Introduction to Bayesian statistics

Part 1 — Concepts

Jorge N. Tendeiro

Department of Psychometrics and Statistics
Faculty of Behavioral and Social Sciences
University of Groningen

j.n.tendeiro@rug.nl
www.jorgetendeiro.com
https://github.com/jorgetendeiro/GSMS-2020

Bayes rule

- $\mathcal{D}$ = data
- θ = unknown parameter

$$\boxed{p(\theta|\mathcal{D}) = \frac{p(\theta)p(\mathcal{D}|\theta)}{p(\mathcal{D})}}$$

In words,

$$\boxed{\text{posterior} = \frac{\text{prior} \times \text{likelihood}}{\text{evidence}}}$$

The *evidence* does not depend on θ; let's hide it:

$$\boxed{\text{posterior} \propto \text{prior} \times \text{likelihood}}$$

The symbol $\propto$ means "proportional to".

Bayes rule

$$\boxed{\text{posterior} \propto \text{prior} \times \text{likelihood}}$$

- *Prior*: Belief about the 'true' value of θ, *before looking at the data*.
- *Likelihood*: The statistical model, linking θ to data.
- *Posterior*: Updated knowledge about θ, in light of the observed data.

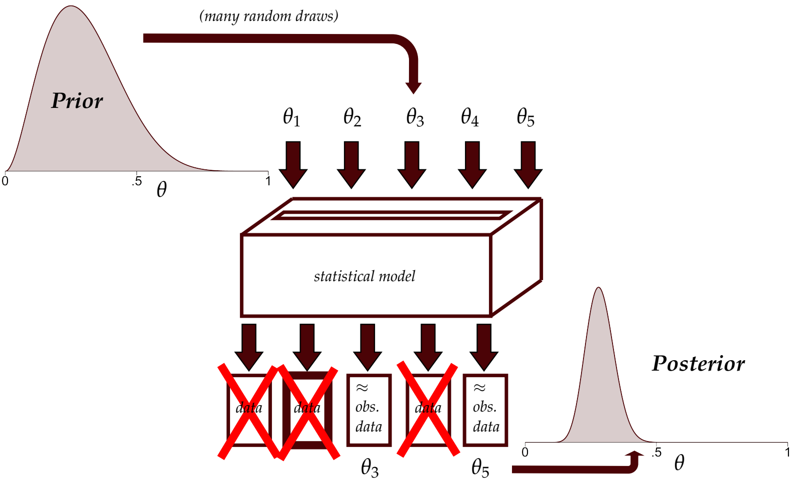
Let's look at the Bayes rule from various angles.

Bayes rule – ABC

One useful way to think about the Bayes rule is by considering *Approximate Bayesian Computation* (ABC; see Wiki).

- ABC is actually computationally *very* inefficient.
- But, it is *conceptually* very clear!

Bayes rule – ABC



Bayes rule – ABC

The Bayes rule from the ABC perspective:

> *Find the values of θ that allow the model to predict data pretty much like our observed data.*

Humm…

Maximum likelihood estimation, anyone?

Bayesian inference can be thought of as an extension of MLE!

Bayes rule – Inverse probability

Bayes rule allows reversing conditional probabilities.

$$\boxed{p(\mathcal{A}|\mathcal{B}) = \frac{p(\mathcal{A})p(\mathcal{B}|\mathcal{A})}{p(\mathcal{B})}}$$

Consider the canonical example:

- $\mathcal{A}$: Have disease.
- $\mathcal{B}$: Test positive.

Then:

- $p(\mathcal{B}|\mathcal{A})$: Probability of testing positive given that one has the disease.
  Test's sensitivity.
- $p(\mathcal{A}|\mathcal{B})$: Probability of having the disease given that (s)he tests positive.
  What patients really want to know.

Bayes rule – Updating beliefs

Definition of probability:

- *Frequentist:* Long-run relative frequency.
  (Problem: $p(\text{Trump winning 2020 election})?\ldots)$
- *Bayesian:* Degree of subjective belief.

$$\boxed{\text{posterior} \propto \text{prior} \times \text{likelihood}}$$

- Bayes rule is a rational means of updating our current belief (*prior*) by means of observed data (*likelihood*).
- *"Today's posterior is tomorrow's prior"* – Lindley (1970)

Bayes rule – Summary

$$\boxed{\text{posterior} \propto \text{prior} \times \text{likelihood}}$$

Bayesian modelling requires three ingredients:

- Data.
- Priors, reflecting our subjective belief about the parameters.
- A statistical model, relating parameters to data.

