take the first column of the determinant above to obtain the coefficients of the minors:

$$\begin{aligned}
0 &= -(\frac{1}{2}\gamma_++\lambda)\begin{vmatrix} -\frac{1}{2}\gamma_+-\lambda & -i\Omega/2 & i\Omega/2 \\ -i\Omega/2 & -\gamma_+-\lambda & 0 \\ i\Omega/2 & \gamma_- & -\gamma_a-\lambda \end{vmatrix} + 0 \\
&+i\frac{\Omega}{2}\begin{vmatrix} 0 & i\Omega/2 & -i\Omega/2 \\ -\frac{1}{2}\gamma_+-\lambda & -i\Omega/2 & i\Omega/2 \\ i\Omega/2 & \gamma_- & -\gamma_a-\lambda \end{vmatrix} - \left(-i\frac{\Omega}{2}\right)\begin{vmatrix} 0 & i\Omega/2 & -i\Omega/2 \\ -\frac{1}{2}\gamma_+-\lambda & -i\Omega/2 & i\Omega/2 \\ -i\Omega/2 & -\gamma_+-\lambda & 0 \end{vmatrix}.
\end{aligned}$$

We chose the third, first and first columns, respectively, of the above minors to obtain the next minors:

$$\begin{aligned}
0 &= -(\frac{1}{2}\gamma_++\lambda)\left[i\frac{\Omega}{2}\begin{vmatrix} -i\Omega/2 & -\gamma_+-\lambda \\ i\Omega/2 & \gamma_- \end{vmatrix} - (\gamma_a+\lambda)\begin{vmatrix} -\frac{1}{2}\gamma_+-\lambda & -i\Omega/2 \\ -i\Omega/2 & -\gamma_+-\lambda \end{vmatrix}\right] \\
&+i\frac{\Omega}{2}\left[(\frac{1}{2}\gamma_++\lambda)\begin{vmatrix} i\Omega/2 & -i\Omega/2 \\ \gamma_- & -\gamma_a-\lambda \end{vmatrix} + i\frac{\Omega}{2}\begin{vmatrix} i\Omega/2 & -i\Omega/2 \\ -i\Omega/2 & i\Omega/2 \end{vmatrix}\right] \\
&+i\frac{\Omega}{2}\left[(\frac{1}{2}\gamma_++\lambda)\begin{vmatrix} i\Omega/2 & -i\Omega/2 \\ -\gamma_+-\lambda & 0 \end{vmatrix} - i\frac{\Omega}{2}\begin{vmatrix} i\Omega/2 & -i\Omega/2 \\ -i\Omega/2 & i\Omega/2 \end{vmatrix}\right]
\end{aligned}$$

$$\begin{aligned}
0 &= (\frac{1}{2}\gamma_+ + \lambda)\left\{-i\frac{\Omega}{2}\left[-i\frac{\Omega}{2}\gamma_- + i\frac{\Omega}{2}(\gamma_+ + \lambda)\right] + (\gamma_a + \lambda)\left[(\frac{\gamma_+}{2} + \lambda)(\gamma_+ + \lambda) + \frac{\Omega^2}{4}\right]\right\} \\
&\quad + (\frac{1}{2}\gamma_+ + \lambda)\left\{i\frac{\Omega}{2}\left[-i\frac{\Omega}{2}(\gamma_a + \lambda) + i\frac{\Omega}{2}\gamma_-\right] + 0\right\} + (\frac{1}{2}\gamma_+ + \lambda)\left\{i\frac{\Omega}{2}\left[0 - i\frac{\Omega}{2}(\gamma_+ + \lambda)\right] + 0\right\}
\end{aligned}$$

We readily find the eigenvalue $\lambda_1 = -\gamma_+/2$. Now we calculate the corresponding eigenvector:

$$(\mathbf{M} - \lambda_1 \mathbf{1})\mathbf{r}_1 = \begin{pmatrix} 0 & 0 & i\Omega/2 & -i\Omega/2 \\ 0 & 0 & -i\Omega/2 & i\Omega/2 \\ i\Omega/2 & -i\Omega/2 & -\gamma_+/2 & 0 \\ -i\Omega/2 & i\Omega/2 & \gamma_- & (\gamma_+ - 2\gamma_a)/2 \end{pmatrix} \begin{pmatrix} r_{11} \\ r_{12} \\ r_{13} \\ r_{14} \end{pmatrix} = 0$$

$$\begin{aligned}
0 &= 0(r_{11} + r_{12}) + i\frac{\Omega}{2}(r_{13} - r_{14}), \qquad \rightarrow r_{11} = r_{12} = 1, \qquad r_{13} = r_{14}, \\
0 &= i\frac{\Omega}{2}(r_{11} - r_{12}) - \frac{\gamma_+}{2}r_{13} + 0r_{14}, \qquad \rightarrow r_{13} = r_{14} = 0, \\
0 &= -i\frac{\Omega}{2}(r_{11} - r_{12}) + \gamma_- r_{13} + (\frac{\gamma_+}{2} - \gamma_a)r_{14},
\end{aligned}$$

We write the first eigenvector as $\mathbf{r}_1 = (1, 1, 0, 0)^T$.

