Chapter 7

Bonds

7a Bonds and Other Investments

This chapter covers how to determine the price of a bond to earn a given yield rate, and how to determine the yield rate for a bond selling at a given price.

The next chapter gives an overview of different types of investments, such as bonds, stocks, mutual funds, certificates of deposit (CDs), etc.

For this chapter, the main thing you have to know about bonds is this: They are a means of borrowing money that is frequently used by the federal government, corporations, states, and municipalities. In this form of borrowing, the lenders or investors (i.e., the people or institutions who buy the bonds) receive interest payments (usually called "coupons") for a fixed period of years (the term of the bond). At the end of the term, the lenders receive the original amount of the loan back (or an amount close to it).

7b Finding the Price of a Bond

As noted above, a bond is basically a loan to a governmental entity or corporation on which you typically receive interest payments called "coupons," and then receive the redemption value on the redemption date. The interest payments are called "coupons" because there was a time when many bonds were issued with coupons physically attached to them. Each coupon represented one interest payment to be received on a specific date. The owner of the bond would clip a coupon and cash it in on the due date. Curiously, even though bonds are no longer issued with coupons (although some unmatured coupon bonds are still out there), the interest payments are still referred to as "coupons."

To make sure that you understand bond terminology, here is a description of two bonds. The question is, if you purchased these bonds, what future payments would you receive?

Bond #1: A \$1,000 bond with 8% semiannual coupons maturing in 10 years.

Bond #2: A \$100 par value bond with $5\frac{1}{2}\%$ annual coupons maturing at \$110 in 10 years.

Answers:

Bond #1: You would receive coupons of \$40 at the end of every 6-months for 10 years (***including*** a coupon at the end of 10 years) and the redemption value of \$1,000 at the end of 10 years.

A few comments:

1. The coupon rate is always given as an ***annual*** rate, so "8% semiannual coupons" means a total coupon of \$80 (8% × \$1,000) in each year but payable semiannually, or \$40 every 6 months.
2. Most bonds have semiannual coupons.
3. A coupon is always payable on the redemption date, in addition to the redemption value.
4. The ***par*** or ***face value*** is the unit in which the bond is issued (normally a round amount like \$100 or \$1,000). This is the value that should be multiplied by the coupon rate to determine the amount of coupon.

5. The par value of Bond #1 is given as \$1,000. The description of Bond #1 did not explicitly say that the bond is redeemable at par. However, here is what "Notation and terminology used for Exam FM," available at `BeAnActuary.org`, says:

   > Unless otherwise stated in the examination question, the redemption value of a bond is equal to the face amount (par value) of the bond.

Bond #2: You would receive annual coupons of \$5.50 ($5\frac{1}{2}\% \times \$100$) for 10 years, and the redemption value of \$110 at the end of 10 years.

A few comments:

1. This bond is different from most because coupons are paid annually.
2. The coupon is always determined by multiplying the ***par or face value*** by the coupon rate. So the coupon for Bond #2 is $5\frac{1}{2}\% \times \$100$, ***not*** $5\frac{1}{2}\% \times \$110$.

At this point, it would be very helpful to have symbols for the various bond values:

- P is the price of the bond.
- F is the face amount (or par value).
- C is the redemption value. (C is equal to F unless stated otherwise.)
- r is the coupon rate per payment period, i.e., r is applied to F to determine the amount of the coupon. Thus, the amount of the coupon is Fr.
- g is a special coupon rate that is useful in mathematical formulas. It is the coupon rate applied to C to determine the amount of the coupon, i.e., $Cg = Fr$. (If a bond is redeemable at par, $C = F$ and $g = r$.)
- n is the number of remaining coupon payments.
- i is the effective rate of interest per payment period (the "yield-to-maturity") for a bond selling at price P. In other words, i is the interest rate at which P is equal to the present value of the bond payments.

Some comments about this notation:

1. Note that different symbols are used for par or face value (F) and redemption value (C), since a bond is not necessarily redeemable at par.
2. The coupon rate r is the rate *per coupon payment period*. For example, for Bond #1, the coupon rate is quoted as "8% semiannual" which means that the amount of each semiannual coupon is 4% of the par value. So in this case, $r = 4\%$.
3. The yield rate i is the effective rate of interest earned *per coupon payment period*.
4. Be careful to distinguish between r and i. Note that r is a *fixed feature* of the bond that determines the coupon payments. It does not change, regardless of the bond's selling price. On the other hand, i is the *yield rate* you would earn on the bond if you purchased it at price P. Obviously, the yield rate varies as the selling price of the bond varies.
5. A special coupon rate g is defined as the coupon rate per unit of *redemption value*, rather than per unit of par value. g is used strictly for formula work: You will never see it in a broker's or newspaper's description of a bond. In the case of Bond #1, $r = g = 4\%$, since it is redeemable at par. In the case of Bond #2, the coupon rate g, expressed as a rate per unit of redemption value, is the coupon divided by the redemption value: $g = \frac{5.50}{110} =$ 5%.
6. By definition of r and g, we have: coupon = $Fr = Cg$.

Why invent a special coupon rate? We will see below that it is *very* useful in writing bond formulas and solving certain types of problems.

Bond Price Formulas

Several formulas can be derived for the price of a bond. Two of these are the most useful formulas for solving problems (at least for the type of problems that are on the syllabus). Naturally, all of the formulas must be algebraically equivalent.

Basic Formula

The price of a bond to yield an effective rate i is the PV of the bond payments at that rate. The PV of the payments is the PV of the coupons plus the PV of the redemption value:

$$P = Fr\, a_{\overline{n}|} + Cv^n$$

or

$$P = Cg\, a_{\overline{n}|} + Cv^n$$

Let's apply this formula to Bond #1 at a yield rate of 6% compounded semiannually and Bond #2 at a yield rate of 4% effective per annum.

Bond #1:

$$P = 40a_{\overline{20}|} + 1{,}000v^{20} \quad \text{at } 3\%$$

Using the calculator: 20 $\boxed{\text{N}}$ 3 $\boxed{\text{Y}}$ 40 $\boxed{\text{PMT}}$ 1,000 $\boxed{\text{FV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{PV}}$

The answer is $P = \$1{,}148.77$ (displayed as a negative).

Bond #2:

$$P = 5.50a_{\overline{10}|} + 110v^{10} \quad \text{at } 4\%$$

Using the calculator: 10 $\boxed{\text{N}}$ 4 $\boxed{\text{I/Y}}$ 5.50 $\boxed{\text{PMT}}$ 110 $\boxed{\text{FV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{PV}}$

The answer is $P = \$118.92$ (displayed as a negative).

Stepping Stones

Example 1
A \$1,000 bond maturing in 7 years has $X\%$ semiannual coupons and sells for \$1,058.45 to yield a nominal annual rate of 5% compounded semiannually. Determine X.

Solution

$$1{,}058.45 = 5X a_{\overline{14}|.025} + 1{,}000v^{14}$$

Note: Since $X\%$ is the annualized coupon rate, the semiannual coupon is equal to:

$$\tfrac{1}{2} \times \frac{X}{100} \times 1{,}000 = 5X$$

Using the calculator: 14 $\boxed{\text{N}}$ 2.50 $\boxed{\text{I/Y}}$ 1,058.45 $\boxed{\text{PV}}$ 1,000 $\boxed{\text{+/-}}$ $\boxed{\text{FV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{PMT}}$

PMT = \$30, so $5X = 30$ and $X = 6$.

Example 2
A \$500 bond with 4.5% annual coupons matures in 12 years with a redemption value X. It is purchased for \$505.68 at an effective annual yield rate of 5%. Determine X.

Solution
The annual coupon is $.045 \times 500 = \$22.50$.

12 $\boxed{\text{N}}$ 5 $\boxed{\text{I/Y}}$ 505.68 $\boxed{\text{PV}}$ 22.50 $\boxed{\text{+/-}}$ $\boxed{\text{PMT}}$ $\boxed{\text{CPT}}$ $\boxed{\text{FV}}$

The redemption value is \$550.

At this point, turn to Calculator Notes #9 immediately following this section for practice in computing the price of a bond (or the yield rate, given the price).

The Premium/Discount Formula

A simple algebraic substitution in the basic formula leads us to the next formula. We know that $a_{\overline{n}|} = \frac{1-v^n}{i}$, so $v^n = 1 - ia_{\overline{n}|}$:

$$\begin{aligned} P &= Fra_{\overline{n}|} + Cv^n \\ &= Fra_{\overline{n}|} + C(1 - ia_{\overline{n}|}) \\ &= C + (Fr - Ci)a_{\overline{n}|} \end{aligned}$$

or

$$P = C + (Cg - Ci)a_{\overline{n}|}$$

We want to emphasize that this formula is *very* useful for solving various types of bond problems. While it may not seem as intuitive as the basic formula at first, you should get to know it as thoroughly as you can.

A verbal explanation of this formula, which may help you remember it, is as follows:

Consider a hypothetical bond with coupons at the special coupon rate g equal to i, the same redemption value C and the same term n as the actual bond. (The amount of each coupon is *Ci*.) What price should you pay for this hypothetical bond to get a yield of i?

The answer is C. It's like buying a $5\frac{1}{2}\%$ bond with redemption value $110 and $5.50 coupons. The special coupon rate $g = \frac{5.50}{110} = 5\%$. If you pay $110 for this bond, you will earn 5% effective, since you get 5% a year on your investment of $110 ($5.50) and you receive your $110 back at the end of the term.

Why did we construct this hypothetical bond? Because its price C can be determined easily without any computations and because we can then compare this price with the price of the actual bond. The only difference between the actual and hypothetical bonds is their coupons: *Fr* (or *Cg*) for the actual bond and *Ci* for the hypothetical bond. So the only difference in price is the PV of the excess (positive or negative) of the actual coupons over the hypothetical coupons:

$$\begin{aligned} P - C &= (Fr - Ci)a_{\overline{n}|} \\ P &= C + (Fr - Ci)a_{\overline{n}|} \end{aligned}$$

or

$$P = C + (Cg - Ci)a_{\overline{n}|}$$

We will now verify that this formula gives the same numerical answers for Bond #1 and Bond #2 that the Basic Formula does:

Bond #1:

$$\begin{aligned} P &= 1{,}000 + [40 - (1{,}000)(.03)]a_{\overline{20}|} \quad \text{at } 3\% \\ &= 1{,}000 + 10a_{\overline{20}|} = 1{,}148.77 \end{aligned}$$

which agrees with the previous result for Bond #1.

Bond #2:

$$\begin{aligned} P &= 110 + [5.50 - (110)(.04)]a_{\overline{10}|} \text{ at } 4\% \\ &= 110 + 1.10a_{\overline{10}|} = 118.92, \end{aligned}$$

which agrees with the previous result for Bond #2.

Trap Alert!

There is a trap that is easy to fall into in applying the premium/discount formula: Using F instead of C for the term *Ci*. For example, in calculating the price of Bond #2 (just above), the trap would be:

$$\begin{aligned} P &= 110 + [5.50 - (100)(.04)]a_{\overline{10}|} \\ &= 110 + 1.50a_{\overline{10}|} \\ &= 122.17 \end{aligned}$$

Make sure that you always plug C into this formula. (Of course, if the bond matures at par, there is no trap to fall into.)

One use of the premium/discount formula is to determine the "premium" or "discount" paid for a bond (hence the name of the formula). We will define these terms and cover this subject in the next section.

Example 3
A \$1,000 bond with 6% annual coupons matures in 15 years and sells at a price P that yields 5% effective. Without actually calculating P, quickly determine whether P is less than or greater than C.

Solution
Use the premium/discount formula: $P = C + (Fr - Ci)a_{\overline{n}|}$. Since the bond matures at par (which we can assume since the example does not state otherwise), $C = F$, $Fr = (.06)(1{,}000) = \$60$, and $Ci = (1{,}000)(.05) = \$50$. Since $Fr > Ci$, $P = C$ plus a positive amount, i.e., $P > C$.

Example 4
A \$1000 bond with 6% semiannual coupons matures in 20 years with a redemption value of \$1,100. It sells at a price P to yield a nominal rate of interest of 5.80%, compounded semiannually. Without actually calculating P, quickly determine whether P is less than or greater than C.

Solution
Again, use the premium/discount formula. We have:

$$\begin{aligned} Fr &= (1{,}000)(.03) = \$30 \\ Ci &= (1{,}100)(.029) = \$31.90 \end{aligned}$$

Since $Fr < Ci$, $P = C$ plus a negative amount, i.e., $P < C$.

The Makeham Formula

The major advantage of the Makeham formula is its power in handling "serial bond" problems. A serial bond can be described as a group of bonds with the same coupon rate but redeemable in installments, i.e., with staggered redemption dates. However, serial bonds are no longer on the syllabus for Exam FM. For completeness, we will briefly cover it in this manual.

We start with the basic formula in the "*Cg*" form:

$$P = Cga_{\overline{n}|} + Cv^n$$

Substitute $a_{\overline{n}|} = \frac{1-v^n}{i}$:

$$P = Cg\left(\frac{1-v^n}{i}\right) + Cv^n = \frac{g}{i}(C - Cv^n) + Cv^n$$

Let $K = PV$ of the redemption value, i.e.:

$$K = Cv^n$$

Then $P = \frac{g}{i}(C-K) + K$, usually written as $P = K + \frac{g}{i}(C-K)$.

The usefulness of this formula is certainly not evident. In fact, it seems to be unrecognizable as an expression for the price of a bond (although it has a verbal explanation). But as we said before, this formula can only be appreciated in the context of serial bonds, which are not on the syllabus.

We'll apply the Makeham formula to Bond #1 at a yield rate of 6% compounded semiannually and to Bond #2 at a yield rate of 4%.

Bond #1:

$$\begin{aligned} P &= K + \frac{g}{i}(C-K) \\ K &= Cv^n = 1{,}000v_{.03}^{20} = 553.68 \\ g &= \frac{Fr}{c} = \frac{(1{,}000)(.04)}{1{,}000} = .04 \\ P &= 553.68 + \frac{.04}{.03}(1{,}000 - 553.68) \\ &= \$1{,}148.77 \end{aligned}$$

Bond #2:

$$\begin{aligned} K &= Cv^n = 110v^{10} \text{ at } 4\% = 74.31 \\ g &= \frac{Fr}{C} = \frac{5.50}{110} = .05 \\ P &= 74.31 + \frac{.05}{.04}(110 - 74.31) = \$118.92 \end{aligned}$$

These results for Bond #1 and Bond #2 are the same as those given by the first two formulas.

Calculator Notes #9: Bonds

We can find the price of a bond to earn a given yield rate (or the yield rate that would be earned at a given price) using the TVM keys or the bond worksheet of the BA II Plus calculator.

48. Find the price of the two bonds described in Section 7b.

    Bond #1: Price to yield a nominal rate of 6% compounded semiannually for a \$1,000 bond with 8% semiannual coupons redeemable at par in 10 years.

    Bond #2: Price to yield 4% effective for a \$100 bond with 5.5% annual coupons redeemable at \$110 in 10 years.

a. Using the TVM keys

Bond #1: Enter 20 [N] 3 [I/Y] 40 [PMT] 1,000 [FV] [CPT] [PV]

The price is 1,148.77

Bond #2: Enter 10 [N] 4 [I/Y] 5.50 [PMT] 110 [FV] [CPT] [PV]

The price is 118.92. Note that this agrees with the answer in Section 7b.

b. Using the bond worksheet

The bond worksheet is accessed by pressing [2nd] [BOND]. (The [BOND] key is secondary to the 9 key.)

If you scroll down using the arrow key, you will see a number of labels (SDT, CPN, etc.) that are specific to bonds. In my opinion, it's just as easy (maybe even a little easier) to use the TVM keys to do bond calculations. But if you would like to know how to use the bond worksheet, by all means learn how to do it. A summary of the labels in the worksheet and what they mean appears in the TI booklet that comes with the calculator. The booklet then shows how to enter data and do bond calculations. You will find an additional example of using the bond worksheet in "Review of Calculator Functions for the TI BA II Plus" on pages 19–22, available at `BeAnActuary.org`.

The bond worksheet can be used to compute prices between coupon payment dates, as well as on a coupon due date. (There have been very few questions of this type on past exams, but this is definitely on the syllabus.) This topic is covered in Section 7d of this manual. As you will see in that section, you can calculate prices between coupon due dates without using the bond worksheet.

49. A \$1,000 bond with 6% semiannual coupons is redeemable at \$1,050 in 20 years. The bond is selling for \$975.00.

    (a) What is the nominal annual yield rate convertible semiannually to a purchaser who buys the bond at this price?

    (b) What price should be paid to earn a nominal annual rate of 7% convertible semiannually?

    (a) To determine the earned rate, we enter:

    40 [N] 975 [+/-] [PV] 30 [PMT] 1,050 [FV] [CPT] [I/Y] [×] 2 [=]

    The answer is 6.35% to 2 decimal places. (We computed the effective semiannual rate and multiplied by 2 to get the nominal rate convertible semiannually.)