Bayes rule is a mathematically rigorous means to combine prior information on *parameters* with the *data*, using the *statistical model* as the bridge between both.

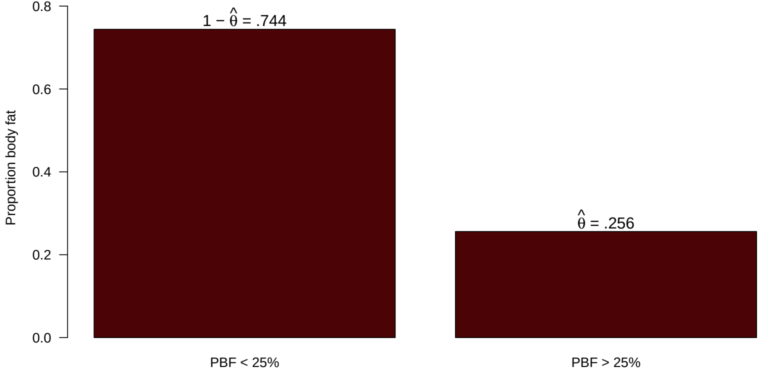
Bayes rule – Example

Data here:
https://dasl.datadescription.com/datafile/bodyfat/.

- Various measurements of 250 men.
- To keep it simple: I dichotomize the percentage of body fat (PBF).
- 0 = PBF lower than 25%;
  1 = PBF larger than 25%.
- *Goal*: Infer the proportion of obese men in the population.

Let's denote the population proportion by θ.

Bayes rule – Example



Bayes rule – Example

Let's use the Bayesian machinery.

Recall that we need three ingredients:

- Data.
- Prior.
- Model.

Bayes rule – Example

Data. For now, let's only use the first 10 scores.

- Sample size: 10
- Number of men with PBF $> 25\%$: 2
- Sample proportion: $\widehat{\theta} = \frac{2}{10} = .20$

0	0	1	0	1	0	0	0	0	0

Bayes rule – Example

Model. We'll use the binomial model. Assumptions:

- Independence between measurements.
- One population with underlying rate θ.
- Random sample.

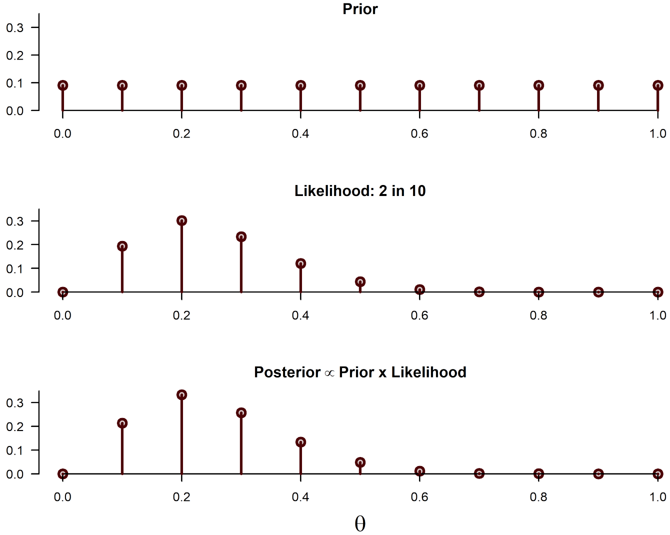
Bayes rule – Example

Prior. We'll try several.

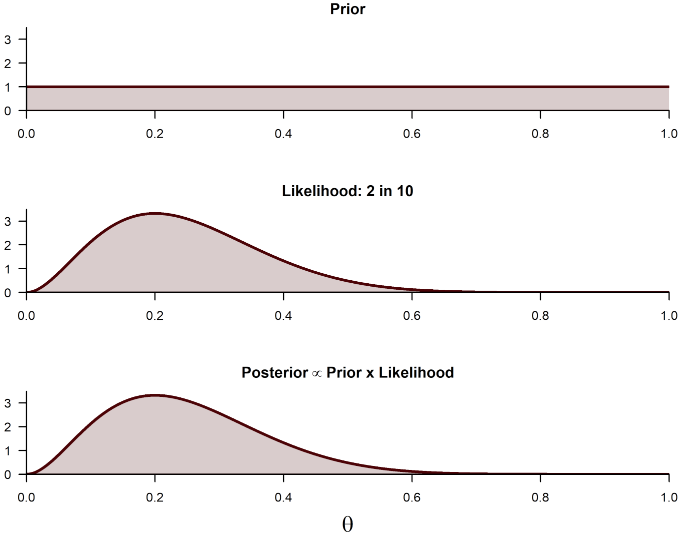
Bayes rule – Example

What happens if the prior is ‘uninformative’?