The characteristic equation for the other three eigenvalues is

$$\begin{aligned}
0 &= \left\{-i\frac{\Omega}{2}\left[-i\frac{\Omega}{2}(\gamma - \gamma_a) + i\frac{\Omega}{2}(\gamma + \gamma_d + \lambda)\right] + (\gamma_a + \lambda)\left[(\frac{\gamma + \gamma_d}{2} + \lambda)(\gamma + \gamma_d + \lambda) + \frac{\Omega^2}{4}\right]\right\} \\
&\quad + \left\{i\frac{\Omega}{2}\left[-i\frac{\Omega}{2}(\gamma_a + \lambda) + i\frac{\Omega}{2}(\gamma - \gamma_a)\right]\right\} + \left\{i\frac{\Omega}{2}\left[-i\frac{\Omega}{2}(\gamma + \gamma_d + \lambda)\right]\right\} \\
&= \frac{\Omega^2}{4}(2\gamma_a + \gamma_d + 2\lambda) + (\gamma_a + \lambda)(\frac{\gamma + \gamma_d}{2} + \lambda)(\gamma + \gamma_d + \lambda) + \frac{\Omega^2}{4}(2\gamma_a + \gamma_d + 2\lambda) \\
&= \Omega^2(\lambda + \gamma_a + \frac{\gamma_d}{2}) + (\gamma_a + \lambda)\left[\frac{(\gamma + \gamma_d)^2}{2} + 3\lambda\frac{(\gamma + \gamma_d)}{2} + \lambda^2\right] \\
0 &= \lambda^3 + \lambda^2\left[\gamma_a + \frac{3\gamma_+}{2}\right] + \lambda\left[\Omega^2 + \frac{3\gamma_a\gamma_+}{2} + \frac{\gamma_+^2}{2}\right] + \left[\Omega^2(\gamma_a + \frac{\gamma_d}{2}) + \frac{\gamma_a\gamma_+^2}{2}\right].
\end{aligned}$$

Mathematica gives exact expressions for $\lambda_{2,+,-}$ (and their eigenvectors) but they are too complicated. The approximated expressions Eqs.(**??**b,c), obtained in Sections VI-A and IX are, however, very good. PLOT comparisons ...

NEXT: link and order references, etc,

IX. REPLACING THE LASER WITH AN INCOHERENT FIELD

Here we revisit the model studied in the previous sections by replacing the laser driving the $|g\rangle - |e\rangle$ transition with a broadband incoherent field. Instead of the atom-laser Hamiltonian we have now a new term in the Liouvillian describing excitation by blackbody radiation of average photon number $\bar{n}$. In fact, this would be produced by a laser when its phase has been scrambled. The spontaneous emission scheme is the same as in the laser problem.

The master equation is

$$\begin{aligned}
\dot{\rho}(t) &= \kappa(2\sigma_{eg}\rho\sigma_{ge} - \sigma_{ge}\sigma_{eg}\rho - \rho\sigma_{ge}\sigma_{eg}) \\
&\quad + \frac{\gamma}{2}(2\sigma_{ge}\rho\sigma_{eg} - \sigma_{eg}\sigma_{ge}\rho - \rho\sigma_{eg}\sigma_{ge}) \\
&\quad + \frac{\gamma_d}{2}(2\sigma_{ae}\rho\sigma_{ea} - \sigma_{ea}\sigma_{ae}\rho - \rho\sigma_{ea}\sigma_{ae}) \\
&\quad + \frac{\gamma}{2}(2\sigma_{ga}\rho\sigma_{ag} - \sigma_{ag}\sigma_{ga}\rho - \rho\sigma_{ag}\sigma_{ga}), \quad (9.1)
\end{aligned}$$

where $\kappa = \gamma\bar{n}/2$ is the incoherent excitation rate. This can be simplified as

$$\begin{aligned}
\dot{\rho}(t) &= \kappa(2\rho_{gg}\sigma_{ee} - \sigma_{gg}\rho - \rho\sigma_{gg}) \\
&\quad - \frac{\gamma_+}{2}(\sigma_{ee}\rho + \rho\sigma_{ee}) - \frac{\gamma_a}{2}(\sigma_{aa}\rho + \rho\sigma_{aa}) \\
&\quad + \gamma\rho_{ee}\sigma_{gg} + \gamma_d\rho_{ee}\sigma_{aa} + \gamma_a\rho_{aa}\sigma_{gg}. \quad (9.2)
\end{aligned}$$