    (b) Leave the entries in the TVM registers as they were in (a), then enter 3.5 [I/Y] [CPT] [PV]. The price to earn 7% compounded semiannually is \$905.85. (Note that this is lower than \$975.00, which is the price that would give us a yield rate of 6.35% convertible semiannually. This makes sense: the less we pay for an investment that returns fixed payments (like a bond), the greater the yield rate.)

Summary of Concepts and Formulas in Sections 7a and 7b

1. Issuing bonds is a technique for governments and corporations to borrow money
   (a) Issuer of the bond is the borrower
   (b) Purchaser of the bond is the lender
   (c) Interest paid to the lender is in the form of periodic coupon payments
      i. A bond that does not pay coupons is instead typically purchased at a discount, i.e., for an amount less than the face value to be received at the time the bond comes to maturity
   (d) At the end of the life of the bond, the lender also receives back the original principal (or an amount close to it)
2. Bond parameters and notation:
   (a) n = number of payments (term of bond)
   (b) i = interest rate per payment period (yield, IRR, yield-to-maturity)
   (c) r = coupon rate per payment period
   (d) F = face value of bond
   (e) C = redemption value of bond
      i. C and F are generally equal (unless otherwise stated)
3. ***Bond price***: as with any asset or liability, the value or price (in theory) is the present value of the future cash flows
4. ***Bond pricing formulas***:
   (a) ***Basic formula***: $P = Fra_{\overline{n}|i} + Cv_i^n$
   (b) ***Premium/Discount formula***: $P = C + (Fr - Ci)a_{\overline{n}|i}$
   (c) ***Makeham formula***: $P = K + \frac{g}{i}(C - K)$, where $K = Cv_i^n$, $g = \frac{Fr}{C}$

Past Exam Questions on Sections 7a and 7b

1. You have decided to invest in two bonds. Bond X is an n-year bond with semiannual coupons, while bond Y is an accumulation bond redeemable in $\frac{n}{2}$ years. The desired yield rate is the same for both bonds. You also have the following information:

   Bond X

   (i) Par value is 1000.

   (ii) The ratio of the semiannual bond rate to the desired semiannual yield rate, $\frac{r}{i}$, is 1.03125.

   (iii) The present value of the redemption value is 381.50.

   Bond Y

   (i) Redemption value is the same as the redemption value of bond X.

   (ii) Price to yield an effective rate i per half year is 647.80.

   What is the price of bond X? [11/01 #31]

   (A) 1019 (B) 1029 (C) 1050 (D) 1055 (E) 1072

2. Bill buys a 10-year 1000 par value 6% bond with semiannual coupons. The price assumes a nominal yield of 6%, compounded semiannually. As Bill receives each coupon payment, he immediately puts the money into an account earning interest at an annual effective rate of i. At the end of 10 years, immediately after Bill receives the final coupon payment and the redemption value of the bond, Bill has earned an annual effective yield of 7% on his investment in the bond. Calculate i. [5/01 #41]

   (A) 9.50% (B) 9.75% (C) 10.00% (D) 10.25% (E) 10.50%

3. A 1000 par value 20-year bond with annual coupons and redeemable at maturiy at 1050 is purchased for P to yield an annual effective rate of 8.25%. The first coupon is 75. Each subsequent coupon is 3% greater than the preceding coupon. Determine P. [11/00 #30]

   (A) 985 (B) 1000 (C) 1050 (D) 1075 (E) 1115

4. A firm has proposed the following restructuring for one of its 1000 par value bonds. The bond presently has 10 years remaining until maturity. The coupon rate on the existing bond is 6.75% per annum paid semiannually. The current nominal semiannual yield on the bond is 7.40%. The company proposes suspending coupon payments for four years with the suspended coupon payments being repaid, with accrued interest, when the bond comes due. Accrued interest is calculated using a nominal semiannual rate of 7.40%. Calculate the market value of the restructured bond. [5/00 #29]

   (A) 755 (B) 805 (C) 855 (D) 905 (E) 955

5. A 1000 par value 10-year bond with semiannual coupons and redeemable at 1100 is purchased at 1135 to yield 12% convertible semiannually. The first coupon is X. Each subsequent coupon is 4% greater than the preceding coupon. Determine X. [SOA 5/98 #17]

   (A) 40 (B) 42 (C) 44 (D) 48 (E) 50

6. Tina buys a 1000 par value 10-year bond with 10% annual coupons at a price to yield an annual effective rate of 10%. The coupons were reinvested at an annual effective rate of 8%. Immediately after receiving the 4th coupon payment, Tina sells the bond to Joe for a price of P. P assumes an annual effective yield of i to the buyer. Tina's annual effective yield from the date of purchase until the date of sale was 8%. Calculate i. [SOA 11/96 #16]

   (A) 11.6% (B) 11.8% (C) 12.0% (D) 12.2% (E) 12.4%

7. John purchases a 1000 par value 10-year bond with coupons at 8% convertible semiannually which will be redeemed for R. The purchase price is 800 and the present value of the redemption value is 301.51. Calculate R. [SOA 11/96 #18]

   (A) 775 (B) 800 (C) 825 (D) 850 (E) 875

8. Two 1000 par value bonds are purchased. The $2n$-year bond costs 250 more than the n-year bond. Each has 13% annual coupons and each is purchased to yield 6.5% annual effective. Calculate the price of the n-year bond. [SOA 5/95 #15]

   (A) 1200 (B) 1300 (C) 1400 (D) 1500 (E) 1600

9. A 30-year bond has 10% annual coupons and a par value of 1000. Coupons can be reinvested at a nominal annual rate of 6% convertible semiannually. X is the highest price that an investor can pay for the bond and obtain an effective yield of at least 10%. Calculate X. [SOA 5/95 #17]

   (A) 518 (B) 618 (C) 718 (D) 818 (E) 918

10. An n-year 1000 par value bond with 4.20% annual coupons is purchased at a price to yield an annual effective rate of i. You are given:

    (i) If the annual coupon rate had been 5.25% instead of 4.20%, the price of the bond would have increased by 100.

    (ii) At the time of purchase, the present value of all the coupon payments is equal to the present value of the bond's redemption value of 1000.

    Calculate i. [SOA 11/93 #13]

    (A) 5.0% (B) 5.5% (C) 5.9% (D) 6.3% (E) 6.5%

11. A 1,000 bond with annual coupons is redeemable at par at the end of 10 years. At a purchase price of 870, the yield rate is i. The coupon rate is $i - .02$. Calculate i. [SOA 5/93 #15]

    (A) 6.7% (B) 7.2% (C) 7.7% (D) 8.2% (E) 8.7%

12. An n-year zero coupon bond with par value of 1,000 was purchased for 600. An n-year 1,000 par value bond with semiannual coupons of X was purchased for 850. A $3n$-year 1,000 par value bond with semiannual coupons of X was purchased for P. All three bonds have the same yield rate. Calculate P. [SOA 5/93 #16]

    (A) 686 (B) 696 (C) 706 (D) 716 (E) 726

13. Jim buys a 10-year bond with par value of 10000 and 8% semiannual coupons. The redemption value of the bond at the end of 10 years is 10500. Calculate the purchase price to yield 6% convertible quarterly. [SOA 11/90 #5]

    (A) 11700 (B) 14100 (C) 14600 (D) 15400 (E) 17700

14. Bart buys a 28-year bond with a par value of 1200 and annual coupons. The bond is redeemable at par. Bart pays 1968 for the bond, assuming an annual effective yield rate of i. The coupon rate on the bond is twice the yield rate. At the end of 7 years, Bart sells the bond for P, which produces the same annual effective yield rate of i to the new buyer. Calculate P. [SOA 11/90 #15]

    (A) 1470 (B) 1620 (C) 1680 (D) 1840 (E) 1880

15. A 10-year bond with coupons at 8% convertible quarterly will be redeemed at 1600. The bond is bought to yield 12% convertible quarterly. The purchase price is 860.40. Calculate the par value. [SOA 11/90 #19]

    (A) 800 (B) 1000 (C) 1200 (D) 1400 (E) 1600

16. On June 1, 1990, an investor buys three 14-year bonds, each with par value 1000, to yield an effective annual interest rate of i on each bond. Each bond is redeemable at par.

    You are given:

    (i) the first bond is an accumulation bond priced at 195.63;

    (ii) the second bond has 9.4% semiannual coupons and is priced at 825.72; and

    (iii) the third bond has 10% annual coupons and is priced at P.

    Calculate P. [SOA 5/90 #14]

    (A) 825 (B) 835 (C) 845 (D) 855 (E) 865

17. A 10-year bond with par value 1000 and annual coupon rate r is redeemable at 1100.

    You are given:

    (i) the price to yield an effective annual interest rate of 4% is P;

    (ii) the price to yield an effective annual interest rate of 5% is $P - 81.49$; and

    (iii) the price to yield an effective annual interest rate of r is X.

    Calculate X. [SOA 5/90 #15]

    (A) 1061 (B) 1064 (C) 1068 (D) 1071 (E) 1075

18. John buys a 10-year, 1,000 par value bond with 8% semiannual coupons. The price of the bond to earn a yield of 6% convertible semiannually is 1,204.15. The redemption value is more than the par value. Calculate the price John would have to pay for the same bond to yield 10% convertible semiannually. [SOA 11/89 #13]

    (A) 875 (B) 913 (C) 951 (D) 989 (E) 1,027

19. A 1,000 par value 3-year bond with annual coupons of 50 for the first year, 70 for the second year and 90 for the third year is bought to yield a force of interest

$$\delta_t = \frac{2t-1}{2(t^2-t+1)} \quad \text{for } t \geq 0.$$

    Calculate the price of this bond. [SOA 11/89 #14]

    (A) 500 (B) 550 (C) 600 (D) 650 (E) 700

20. Henry has a five-year 1,000,000 bond with coupons at 6% convertible semiannually. Fiona buys a 10-year bond with face amount X and coupons at 6% convertible semiannually. Both bonds are redeemable at par. Henry and Fiona both buy their bonds to yield 4% compounded semiannually and immediately sell them to an investor to yield 2% compounded semiannually. Fiona earns the same amount of profit as Henry. Calculate X. [SOA 5/89 #14]

    (A) 500,000 (B) 502,000 (C) 505,000 (D) 571,000 (E) 574,000

21. A 1,000 bond with coupon rate c convertible semiannually will be redeemed at par in n years. The purchase price to yield 5% convertible semiannually is P. If the coupon rate were $c-0.02$, the price of the bond would be $P-300$. Another 1,000 bond is redeemable at par at the end of $2n$ years. It has a coupon rate of 7% convertible semiannually and the yield rate is 5% convertible semiannually. Calculate the price of this second bond. [SOA 11/88 #13]

    (A) 1,300 (B) 1,375 (C) 1,475 (D) 2,100 (E) 2,675

22. A 100 par value 6% bond with semiannual coupons is purchased at 110 to yield a nominal rate of 4% convertible semiannually. A similar 3% bond with semiannual coupons is purchased at P to provide the buyer with the same yield. Calculate P. [SOA 11/88 #15]

    (A) 90 (B) 95 (C) 100 (D) 105 (E) 110

23. A 700 par value five-year 10% bond with semiannual coupons is purchased for 670.60. The present value of the redemption value is 372.05 Calculate the redemption value. [SOA 5/88 #15]

    (A) 500 (B) 599 (C) 606 (D) 700 (E) 1,000

24. An insurance company owns a 1,000 par value 10% bond with semiannual coupons. The bond will mature for 1,000 at the end of 10 years. The company decides that an 8-year bond would be preferable. Current yield rates are 7% compounded semiannually. The company uses the proceeds from the sale of the 10% bond to purchase a 6% bond with semiannual coupons, maturing at par at the end of 8 years. Calculate the par value of the 8-year bond. [SOA 11/87 #5]

    (A) 1,000 (B) 1,291 (C) 1,306 (D) 1,419 (E) 1,497

25. Two bonds are both redeemable at their par value of $100 in t years. Bond A has 3.5% semiannual coupons and cost $88. Bond B has 4% semiannual coupons and cost $92. The bonds were purchased to produce the same yield rate. What is the yield rate per annum convertible semiannually? [CAS 5/87 #14]

    (A) Less than 4.8%

    (B) At least 4.8%, but less than 5.2%

    (C) At least 5.2%, but less than 5.6%

    (D) At least 5.6%, but less than 6.0%

    (E) 6.0% or more

26. A 30-year bond has an annual coupon rate of 6% for the first ten years, 7% for the next ten years, 8% for the last ten years, and matures at its par value of 100. The bond is bought to produce an effective annual yield rate of 7%. Determine an expression for the price of the bond. (All interest functions are at 7%.) [SOA 5/87 #5]

    (A) $6a_{\overline{10}|} + 7v^{10}a_{\overline{10}|} + 8v^{20}a_{\overline{10}|} + \frac{100}{(1.06)(1.07)(1.08)}$

    (B) $100 - v^{20}a_{\overline{10}|} + a_{\overline{10}|}$

(C) 100

(D) $7a_{\overline{30}|} + 100v^{30}$

(E) $100 + v^{20}a_{\overline{10}|} - a_{\overline{10}|}$

27. A bond with coupons equal to 40 sells for P. A second bond with the same maturity value and term has coupons equal to 30 and sells for Q. A third bond with the same maturity value and term has coupons equal to 80. All prices are based on the same yield rate, and all coupons are paid at the same frequency. Determine the price of the third bond. [SOA 5/87 #10]

(A) $4P - 4Q$ (B) $4P + 4Q$ (C) $4Q - 3P$ (D) $5P - 4Q$ (E) $5Q - 4P$

28. Two bonds are purchased for the same price to yield 5%. Bond X has 4% annual coupons and matures for its face value of 100. Bond Y has annual coupons of 3 and matures for 180. Both bonds mature at the end of n years. Calculate n. [SOA 11/86 #12]

(A) 14 (B) 20 (C) 26 (D) 33 (E) 40

29. On January 1, 1987, three 100 par value bonds with 6% annual coupons will mature at the end of 1, 2, and 3 years, respectively. The redemption value of each bond is 100. You are given that the prices for these bonds on January 1, 1987 are:

Maturity Date	**Price**
December 31, 1987	101.92
December 31, 1988	102.84
December 31, 1989	105.51

These prices are based on an interest rate of i in 1987, j in 1988, and k in 1989. Determine j. [SOA 5/86 #2]

(A) 2% (B) 3% (C) 4% (D) 5% (E) 6%

30. A 100 par value 100-year bond with a redemption value of 100 has annual coupons of 10% for the first 10 years, 9% for the next 10 years, 8% for the next 10 years, . . . , 1% for the last 10 years. Calculate the price of the bond to yield i. [SOA 5/86 #8]

(A) $\dfrac{10s_{\overline{10}|} - a_{\overline{10}|}}{s_{\overline{10}|}} + 100v^{100}$ (B) $\dfrac{10s_{\overline{10}|} - a_{\overline{10}|}}{i(s_{\overline{10}|})} + 100v^{100}$ (C) $\dfrac{10s_{\overline{10}|} - v^{10}a_{\overline{10}|}}{s_{\overline{10}|}} + 100v^{100}$

(D) $\dfrac{10s_{\overline{10}|} - a_{\overline{100}|}}{s_{\overline{10}|}} + 100v^{100}$ (E) $\dfrac{10s_{\overline{10}|} - a_{\overline{100}|}}{i(s_{\overline{10}|})} + 100v^{100}$

31. You are given the following information on a bond:

(i) Par value = 1000.

(ii) Redemption value = 1000.

(iii) Coupon rate = 12%, convertible semiannually.

(iv) It is priced to yield 10%, convertible semiannually.