Bayes rule – Example



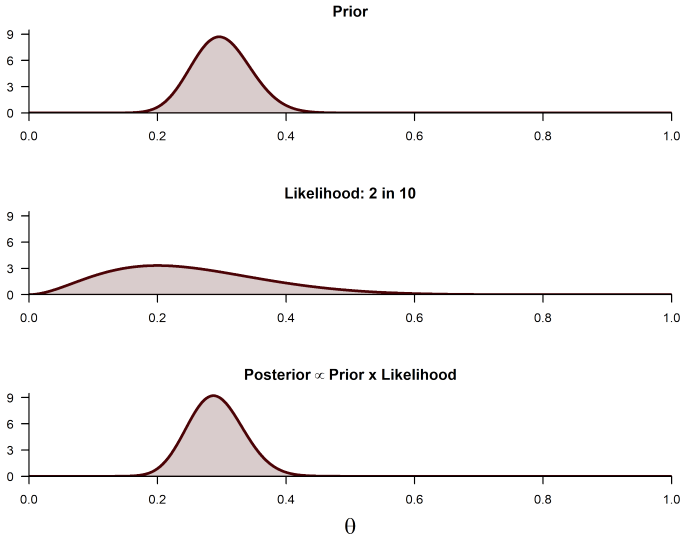
Bayes rule – Example



Bayes rule – Example

What happens if the prior is ‘very informative’?

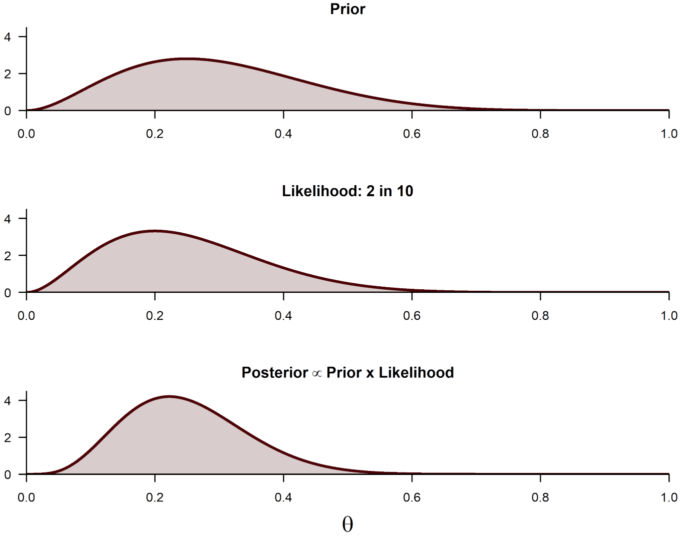
Bayes rule – Example



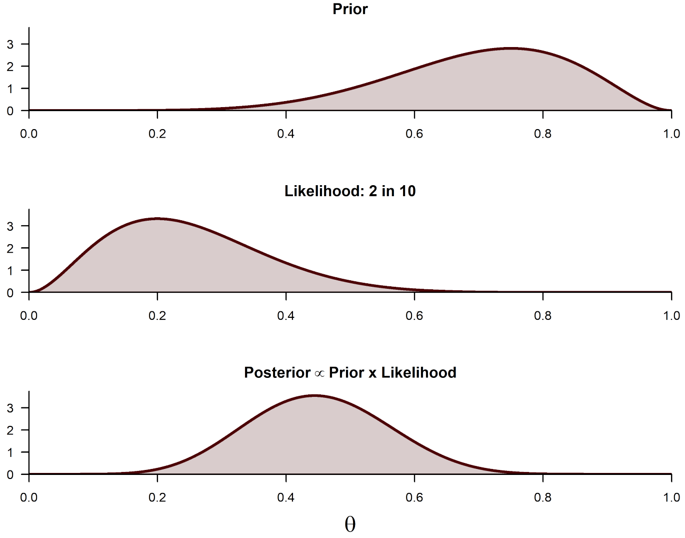
Bayes rule – Example

What happens if neither the prior nor the likelihood dominates?

