This topic is to measure the impact of interest rate change on their portolio of fixed income asset value.

1. Determine the Macaulay duration for each of the following assets at an effective annual rate of interest of 5%.
(b) A 10 year zero coupon bond

Let A_t = cash inflow from an asset at time t.

$$D_{mac}(i) = \frac{\Sigma (v^t A_t) \cdot t}{\Sigma v^t A_t}.$$

Since 10 year zero coupon bond is a single cash flow,

$$D_{mac}(i) = \frac{X \cdot v^{10} \cdot 10}{X \cdot v^{10}} = 10.$$

2. Determine the Macaulay duration for each of the following assets at an effective annual rate of interest of 5%.
(a) A perpetuity-immediate with level annual payments

$$D_{mac}(i) = \frac{\Sigma v^t \cdot A_t \cdot t}{\Sigma v^t A_t} = \frac{A_t \, Ia_{\overline{\infty}|i\%}}{A_t \, a_{\overline{\infty}|i\%}} = \frac{Ia_{\overline{\infty}|i\%}}{a_{\overline{\infty}|i\%}} = \frac{\frac{1}{i^2} + \frac{1}{i}}{\frac{1}{i}}$$

The numerator: Increasing perpuity formula.

The denominator: The normal perpuity formula.

3. Determine the Macaulay duration for each of the following assets at an effective annual rate of interest of 5%.
c) A 10 year bond with 10% annual coupons maturing at par

Assume $F = C = 100$. $Fr = 10$.

$$D_{mac}(i) = \frac{10\, Ia_{\overline{10}|5\%} + 10(100)v^{10}}{10\, a_{\overline{10}|5\%} + 100\, v^{10}} = 7.27.$$

$$Ia_{\overline{10}|5\%} = \frac{\ddot{a}_{\overline{10}|} - 10v^{10}}{i}$$

4. Determine the Modified duration for each of the following assets at an effective annual rate of interest of 5.0625%.
(d) A 10 year bond with 5% semiannual coupons maturing at par

$D_{mod}(j) = v \cdot D_{mac}(j)$. j represents the effective rate of half year.

$(1+j)^2 = 1.050625 \Rightarrow j = 2.5\%$.

Assume that $F = C = 100$, $r = 2.5\%$.

Since the bond is selling at par, the yield rate j is equal to the coupon rate.

the Macaulay duration is $\ddot{a}_{\overline{20}|2.5\%}$

$$D_{mac}(j) = \frac{\sum t\,v^t \cdot A_t}{\sum v^t A_t} = \frac{\sum t \cdot \frac{\text{Coupon}}{(1+j)^t} + 20 \cdot \frac{\text{Face Value}}{(1+j)^t}}{\text{Price of bond}} = 8 \text{ years.}$$

At the same time, it is special case since $r = j$, $P = C = F$.

$$D_{mac}(j) = \frac{Fr \cdot (Ia)_{\overline{n}|j} + n \cdot C \cdot v^n}{P} = \frac{\left(\frac{\ddot{a}_{\overline{n}|j} - nv^n}{j}\right) \cdot Fr + nC \cdot v^n}{P} = \ddot{a}_{\overline{n}|j}.$$

5. There are 3 bonds in a portfolio of assets. The Macaulay duration of the entire portfolio is 10 years. The duration of the first bond is 8 years and the duration of the second bond is 6 years. The price of the first bond is twice the price of the second bond and half the price of the third bond. What is the duration of the third bond?

· This question tells you duration of a portfolio.

Suppose a company has a portfolio of assets, such as a group of bonds with different remaining terms, different coupon rates and different maturity values. How can we determine the duration of the entire portfolio?

One way would be to determine the total cash inflow of the entire portfolio at each time t and then apply the D_{mac} or D_{mod} formula. This could be a little cumbersome because the total cash flows would not form a convenient series. (The bonds would mature at different times, coupons would drop off as the bonds matured, etc.)