The bond has a term of n years. If the term of the bond is doubled, the price will increase by 50. Calculate the price of the n-year bond. [SOA 11/85 #6]

(A) 1050 (B) 1100 (C) 1150 (D) 1200 (E) 1250

32. A \$1,000 bond bearing coupons at an annual rate of 5.5% payable semiannually and redeemable at \$1,100 is bought to yield a nominal annual rate of 4% convertible semiannually. If the present value of the redemption value at this yield is \$140, what is the purchase price? [CAS 5/85 #15]

(A) Less than \$1,310

(B) At least \$1,310, but less than \$1,330

(C) At least \$1,330, but less than \$1,350

(D) At least \$1,350, but less than \$1,370

(E) \$1,370 or more

33. Smith purchases a \$1000 par value, ten year, 5% bond with semiannual coupons. Smith pays a price of P_1. After six years, i.e., following the twelfth coupon, Smith sells the bond to Jones at a price of P_2. Jones retains the bond to maturity. The yield rate on Smith's investment is 7% convertible semiannually. The yield rate on Jones's investment is 6% convertible semiannually. What is P_1? [CAS 5/84 #6]

    (A) \$865 or less

    (B) At least \$865, but less than \$875

    (C) At least \$875, but less than \$885

    (D) At least \$885, but less than \$895

    (E) \$895 or more

34. A 4% \$100 bond matures in 25 years. It bears semiannual coupons and is purchased for \$117.50 to yield $i^{(2)}$. A 5% \$100 bond also matures in 25 years. It also bears semiannual coupons, but is purchased for \$135.00 to yield $i^{(2)}$. What is $i^{(2)}$? [CAS 5/84 #7]

    (A) Less than 1.75%

    (B) At least 1.75%, but less than 2.25%

    (C) At least 2.25%, but less than 2.75%

    (D) At least 2.75%, but less than 3.25%

    (E) 3.25% or more

35. A bond of amount 1 sells for $1+p$ at a certain fixed yield rate. If the bond's coupon rate were halved, the price would be $1+q$. What would the price be if the coupon rate were doubled? Throughout, assume the bond is unchanged in all other respects. [CAS 5/84 #9]

    (A) $1+3p-2q$ (B) $(1+p)^2/(1+q)$ (C) $1+p+2q$ (D) $1+2p-q$ (E) $1+4p-4q$

36. An n-year bond with par value \$1,000 has annual coupons of \$82. At a price of \$891.62, the yield to the purchaser is 10%, convertible semiannually. Find n. [SOA SAMPLE/84 #5]

    (A) 8 (B) 10 (C) 12 (D) 16 (E) 20

37. A \$1,000 par value bond with 9% coupons payable semiannually is purchased for \$1,300. The yield to the purchaser is 6%, convertible semiannually. If the same bond were redeemable at 120% of par, what price would have been paid to obtain the same yield? [SOA SAMPLE/84 #10]

    (A) \$1,260 (B) \$1,320 (C) \$1,380 (D) \$1,440 (E) \$1,500

38. A 9% bond with a 1,000 par value and coupons payable semiannually is redeemable at maturity for 1,100. At a purchase price of P, the bond yields a nominal annual interest rate of 8%, compounded semiannually, and the present value of the redemption amount is 190. Determine P. [SOA SAMPLE/83 #11]

    (A) 1,050 (B) 1,085 (C) 1,120 (D) 1,165 (E) 1,215

39. Two 15-year bonds with \$100 redemption values are each purchased to yield an effective annual interest rate of 4%. The first bond bears annual g% coupons and is purchased at a premium of \$11.12. The second bond bears annual $(g+2)$% coupons. Which of the following is closest to the purchase price of the second bond? [CAS 5/83 #5]

    (A) \$125 (B) \$127 (C) \$129 (D) \$131 (E) \$133

Solutions to Past Exam Questions on Sections 7a and 7b

1. Let X = price of Bond X and Y = price of Bond Y. Using the basic formula,

$$X = Fra_{\overline{2n}|} + Cv^{2n} = \frac{Fr}{i}\left(1 - v^{2n}\right) + 381.50 = 1{,}000(1.03125)\left(1 - v^{2n}\right) + 381.50$$

$$X = 1{,}412.75 - 1{,}031.25v^{2n}$$
$$Y = 647.80 = Cv^{n}$$

Since Cv^{2n} (PV of X's redemption value) = 381.50, by division we have

$$\frac{Cv^{2n}}{Cv^{n}} = v^{n} = \frac{381.50}{647.80} \text{ and } v^{2n} = .346822$$

Substituting in the above expression for X:

$$X = 1{,}412.75 - (1{,}031.25)(.346822)$$
$$= 1{,}055.09 \qquad \text{ANS. (D)}$$

2. Bill's AV at the end of 10 years is the AV of the coupons plus 1,000:

$$AV = 30s_{\overline{20}|j} + 1{,}000, \text{ where } j = (1+i)^{\frac{1}{2}} - 1 \text{ (the effective semiannual rate).}$$

Since the price assumes 6% compounded semiannually and the coupon rate is also 6% semiannually, Bill pays 1,000 for the bond.

$$1{,}000(1.07)^{10} = 30s_{\overline{20}|j} + 1{,}000$$
$$s_{\overline{20}|j} = 32.2384 \text{ and } j = 4.7597\%$$
$$i = (1+j)^2 - 1 = 9.75\% \qquad \text{ANS. (B)}$$

3.

$$P = 75\left(v + 1.03v^2 + \cdots + 1.03^{19}v^{20}\right) + 1{,}050v^{20}$$
$$= \frac{75}{1.0825}\left[1 + \frac{1.03}{1.0825} + \left(\frac{1.03}{1.0825}\right)^2 + \cdots + \left(\frac{1.03}{1.0825}\right)^{19}\right] + 1{,}050v^{20}$$
$$= \frac{75}{1.0825}\left[\frac{1 - \left(\frac{1.03}{1.0825}\right)^{20}}{1 - \frac{1.03}{1.0825}}\right] + 215 = 900 + 215$$
$$= 1{,}115 \qquad \text{ANS. (E)}$$

4. Since the suspended coupons will be repaid with accrued interest at the market rate of 3.70% effective per $\frac{1}{2}$-year, the market value of the bond is exactly the same as the original market value based on all coupons being paid:

$$P = 33.75a_{\overline{20}|} + 1000v^{20} \text{ at } 3.7\%$$

Enter 20 $\boxed{\text{N}}$ 3.7 $\boxed{\text{I/Y}}$ 33.75 $\boxed{\text{PMT}}$ 1,000 $\boxed{\text{FV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{PV}}$. The answer is 954.63. ANS. (E)

5.

$$1{,}135 = X\left[v + 1.04v^2 + \cdots + 1.04^{19}v^{20}\right] + 1{,}100v^{20}$$
$$\text{where } v = \frac{1}{1.06}$$
$$1{,}135 = \frac{X}{1.04}\left[\frac{1.04}{1.06} + \left(\frac{1.04}{1.06}\right)^2 + \cdots + \left(\frac{1.04}{1.06}\right)^{20}\right] + 342.99$$
$$1{,}135 = \frac{X}{1.04}\left\{\frac{\left(\frac{1.04}{1.06}\right)\left[1 - \left(\frac{1.04}{1.06}\right)^{20}\right]}{1 - \frac{1.04}{1.06}}\right\} + 342.99$$
$$X = 50.00 \qquad \text{ANS. (E)}$$

Note: The expression in brackets is $a_{\overline{20}|i'}$, where i' is an adjusted rate of interest given by $1 + i' = \frac{1.06}{1.04}$, or $i' = \frac{.02}{1.04} = 1.923077\%$. This is an alternative way of getting a value for the sum of the progression.

6. Tina's original investment was the price of the bond: $P = 100a_{\overline{10}|} + 1{,}000v^{10}$ at 10%, or 1,000. (You can skip this computation: If you get 10% on your investment and then 1,000, you should pay 1,000 for the investment at 10%.) Tina got an AV at the end of 4 years equal to the AV of the coupons at 8% plus P: $1{,}000(1.08)^4 = 100s_{\overline{4}|.08} + P$. (Could compute P by entering 4 $\boxed{\text{N}}$ 8 $\boxed{\text{I/Y}}$ 1,000 $\boxed{+/-}$ $\boxed{\text{PV}}$ 100 $\boxed{\text{PMT}}$ $\boxed{\text{CPT}}$ $\boxed{\text{FV}}$) $P = 909.88$, the price that Joe paid. For Joe, $909.88 = 100a_{\overline{6}|i} + 1{,}000v^6$. Solve for i by entering 6 $\boxed{\text{N}}$ 909.88 $\boxed{+/-}$ $\boxed{\text{PV}}$ 100 $\boxed{\text{PMT}}$ 1,000 $\boxed{\text{FV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{I/Y}}$. The answer is 12.2%. ANS. (D)

7.

$$\begin{aligned}800 &= 40a_{\overline{20}|i} + Rv_i^{20} = 40a_{\overline{20}|i} + 301.51\\ a_{\overline{20}|i} &= \frac{800 - 301.51}{40} = 12.46225\end{aligned}$$

and $i = 5.0\%$. $\therefore Rv^{20} = 301.51$, $R = 301.51(1.05)^{20} = 800$ ANS. (B)

8.

$$\begin{aligned}&\text{Price of 2n-year bond: } 130a_{\overline{2n}|} + 1{,}000v^{2n}\\ &\text{Price of n-year bond: } 130a_{\overline{n}|} + 1{,}000v^{n}\\ &\text{Difference } = 250 = 130\left(a_{\overline{2n}|} - a_{\overline{n}|}\right) + 1{,}000v^{2n} - 1{,}000v^n\end{aligned}$$

Substituting $a_{\overline{2n}|} = \frac{1-v^{2n}}{.065}$ and $a_{\overline{n}|} = \frac{1-v^n}{.065}$, combining terms, etc., we get $1{,}000v^{2n} - 1{,}000v^n + 250 = 0$, a quadratic in v^n. Solving, we get $v^n = \frac{1}{2}$. Price of n-year bond $= 130a_{\overline{n}|} + 1{,}000v^n = 130\left(\frac{1-v^n}{.065}\right) + 500 = 1{,}500$. ANS. (D)

Note: It was not necessary to solve explicitly for n, since all we needed was v^n to determine the price.

9. The AV of the coupons can be computed as $100s_{\overline{30}|j}$, where $j = 1.03^2 - 1 = 6.09\%$. X is the PV of the accumulated coupons plus maturity value at 10% effective:

$$\begin{aligned}X &= \left(100s_{\overline{30}|.0609} + 1{,}000\right)v_{.10}^{30}\\ &= (9{,}032.19)(.057309) = 517.62 \qquad \text{ANS. (A)}\end{aligned}$$

10. PV of increase in coupons from 4.2% to $5.25\% = (52.50 - 42.00)a_{\overline{n}|} = 10.50a_{\overline{n}|} = 100$. $a_{\overline{n}|} = \left(\frac{100}{10.5}\right)$.

$$\begin{aligned}&PV \text{ of coupons } = 42a_{\overline{n}|} = (42)\left(\frac{100}{10.5}\right) = 400\\ &PV \text{ of redemption value } = 1{,}000v^n = 400.\\ &\therefore v^n = .4,\ a_{\overline{n}|} = \frac{1-v^n}{i} = \frac{.6}{i} = \frac{100}{10.5}\\ &\text{and } i = \frac{(.6)(10.5)}{100} = 6.3\% \qquad \text{ANS. (D)}\end{aligned}$$

11. Using the premium/discount formula:

$$\begin{aligned}&P = C + (Cg - Ci)a_{\overline{n}|i}\\ &870 = 1{,}000 + [1{,}000(i - .02) - 1{,}000i]a_{\overline{10}|i}\\ &870 = 1{,}000 - 20a_{\overline{10}|i},\ a_{\overline{10}|i} = 6.5\\ &\text{and } i = 8.71\% \qquad \text{ANS. (E)}\end{aligned}$$

12.

$$\begin{aligned}&600 = 1000v^n,\ \text{so } v^n = .6\\ &\text{Let } Y = \text{ equivalent annual coupon of the other 2 bonds.}\end{aligned}$$

$$\begin{aligned}
850 &= Y a_{\overline{n}|i} + 1{,}000 v^n \\
&= Y\left(\frac{1-.6}{i}\right) + 1{,}000(.6) \\
&= \frac{Y}{i}(.4) + 600 \\
\frac{Y}{i} &= 625 \\
P &= Y a_{\overline{3n}|} + 1{,}000 v^{3n} \\
&= \frac{Y}{i}\left(1-v^{3n}\right) + 1{,}000 v^{3n} \\
&= 625\left(1-.6^3\right) + 1{,}000\left(.6^3\right) \\
&= 490 + 216 = 706 \qquad \text{ANS. (C)}
\end{aligned}$$

13. Let j = effective semiannual rate. $j = 1.015^2 - 1 = 3.0225\%$ and the PV of the coupons = $400 a_{\overline{20}|j} = 5{,}938.63$.

PV of the redemption value = $10{,}500 v_{.015}^{40} = 5{,}788.25$.

Price = $5{,}938.63 + 5{,}788.25 = 11{,}726.88$ ANS. (A)

Note: PV of coupons could also be computed as $400\frac{a_{\overline{40}|.015}}{s_{\overline{2}|.015}}$

14.

$$\begin{aligned}
1{,}968 &= (1{,}200)(2i) a_{\overline{28}|i} + 1{,}200 v^{28} \\
&= (1{,}200)(2i)\left(\frac{1-v^{28}}{i}\right) + 1{,}200 v^{28}
\end{aligned}$$

Solving for v^{28} gives $v^{28} = .36$, $v^7 = .7746$ and $v^{21} = .4648$. (Contrived question when financial calculators were not used on the exam.)

$$\begin{aligned}
P &= (1{,}200)(2i) a_{\overline{21}|} + 1{,}200 v^{21} \\
&= 2{,}400 - 1{,}200 v^{21} = 2{,}400 - 1{,}200(.4648) \\
&= 1841 \qquad \text{ANS. (D)}
\end{aligned}$$

15.

$$\begin{aligned}
860.40 &= (.02) F a_{\overline{40}|.03} + 1{,}600 v^{40} \\
&= .4623F + 490.49 \\
F &= 800.15 \qquad \text{ANS. (A)}
\end{aligned}$$

16.

$$\begin{aligned}
&1{,}000 v^{14} = 195.63,\ v^{14} = .19563 \\
&1{,}000(.094) a_{\overline{14}|}^{(2)} + 1{,}000 v^{14} \\
&= \frac{94(1-v^{14})}{i^{(2)}} + 195.63 = 825.72
\end{aligned}$$

$$\begin{aligned}
\frac{94(1-v^{14})}{i^{(2)}} &= \frac{94(.80437)}{i^{(2)}} = 630.09 \\
i^{(2)} &= .12 \text{ and } i = .1236
\end{aligned}$$

$$\begin{aligned}
P &= 100 a_{\overline{14}|} + 1{,}000 v^{14} = 100\left(\frac{1-v^{14}}{i}\right) + 195.63 \\
&= 100\left(\frac{1-.19563}{.1236}\right) + 195.63 = 846.41 \qquad \text{ANS. (C)}
\end{aligned}$$

Note: With a financial calculator, i could be determined from the first bond's result that $v^{14} = .19563$.

17.

$$\begin{aligned}&\text{(i) } P = 1{,}000ra_{\overline{10}|.04} + 1{,}100v_{.04}^{10} = 8{,}111r + 743.12\\&\text{(ii) } P - 81.49 = 1{,}000ra_{\overline{10}|.05} + 1{,}100v_{.05}^{10} = 7{,}722r + 675.30\end{aligned}$$

Subtracting, we get $81.49 = 389r + 67.82$ and $r = .035$

$$\begin{aligned}X &= 1{,}000ra_{\overline{10}|r} + 1{,}100v_r^{10} = 1{,}000r\left(\frac{1-v_r^{10}}{r}\right) + 1{,}100v_r^{10}\\&= 1{,}000 + 100v_{.035}^{10} = 1{,}070.89 \qquad \text{ANS. (D)}\end{aligned}$$

18.

$$\begin{aligned}&1{,}204.15 = 40a_{\overline{20}|.03} + Cv^{20}\\&C = \left(1{,}204.15 - 40a_{\overline{20}|}\right)(1.03)^{20} = 1{,}100\end{aligned}$$

$$\begin{aligned}\text{John's price } &= 40a_{\overline{20}|.05} + 1{,}100v_{.05}^{20}\\&= 913.07 \qquad \text{ANS. (B)}\end{aligned}$$

19.

$$\begin{aligned}&a(t) = e^{\int_0^t \frac{2r-1}{2\left(r^2-r+1\right)}\,dr} = e^{\frac{1}{2}\ln\left(r^2-r+1\right)\Big]_0^t} = \left(t^2-t+1\right)^{\frac{1}{2}}\\&a^{-1}(t) = \left(t^2-t+1\right)^{-\frac{1}{2}}\\&a^{-1}(1) = 1,\ a^{-1}(2) = \frac{1}{\sqrt{3}},\ a^{-1}(3) = \frac{1}{\sqrt{7}}\\&P = 50 + \frac{70}{\sqrt{3}} + \frac{90+1{,}000}{\sqrt{7}} = 502.40 \qquad \text{ANS. (A)}\end{aligned}$$

20. Using units of 1,000, Henry pays $30a_{\overline{10}|.02} + 1{,}000v_{.02}^{10} = 1{,}089.83$ for his bond. If he sells it at a price that will yield an investor 1% effective per half year, he sells it at $30a_{\overline{10}|.01} + 1{,}000v_{.01}^{10} = 1{,}189.43$. Thus, his profit is 99.60. Similarly,

$$\begin{aligned}\text{Fiona's profit } &= .03Xa_{\overline{20}|.01} + Xv_{.01}^{20} - \left(.03Xa_{\overline{20}|.02} + Xv_{.02}^{20}\right)\\&= .03X\left(a_{\overline{20}|.01} - a_{\overline{20}|.02}\right) + X\left(v_{.01}^{20} - v_{.02}^{20}\right)\\&= X(.050824 + .146573) = 99.60\end{aligned}$$

$X = 504.567$ in units of 1,000 or 504,567 ANS. (C)