Bayes rule – Example



Bayes rule – Example



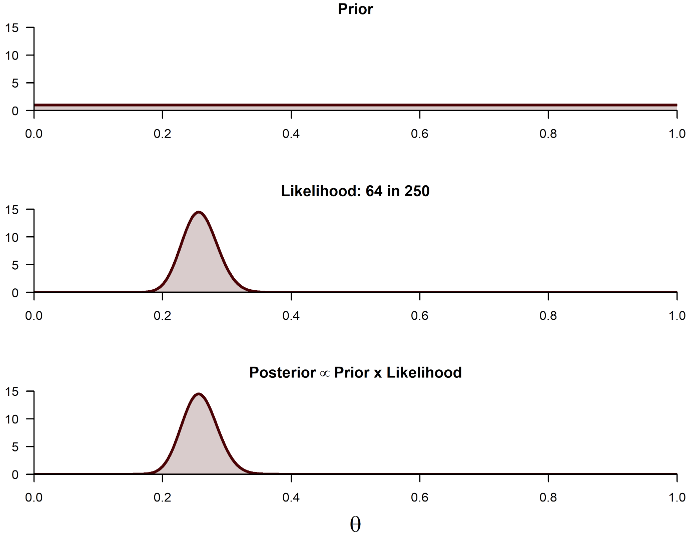
Bayes rule – Example

Let's now use all the data.

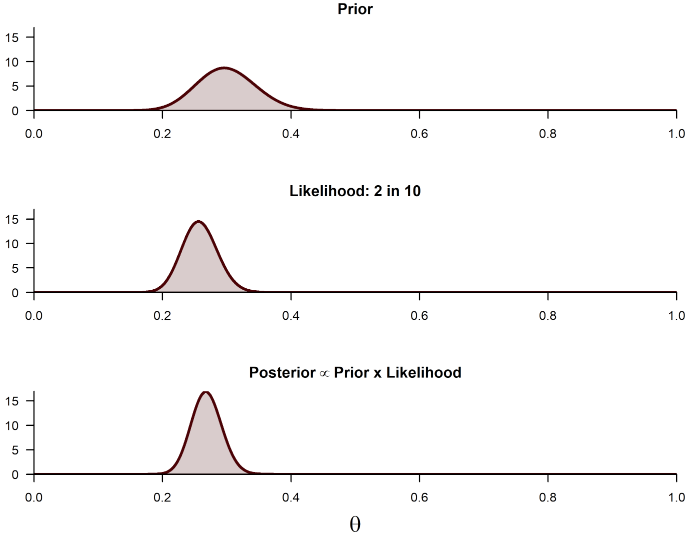
- Sample size: 250
- Number of men with PBF $> 25\%$: 64
- Sample proportion: $\widehat{\theta} = \frac{64}{250} = .256$

How does Bayes updating look like now, when the data dominate?