Another way would be to determine the duration of each bond in the portfolio and then compute the duration of the entire portfolio by taking a weighted average of the individual durations. Think about this for a minute. How would you weight the individual durations of the bonds?

The answer is that the weight would be the present value of the bond's payments, i.e., the price of the bond at the interest rate for which you are calculating duration. The price is the denominator in the formula for duration, so weighting the duration of each bond by its price and then dividing the sum of the weighted durations by the sum of the weights (i.e., by the price of the entire portfolio) will result in the duration of the entire portfolio. (If you need convincing, just write the formula for D_{mac} for Bond 1, Bond 2, etc., and do a couple of lines of simple algebra to show that this is true.)

Let the prices of bonds 1, 2, and 3 be 2, 1 and 4, respectively.

The duration of the entire portfolio (10) is the weighted average of the duration of each bond in the portfolio, with the weights equal to the price of each bond.

Let the unkown duration of the third bond be equal to $(D_{mac})_3$. So:

$$D_{mac} = \frac{P_1 (D_{mac})_1 + P_2 (D_{mac})_2 + P_3 (D_{mac})_3}{P_1 + P_2 + P_3}$$

$$= \frac{(2)(8) + (1)(6) + (4)(D_{mac})_3}{2+1+4} = 10.$$

$\Rightarrow (D_{mac})_3 = 12$. #

· The Condition of Redington immunization.

A sequence of cash flows is said to be in **Redington immunization** if the following three conditions hold:

1. The PV of the assets equals the PV of the liabilities. That is, $P_A(i) = P_L(i)$.
2. The duration of the assets equals the duration of the liabilities. Equivalently, $P_A'(i) = P_L'(i)$.
3. The convexity of the assets is greater than the convexity of the liabilities. Equivalently, $P_A''(i) > P_L''(i)$.

Redington immunization protects the investor from small changes in the interest rate.

The first criteria ensures that the current NPV is zero. The second criteria guarantees that the NPV has a critical point at the current value of i. The third criteria results in that critical point being a local minimum for the NPV.

If a set of cash flows satisfies the first two criteria of Redington immunization, it is said to be **duration matched**.

Using P to stand for the net present value. $P = P_A - P_L = \Sigma(A_t - L_t)\cdot v^t$.

The three conditions become

(1). $P = 0$. (The NPV of assets and Liabilities is 0).

(2). $P' = 0$. (The first derivative of the NPV is 0).

(3). $P'' = 0$ (The second derivative of the NPV is greater than 0).

6. A company must make payments of \$10 annually in the form of a 10-year annuity-immediate. It plans to buy two zero coupon bonds to fund these payments. The first bond matures in 2 years and the second bond matures in 9 years, and both are purchased to yield 10% effective. What face amount of each bond should the company buy in order to be immunized from small changes in the interest rate (Redington immunization)?

Let X = face amount of the 2-year bond and Y = face amount of 9-year bond.

Apply the first two conditions for Redington immunization to solve for X and Y.

(1). $P_A = P_L$ at 10%. $v^2X + v^9Y = 10a_{\overline{10}|}$ at 10%.

that is, $0.8264X + 0.4241Y = 61.4457$.

(2). $P_A' = P_L'$.

Note: $\frac{d}{di}v^n = \frac{d}{di}(1+i)^{-n} = -n(1+i)^{-n-1} = -nv^{n+1}$.

$$-2v^3X - 9v^{10}Y = \frac{d}{di}(10(v+v^2+\cdots+v^{10}))$$

$$= -10(v^2+2v^3+\cdots+10v^{11})$$

$$= -10v(v+2v^2+\cdots+10v^{10})$$

$= -10\nu(Ia)_{\overline{10|}}$

That is, $1.5026X + 3.4699Y = 263.9628$.

So here we have two linear equations in X and Y.

Solve it, we get $X = 45.40$. $Y = 56.41$.