21. Use the premium/discount formula

$$\begin{aligned}&P = C + (Fr - Ci)a_{\overline{n}|} = 1{,}000 + 1{,}000\left(\frac{C}{2} - .025\right)a_{\overline{2n}|} \text{ at } 2.5\%\\&P - 300 = 1{,}000 + 1{,}000\left(\frac{C}{2} - .01 - .025\right)a_{\overline{2n}|} \text{ at } 2.5\%\\&300 = 1{,}000(.01)a_{\overline{2n}|},\ v^{2n} = .25.\\&a_{\overline{4n}|} = \frac{1-\left(v^{2n}\right)^2}{.025} = \frac{1-.0625}{.025} = 37.5\end{aligned}$$

$$\begin{aligned}\therefore \text{ price of 2nd bond } &= 1{,}000 + 1{,}000(.035 - .025)a_{\overline{4n}|}\\&= 1{,}000 + 10(37.5) = 1{,}375 \qquad \text{ANS. (B)}\end{aligned}$$

22.

$$\begin{aligned}&P = C + (Fr - Ci)a_{\overline{n}|}\\&\text{1st Bond: } 110 = 100 + (3-2)a_{\overline{n}|},\ a_{\overline{n}|} = 10\\&\text{2nd Bond: } P = 100 + (1.5-2)(10) = 95 \qquad \text{ANS. (B)}\end{aligned}$$

23.

$$P = Fra_{\overline{n}|} + Cv^n$$

$$\begin{aligned} 670.60 &= 35a_{\overline{10}|} + Cv^{10} = 35a_{\overline{10}|} + 372.05 \\ a_{\overline{10}|} &= 8.53 \text{ and } i = 3\% \\ Cv^{10} &= 372.05,\ C = 372.05(1.03)^{10} = 500 \qquad \text{ANS. (A)} \end{aligned}$$

24.

$$\begin{aligned} &\text{Let } F = \text{ par value of the 8-year bond} \\ &50a_{\overline{20}|} + 1{,}000v^{20} = F\left[(.03)a_{\overline{16}|} + v^{16}\right] \text{ at } 3.5\%. \\ &F = \frac{1{,}213.19}{.362824 + .576706} = 1{,}291.27 \qquad \text{ANS. (B)} \end{aligned}$$

25. Use the premium/discount formula $P = C + (Fr - Ci)a_{\overline{n}|}$.

$$\begin{aligned} \text{Bond A: } & 88 = 100 + (1.75 - 100i)a_{\overline{24}|} \\ & \text{or } (1.75 - 100i)a_{\overline{24}|} = -12, \\ & \text{where } i \text{ is the effective semiannual rate} \\ \text{Bond B: } & 92 = 100 + (2 - 100i)a_{\overline{24}|} \\ & \text{or } (2 - 100i)a_{\overline{24}|} = -8 \\ \text{Dividing: } & \frac{1.75 - 100i}{2 - 100i} = 1.5 \end{aligned}$$

and $i = 2.50\%$. The nominal annual rate $= 2 \times 2.50\% = 5\%$ ANS. (B)

26.

$$\begin{aligned} &P = 8a_{\overline{30}|} - a_{\overline{20}|} - a_{\overline{10}|} + 100v^{30} \\ &v^n = 1 - ia_{\overline{n}|}, \text{ so } 100v^{30} = 100 - 7a_{\overline{30}|} \\ &P = 100 + a_{\overline{30}|} - a_{\overline{20}|} - a_{\overline{10}|} \\ &\text{But } a_{\overline{30}|} - a_{\overline{20}|} = v^{20}a_{\overline{10}|} \\ &P = 100 + v^{20}a_{\overline{10}|} - a_{\overline{10}|} \qquad \text{ANS. (E)} \end{aligned}$$

27.

$$\begin{aligned} &P = C + (40 - Ci)a_{\overline{n}|}, \text{ so } P - C = (40 - Ci)a_{\overline{n}|} \\ &Q = C + (30 - Ci)a_{\overline{n}|}, \text{ so } P - Q = 10a_{\overline{n}|} \\ &\text{Let } R = \text{price of 3rd bond.} \\ &R = C + (80 - Ci)a_{\overline{n}|} = C + 40a_{\overline{n}|} + (40 - Ci)a_{\overline{n}|} \\ &\quad = C + 4(P - Q) + P - C = 5P - 4Q \qquad \text{ANS. (D)} \end{aligned}$$

Notes: (1) The problem could also be done using the Basic Formula. (2) Using the 0% test, the only answer that could be correct is (D).

28.

$$\begin{aligned} &4a_{\overline{n}|} + 100v^n = 3a_{\overline{n}|} + 180v^n \\ &a_{\overline{n}|} = 80v^n \text{ or } s_{\overline{n}|} = 80 \text{ at } 5\% \\ &n = 32.99 \qquad \text{ANS. (D)} \end{aligned}$$

29.

$$\begin{aligned} &101.92 = \frac{100 + 6}{1 + i}, \text{ so } i = .04 \\ &102.84 = \frac{6}{1.04} + \frac{100 + 6}{(1.04)(1 + j)}, \text{ so } j = .05 \qquad \text{ANS. (D)} \end{aligned}$$

30.

$$\begin{aligned}
&\text{Let } X = PV \text{ of coupons}\\
&X = 10a_{\overline{10}|} + 9v^{10}a_{\overline{10}|} + \cdots + v^{90}a_{\overline{10}|}\\
&(1+i)^{10}X = 10s_{\overline{10}|} + 9a_{\overline{10}|} + \cdots + v^{80}a_{\overline{10}|}
\end{aligned}$$

$$\begin{aligned}
\left[(1+i)^{10} - 1\right] X &= 10s_{\overline{10}|} - a_{\overline{10}|} - v^{10}a_{\overline{10}|} - \cdots - v^{90}a_{\overline{10}|}\\
&= 10s_{\overline{10}|} - a_{\overline{100}|}
\end{aligned}$$

$$\begin{aligned}
X &= \frac{10s_{\overline{10}|} - a_{\overline{100}|}}{(1+i)^{10} - 1} = \frac{10s_{\overline{10}|} - a_{\overline{100}|}}{is_{\overline{10}|}}\\
\text{Price} &= X + 100v^{100} \qquad \textbf{ANS. (E)}
\end{aligned}$$

31.

$$\begin{aligned}
\text{Price of } n\text{-year bond} &= 1{,}000 + (60 - 50)a_{\overline{2n}|}\\
&= 1{,}000 + 10a_{\overline{2n}|}\\
\text{Price of } 2n\text{-year bond} &= 1{,}000 + 10a_{\overline{4n}|}
\end{aligned}$$

$$\begin{aligned}
\text{Difference } &= 50 = 10\left(a_{\overline{4n}|} - a_{\overline{2n}|}\right)\\
&= 10\left(\frac{1 - v^{4n}}{.05} - \frac{1 - v^{2n}}{.05}\right)
\end{aligned}$$

This is a quadratic equation in v^{2n} that reduces to $4v^{4n} - 4v^{2n} + 1 = 0$, so $v^{2n} = .5$. Price of n-year bond

$$\begin{aligned}
&= 1{,}000 + 10a_{\overline{2n}|} = 1{,}000 + 10\left(\frac{1 - v^{2n}}{.05}\right)\\
&= 1{,}000 + 10\left(\frac{1 - .5}{.05}\right) = 1{,}100 \qquad \textbf{ANS. (B)}
\end{aligned}$$

32. If n is the term of the bond in $\frac{1}{2}$-year periods, we are given that $1{,}100v^n = 140$, so $v^n = .127273$. The price of the bond is $27.50a_{\overline{n}|} + 1{,}100v^n$

$$\begin{aligned}
&= 27.50\left(\frac{1 - v^n}{.02}\right) + 140\\
&= 27.50\left(\frac{1 - .127273}{.02}\right) + 140 = 1{,}340 \qquad \textbf{ANS. (C)}
\end{aligned}$$

33. Jones buys the bond to yield 6% convertible semiannually, so $P_2 = 25a_{\overline{8}|} + 1{,}000v^8$ at 3%, $P_2 = 964.90$. Smith's price to yield 7% convertible semiannually was

$$\begin{aligned}
P_1 &= 25a_{\overline{12}|} + 964.90v^{12} \text{ at } 3.5\%,\\
P_1 &= 880.14 \qquad \textbf{ANS. (C)}
\end{aligned}$$

34.

$$\begin{aligned}
117.50 &= 2a_{\overline{50}|} + 100v^{50} \text{ at } \frac{i^{(2)}}{2}\\
135.00 &= 2.50a_{\overline{50}|} + 100v^{50}\\
17.50 &= .5a_{\overline{50}|},\ a_{\overline{50}|} = 35,\\
\frac{i^{(2)}}{2} &= 1.5\%,\ i^{(2)} = 3\% \qquad \textbf{ANS. (D)}
\end{aligned}$$

35. Since the par value is 1, we have $1+p = ra_{\overline{n}|} + v^n$ and $1+q = \frac{r}{2}a_{\overline{n}|} + v^n$. Solve for $ra_{\overline{n}|}$ and v^n in terms of p and q by subtracting: $p-q = \frac{r}{2}a_{\overline{n}|}$, $ra_{\overline{n}|} = 2(p-q)$, and $v^n = 1+p-ra_{\overline{n}|} = 1+p-2(p-q) = 1-p+2q$. The price of the bond with double coupons is $2ra_{\overline{n}|} + v^n = 4(p-q)+1-p+2q = 3p-2q+1$ ANS. (A)

36.
$$891.62 = 1{,}000 + \left(\frac{82}{s_{\overline{2}|.05}} - 50\right) a_{\overline{2n}|.05},$$

where $\frac{82}{s_{\overline{2}|.05}}$ is the semiannual coupon equivalent to 82 paid annually. Solving this for $a_{\overline{2n}|}$, we get

$$\begin{aligned} a_{\overline{2n}|} &= \frac{1{,}000 - 891.62}{50 - \frac{82}{2.05}} = 10.838 \text{ and} \\ 2n &= 16, \; n = 8 \qquad \text{ANS. (A)} \end{aligned}$$

Note: An alternative approach is to determine the annual effective rate $i' = 1.05^2 - 1 = 10.25\%$. Using the premium/discount formula,

$$\begin{aligned} 891.62 &= 1{,}000 + [82 - 1{,}000(.1025)]a_{\overline{n}|} \\ a_{\overline{n}|} &= 5.286829 \text{ at } 10.25\% \\ n &= 8 \end{aligned}$$

37.

$$\begin{aligned} 1{,}300 &= 1{,}000 + (45-30)a_{\overline{n}|} \\ a_{\overline{n}|} &= \frac{300}{15} = 20 \end{aligned}$$

$$\begin{aligned} P &= 1{,}200 + (45-36)a_{\overline{n}|} \\ &= 1{,}200 + (9)(20) = 1{,}380 \qquad \text{ANS. (C)} \end{aligned}$$

38. Let n = number of $\frac{1}{2}$-year periods in the term of the bond. We are given that $1{,}100v^n = 190$, so $v^n = \frac{190}{1{,}100}$.

$$\begin{aligned} \text{Price } &= 45a_{\overline{n}|} + 1{,}100v^n \\ &= 45\left(\frac{1 - \frac{190}{1{,}100}}{.04}\right) + 190 \\ &= 930.68 + 190 = 1{,}120.68 \qquad \text{ANS. (C)} \end{aligned}$$

39. Using the premium/discount formula $P - C = (Fr - Ci)a_{\overline{n}|}$, we have $11.12 = (100g - 4)a_{\overline{15}|.04}$, $g = \frac{\frac{11.12}{a_{\overline{15}|}} + 4}{100} = 5\%$.

$$\begin{aligned} \text{Price of 2nd bond } &= 7a_{\overline{15}|.04} + 100v^{15} \\ &= 133.36 \qquad \text{ANS. (E)} \end{aligned}$$

7c Premium and Discount

When a bond is purchased for more than its redemption value, the excess of the price over the redemption value is called the "premium." When a bond is purchased for less than its redemption value, the excess of the redemption value over the price is called the "discount." (Discount could be defined as negative premium, and vice versa, but the financial world doesn't like to deal with negatives.)

The premium/discount formula is ideal for quickly determining whether a bond is sold at a premium or discount:

$$P = C + (Cg - Ci)a_{\overline{n}|}$$

We see that if $g > i$, the price is the redemption value C plus a *positive* amount. Thus, $P > C$ and the bond is purchased at a premium equal to $P - C = (Cg - Ci)a_{\overline{n}|}$.

Similarly, if $g < i$, the price is C plus a *negative* amount. Thus, $P < C$ and the bond is purchased at a discount equal to $C - P = (Ci - Cg)a_{\overline{n}|}$. By defining the discount in this manner, we avoid dealing with a negative premium $(Cg - Ci)a_{\overline{n}|}$ when $g < i$.

We will use as an example a bond with only three coupons remaining, in order to illustrate the concepts with a minimum of arithmetic.

Consider a \$1,000 bond with 8% semiannual coupons redeemable at \$1,050 in $1\frac{1}{2}$ years, purchased to yield a nominal annual rate of 6% compounded semiannually.

1. How can we quickly determine whether the bond is purchased at a premium or discount without actually computing the price?

   Answer: Compare g and i. For this bond, $g = \frac{40}{1{,}050}$ and $i = .03$. Since $g > i$, we know that the price is greater than the redemption value. Therefore, the bond sells at a premium.

2. What is the amount of the premium?

   Answer: In general, the amount of the premium is equal to $P - C = (Cg - Ci)a_{\overline{n}|}$ or $(Fr - Ci)a_{\overline{n}|}$. For this bond:

   $$\text{Premium} = [40 - (1{,}050)(.03)]a_{\overline{3}|} \text{ at } 3\% = \$24.04$$

   (Note that we have avoided the trap of using F in the term Ci, which would give Premium = $[40-(1{,}000)(.03)]a_{\overline{3}|} = \$28.29$.)

   Since the bond sells for \$24.04 more than its redemption value, the price is $\$1{,}050 + 24.04 = \$1{,}074.04$. (We could also have directly computed the price using the BA II Plus as follows: 3 $\boxed{\text{N}}$ 3 $\boxed{\text{I/Y}}$ 40 $\boxed{\text{PMT}}$ 1,050 $\boxed{\text{FV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{PV}}$. Again, the answer is 1,074.04.)

Let's consider how much interest the investor earns in the first 6 months. The investment is \$1,074.04 and the effective rate is 3% for a half-year period, so the investor earns $(0.3)(1{,}074.04) = \$32.22$ in interest. But in fact, he receives a coupon of \$40 at the end of 6 months. What does the excess of \$40 over \$32.22 = \$7.78 represent?

Answer: It must represent a return of part of the original investment of \$1,074.04 (the same concept as "principal repaid" when a loan is amortized). So the "outstanding balance" of the investment, so-to-speak, just after the first coupon payment is \$1,074.04 − \$7.78 = \$1,066.26. We can carry this process on for the remaining two one-half year periods, which we shall do in a moment.

You can see that the concepts here are no different from those in Section 6a for a loan being amortized by level payments, with one exception. For a bond, the outstanding balance just after the last coupon is paid (but just before the redemption value is paid) is not 0: It is the *redemption value*. When the redemption value is paid a moment later, the outstanding balance is reduced to 0.

Although the concepts are the same, bond terminology is different:

1. "Book value" is used instead of "outstanding loan balance." This reflects the fact that insurance companies and pension funds normally show the asset values of bonds in their books of account computed as described above.

2. Instead of "principal repaid," the periodic reduction in the book value is called the "amount for amortization of premium." This is because the sum of the principal repaid portions of the coupons "amortizes" the premium, i.e., reduces the book value of the bond from P to C, where $P - C$ is the premium. Another term used for the principal repaid is the "write down," since the asset value of a bond is "written down" by this amount each period.

3. Instead of "payment amount," which is a general term under any loan or investment, we use the specific payment under a bond, namely, the coupon.

Completing the arithmetic for the bond in the example, we construct the following bond amortization schedule:

		Interest Earned	**Amount for Amortization of Premium**	**Book Value**
Period t	Coupon	$I_t = iB_{t-1}$	P_t=Coupon$-I_t$	$B_t = B_{t-1} - P_t$
0				1,074.04
1	40	32.22	7.78	1,066.26
2	40	31.99	8.01	1,058.25
3	40	31.75	8.25	1,050.00
Totals	**120**	**95.96**	**24.04**	

(Note that we can use any of the methods shown in Calculator Notes #8 to construct this schedule, once we have entered the bond data in the TVM registers and pressed CPT PV.)