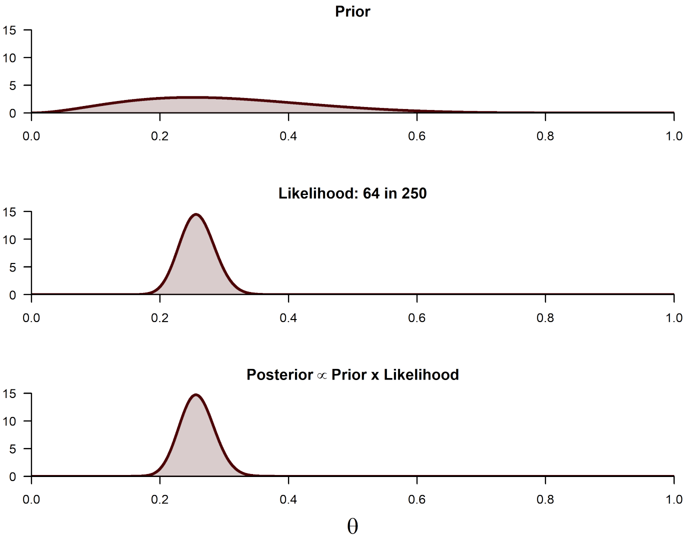
Bayes rule – Example



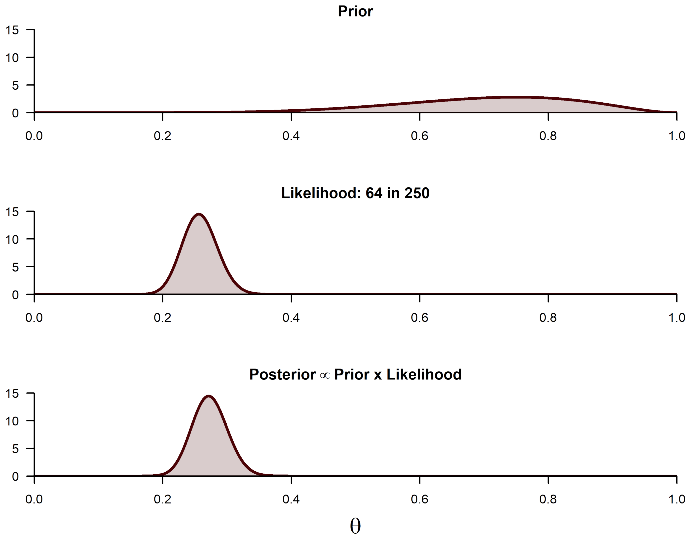
Bayes rule – Example



Bayes rule – Example



Bayes rule – Example



Bayes rule – Some conclusions

- Bayes rule highlights the parameter values that make the observed data look more plausible.
- The posterior distribution is a compromise between the information in the prior and the information in the data.

Bayes rule – Some conclusions

How do priors typically affect posterior distributions?

- For 'uninformative' priors, posterior $\approx$ likelihood.
- For 'very informative' priors, posterior $\approx$ prior.

Bayes rule – Some conclusions

How do data typically affect posterior distributions?

- For small sample sizes, posterior $\approx$ prior.
- For large sample sizes, posterior $\approx$ likelihood.

Bayesian inference – Some criticism

I think I can hear some of you thinking right now…

> *"Hey, but there are sooo many posterior distributions!"*
> *"This seems all sooo subjective!"*
>
> *"I sooo don't like it!"*
> *:-(*

Fair points.

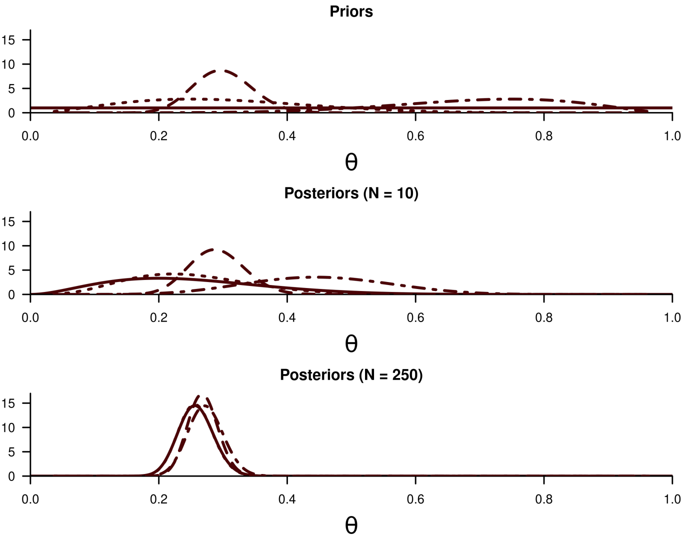
Let me offer several counter-arguments.

Bayesian inference – Counterarguments to criticism

1. Posterior distributions are fairly stable across a wide range of reasonable priors, *for large data sets*.

More data $\Rightarrow$ more information $\Rightarrow$ more certainty.

Bayesian inference – Counterarguments to criticism



Bayesian inference – Counterarguments to criticism

2. *Illusion of certainty:* Pretending that results tell us more than is actually possible.

Bayesian inference – Counterarguments to criticism

There is subjectivity in each step of the scientific way:

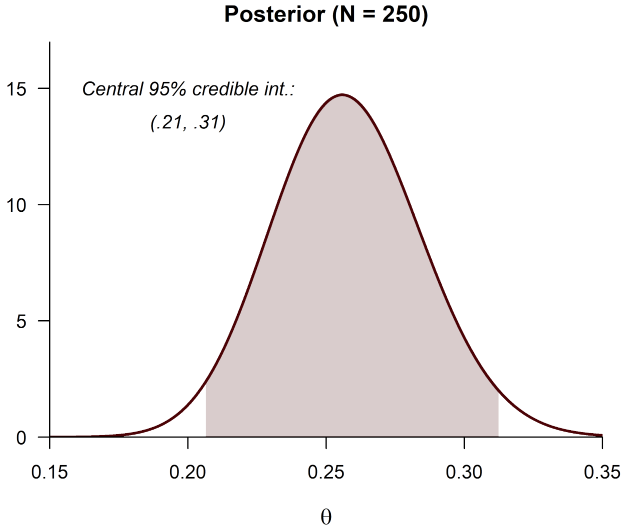
- Selection of participants.
- Number of assessments.
- Variables to measure.
- Variables to control.
- Variability across researchers / labs.
- Statistical model to use.
- Variables to (not) include in the model.
- …

Then, try topping it up with few and noisy data…

It is *fair*, *logical*, *necessary*, that statistical inference reflects uncertainty.