The equations for the populations are decoupled from those of the coherences. Moreover, the latter are zero at all times. The relevant equations are ($\gamma_+ = \gamma + \gamma_d$)

$$\begin{aligned}
\dot{\rho}_{ee} &= 2\kappa\rho_{gg} - \gamma_+\rho_{ee}\,, && (9.3a)\\
\dot{\rho}_{gg} &= -2\kappa\rho_{gg} + \gamma\rho_{ee} + \gamma_a\rho_{aa}\,, && (9.3b)\\
\dot{\rho}_{aa} &= \gamma_d\rho_{ee} - \gamma_a\rho_{aa}\,. && (9.3c)
\end{aligned}$$

Using the conservation of population, $\rho_{aa} = 1 - \rho_{ee} - \rho_{gg}$, the system of equations is reduced to

$$\begin{aligned}
\dot{\rho}_{ee} &= 2\kappa\rho_{gg} - \gamma_+\rho_{ee}\,, && (9.4a)\\
\dot{\rho}_{gg} &= -(2\kappa + \gamma_a)\rho_{gg} + \gamma_-\rho_{ee} + \gamma_a\,. && (9.4b)
\end{aligned}$$

where $\gamma_- = \gamma - \gamma_a$.

These equations can be written in matrix form:

$$\dot{\mathbf{r}}(t) = \mathbf{M}\mathbf{r}(0) + \mathbf{b}\,, \qquad (9.5)$$

where

$$\mathbf{r} = \begin{pmatrix} \rho_{ee} \\ \rho_{gg} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ \gamma_a \end{pmatrix},$$
$$\mathbf{M} = \begin{pmatrix} -\gamma_+ & 2\kappa \\ \gamma_- & -(2\kappa + \gamma_a) \end{pmatrix}.$$

The steady state solutions (where $\dot{\rho}_{jj} = 0$) are:

$$\begin{aligned}
\rho_{ee}^{st} &= \frac{2\kappa\gamma_a}{\gamma_+\gamma_a + 2\kappa(\gamma_d + \gamma_a)}\,, && (9.6a)\\
\rho_{gg}^{st} &= \frac{\gamma_+\gamma_a}{\gamma_+\gamma_a + 2\kappa(\gamma_d + \gamma_a)}\,, && (9.6b)\\
\rho_{aa}^{st} &= \frac{2\kappa\gamma_d}{\gamma_+\gamma_a + 2\kappa(\gamma_d + \gamma_a)}\,. && (9.6c)
\end{aligned}$$

It can be verified that the populations add to one.

Using Mathematica, with the FullSimplify option, the eigenvalues of $\mathbf{M}$ are found to be

$$\begin{aligned}
\lambda_\pm &= -\frac{2\kappa + \gamma_+ + \gamma_a}{2} \\
&\quad \pm\frac{1}{2}\sqrt{8\kappa\gamma_- + (2\kappa - \gamma_+ + \gamma_a)^2}\,, && (9.7)\\
\lambda_+ &\ll \lambda_-\,. && (9.8)
\end{aligned}$$

(plot vs κ). Try to simplify and/or write special cases.

$$f(t) = Ae^{\lambda_+ t} + Be^{\lambda_- t}$$

RIVE: (a) Hacer una figura del nuevo sistema de tres niveles. (b) Verifica las ecuaciones y soluciones en estado estacionario. (c) Graficar ρ_{ee}^{st}, ρ_{gg}^{st} y ρ_{aa}^{st} en función de κ/γ. (d) Obtener las soluciones para $\rho_{ee}(t)$ y $\rho_{gg}(t)$ (T. de Laplace o con Mathematica). (e) Obtener y graficar T_B en función de κ/γ. (f) Comparar poblaciones entre casos láser y lámpara. (g) Graficar $\lambda_\pm$.

[1] J. Evers and Ch. H. Keitel, Phys. Rev. A **65**, 033813 (2002).

[2] H. M. Castro-Beltran, R. Roman-Ancheyta, and L. Gutierrez, Phys. Rev. A **93**, 033801 (2016).

[3] H. J. Carmichael and D. F. Walls, J. Phys. B **9**, 1199 (1976).

[4] A. Denisov, H. M. Castro-Beltran, and H. J. Carmichael, Phys. Rev. Lett. **88**, 243601 (2002).

[5] H. J. Carmichael, *Statistical Methods in Quantum Optics 1: Master Equations and Fokker-Planck Equations* (Springer-Verlag, Berlin, 1999).

[6] D. T. Pegg and P. L. Knight, "Interrupted fluorescence, quantum jumps, and wave-function collapse," Phys. Rev. A **37**, 4303 (1988).

[7] M. B. Plenio and P. L. Knight, Rev. Mod. Phys. **70**, 101 (1997).

[8] V. Bühner and Chr. Tamm, Phys. Rev. A **61**, 061801 (2000).

[9] G. C. Hegerfeldt and M.B. Plenio, Phys. Rev. A **52**, 3333 (1995).

[10] B. M. Garraway, M. S. Kim, and P. L. Knight, Optics Commun. **117**, 550 (1995).

[11] M. Merz and A. Schenzle, Appl. Phys. B **50**, 115 (1990).