A few comments:

1. The total interest earned is defined in the same way as for a loan: The total payments received minus the amount invested. For this bond, the total payments are 3 coupons plus the redemption value: $3 \times \$40 + \$1{,}050 = \$1{,}170$. The amount invested is \$1,074.04. The total interest earned is \$1,170 – \$1,074.04 = \$95.96, which agrees with the total of the "Interest Earned" column.
2. The total of the amounts for amortization of premium column is equal to the premium of \$24.04. That's exactly what we mean by "amortizing" (i.e., paying back this part of the investment).
3. Although we haven't proven it, the amounts for amortization of premium are in geometric progression with common ratio $(1+i)$. This can be seen from the formula for P_t:

$$\text{Amount for Amortization of Premium} = P_t = (Fr - Ci)v^{n-t+1} \text{ or}$$
$$P_t = (Cg - Ci)v^{n-t+1}$$

   You should be able to prove this in one step, if you express P_t as $(B_{t-1} - B_t)$, where B_t is the book value of the bond just after the t^{th} coupon is paid. Prospectively, B_t can be expressed as the price of the bond at the original yield rate, considering only the remaining coupons and the redemption value. Use the premium/discount formula for the easiest derivation of P_t.

What if a bond is purchased at a discount? Take the same bond as the one in the last example, but instead of a yield rate of 6% nominal, compounded semiannually, let's say the yield rate is a nominal 10% compounded semiannually, or 5% effective per one-half year coupon period. We immediately see that the bond is purchased at a discount (since $\frac{40}{1{,}050} < .05$). The price of the bond is:

$$P = 1{,}050 + [40 - (1{,}050)(.05)]a_{\overline{3}|} \text{ at } 5\% = 1{,}050 - 12.5a_{\overline{3}|} = \$1{,}015.96$$

Discount $= C - P = \$1{,}050 - \$1{,}015.96 = \$34.04$.

The interest earned in the first 6 months is (.05)(1,015.96) = \$50.80. This is odd: The coupon is only \$40. Can the interest earned be \$10.80 more than the coupon payment actually received by the investor?

The answer is yes. Look at it this way: The arithmetic says that the investor should earn interest of \$50.80 at 5%. If he/she gets only \$40, the bond "owes" the investor \$10.80 in interest at that point. As long as the investor eventually receives this amount from the bond, *with interest from the time it was due*, the investor should be satisfied. (Note that this is the same concept of *negative amortization* that was illustrated in Example 2 of Section 6b.)

It turns out that when a bond is purchased at a discount, each coupon is short of the amount of interest that should be earned in the period. But all of these shortages are made up, *with interest*, when the bond is redeemed.

Looking at the bond in our example again, the principal repaid in the first period is, as usual, the coupon minus the interest earned. So $P_1 = 40 - 50.80 = -10.80$. This means that when we compute B_1 as the price minus P_1, the outstanding balance (i.e., the book value) actually *increases*. This is in accord with what we said before: The bond "owes" us \$10.80, so this amount is *added* to the outstanding balance as a kind of additional investment that will be repaid with interest on the redemption date.

We will set up a bond amortization schedule to show that this all works out. But to avoid using negative numbers (remember, the financial world abhors negatives!), we will write the entries in the principal repaid column as positive amounts and remember to *add* each of them to the previous book value. (This is the same as subtracting the actual negative principal repaid amounts.) These principal repaid amounts are called "amounts for accumulation of discount," since their sum must equal the discount in order for the book value to increase from P to C. They are also called "write-ups," since the book value of the bond is "written up" by these amounts.

		Interest Earned	**Amount for Amortization of Discount**	**Book Value**
Period t	Coupon	$I_t = iB_{t-1}$	$P_t = I_t -$Coupon	$B_t = B_{t-1} + P_t$
0				1,015.96
1	40	50.80	10.80	1,026.76
2	40	51.34	11.34	1,038.10
3	40	51.90	11.90	1,050.00
Totals	**120**	**154.04**	**34.04**	

A few comments:

1. The total interest earned is equal to the total payments received minus the amount invested. In this case, total interest $= 3 \times \$40 + \$1{,}050 - \$1{,}015.96 = \$154.04$. This agrees with the total of the "Interest Earned" column.
2. The total of the "amounts for accumulation of discount" column ($34.04) is equal to the discount.
3. The amounts for accumulation of discount are in geometric progression. This can be seen from the following formula:
$$P_t = (Ci - Cg)v^{n-t+1}$$
Note that in the case of a bond purchased at a discount, we have defined the amount for accumulation of discount P_t as the *negative* of the "normal" principal repaid, to avoid negative signs. You must be careful, when dealing with a bond purchased at a discount, to be aware of this.

The importance of determining the interest earned included in each coupon is that this is the amount normally reported as investment income by a financial entity, such as an insurance company or pension fund. As we have seen:

1. For a bond purchased at a premium, the interest earned each year is *less* than the amount of the coupon. The balance of the coupon (i.e., the "amount for amortization of premium," or "write down") is a partial repayment of the premium paid for the bond. Accordingly, it acts to reduce the book value each period, until the book value is equal to the redemption value.
2. For a bond purchased at a discount, the interest earned each year is *more* than the amount of the coupon. The excess of the interest earned over the coupon (i.e., the "amount for accumulation of discount," or "write up") is like an additional investment in the bond that acts to increase the book value each period, until the book value is equal to the redemption value.

Stepping Stones

Example 1
A $100 bond with 5% semiannual coupons that matures at $105 in 10 years is purchased to yield a nominal annual rate of 4% compounded semiannually. Determine the premium paid for the bond.

Solution
From the premium/discount formula, we have:

$$\begin{aligned}\text{Premium} &= P - C = (Fr - Ci)a_{\overline{n}|} \\ &= [2.50 - (105)(.02]a_{\overline{20}|.02} \\ &= 0.4a_{\overline{20}|.02} \\ &= \$6.54\end{aligned}$$

Example 2
For the bond in Example 1, determine the amount for amortization of premium (write down) in the 9th coupon.

Solution

$$\begin{aligned}P_t &= (Fr - Ci)v^{n-t+1} \\ P_9 &= (2.50 - 2.10)v^{20-9+1} \\ &= 0.4v^{12} \quad \text{at } 2\% \\ &= \$.32 \quad \text{(nearest \$.01)}\end{aligned}$$

The write down included in the 9th coupon could also be determined as the excess of the book value just after the 8th coupon is paid over the book value just after the 9th coupon is paid, i.e., $B_8 - B_9$.

Example 3
We are given the following facts about a 15-year bond with semiannual coupons:

1. The bond is purchased at a price that results in a yield rate of 8% compounded semiannually.
2. The amount for accumulation of discount in the 11th coupon is \$1.50.

Determine the amount of discount at which the bond was purchased.

Solution
The quickest way is to make use of the fact that the write ups are in geometric progression with common ratio $(1+i)$. If the 11th write up is \$1.50, the sum of the write ups (which is equal to the total discount) is:

$$\begin{aligned}&\text{Sum of the write ups included in the 1st to 30th coupons (i.e., the discount)} \\ &= 1.50(v^{10} + v^9 + \cdots + v + 1 + 1.04 + 1.04^2 + \cdots + 1.04^{19}) \\ &= 1.50(a_{\overline{10}|} + s_{\overline{20}|}) \quad \text{at } 4\% \\ &= \$56.83\end{aligned}$$

Summary of Concepts and Formulas in Section 7c

1. For P = bond price and C = bond redemption value:
   (a) $P > C$ means the bond is purchased at a ***premium***
      i. Premium = $P - C$ = excess of price over redemption value
      ii. Also, $g > i$
   (b) $P < C$ means the bond is purchased at a ***discount***
      i. Discount = $C - P$ = excess of redemption value over price
      ii. Also, $i > g$
2. Bond terminology (relative to loan terminology):
   (a) Book value
      i. Used instead of "outstanding loan balance"
   (b) Amortization of premium
      i. Instead of "principal repaid"
      ii. "Writing down"
   (c) Accumulation of discount
      i. "Writing up"
   (d) Coupon
      i. Instead of "payment amount"
3. Bond amortization schedule:
   (a) When bond is sold at a premium:
      i. Book value is "written down" over time
      ii. Interest earned = $I_t = i \cdot B_{t-1}$
      iii. Amount for amortization of premium = $P_t = \mathit{Coupon} - I_t$
      iv. Amortization reduces the book value from P to C over the life of the bond
      v. Book value = $B_t = B_{t-1} - P_t$
   (b) When bond is sold at a discount:
      i. Book value is "written up" over time
      ii. Interest earned = $I_t = i \cdot B_{t-1}$
      iii. Amount for accumulation of discount = $P_t = I_t - \mathit{Coupon}$
      iv. Accumulation of discount increases the book value from P to C over the life of the bond
      v. Book value = $B_t = B_{t-1} + P_t$

Past Exam Questions on Section 7c

1. A 10,000 par value 10-year bond with 8% annual coupons is bought at a premium to yield an annual effective rate of 6%. Calculate the interest portion of the 7th coupon. [5/03 #42]

   (A) 632 (B) 642 (C) 651 (D) 660 (E) 667

2. Among a company's assets and accounting records, an actuary finds a 15-year bond that was purchased at a premium. From the records, the actuary has determined the following:

   (i) The bond pays semiannual interest.

   (ii) The amount for amortization of the premium in the 2nd coupon payment was 977.19.

   (iii) The amount for amortization of the premium in the 4th coupon payment was 1046.79.

   What is the value of the premium? [11/00 #40]

   (A) 17,365 (B) 24,784 (C) 26,549 (D) 48,739 (E) 50,445

3. A 1000 par value 5-year bond with 8.0% semiannual coupons was bought to yield 7.5% convertible semiannually. Determine the amount of premium amortized in the 6th coupon payment. [5/00 #43]

   (A) 2.00 (B) 2.08 (C) 2.15 (D) 2.25 (E) 2.34

4. Laura buys two bonds at time 0. Bond X is a 1000 par value 14-year bond with 10% annual coupons. It is bought at a price to yield an annual effective rate of 8%. Bond Y is a 14-year par value bond with 6.75% annual coupons and a face amount of F. Laura pays P for the bond to yield an annual effective rate of 8%. During year 6, the writedown in premium (principal adjustment) on bond X is equal to the writeup in discount (principal adjustment) on bond Y. Calculate P. [SAMPLE/00 #38]

   (A) 1415 (B) 1425 (C) 1435 (D) 1445 (E) 1455

5. A 1000 par value 18-year bond with annual coupons is bought to yield an annual effective rate of 5%. The amount for amortization of premium in the 10th year is 20. The book value of the bond at the end of 10 years is X. Calculate X. [SAMPLE/00 #39]

   (A) 1180 (B) 1200 (C) 1220 (D) 1240 (E) 1260

6. Bryan buys a $2n$-year 1000 par value bond with 7.2% annual coupons at a price of P. The price assumes an annual effective yield of 12%. At the end of n years, the book value of the bond, X, is 45.24 greater than the purchase price, P. Assume $v_{12\%}^n < 0.5$. Calculate X. [SAMPLE/99 #11]

   (A) 645 (B) 652 (C) 659 (D) 666 (E) 675

7. You are given:

   (i) A 10-year 8% semiannual coupon bond is purchased at a discount of X.

   (ii) A 10-year 9% semiannual coupon bond is purchased at a premium of Y.

   (iii) A 10-year 10% semiannual coupon bond is purchased at a premium of $2X$.

   (iv) All bonds were purchased at the same yield rate and have par values of 1000.

   Calculate Y. [SOA 11/93 #11]

   (A) $\frac{1}{3}X$ (B) $\frac{2}{5}X$ (C) $\frac{1}{2}X$ (D) $\frac{2}{3}X$ (E) X

8. Becky buys an n-year 1,000 par value bond with 6.5% annual coupons at a price of 825.44. The price assumes an annual effective yield rate of i. The total write-up in book value of the bond during the first 2 years after purchase is 23.76. Calculate i. $(i > 0)$ [SOA 5/93 #14]

   (A) 8.50% (B) 8.75% (C) 9.00% (D) 9.25% (E) 9.50%

9. An n-year 1000 par value bond with 8% annual coupons has an annual effective yield of i, $i > 0$. The book value of the bond at the end of the year 3 is 1099.84 and the book value at the end of year 5 is 1082.27. Calculate the purchase price of the bond. [SOA SAMPLE/93 #1]

   (A) 1112 (B) 1122 (C) 1132 (D) 1142 (E) 1152

10. Dick purchases an n-year 1,000 par value bond with 12% annual coupons at an annual effective yield of i, $i > 0$. The book value of the bond at the end of year 2 is 1479.65, and the book value at the end of year 4 is 1439.57. Calculate the purchase price of the bond. [SOA 11/92 #16]

    (A) 1510 (B) 1515 (C) 1519 (D) 1523 (E) 1527

11. A bond with a par value of 1000 and 6% semiannual coupons is redeemable for 1100.

    You are given:

    (i) the bond is purchased at P to yield 8%, convertible semiannually; and

    (ii) the amount of principal adjustment for the 16th semiannual period is 5.

    Calculate P. [SOA 11/90 #11]

    (A) 760 (B) 770 (C) 790 (D) 800 (E) 820

12. On May 1, 1985, a bond with par value 1000 and annual coupons at 5.375% was purchased to yield an effective annual interest rate of 5%. On May 1, 2000, the bond is redeemable at 1100. The book value of the bond is adjusted each year so that it equals the redemption value on May 1, 2000. Calculate the amount of write-up or write-down in the book value in the year ending May 1, 1991. [SOA 5/90 #13]

    (A) 1.25 write-down (B) 0.81 write-down (C) 0.77 write-down (D) 0.81 write-up (E) 0.77 write-up

13. A ten-year bond with par value of $1,000 is purchased to yield 8% convertible semiannually. Par value equals redemption value. The interest-paid portion of the first semiannual coupon is $44.50. At what nominal rate of interest convertible semiannually are the coupons paid? [CAS 5/88 #13]

    (A) Less than 7.75%

    (B) At least 7.75%, but less than 8.50%

    (C) At least 8.50%, but less than 9.25%

    (D) At least 9.25%, but less than 10.00%

    (E) 10.00% or more

14. A 30-year 10,000 bond that pays 3% annual coupons matures at par. It is purchased to yield 5% for the first 15 years and 4% thereafter. Calculate the amount for accumulation of discount for year 8. [SOA 11/87 #15]

    (A) 78 (B) 83 (C) 88 (D) 93 (E) 98

15. A $1000 par value ten-year 8% bond has semiannual coupons. The redemption value equals the par value. The bond is purchased at a premium to yield 6% convertible semiannually. What is the amount for amortization of the premium in the tenth coupon? [CAS 5/87 #15]

    (A) Less than $7.20

    (B) At least $7.20, but less than $7.25

    (C) At least $7.25, but less than $7.30

    (D) At least $7.30, but less than $7.35

    (E) $7.35 or more.

16. A ten-year 5% bond with semiannual coupons is purchased to yield 6% compounded semiannually. The par value and redemption value are both $1,000. What is the book value of the bond six years after issue of the bond? [CAS 5/86 #15]

    (A) Less than $960

    (B) At least $960, but less than $965

    (C) At least $965, but less than $970

    (D) At least $970, but less than $975

    (E) At least $975

17. A 15-year bond with semiannual coupons has a redemption value of $100. It is purchased at a discount to yield 10% compounded semiannually. If the amount for accumulation of discount in the 27th coupon payment is $2.25, which of the following is closest to the total amount of discount in the original purchase price? [CAS 5/85 #14]

    (A) $39 (B) $40 (C) $41 (D) $42 (E) $43

18. A $1,000 par value bond bearing 4% annual coupons is purchased at a discount to yield an effective annual rate of 5%. The write-up in value during the first year is $4.36. Which of the following is closest to the purchase price? [CAS 5/83 #4]

    (A) $857 (B) $867 (C) $877 (D) $887 (E) $897

19. A \$1,000 par value bond with 4% annual coupons is purchased at a discount ten years prior to the maturity date. The proceeds of the coupons are invested in a savings account with a 5% effective annual rate of interest. The effective yield on the ten year investment—including the bond and the savings account—is 6%. What is the book value of the bond one year after purchase? [CAS 5/83 #9]

    (A) Less than \$848

    (B) At least \$848 but less than \$852

    (C) At least \$852 but less than \$856

    (D) At least \$856 but less than \$860

    (E) \$860 or more

20. A ten-year bond bears semiannual coupons of \$4 each and has a redemption value of \$100. The bond is purchased to yield 10% compounded semiannually. To the nearest .02, find the amount of increase in the book value at the time of the tenth coupon payment. [CAS 11/82 #11]

    (A) .50 (B) .52 (C) .54 (D) .56 (E) .58

Solutions to Past Exam Questions on Section 7c

1. Compute the book value at the end of 6 years (B_6) and multiply by 6%:

$$B_6 = 800a_{\overline{4}|} + 10{,}000v^4 = 10{,}693.02$$
$$I_7 = (.06)(10{,}693.02) = 641.58 \qquad \text{ANS. (B)}$$

2. The write-downs are in geometric progression with common ratio $(1+i)$: $\frac{1{,}046.79}{977.19} = (1+i)^2$, where i is the effective rate per $\frac{1}{2}$-year. $\therefore i = 3.5\%$. The premium = sum of the write-downs

$$= 977.19\left[v + 1 + (1+i) + \cdots + (1+i)^{28}\right] = 977.19\left(a_{\overline{1}|} + s_{\overline{29}|}\right) = 48{,}739.29$$

ANS. (D)

3. In general, the amount of principal repaid (write-down) in t^{th} coupon $P_t = (Fr - Ci)v^{n-t+1}$. In this question,

$$P_6 = (40 - 37.50)v^{10-6+1} = 2.50v^5 = 2.08 \qquad \text{ANS. (B)}$$

Note: As an alternative, $B_5 = 40a_{\overline{5}|} + 1{,}000v^5 = 1{,}011.21$. $I_6 = .0375B_5 = 37.92$. $P_6 = Fr - I_6 = 40 - 37.92 = 2.08$.