Do embrace uncertainty!

Bayesian inference – Counterarguments to criticism



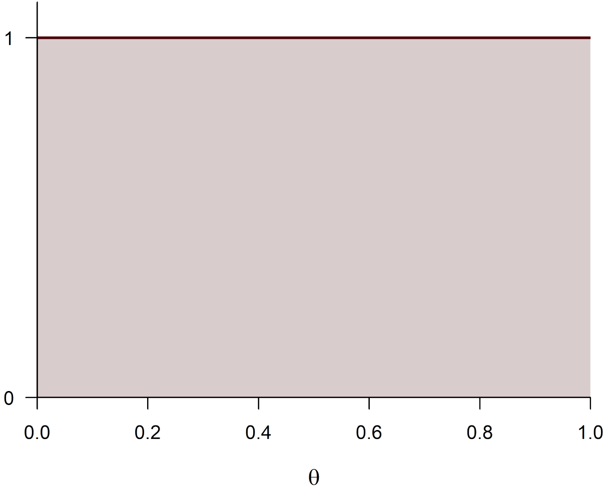
Bayesian inference – Counterarguments to criticism

3. Priors allow incorporating useful information.

- What is known about the parameter?

Let’s not pretend we do not know anything.

Bayesian inference – Counterarguments to criticism



Bayesian inference – Counterarguments to criticism

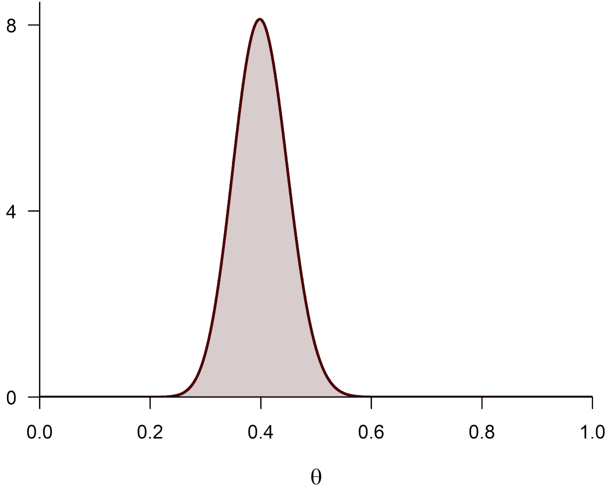
About PBF:

- It is actually known that the proportion of obese men in (some...) population is about 40%.
- We can (we *should*!) take this into account.
- And *that* is what the prior is for.

(Do you know how much variability to expect?
Then include this in the prior too!!)

One of Bayes' advantages: Accummulation of evidence.
Use it!

Bayesian inference – Counterarguments to criticism



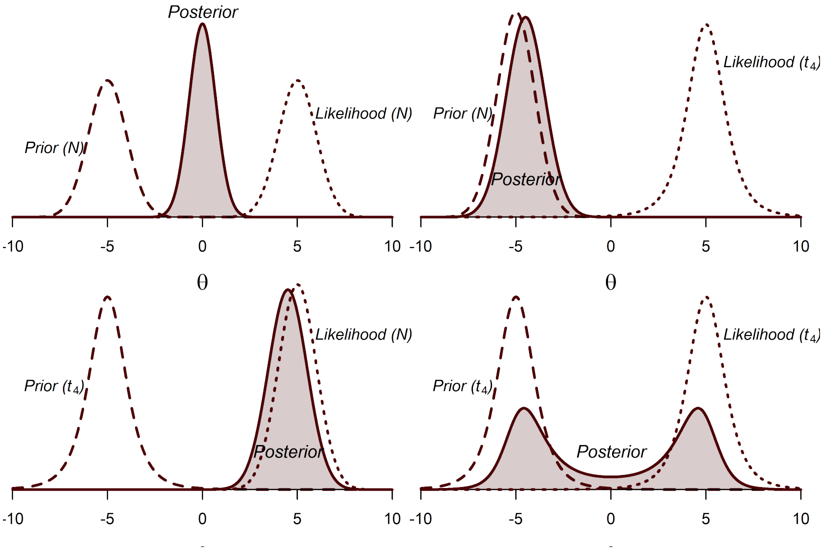
Bayesian inference – Counterarguments to criticism

Of course, strange things can occur.