7. Exact matching:

An insurance company must pay liabilities of \$50,000 due one year from now and \$100,000 due two years from now.

There are two available investments:

• a 1-year bond with face amount of \$1,000 that pays a 5 % annual coupon rate and has a 7% annual yield rate

• a 2-year bond with face amount of \$1,000 that pays a 6 % annual coupon rate and has a 4 % annual yield rate

Determine the total cost of purchasing the above two bonds in amounts required to match the liability payments exactly.

- [ ] 134,073
- [ ] 139,337
- [ ] 131,016
- [ ] 135,421

	0	1	2
Liability:		50 000	100 000
Asset:		5660.37735 42228.21204 + 2111.410602	94339.62264 + 5660.377358
1-year bond		$\boxed{1000 + 50}$	
2-year bond		$\boxed{60}$	$\boxed{1060}$

Work backward, starting with 2-year bond, then 1-year bond.

At time 2, We have cash inflow 1060 but liability is 100,000.

So matching 100000 cash flow out and 1060 cash inflow.

The company should by a 2-year bond with par value of: $\frac{100000}{1060}\cdot(1000)$

$= 94339.62264$.

This bond has coupons of $6\% \cdot (94339.62264) = 5660.377358$ at time 1.

At time 1, we need an additional cash inflow of $(50000 - 5660.377358 = 44339.62264)$.

So the company should by a 1-year bond with par value of: $\frac{44339.62264}{1050}(1000)$

$= 42228.21204.$

This bond has coupons of $5\% \cdot (42228.21204) = 2111.410602.$

To sum up, For the 1-year bond,

the cost of it $= (42228.21204 + 2111.410602) \cdot (1.07)^{-1} = 41438.89967.$

The cost of the 2-year bond

$= 5660.377358 \cdot (1.04)^{-1} + (5660.377358 + 94339.62264)(1.04)^{-2}$

$= 97898.29553.$

Total Cost of them $= 97898.29553 + 41438.89967 = 139337.1952$

8. You are given the following facts about an asset:

(i) The present value of the future cash flows at an _effective annual interest rate of 5% is 150,000.

(ii) Dmac(5%) = 14. 7

(iii) The present value of the future cash flows at an effective annual interest rate of X% using the first-order modified approximation is 152,100.

Determine X

- [ ] 4.5%
- [ ] 4.6%
- [ ] 4.7%
- [ ] 4.9%
- [ ] 4.8%

First-Order Modified Approximation:

$$\boxed{P(i) - P(i_0) \approx -P(i_0)(i - i_0)\, D_{mod}(i_0) = (i - i_0) \cdot P'(i_0).}$$

So determine $D_{mod}(5\%) = v \cdot D_{mac}(5\%) = \frac{14.7}{1.05} = 14.$

Substituting: $152100 - 150000 = -150000(i - 5\%) \cdot (14)$

$i = 4.9\%.$

$\rightarrow P(i) \approx P(i_0)[1 - (i - i_0) D_{mod}(i_0)].$ Remeber it!

9. You are given the following facts about a liability:

(i) The present value of the future cash flows at an effective annual interest rate of 6% is 1,200,000.

(ii) Dmod(6%) = 8.25

(iii) The present value of the future cash flows at an effective annual interest rate of X% using the first-order Macaulay approximation is 1,180,381.

Determine X

- [ ] 6.1
- [ ] 6.5
- [ ] 6.4
- [ ] 6.3
- [ ] 6.2

First - order Macaulay approximation. $P(i) \approx P(i_0)\left(\frac{1+i_0}{1+i}\right)^{D_{mac}(i_0)}$.

$D_{mac}(6\%) = (1+i_0)\cdot D_{mod}(6\%) = (1.06)(8.25) = 8.745$.

Substituting: $1180381 = 1200000\left(\frac{1.06}{1+i}\right)^{8.745}$.

$i = 6.2\%$.