4. Write-down in 6th year on Bond X $= (100 - 80)v^9 = 20v^9$. Write-up in 6th year on Bond Y (where F = face amount) $= (.08F - .0675F)v^9 = .0125Fv^9 = 20v^9$.

$$F = \frac{20}{.0125} = 1{,}600 \text{ and } P = 108a_{\overline{14}|} + 1{,}600v^{14}$$
$$= 1{,}435 \qquad \text{ANS. (C)}$$

5.

$$P_{10} = 20 = (Fr - 50)v^{18-10+1} = (Fr - 50)v^9$$
$$Fr = 20(1.05)^9 + 50 = 81.03$$
$$X = B_{10} = 81.03a_{\overline{8}|} + 1{,}000v^8 = 1{,}200.55 \qquad \text{ANS. (B)}$$

6.

$$P = 72a_{\overline{2n}|} + 1{,}000v^{2n} \text{ at } 12\%$$
$$B_n = X = P + 45.24 = 72a_{\overline{n}|} + 1{,}000v^n$$
$$72a_{\overline{2n}|} + 1{,}000v^{2n} + 45.24 = 72a_{\overline{n}|} + 1{,}000v^n$$
$$72\left(\frac{1 - v^{2n}}{.12}\right) + 1{,}000v^{2n} + 45.24 = 72\left(\frac{1 - v^n}{.12}\right) + 1{,}000v^n$$

This simplifies to the following quadratic in v^n: $400v^{2n} - 400v^n + 45.24 = 0$. The positive root is $v^n = .13$.

$$X = 72a_{\overline{n}|} + 1{,}000v^n$$
$$= 72\left(\frac{1 - .13}{.12}\right) + 1{,}000(.13) = 652 \qquad \text{ANS. (B)}$$

7. $X = (1{,}000i - 40)a_{\overline{20}|}$, $Y = (45 - 1{,}000i)a_{\overline{20}|}$, $Z = (50 - 1{,}000i)a_{\overline{20}|}$. Given that $Z = 2X$, $i = 4\frac{1}{3}\%$, so $Z = 6\frac{2}{3}a_{\overline{20}|}$ and $Y = 1\frac{2}{3}a_{\overline{20}|}$. $\therefore Y = \frac{1}{4}Z = \frac{1}{2}X$ ANS. (C)

8. The book value at the end of the 2nd year $= 825.44(1+i)^2 - 65s_{\overline{2}|}$ (retrospectively) $= 825.44(1+i)^2 - 65(2+i)$. The total write up is the excess of the book value over the purchase price $= 825.44(1+i)^2 - 65(2+i) - 825.44 = 23.76$. This is a quadratic in $(1+i) : 825.44(1+i)^2 - 65(1+i) - 914.20 = 0$. The solution gives $i = 9.25\%$. ANS. (D)

9. Retrospectively, $B_5 = B_3(1+i)^2 - 80s_{\overline{2}|}$. Thus $1{,}082.27 = 1{,}099.84(1+i)^2 - 80(2+i)$. This is the following quadratic in $(1+i)$: $1{,}099.84(1+i)^2 - 80(1+i) - 1{,}162.27 = 0$. $1+i$ solves to 1.065002 and $i = 6.5\%$. The original price can be computed as $80a_{\overline{3}|} + (PV$ of book value end of 3rd year$) = 80a_{\overline{3}|} + 1{,}099.84v^3 = 1{,}122.38$ ANS. (B)

(Just enter 3 $\boxed{\text{N}}$ 6.5 $\boxed{\text{I/Y}}$ 80 $\boxed{\text{PMT}}$ 1,099.84 $\boxed{\text{FV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{PV}}$.)

10. This is the same type of question as the previous one. $1{,}439.57 = 1{,}479.65(1+i)^2 - 120s_{\overline{2}|}$, which reduces to the quadratic $1{,}479.65(1+i)^2 - 120(1+i) - 1{,}559.57 = 0$, so $i = 6.800187\%$.

$$\begin{aligned}\text{Price } &= 120a_{\overline{2}|} + 1{,}479.65v^2 \text{ at } i\\ &= 1{,}514.79 \qquad \text{ANS. (B)}\end{aligned}$$

11. $P_{16} = [30-(1{,}100)(.04)]v^{n-16+1} = -14v^{n-15} = -5$. (The principal adjustment is a write-up, since $g = \frac{30}{1{,}100}$, $i = .04$ and $g < i$.)

$$v^{n-15} = \frac{5}{14},\ v^n = \frac{5}{14}v^{15} = .198309$$

The price can be found without explicitly solving for n, etc.:

$$\begin{aligned}P = 30a_{\overline{n}|} + 1{,}100v^n &= 30\left(\frac{1-.198309}{.04}\right) + 1{,}100(.198309)\\ &= 601.27 + 218.14 = 819.41 \qquad \text{ANS. (E)}\end{aligned}$$

12. $g = \frac{53.75}{1{,}100} = 4.886\%$, so $g < i$ and the principal adjustments are write-ups. Write-up in the t^{th} period $P_t = (Ci - Fr)v^{n-t+1}$. $\therefore P_6 = [1{,}100(.05) - 53.75]v^{15-6+1} = 1.25v^{10} = .77$ ANS. (E)

13.

$$\begin{aligned}\text{Price } &= P = 1{,}000ra_{\overline{20}|} + 1{,}000v^{20} \text{ at } 4\%\\ I_1 &= .04P = 40ra_{\overline{20}|} + 40v^{20} = 44.50\\ r &= \frac{44.50 - 40v^{20}}{40a_{\overline{20}|}} = 4.8278\%\end{aligned}$$

This is the effective semiannual coupon rate. Nominal annual rate $= 2r = 9.66\%$ ANS. (D)

14. Determine the book values at the end of 7 and 8 years. The write-up $= B_8 - B_7$.

$$\begin{aligned}B_7 &= 300a_{\overline{8}|.05} + v_{.05}^{8}\left[300a_{\overline{15}|.04} + v_{.04}^{15}(10{,}000)\right]\\ &= 1{,}938.96 + (.676839)(8{,}888.16)\\ &= 7{,}954.81\end{aligned}$$

$$\begin{aligned}B_8 &= B_7(1.05) - 300 = 8{,}052.55\\ \text{Write-up} &= B_8 - B_7 = 8{,}052.55 - 7{,}954.81 = 97.74 \qquad \text{ANS. (E)}\end{aligned}$$

15.

$$\begin{aligned}P_t &= (Fr - Ci)v^{n-t+1}\\ P_{10} &= (40-30)v^{20-10+1} = 10v^{11} = 7.22 \qquad \text{ANS. (B)}\end{aligned}$$

16. Six years after issue, there are 8 coupons remaining. $B_{12} = 25a_{\overline{8}|} + 1{,}000v^8$ at $3\% = 964.90$ ANS. (B)

17. The write-ups are in geometric progression with common ratio $(1+i)$. The total discount is equal to the sum of the write-ups. If the 27th write-up $= 2.25$, the sum of the write-ups from the 1st to the 30th =

$$2.25\left[v^{26} + v^{25} + \cdots + v + 1 + (1+i) + (1+i)^2 + (1+i)^3\right]$$

at $5\% = 2.25\left(a_{\overline{26}|} + s_{\overline{4}|}\right) = 42.04$. ANS. (D)

18. Let P = purchase price. The interest in the 1st coupon = I_1 = $.05P$. The write-up is the excess of the interest over the coupon = $.05P - 40 = 4.36$.

$$P = \frac{44.36}{.05} = 887.20 \qquad \text{ANS. (D)}$$

19. The AV of the coupons $= 40s_{\overline{10}|.05} = 503.12$. The total received at the end of 10 years $= 1{,}503.12$. If P was the purchase price, $P = 1{,}503.12v^{10}$ at 6%, so $P = 839.33$. $B_1 = P(1+i) - 40 = 839.33(1.06) - 40 = 849.69$ ANS. (B)

    Note: This question is somewhat ambigious, because it doesn't say whether book values are based on the overall yield rate of 6% (which takes into account the reinvestment of the coupons at 5%), or whether they are based on the yield rate ignoring the reinvestment of coupons.

    Ignoring the reinvestment of coupons, the yield rate is the solution of $839.33 = 40a_{\overline{10}|} + 1{,}000v^{10}$ at i. The calculator gives $i = 6.204154\%$, so $B_1 = (839.33)(1.06204154) - 40 = 851.40$. This answer is in the same range as (B).

20. Write-up in t^{th} coupon = $(Ci - Fr)v^{n-t+1}$. Write-up in 10th coupon = $(5-4)v^{20-10+1} = v^{11} = .5847$ ANS. (E)

7d Price Between Coupon Dates

Up to now, we have determined the price of a bond on a date that is exactly one period before the next coupon date (or, what is the same thing, just after the payment of a coupon). In this section, we will consider the price on a date that is a fraction, k, of a coupon period after the last coupon date.

Unfortunately, the notation and terminology for this topic are not the same in the various suggested text books for Exam FM.

There are two types of bond prices between coupon due dates that you should be familiar with:

1. The price that is actually paid for the bond on the date of purchase (or on the "settlement date").[1]
2. The price that is quoted for the bond in the financial press. The practice is to quote this price *excluding* the value of the coupon that has accrued by the date of purchase.[2]

The accrued coupon is commonly called ***accrued interest*** in the financial world, so we will use that terminology in the rest of this section.

Why does the financial press quote the price excluding accrued interest? We will answer this question below.

Now let's consider how these two types of prices are calculated.

Price Actually Paid on the Date of Purchase

Assume that the last coupon was paid at time t (measured in coupon periods) and that the bond will be sold at time $t+k$, where k is the fraction of a coupon period since time t $(0 \le k \le 1)$. The price actually paid at time $t+k$ is simply equal to the price on the last coupon date brought up with interest to time $t+k$.

We will use the symbol B_t for the price of the bond just after the last coupon was paid and B_{t+k} for the actual price at time $t+k$. If the effective rate of interest per coupon period is i, the price that should be paid for the bond at time $t+k$ is:

$$B_{t+k} = (1+i)^k B_t$$

This could also be expressed as the present value at time $t+k$ of the price that would be paid on the *next* coupon due date (i.e., at time $t+1$), including the coupon due on that date:

$$B_{t+k} = v^{1-k}(B_{t+1} + Fr)$$

Why is it necessary to add the coupon to B_{t+1}? Remember that B_{t+1} is the price at time $t+1$ just *after* the coupon due at that time has been paid. Since the purchaser of the bond will receive that coupon, its present value at time $t+k$ should be included in the price that the purchaser pays.

Of course, it can be shown that these two ways of calculating the actual price are equivalent.

Let's take a specific example of a bond purchased between coupon due dates. We will use the first bond illustrated in Section 7c, i.e., a $1,000 bond with 8% semiannual coupons, redeemable at $1,050 in $1\frac{1}{2}$ years and purchased to yield a nominal annual rate of 6% compounded semiannually. (In the above equations for B_{t+k}, $i = 3\%$.)

Here are the book values for this bond, taken from the first table in Section 7c:

Period t	**Book Value**
0	1,074.04
1	1,066.26
2	1,058.25
3	1,050.00

[1] The various textbooks use different terms for the actual price paid, among them being "price-plus-accrued," "full price," "dirty price," and "flat price."

[2] Once again, the various textbooks use different terms for the price excluding the value of the accrued coupon, among them being "price," "clean price," and "market price."

We will assume that the yield rate stays constant at $i = 3\%$ per half year. Then the book values in this table also represent the prices that would be paid for this bond on each coupon due date, just after the coupon due on that date was paid.

Suppose the bond is purchased $1\frac{1}{2}$ months after issue, so that $k = 0.25$. (k represents the fraction of a coupon period since the last coupon was paid; $1\frac{1}{2}$ months is one-fourth of a six-month coupon period.) Then the price $1\frac{1}{2}$ months after issue is determined as follows:

$$B_{0.25} = (1.03)^{0.25}B_0 = (1.03)^{0.25}(1{,}074.04) = 1{,}082.01$$

We can also calculate the price as the present value of the price on the next coupon date, including the coupon then due:

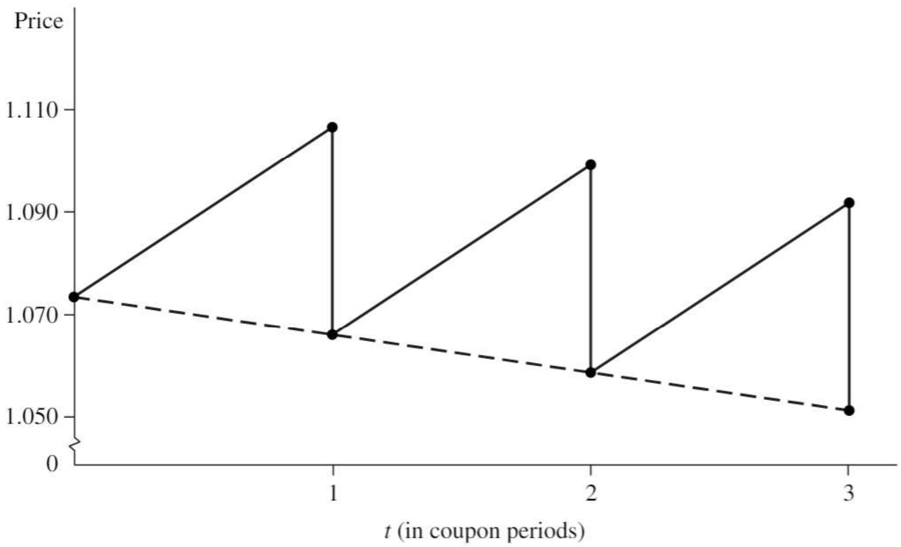
$$B_{0.25} = v^{0.75}(B_1 + 40) = v^{0.75}(1{,}066.26 + 40) = 1{,}082.01$$

As expected, we get the same answer whether we work from the price on the last coupon date or from the price on the next coupon date.

Now let's consider the progression of the prices as we move along from the issue of the bond at $t = 0$ to the redemption of the bond at $t = 3$.

As k varies from 0 to 1 between each coupon due date, the price increases by the factor $(1.03)^k$. At $t = 1$, an instant *before* the first coupon is paid, the price is $(1.03)(1{,}074.04) = 1{,}106.26$. At $t = 1$, an instant *after* the first coupon is paid, the price is 1,066.26. (We can get this price from the above table or by subtracting the $40 coupon from 1,106.26.) Obviously, there is a drop of $40 in the price of the bond at the instant this coupon is paid. This drop in value occurs on each coupon due date.

We can graph the progression of prices as follows:



The graph has a saw-toothed appearance, with a "tooth" on each coupon due date. (In this example, the bond is sold at a premium, so the teeth curve downward. If a bond were sold at a discount, the teeth would curve upward; if a bond were sold at par, the teeth would be horizontal.)

The price between coupon dates may appear to be a straight line, but it is actually an exponential curve.

We have also drawn a dotted line on the graph that connects the actual prices on coupon dates just after a coupon has been paid. We'll explain this dotted line below.

Price Quoted in the Financial Press (i.e., Excluding Accrued Interest)

By convention, the price of a bond quoted in the financial press is the actual price *minus* the accrued interest (or accrued coupon). The reason for this is that bond traders prefer to see a smooth progression of prices, rather than the saw-toothed progression pictured in the graph above. Of course, the actual price paid for a bond is the price quoted in the financial press *plus* accrued interest.

There are several methods of creating a smooth progression of prices but we will only consider the method that is normally used in practice.[3]

[3] The method described herein is called the "semi-theoretical method" by some of the suggested textbooks and the "semipractical clean price" by others.

The accrued interest is clearly 0 at time t (where t is any coupon date just after the coupon has been paid). The accrued interest gradually increases to the full amount of the coupon Fr just before the coupon is paid at time $t+1$. A simple approximation to the accrued interest at time $t+k$ is kFr. On this basis, the priced quoted in the financial press is the actual price minus kFr:

$$\text{Price quoted in the financial press} = B_{t+k} - kFr = B_t(1+i)^k - kFr$$

Taking the same example as above, the price quoted in the financial press $1\frac{1}{2}$ months after issue would be $1{,}074.04(1.03)^{0.25} - (0.25)(40) = 1{,}082.01 - 10 = 1{,}072.01$. To get the actual price at this point, we would add the accrued coupon of \$10 to the quoted price.

The dotted line in the graph above represents the smooth progression of prices that would be quoted in the financial press.