> *"A Bayesian is one who, vaguely expecting a horse, and catching a glimpse of a donkey, strongly believes he has seen a mule." (Stephen Senn)*

(*Inspired by Aki Vehtari and John Kruschke.*)

Bayesian inference – Counterarguments to criticism



Bayesian inference – Summarize results

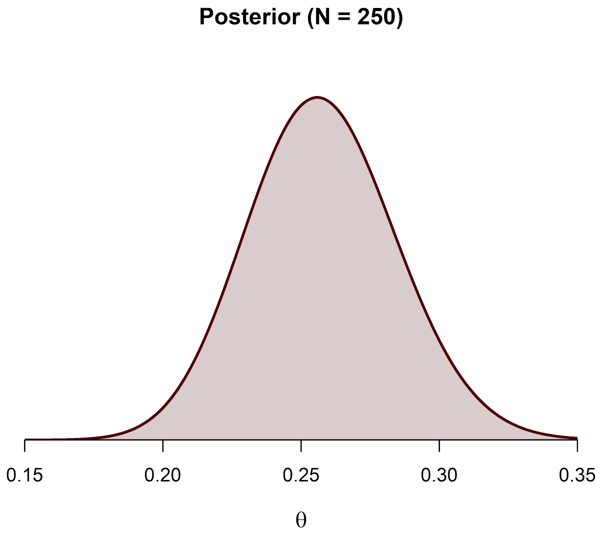
The posterior distribution is the *Holy Grail* in Bayesian statistics.

It reflects our current knowledge of the world, conditional on:

- The chosen model.
- The chosen prior(s).
- The observed data.

How can we summarize the information in the posterior distribution?

Bayesian inference – Summarize results



Bayesian inference – Summarize results

Point estimates.
Commonly used:

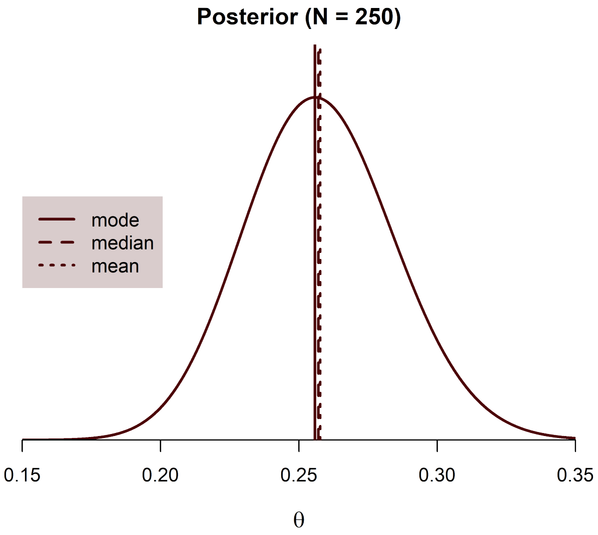
- posterior mean
- posterior mode
- posterior median.

For the PBF data based on 250 scores:

$$\text{post. mean} \approx \text{post. mode} \approx \text{post. median} \approx .26.$$

(Recall: $\widehat{\theta} = .256$.)

Bayesian inference – Summarize results



Bayesian inference – Summarize results

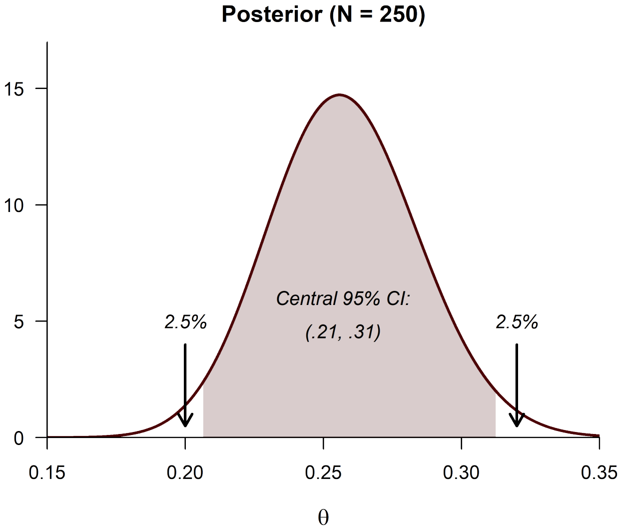
Interval estimates.
I will focus on the 95% credible interval throughout.