Counting the Days

In the above example, we considered a bond purchased $1\frac{1}{2}$ months after issue. We set $k = 0.25$, since $1\frac{1}{2}$ months is one-quarter of the six-month coupon period. However, in actual practice, k is based on counting the number of days since the last coupon payment and the number of days between coupon payments.

There are two methods for counting the days:

"Actual/Actual" Method

Under the "actual/actual" method, which is used for *government* bonds, the actual number of days is used for the numerator and denominator of k. An example will make this clear.

Suppose a semiannual coupon bond pays coupons every July 23 and January 23. Say the bond will be purchased on November 3. Then k would be determined as follows:

$$\begin{aligned}
\text{Numerator of } k &= 8+31+30+31+3 = 103\\
\text{Denominator of } k &= 8+31+30+31+30+31+23 = 184\\
&\qquad \text{(The next coupon date is January 23.)}\\
k &= 103/184 = .55978
\end{aligned}$$

"30/360" Method

The "30/360" method is used for *corporate* and *municipal* bonds. Under this method, each month is considered to have exactly 30 days, so a year is assumed to consist of 360 days. Taking the same example as the bond purchased on November 3:

$$\begin{aligned}
\text{Numerator of } k &= 7+30+30+30+3 = 100\\
\text{Denominator of } k &= 7+30+30+30+30+30+23 = 180\\
&\qquad \text{(Under this method, the denominator is always 180 for a}\\
&\qquad \text{semiannual coupon bond.)}\\
k &= 100/180 = .55556
\end{aligned}$$

Note that the number of days from July 23 to the end of the month is counted as 7, since each month is considered to have 30 days under this method.

Stepping Stones

Example 1
A \$1,000 par value bond has 7.5% semiannual coupons and matures on July 1, 2017 at \$1,050. Find the actual selling price of this bond on November 15, 2013 and the price that would be quoted in a financial newspaper on the same date, based on a nominal annual yield rate of 5.80% compounded semiannually. Use the actual number of days to compute the accrued interest.

Solution
The actual price can be determined by finding the price on the last coupon date (7/1/2013) and accumulating this price with interest:

$$\begin{aligned}\text{Price on } 7/1/2013 &= Fra_{\overline{n}|} + Cv^n \\ &= 37.50a_{\overline{8}|} + 1050v^8 \text{ at } 2.9\% \\ &= \$1{,}099.70\end{aligned}$$

The actual number of days from 7/1 to 11/15 is 137 and the actual number of days in the coupon period from 7/1 to 1/1 is 184. Thus, we have:

$$\begin{aligned}\text{Actual price on } 11/15/2013 &= (1{,}099.70)(1.029)^{\frac{137}{184}} \\ &= \$1{,}123.36\end{aligned}$$

The accrued interest on 11/15/2013 is $\left(\frac{137}{184}\right)(37.50) = \$27.92$.

Thus, the price quoted in a financial newspaper on 11/15/2013 (i.e., the price excluding accrued interest) is $1{,}123.36 - 27.92 = \$1{,}095.44$

Note: We have noted that different terminology is used by the various suggested textbooks for the actual price and the quoted price of a bond purchased between coupon due dates. Very few questions have appeared on this topic in the past. Hopefully, if they do appear in the future, it will be made very clear what a question is asking for, without relying on the terminology used in any particular textbook.

7e Determination of Yield Rates

Up to now, we have usually assumed that the yield rate is known and that the price has to be computed. Of course, in the real world, the price of the bond in the marketplace would be the known quantity and the yield rate at that price would have to be computed.

You may recall from the discussion of unknown interest rates in Section 2b that solving for an unknown interest rate "by hand" can be a difficult problem. In the case of a bond, using the Basic Formula, we are looking for the solution of the following equation for i:

$$P = Fra_{\overline{n}|} + Cv^n$$

where P, Fr, C, and n are known. With the BA II Plus, you would enter P in $\boxed{\text{PV}}$, Fr in $\boxed{\text{PMT}}$, C in $\boxed{\text{FV}}$, n in $\boxed{\text{N}}$ then $\boxed{\text{CPT}}$ $\boxed{\text{I/Y}}$. (P would be entered with the opposite sign from Fr and C.) If you did not have a financial calculator, you might have to resort to an iterative technique, such as Newton-Raphson.

It is also possible to develop a simple approximation formula for the yield rate that doesn't require a financial calculator or iteration. This formula is called the ***Bond Salesman's Method,*** which is not on the syllabus for this exam. We are including it here for your general information as a simple method for estimating the yield rate.

We will develop the Bond Salesman's Method from a common sense point of view that makes it very easy to remember.

Stepping Stones

Example 1
A $100 bond with 6% semiannual coupons matures at the end of 8 years at $103. If it sells at $95, approximate the yield rate using the Bond Salesman's Method.

Solution
The Bond Salesman's Method approximates the yield rate by dividing (1) the average amount of interest earned per period by (2) the average amount invested.

The ***total*** interest earned over the entire 8 year period is simply the sum of the returns minus the amount invested. For this bond, we would receive 16 semiannual coupons of $3 each plus the redemption value of $103 on an investment of $95, so the total interest earned is $16 \times \$3 + \$103 - \$95 = \$56$. Thus, item (1), the average amount of interest earned per period, is $\frac{56}{16} = \$3.50$ per half year.

A reasonable approximation of (2), the average investment, is the mean of the original investment (\$95) and the value of the investment at the end of the term (\$103). Thus, item $(2) = \frac{1}{2}(95 + 103) = \$99$.

Finally, the Bond Salesman's Method gives the approximate value of i as $(1) \div (2)$:

$$i \approx \frac{3.50}{99} = 3.54\% \text{ to 2 decimals}$$

This is the approximate effective rate for a half-year period.

Example 2
Determine the exact yield rate to 2 decimals for the bond in Example 1.

Solution
16 $\boxed{\text{N}}$ 95 $\boxed{\text{PV}}$ 3 $\boxed{+/-}$ $\boxed{\text{PMT}}$ 103 $\boxed{+/-}$ $\boxed{\text{FV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{I/Y}}$

The answer is 3.56%. In this particular case, the Bond Salesman's Method gives a very good approximation to the exact yield rate.

Example 3
Develop the Bond Salesman's Method in terms of symbols.

Solution
For a bond selling at P, with coupons of Fr or Cg per period and redeemable at C in n periods, we have the following:

$$\text{Total interest} = nCg + C - P$$
$$\text{Average interest per period} = \frac{nCg + C - P}{n}$$
$$\text{Average investment} = \frac{1}{2}(P + C)$$
$$\text{Approximate yield rate per period } i \approx \frac{nCg + C - P}{\frac{n}{2}(P + C)}$$

In this form, it's very clear that we are simply dividing the average interest per period by the average investment. You hardly have to memorize this formula.

Some of the textbooks state the above formula in different (but equivalent) ways. Broverman uses j instead of i and r instead of g, and states the formula as:

$$j = \frac{r - \frac{P-C}{n \cdot C}}{1 + \frac{P-C}{2C}}$$

Kellison defines k as $\frac{P-C}{C}$ and states the formula as:

$$i = \frac{g - \frac{k}{n}}{1 + \frac{k}{2}}$$

These versions of the Bond Salesman's Method make it more difficult to see the approximation as simply the average interest earned per period divided by the average investment.

Summary of Concepts and Formulas in Sections 7d and 7e

1. Price between coupon dates:
   (a) Full price (a.k.a. price-plus-accrued, theoretical dirty price):
      i. Money that actually changes hands when bond is sold
      ii. $B_{t+k} = B_t(1+i)^k$
   (b) Price excluding accrued interest (price quoted in the financial press):
      i. $= B_{t+k} - kFr = B_t(1+i)^k - kFr$
2. Determination of yield rates
   (a) E.g., solve for i in the basic formula: $P = Fra_{\overline{n}|i} + Cv_i^n$
   (b) Calculable with exam calculator

Past Exam Questions on Sections 7d and 7e

1. Dan purchases a 1000 par value 10-year bond with 9% semiannual coupons for 925. He is able to reinvest his coupon payments at a nominal rate of 7% convertible semiannually. Calculate his nominal annual yield rate convertible semiannually over the 10-year period. [SOA 11/05 #16]

   (A) 7.6% (B) 8.1% (C) 9.2% (D) 9.4% (E) 10.2%

2. Tina buys a 1000 par value 10-year bond with 10% annual coupons at a price to yield an annual effective rate of 10%. The coupons are reinvested at an annual effective rate of 8%. Immediately after receiving the 4th coupon payment, Tina sells the bond to Joe for a price P at an annual effective yield of i to the buyer. Tina's annual effective yield from the date of purchase until the date of sale was 8%. Calculate i. [SOA 11/96 #16]

   (A) 11.6% (B) 11.8% (C) 12.0% (D) 12.2% (E) 12.4%

3. A 1000 par value 5-year bond with semiannual coupons of 60 is purchased to yield 8% convertible semiannually. Two years and two months after purchase, the bond is sold at a price which maintains the same yield for the buyer. Calculate this price. [SOA 5/95 #16]

   (A) 1089 (B) 1099 (C) 1105 (D) 1113 (E) 1119

4. John buys a bond that is due to mature at par in 1 year. It has a 100 par value and coupons at 4% convertible semiannually. John pays 98.51 to obtain a yield rate i convertible semiannually, $i > 0$. Calculate i. [SOA 5/91 #11]

   (A) 4.55% (B) 4.80% (C) 5.05% (D) 5.30% (E) 5.55%

5. (No longer on the syllabus) A 1,000 par value 10-year bond with coupons at 5% convertible semiannually is selling for 1,081.78. The bond salesman's method is used to approximate the yield rate convertible semiannually. Calculate the difference between this approximation and the exact value. [SOA 11/88 #14]

   (A) 0.00003 (B) 0.00006 (C) 0.00012 (D) 0.00018 (E) 0.00024

6. (No longer on the syllabus) You are given a 10-year bond with semiannual coupons, where:

   (i) The purchase price is 650.

   (ii) The par value is 1,000.

   (iii) The redemption value is 1,050.

   (iv) The coupon rate is 12%.

   Using the bond salesman's method, calculate the nominal yield rate, convertible semiannually. [SOA 5/88 #13]

   (A) 0.0941 (B) 0.0952 (C) 0.1882 (D) 0.1904 (E) 0.1988

7. A $1000 par value 5 year bond with 6% semiannual coupons is selling for $1112. What is the annual effective yield rate? [CAS 5/84 #3]

   (A) Less than 3.6%

   (B) At least 3.6%, but less than 3.7%

   (C) At least 3.7%, but less than 3.8%

   (D) At least 3.8%, but less than 3.9%

   (E) 3.9% or more

8. A $100 par value 10 year bond provides 5% semiannual coupons. The yield rate is 4% convertible semiannually. What is the flat price (i.e., the money that actually changes hands if the bond is sold, ignoring expenses) 8.4 years after issue at the same yield rate? [CAS 5/84 #14]

   (A) Less than $102.20

   (B) At least $102.20, but less than $102.80

   (C) At least $102.80, but less than $103.40

   (D) At least $103.40, but less than $104.00

   (E) $104.00 or more

9. A 1,000 20-year 8% bond with semiannual coupons is purchased for 1,014. The redemption value is 1,000. The coupons are reinvested at a nominal annual rate of 6%, compounded semiannually. Determine the purchaser's annual effective yield rate over the 20 year period. [SOA SAMPLE/83 #5]

   (A) 6.9% (B) 7.0% (C) 7.1% (D) 7.2% (E) 7.3%

10. A 10,000 par value bond with 8% semiannual coupons is sold 3 years and 4 months before the bond matures. The purchase will yield 6% convertible semiannually to the buyer. The price at the most recent coupon date, immediately after the coupon payment, was 5,640. Calculate the quoted price of the bond, i.e., the price excluding accrued interest. [Modified SOA Sample #43]

    (A) 5,500 (B) 5,520 (C) 5,540 (D) 5,560 (E) 5,580

11. A 1000 bond with 6% semiannual coupons matures at par on October 15, 2020. The bond was purchased on June 28, 2005 to yield the investor 7% convertible semiannually. Caculate the purchase price. (Note that April 15 is the 105th day of the year, June 28 is the 179th day of the year, and October 15th is the 288th day of the year) [Modified SOA Sample #50]

    (A) 906 (B) 907 (C) 908 (D) 919 (E) 925

Solutions to Past Exam Questions on Sections 7d and 7e

1. Dan receives a total of $45s_{\overline{20}|.035} + 1{,}000 = 2{,}272.59$ at the end of 10 years. The investment was 925, so if i is the effective annual yield rate, we have $925(1+i)^{10} = 2{,}272.59$ and $i = 2.45685^{1/10} - 1 = 9.4052\%$. Thus, the nominal annual yield convertible semiannually is $i^{(2)} = 2(1.094052^{0.5} - 1) = 9.194\%$

   (or compute the semiannual rate j by setting $925(1+j)^{20} = 2{,}272.59$, then $i^{(2)} = 2j$) ANS. (C)

2. Since the coupon rate equals the yield rate, the bond is bought at par and the price is 1,000. Tina receives a total of $100s_{\overline{4}|.08} + P$ at the end of 4 years. Her annual effective yield from the date of purchase until the date of sale was 8%, so $1,000(1.08)^4 = 100s_{\overline{4}|.08} + P$ which gives $P = 909.88$. Joe's yield rate is i, so $909.88 = 100a_{\overline{6}|i} + 1,000v^6$. Using the calculator, we get $i = 12.2\%$. ANS. (D)

3. The book value just after the 4th coupon payment (i.e., the PV of future payments) at the original yield rate $= 60a_{\overline{6}|} + 1{,}000v^6$ at $4\% = 1{,}104.84$. The price to maintain the same yield rate $= (1{,}104.84)(1.04)^{\frac{1}{3}} = 1{,}119.38$. (2 months after the end of the 2nd year is $\frac{2}{6} = \frac{1}{3}$ of a semiannual interest period.) ANS. (E)

4. If j = effective semiannual rate, $P = 2a_{\overline{2}|j} + 100v^2 = 98.51$. Enter $n = 2$, $PV = 98.51$, $PMT = 2$, $FV = 100$, then $CPT\%i = 2.776\%$. The yield rate convertible semiannual is $2j = 5.55\%$. ANS. (E)

   Note: This was a pre-calculator question. The intention was to solve the quadratic $98.51 = 2v_j + 2v_j^2 + 100v_j^2$ for j.

5. Exact value of semiannual effective rate is solution of $1.081.78 = 25a_{\overline{20}|j} + 1{,}000v^{20}$. Using the calculator, $j = 1.999867\%$. Using the bond salesman's method:

$$j' = \frac{(25)(20) + 1{,}000 - 1{,}081.78}{\frac{20}{2}(1{,}081.78 + 1{,}000)} = 2.008954\%$$

   Difference in semiannual effective rate $= .02008954 - .01999867 = .000091$
   Difference in nominal annual rates $= (2)(.000091) = .000182$ ANS. (D)

6. By the bond salesman's method, the effective semiannual rate j is given by:

$$\begin{aligned} j &= \frac{(20)(60) + 1{,}050 - 650}{\frac{20}{2}(650 + 1{,}050)} \\ &= 9.411765\% \end{aligned}$$

   Nominal annual rate $= 2j = .1882$ ANS. (C)

7. $1{,}112 = 30a_{\overline{10}|} + 1{,}000v^{10}$ at rate j

   Enter the appropriate values in the calculator to get $j = 1.768216\%$.

   The annual effective rate $i = (1+j)^2 - 1 = 3.57\%$. ANS. (A)

   (This is another pre-calculator question. Could be solved by trial-and-error, or bond salesman's method gives $j = 1.78\%$, $i = 3.56\%$.)

8. First, find the price at the end of 8 years, just after the 16th coupon has been paid:

$$P = 2.50a_{\overline{4}|} + 100v^4 = 101.90 \text{ at } 2\%$$
$$\text{Flat price} = 101.90(1.02)^{0.8} = 103.53.$$

   (0.8 is the fraction of one semiannual interest period after 8 years on which the bond is sold.) ANS. (D)

9. The purchaser receives a total of $40s_{\overline{40}|.03} + 1{,}000 = 4{,}016.05$ at the end of 20 years. The investment was 1,014, so if i is the effective annual rate, we have $(1{,}014)(1+i)^{20} = 4{,}016.05$ and $i = 7.12\%$. ANS. (C)

10. The quoted price is equal to $B_{t+k} - kFr = B_t(1+i)^k - kFr$ with $k = 2/6 = 1/3$, since the bond is sold 2 months after the last coupon payment. The quoted price $= 5,640(1.03)^{1/3} - (1/3)(400) = 5,562.52$. ANS. (D)

11. The price of the bond on April 15, 2005 just after the last coupon payment is $B_t = 30a_{\overline{31}|.035} + 1000v^{31} = 906.32$. The number of days between April 15 and June 28 is $179 - 105 = 74$ and the number of days between April 15 and October 15 is $288 - 105 = 183$. On June 28, 2005, the price is $B_{t+k} = B_t(1+i)^k = 906.32(1.035)^{74/183} = 919.02$. ANS. (D)

7f Callable Bonds

You or someone you know may have refinanced their home mortgage in recent years. Of course, the main reason for doing this is to take advantage of a decline in interest rates since the original mortgage was obtained.

A corporation or governmental entity that borrows money by issuing bonds might also want to take advantage of a decline in interest rates after the bonds are issued. For this reason, many bonds have an option that permits the borrower to "call" the bonds, i.e., to redeem them prior to the normal maturity date. If interest rates decline after the bonds are issued, the issuer could call the bonds (pay off their redemption values) and then issue new bonds paying lower coupons. These ***callable bonds*** generally have an earliest ***call date*** that is several years after issue.

Callable bonds present potential investors with a dilemma: if they don't know when the bond will be redeemed, they can't determine exactly what their yield rate will be at the current selling price. (They would also face the possibility of having to reinvest the redemption value earlier than expected, at a time when interest rates were lower.)

Let's consider the following example.

Stepping Stones

Example 1
A $100 bond with 6% annual coupons and a maturity date 20 years from now can be called (i.e., redeemed by the issuer) at par on any coupon due date starting 10 years from now. What price should an investor pay to get a yield rate of exactly 4% effective?

Solution
Actually, it is not possible to answer this question in its present form. This is because at any given price, the yield rate will vary, depending on when the bond is redeemed (i.e., 10 to 20 years from issue). So we cannot find a single price that will always result in a yield rate of exactly 4%. This will become clearer as we go along.

To revise the question so that it can be answered, we would have to say something like this: "What is the highest price an investor can pay and still be certain of a yield rate of at least 4%?"

The key to this problem is to determine the price based on the ***worst possible redemption date***. If redemption occurs on any other date, we will be better off. But what is the worst possible redemption date?

The answer is that it depends on whether the bond sells at a premium or a discount. You may recall from Section 7c that when $g > i$, the bond is said to sell at a premium, and when $g < i$, it sells at a discount. The bond in Example 1 sells at a premium, since $.06 > .04$.

The worst thing that could happen if a bond sells at a premium is that it is redeemed on the earliest possible date. You don't have to memorize this; just consider any bond selling at a premium, say a $100 bond that sells for $120 with 6% annual coupons. Here is a quick sketch for redemption in a year:

On an investment of $120, you get back only $106 at the end of a year. There is a "loss" of $20, the excess of the price over the redemption value (i.e., the premium). This is offset by the $6 coupon, for a net loss of $14 in one year on an investment of $120. This represents an effective rate of $-\frac{14}{120} = -11.67\%$.

Intuition tells us that the more years that we can spread the $20 loss over, the better the yield rate. For example, the yield rate would be negative if the bond were called within 3 years (we would get back less than our $120 investment). The yield rate would become positive if the bond were called at the end of 4 years. (By the end of 4 years, we would receive a total of $24 in coupons and the $100 redemption value, for a return of $124 on an investment of $120. This means that the yield rate would be positive.)

We can conclude that for a bond selling at a ***P***remium, the ***E***arliest redemption date is the ***W***orst. We will call this the ***PEW*** principle. There is no need to memorize this; the one-year sketch will tell you this immediately.

So for the bond in Example 1, we should compute the price based on the earliest possible redemption date, which is 10 years from now. The price at 4% effective is:

$$P = 100 + (6 - 4)a_{\overline{10}|} = \$116.22$$

If we buy the bond at this price and it is called in 10 years, we would earn an effective yield rate of 4%. If the bond is called on any later date, the yield rate would be higher. For example, if the bond is called at the latest possible date, in 20 years, you can use your calculator to compute the yield rate as 4.73% at a price (*PV*) of $116.22. You will find that the yield rate increases from 4% to 4.73% as the term of the bond increases from 10 years to 20 years.

Example 2
For the same bond as in Example 1 ($100 bond with 6% annual coupons that can be called on any coupon date 10 years to 20 years from now), what price should an investor pay to get a minimum yield rate of 8% effective? If this price is paid, what is the maximum yield rate the investor can earn?

Solution
This bond sells at a discount, since $g < i$ $(.06 < .08)$. It should come as no surprise that the worst possible redemption date is the latest date, i.e., 20 years from now. Let's see if we can reason this out.

Take any bond selling at a discount, for example, a $100 bond with 6% annual coupons selling for $80. Assume the bond is redeemed in a year:

The "gain" of $20 (the discount $= 100 - 80$) is received all in one year, plus a $6 coupon. Thus, there is a total return of $26 on an investement of $80, for an exact effective yield rate of $\frac{26}{80} = 32.5\%$. If the bond is redeemed later, the discount of $20 is spread over more years, which dilutes the effective rate. Clearly, in the discount situation, the ***latest*** redemption date is the worst.

Applying this to Example 2:

$$\begin{aligned} P &= 100 + (6 - 8)a_{\overline{n}|} \text{ at } 8\% \\ &= 100 - 2a_{\overline{n}|} \end{aligned}$$

Use $n = 20$ to determine the price:

$$P = \$80.36$$

The question also asks for the maximum yield rate the investor could earn if he bought the bond at this price, i.e., $80.36. We have already concluded that early redemption is a good thing in the discount situation, so we plug in a term of 10 years (N), a price (*PV*) of $80.36, coupons of $6 (*PMT*) and a redemption value (*FV*) of $100, and compute i. You should get $i = 9.07\%$.

Example 3
A $1,000 bond with 5% semiannual coupons and redeemable at $1,100 in 20 years is purchased for $1,250. It can be called by the issuer on any coupon due date from the 10^{th} to the 39^{th}. On how many of the 20 possible call dates will the yield rate be at least 3% compounded semiannually?

Solution
First, determine n for which the yield rate would be exactly 1.5% per half year:

1.5 $\boxed{\text{I/Y}}$ 1,250 $\boxed{\text{PV}}$ 25 $\boxed{\text{+/-}}$ $\boxed{\text{PMT}}$ 1,100 $\boxed{\text{+/-}}$ $\boxed{\text{FV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{N}}$

The result is $n = 20.65$. Since the bond is purchased at a premium ($\$1{,}250 > \$1{,}100$), the worst redemption date is the earliest (PEW). So any call date from the 21^{st} to the 39^{th} will result in a yield rate greater than 3% compounded semiannually. There are 19 such call dates.

A More General Principle

It is possible to state a more general principle that covers both the premium and discount situations. This principle also covers more complex situations where it may not be clear whether the earliest or latest redemption date should be used, because the redemption values are not the same on different call dates.

The more general principle is very simple: to ensure that we earn a specified minimum yield rate, compute the ***lowest price*** for all of the possible redemption dates at this yield rate.

To show that this covers both the premium and discount situations, first consider Example 1. We express the price of the bond using the premium/discount formula:

$$\begin{aligned} P &= C + (Fr - Ci)a_{\overline{n}|} \\ &= 100 + (6-4)a_{\overline{n}|} \text{ at } 4\% \\ &= 100 + 2a_{\overline{n}|} \end{aligned}$$

The possible values of n are from $n = 10$ to $n = 20$. The lowest price is the price when $n = 10$. This is the same result we obtained earlier: when a bond sells at a premium, assume the earliest date. (PEW principle).

In Example 2, the premium/discount formula gives:

$$P = 100 - 2a_{\overline{n}|} \text{ at } 8\%$$

Again, the possible values of n are from $n = 10$ to $n = 20$. The lowest price is obviously the price when $n = 20$. Thus, we reach the same conclusion as before: when a bond sells at a discount, use the latest date.

The results for the premium and discount situations can be viewed as corollaries to the general principle of "lowest price." Here is an example of a more complex question which can be answered using the general principle.

Example 4
A 5% semiannual coupon $100 bond maturing in 15 years is callable on any coupon date after the 10th. If called on the 11th through 20th coupon date, the redemption value would be $110. If called on the 21st through 30th coupon date, redemption would be at par. Find the price that would ensure an investor a minimum yield of 3% per annum compounded semiannually.

Solution
The approach is to first determine (1) the lowest price for redemption on the 11th to 20th coupon dates and (2) the lowest price for redemption on the 21st to 30th coupon dates. Then take the lower of prices (1) and (2). This is the price that would ensure the desired yield rate.

Note that the bond would sell at a premium for all possible redemption dates. (In the period from 11 to 20, $g = \frac{2.50}{110} = 2.27\%$. In the period from 21 to 30, $g = \frac{2.50}{100} = 2.50\%$. Since $i = .015$, $g > i$ in all cases.)

For redemption on the 11th to 20th coupon dates:

$$\begin{aligned} P &= C + (Fr - Ci)a_{\overline{n}|} \text{ at } 1.5\% \\ &= 110 + [(2.50 - (110)(.015)]\, a_{\overline{n}|} \\ &= 110 + .85a_{\overline{n}|} \end{aligned}$$

The lowest price is for $n = 11$:

$$P = 110 + .85a_{\overline{11}|} = \$118.56$$

For redemption on the 21st to 30th coupon dates:

$$\begin{aligned} P &= 100 + (2.50 - 1.50)a_{\overline{n}|} \\ &= 100 + a_{\overline{n}|} \end{aligned}$$

Again, the lowest price is for the smallest possible n, i.e., $n = 21$:

$$P = 100 + a_{\overline{21}|} = \$117.90$$

We take the lower of the two prices $118.56 and $117.90, or $117.90.

Notes:

1. This bond sells at a premium. In spite of this, the redemption date that gives the lowest price is not the earliest possible call date (11th coupon date in this case). This can happen when the redemption value is not the same on all possible redemption dates.
2. It is not unusual for a callable bond to have redemption values that decrease with time. The excess of the redemption value over the par value ($10 on the 11th to 20th coupon dates in this example) is called the ***call premium***.

Summary of Concepts and Formulas in Section 7f

1. Call provision (call "option") in bonds—***callable bonds***
   (a) Borrower (issuer) can redeem (call back, or buy back) bond prior to maturity
   (b) Bonds generally have a call date that reflects the earliest time in the life of the bond at which the issuer can call back the bond
      i. Prior to that, the lender/investor is protected from bond being called
   (c) The ***call price*** of a bond is the redemption price the issuer must pay to the lender if the bond is called
      i. May differ from the regular redemption price C
2. Analytical considerations
   (a) For issuer
      i. May find calling a bond to be an advantage if interest rates decline after issuing bond
   (b) For investor
      i. Term of bond is uncertain
      ii. Hard to determine yield relative to selling price
3. Guideline for investors trying to determine appropriate bond prices
   (a) For bond selling at a premium
      i. Assume redemption is at the earliest possible redemption date
   (b) For bond selling at a discount
      i. Assume redemption is at the latest possible redemption date
   (c) More general rule: To ensure that we earn a specified minimum yield rate, determine the lowest price for all of the possible redemption dates at this yield rate

Past Exam Questions on Section 7f

1. Matt purchases a 20-year par value bond with 8% semiannual coupons at a price of 1,722.25. The bond can be called at par value X on any coupon date starting at the end of year 15. The price guarantees that Matt will receive a nominal semiannual yield of at least 6%. Bert purchases a 20-year par value bond identical to the one purchased by Matt except it is not callable. Assuming a nominal semiannual yield of 6%, the cost of the bond purchased by Bert is P. Calculate P. [SOA 5/93 #17]

   (A) 1,700 (B) 1,725 (C) 1,750 (D) 1,775 (E) 1,800

2. An inverstor purchases a 1,000 bond redeemable at par that pays 8% semiannual coupons and matures in 10 years. The bond will yield 7% convertible semiannually to maturity. If the bond is called in five years, the minimum redemption value the investor needs to realize the same yield is X. Determine X. [SOA 11/92 #17]

   (A) 1,036 (B) 1,042 (C) 1,048 (D) 1,054 (E) 1,060

3. An investor bought a 15-year bond with par value of 100000 and 8% semiannual coupons. The bond is callable at par on any coupon date beginning with the 24th coupon. Find the highest price paid that will yield a rate not less than $i^{(2)} = 10\%$. [SOA 5/91 #15]

   (A) 85800 (B) 85400 (C) 85000 (D) 84600 (E) 84200

4. An investor bought a \$1000 par value bond with 6% *annual* coupons. The maturity date was exactly ten years after the purchase date and the redemption value was to be equal to the par value. The bond was purchased at a premium to yield 5% per annum. One year later, just after payment of the coupon, the bond was called in at 104% of par value. What was the investor's rate of return on his investment? [CAS 5/88 #14]

   (A) Less than 2.0%

   (B) At least 2.0%, but less than 2.2%

   (C) At least 2.2%, but less than 2.4%

   (D) At least 2.4%, but less than 2.6%

   (E) 2.6% or more

5. A 1000 par value, 8% bond with quarterly coupons is callable five years after issue. The bond matures for 1000 at the end of ten years and is sold to yield a nominal rate of 6% compounded quarterly under the assumption that the bond will not be called. Calculate the redemption value, at the end of five years, that will yield the purchaser the same nominal rate of 6% compounded quarterly. [SOA 5/87 #19]

   (A) 1060 (B) 1067 (C) 1073 (D) 1080 (E) 1086

6. A 1000-par value 23-year bond with 5% semiannual coupons is callable on any coupon date in the 13th to the 23rd year.

   (i) In the 13th to the 18th year, the bond is callable at a redemption value of 1260

   (ii) In the 19th to the 23rd year, the bond is callable at a redemption value of 1080

   Calculate the maximum purchase price for this bond that will guarantee an annual nominal yield rate of at least 4% convertible semiannually. [SOA SAMPLE #166-Modified]

   (A) 1152.37 (B) 1168.30 (C) 1170.00 (D) 1181.63 (E) 1254.90

Solutions to Past Exam Questions on Section 7f

1. The bond that Matt buys is sold at a premium, since $g > i(.04 > .03)$. Thus, the price that guarantees an effective rate of 3% per $\frac{1}{2}$-year period is based on assuming the earliest possible redemption date. This is 15 years, or 30 coupons, after purchase:

$$\begin{aligned} 1{,}722.25 &= .04Xa_{\overline{30}|.03} + Xv^{30} \\ &= (.7840 + .4120)X = 1.1960X \\ X &= 1{,}440 \end{aligned}$$

$$\begin{aligned} \text{Bert's bond: } P &= (.04)(1{,}440)a_{\overline{40}|.03} + 1{,}440v^{40} \\ &= 1{,}772.85 \qquad \text{ANS. (D)} \end{aligned}$$

2. The investor originally paid $40a_{\overline{20}|.035} + 1{,}000v^{20} = 1{,}071.06$. If X is the redemption value at the end of 5 years, it must satisfy the following at 3.5%: $1{,}071.06 = 40a_{\overline{10}|} + Xv^{10}$. This can be solved for X by entering 10 $\boxed{\text{N}}$ 3.5 $\boxed{\text{I/Y}}$ 1,071.06 $\boxed{+/-}$ $\boxed{\text{PV}}$ 40 $\boxed{\text{PMT}}$ $\boxed{\text{CPT}}$ $\boxed{\text{FV}}$.

   The answer is 1,041.58. ANS. (B)

3. Since $g < i$, the bond is purchased at a discount. The latest call date should be assumed:

$$\begin{aligned} P &= 4{,}000a_{\overline{30}|} + 100{,}000v^{30} \text{ at } 5\% \\ &= 84{,}627.55 \qquad \text{ANS. (D)} \end{aligned}$$

4. $P = 60a_{\overline{10}|.05} + 1{,}000v^{10} = 1{,}077.22$

   One year later, the investor receives a coupon of 60 plus redemption value of $104\% \times 1{,}000 = 1{,}040$, for a total of 1,100. Thus, $1{,}077.22(1 + i) = 1{,}100$ and $i = 2.1\%$. ANS. (B)

5. $P = 20a_{\overline{40}|1.5\%} + 1{,}000v^{40} = 1{,}149.58$

   Let X = unknown redemption value at end of 5 years. Then:

$$1{,}149.58 = 20a_{\overline{20}|} + Xv^{20}$$

   Enter 20 $\boxed{\text{N}}$ 1.5 $\boxed{\text{I/Y}}$ 1,149.58 $\boxed{\text{PV}}$ 20 $\boxed{+/-}$ $\boxed{\text{PMT}}$ $\boxed{\text{CPT}}$ $\boxed{\text{FV}}$. The answer is 1,085.84. ANS. (E)

6. In the 13th-18th year, $C = 1260$, $g = 25/1260 = .0198 < i$, the bond sells at a discount and the worst case for the buyer is a late call. Assume $n = 36$. The price $P = 25a_{\overline{36}|2\%} + 1{,}260v^{36} = 1254.90$

   In the 19th-23rd year, $C = 1080$, $g = 25/1080 = .0234 > i$, the bond sells at a premium and the worst case for the buyer is an early call. Assume $n = 37$. The price $P = 25a_{\overline{37}|2\%} + 1{,}080v^{37} = 1168.30$

   We take the lower of the two prices: 1168.30. ANS. (B)