There are some variants (do not overly worry about these nuances):

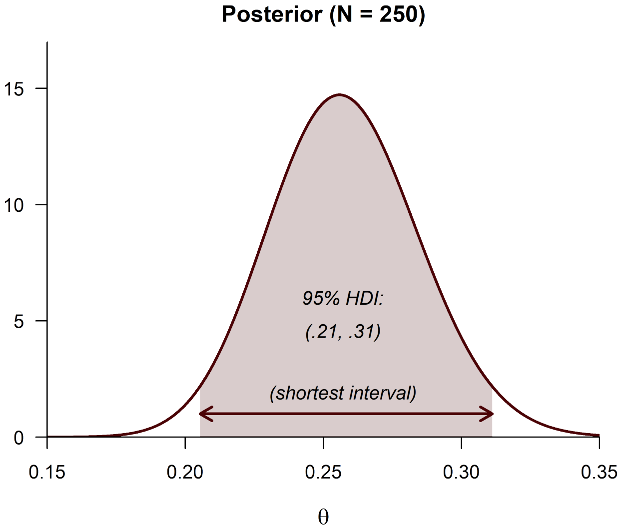
- *Central 95% credible interval.*
  With 2.5% probability out on each tail.
- *95% HDI (highest density interval).*
  The shortest interval covering area .95.

For the PBF data they practically coincide.

Bayesian inference – Summarize results



Bayesian inference – Summarize results



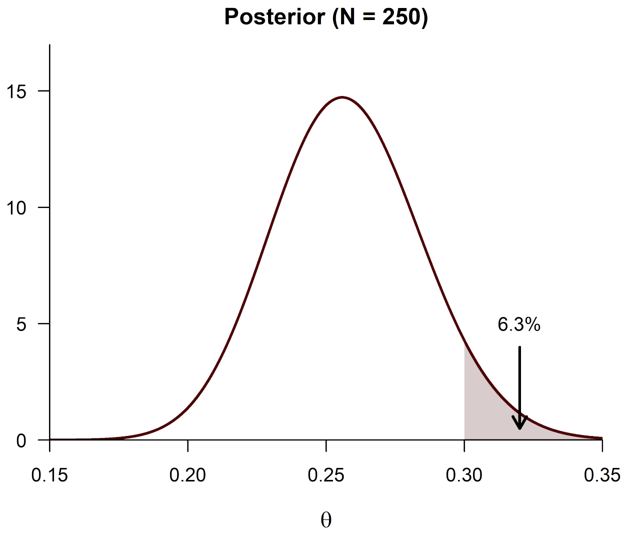
Bayesian inference – Summarize results

The posterior distribution allows computing probabilities for any events involving parameters.

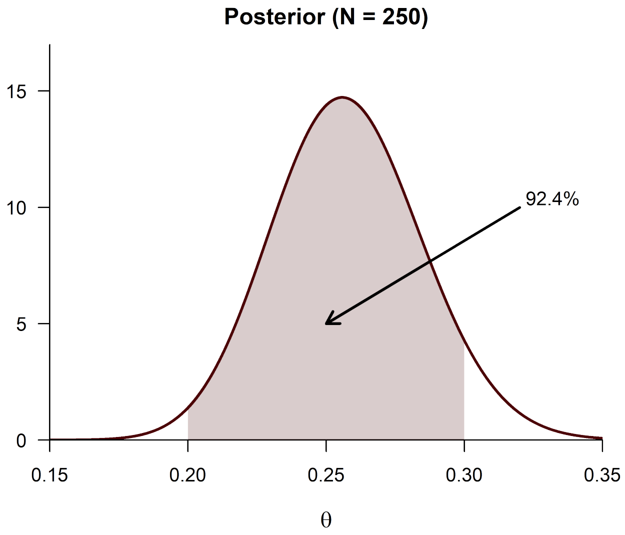
For instance:

- What is the (posterior) probability that the population proportion of obese men is larger than 30%?
- What is the (posterior) probability that the population proportion of obese men is between 20% and 30%?

Bayesian inference – Summarize results



Bayesian inference – Summarize results



Next

Specific examples will be dealt with in Part 2.

More concepts will be introduced as we proceed.

Now where's that cup of coffee?